



ON FREE NETS OVER MINKOWSKI SPACE

HELLMUT BAUMGÄRTEL*

Mathematisches Institut der Universität Potsdam, Am Neuen Palais 10, Postfach 601 553,
14415 Potsdam, Germany

MATTHIAS JURKE†

Technische Universität Berlin, Fachbereich Mathematik, MA 8–5, Straße des 17. Juni 136,
10623 Berlin, Germany (e-mail: jurke@sfb288.math.tu-berlin.de)

and

FERNANDO LLEDÓ‡

Technische Universität Berlin, Institut für Theoretische Physik, PN 7–1, Hardenbergstraße 36,
10623 Berlin, Germany (e-mail: flm0433@w421zrz.physik.tu-berlin.de)

Using standard results on CAR- and CCR-theory and on representation theory of the Poincaré group a direct way to construct nets of local C^* -algebras satisfying Haag–Kastler’s axioms is given. No explicit use of any field operator or of any concrete representation of the algebra is made. The nets are associated to models of mass $m \geq 0$ and arbitrary spin or helicity. Finally, Fock states satisfying the spectrality condition are specified.

1. Introduction

The aim of the present paper is to give an abstract construction of a causal and covariant net of local C^* -algebras [1, 2] associated to free massive and massless models with arbitrary spin and helicity, respectively. The net will be constructed directly, i.e. no field operator or any concrete representation of the algebra will be used explicitly. (Compare with the approach in [3].) The quantization procedure does not use any underlying classical theory either. For this reasons we will call the result of this construction a free net. Its relation to the standard net of von Neumann algebras obtained from the free fields is still open.

*Partly supported by DFG; SFB 288: “Differentialgeometrie und Quantenphysik”.

†Supported by DFG; SFB 288: “Differentialgeometrie und Quantenphysik”.

‡Supported by grant of the Spanish Ministry of Education/CICYT. Present address: Mathematisches Institut der Universität Potsdam, Am Neuen Palais 10, Postfach 601 553, 14415 Potsdam, Germany.

The construction we propose is based on the well-known facts from the CAR- and CCR-theory [4, 5, 6, 7, 8] and on standard results of the classification of the unitary irreducible representation (UIR) of the Poincaré group [9, 10]. Indeed, one can interpret certain combinations of the Hilbert spaces associated to the parameters $\{m \geq 0, s, \pm\}$ as the reference spaces of CAR- and CCR-algebras. Then, the local algebras that satisfy Haag–Kastler’s axioms are specified as C^* -subalgebras of the CAR- and CCR-algebras constructed before. In this construction all spins (or helicities) are treated at the same formal level, so that no concrete spin or helicity degree is emphasized. Within the same strategy alternative construction procedures for the net of local algebras were defined and compared in [11], [12] and [13].

The paper is organized in 9 sections: In Section 2 the construction of local algebras associated to fermionic and bosonic models, that satisfy Haag–Kastler’s axioms, is reduced to the construction of certain nets of linear submanifolds of the respective reference spaces.

Part A corresponds to the study of massive models. In Section A.1 the first definitions that characterize the concrete models are given and some technical lemmas are proved. In Section A.2 the CAR- and CCR-algebras for the models are constructed, and in Section A.3 the two main theorems of existence of the nets of linear submanifolds for the Fermi and Bose models are stated and proved. In the next section Fock states for these models are defined for which the spectrality condition is satisfied.

Part B analyses the massless models: In Section B.1 we describe some peculiarities of the representation theory of massless models. We also present some results concerning the theory of induced representations. In Section B.2 the factor spaces \mathfrak{H}'_{\pm} are constructed and some of its properties are proved. These spaces are associated to massless models with complementary helicities. In the next section the causal and covariant nets associated to fermionic and bosonic models are constructed. In the concluding section Fock states satisfying the spectrality condition are defined for these cases.

2. Nets of local linear submanifolds of the reference spaces

It is a well-known result that once a complex Hilbert space h_F and an anti-unitary involution Γ on h_F are fixed, then an algebraically unique C^* -algebra can be constructed. It is denoted by $\text{CAR}(h_F, \Gamma)$ [4]. Analogously, $\text{CCR}(h_B, \sigma)$ denotes the algebraically unique C^* -algebra over a real-linear space h_B , where a non-degenerated symplectic form σ is defined [8]. It is therefore natural to construct a net of local algebras using a certain net of linear submanifolds of the respective spaces. The precise properties that the net of linear submanifolds must satisfy in order to generate a correct net of local algebras are stated in this section.

We denote by $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^4$ open and bounded subsets of the Minkowski space and by $\widetilde{\mathfrak{B}}_{\pm}$ the universal covering group of the proper orthochronous component of the Poincaré group. The construction of the local algebras will be based on the following facts (see e.g. [7, p. 140 ff.]):

- Let Γ be an anti-unitary involution on a complex Hilbert space h_F and let

$$\mathcal{O} \longmapsto M_F(\mathcal{O}) \tag{1}$$

be a net of complex-linear submanifolds of h_F . V_F is a strongly continuous unitary representation of \mathfrak{P}_+^\uparrow on h_F with $\Gamma V_F(g) = V_F(g)\Gamma$. Assume that for any \mathcal{O} , \mathcal{O}_1 and \mathcal{O}_2 the following properties are satisfied:

- (0)_F $M_F(\mathcal{O})$ is Γ -invariant (i.e. $\Gamma M_F(\mathcal{O}) \subset M_F(\mathcal{O})$),
- (I)_F $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $M_F(\mathcal{O}_1) \subseteq M_F(\mathcal{O}_2)$,
- (II)_F $M_F(g\mathcal{O}) = V_F(g)M_F(\mathcal{O})$, $g \in \mathfrak{P}_+^\uparrow$,
- (III)_F $\mathcal{O}_1 \perp \mathcal{O}_2$ implies $M_F(\mathcal{O}_1) \perp M_F(\mathcal{O}_2)$,

where $\mathcal{O}_1 \perp \mathcal{O}_2$ means that \mathcal{O}_1 and \mathcal{O}_2 are spacelike separated, and $M_F(\mathcal{O}_1) \perp M_F(\mathcal{O}_2)$ means that $M_F(\mathcal{O}_1)$ and $M_F(\mathcal{O}_2)$ are orthogonal in h_F . Let $\mathbb{Z}_2 = \{\iota, \theta\}$, where ι is the identical automorphism and $\theta := \alpha_\vartheta$ is the Bogoliubov automorphism of $\text{CAR}(h_F, \Gamma)$ which corresponds to $\vartheta\varphi := -\varphi$. As usual $A(\varphi)$, $\varphi \in h_F$, denote the ‘‘canonical generators’’ of $\text{CAR}(h_F, \Gamma)$. $\mathcal{B}^{\mathbb{Z}_2}$ means a fixed point subalgebra of \mathcal{B} w.r.t. the automorphism group \mathbb{Z}_2 . Then the algebras

$$\mathcal{A}_F(\mathcal{O}) := C^*(\{A(\varphi) : \varphi \in M_F(\mathcal{O})\})^{\mathbb{Z}_2} \tag{2}$$

form an isotone, covariant and causal net of C^* -algebras, i.e. for any \mathcal{O} , \mathcal{O}_1 and \mathcal{O}_2 one has:

- (i)_F $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$,
- (ii)_F $\mathcal{A}(g\mathcal{O}) = \alpha_g \mathcal{A}(\mathcal{O})$, $g \in \mathfrak{P}_+^\uparrow$,
- (iii)_F $\mathcal{O}_1 \perp \mathcal{O}_2$ implies that $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute elementwise,

where α_g denotes the Bogoliubov automorphism that corresponds to $V_F(g)$.

- Let $\sigma(\cdot, \cdot)$ be a non-degenerated symplectic form on a real-linear space h_B . Let

$$\mathcal{O} \longmapsto M_B(\mathcal{O}) \tag{3}$$

be a net of real-linear submanifolds of h_B and let V_B be a symplectic representation of \mathfrak{P}_+^\uparrow . Assume that for any \mathcal{O} , \mathcal{O}_1 and \mathcal{O}_2 the following conditions are satisfied:

- (I)_B $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $M_B(\mathcal{O}_1) \subseteq M_B(\mathcal{O}_2)$,
- (II)_B $M_B(g\mathcal{O}) = V_B(g)M_B(\mathcal{O})$, $g \in \mathfrak{P}_+^\uparrow$,
- (III)_B $\mathcal{O}_1 \perp \mathcal{O}_2$ implies $M_B(\mathcal{O}_1) \perp M_B(\mathcal{O}_2)$,

where $M_B(\mathcal{O}_1) \perp M_B(\mathcal{O}_2)$ means that $M_B(\mathcal{O}_1)$ and $M_B(\mathcal{O}_2)$ are symplectic orthogonal. As usual $W(\varphi)$, $\varphi \in h_B$, denote the ‘‘Weyl elements’’ of $\text{CCR}(h_B, \sigma)$. In this case the algebras

$$\mathcal{A}_B(\mathcal{O}) := C^*(\{W(\varphi) : \varphi \in M_B(\mathcal{O})\}) \tag{4}$$

form an isotone, covariant and causal net of C^* -algebras, i.e. for any \mathcal{O} , \mathcal{O}_1 and \mathcal{O}_2 one has:

- (i)_B $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$,
- (ii)_B $\mathcal{A}(g\mathcal{O}) = \alpha_g \mathcal{A}(\mathcal{O})$, $g \in \widehat{\mathfrak{P}}_+^\uparrow$,
- (iii)_B $\mathcal{O}_1 \perp \mathcal{O}_2$ implies that $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute elementwise,

where α_g denotes the Bogoliubov automorphism that corresponds to $V_B(g)$.

REMARK: By these propositions the problem of constructing covariant and causal nets of local C^* -algebras (within the framework of CAR- and CCR-theory) is shifted to the construction of corresponding nets of linear submanifolds of the reference spaces.

Part A: The massive case ($m > 0$)

A.1. Representations, embeddings and their mutual relations

We consider the following representations of $\widehat{\mathfrak{P}}_+^\uparrow$ on $C_0^\infty(\mathbb{R}^4, \mathcal{H})$ and on $h_m := L^2(H_m^+, \mathcal{H}, \mu(dp))$:

$$(T_1(g)f)(x) := D_A f(A_A^{-1}(x-a)) \quad \text{for } f \in C_0^\infty(\mathbb{R}^4, \mathcal{H}), \quad (5)$$

$$(V_1(g)\varphi)(p) := e^{-ipa} D_A \varphi(A_A^{-1}p) \quad \text{for } \varphi \in h_m, \quad (6)$$

$$(V_2(g)\varphi)(p) := e^{ipa} D_A \varphi(A_A^{-1}p) \quad \text{for } \varphi \in h_m, \quad (7)$$

where $g = (A, a) \in \widehat{\mathfrak{P}}_+^\uparrow = \text{SL}(2, \mathbb{C}) \otimes \mathbb{R}^4$, $A \mapsto D_A := \otimes A$ is a finite-dimensional representation of $\text{SL}(2, \mathbb{C})$ over $\mathcal{H} := \text{Sym}\left(\otimes \mathbb{C}^2\right)$ of¹ the type $(n, 0)$ and A_A is the corresponding Lorentz transformation for the matrices $\pm A \in \text{SL}(2, \mathbb{C})$. With respect to the scalar product

$$\langle \varphi, \psi \rangle_{\beta_+} := \int_{H_m^+} (\varphi(p), \beta_+(p) \psi(p))_{\mathcal{H}} \mu(dp) \quad \text{for all } \varphi, \psi \in h_m, \quad (8)$$

the representations V_1 and V_2 are unitary and irreducible, where

$$\beta_+(p) := D_{P^+} = \bigotimes_{i=1}^n P^+, \quad P^+ = \frac{1}{m} \left(p_0 \sigma_0 - \sum_{i=1}^3 p_i \sigma_i \right) \quad (9)$$

and σ_μ are the Pauli matrices, i.e.

¹Sym denotes the symmetrization operator.

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (10)$$

LEMMA A.1.1. V_1 and V_2 are anti-unitarily equivalent, i.e. there exists an anti-linear mapping (unique up to a phase) $\Gamma_1: h_m \rightarrow h_m$ that satisfies $\langle \Gamma_1 \varphi, \Gamma_1 \psi \rangle_{\beta_+} = \langle \psi, \varphi \rangle_{\beta_+}$ for all $\varphi, \psi \in h_m$ and $\Gamma_1 V_1 = V_2 \Gamma_1$.

Proof: Let Γ_0 be an anti-unitary involution on \mathcal{H} . Define $E := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SU}(2)$ and $H_\mu := (m + P)/\sqrt{2m(m + p_0)}$, where $P = \sum_\mu p_\mu \sigma_\mu$. By direct computation it can be shown that $\Gamma_1 := D_{H_p} D_E^{-1} \Gamma_0 D_{H_p^{-1}}$ satisfies all the mentioned properties.

The uniqueness (up to a phase) of Γ_1 is a consequence of the irreducibility of V_1 and V_2 . ■

REMARK A.1.2: The existence of Γ_1 is based on the fact that the self-representation and the conjugate representation of $\text{SU}(2)$ are equivalent. E is precisely the unitarity intertwining between both representations. For massless models the corresponding little group will not have this property, so that the following construction of Fermi models in Section A.2 can not be extended directly in order to include the massless case. (Compare with the definition of Γ in Subsection B.3.2.1.)

The proof of the covariance property of the net (see (II)_{B,F} in the preceding section) will be based on the following

REMARK A.1.3: The linear embeddings $\mathfrak{J}_\pm: C_0^\infty(\mathbb{R}^4, \mathcal{H}) \rightarrow h_m$, defined by $(\mathfrak{J}_\pm f)(p) := \widehat{f}(\pm p)$,² satisfy the equations

$$\mathfrak{J}_+ T_1(g) = V_1(g) \mathfrak{J}_+, \quad \mathfrak{J}_- T_1(g) = V_2(g) \mathfrak{J}_- \quad \text{for all } g \in \widetilde{\mathfrak{P}}_+^1. \quad (11)$$

The causality (see (III)_{B,F} in the preceding section) of the net will be based on the following result:

LEMMA A.1.4. Let $x \in \mathbb{R}^4$ be a spacelike vector (i.e. $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0$). With the notation of the beginning of this section one has

$$\int_{H_m^+} (e^{ipx} - e^{-ipx}) \beta_+(p) \mu(dp) = 0, \quad n \text{ even}, \quad (12)$$

$$\int_{H_m^+} (e^{ipx} + e^{-ipx}) \beta_+(p) \mu(dp) = 0, \quad n \text{ odd}. \quad (13)$$

²The hat $\widehat{}$ means the Fourier transformation restricted to the positive mass shell H_m^+ .

Proof: It is well-known that for $x^2 < 0$ the two-point function

$$\Delta(x) = \int_{H_m^+} e^{-ipx} \mu(dp) \quad (14)$$

is an even C^∞ function, i.e. $\Delta(x) = \Delta(-x)$ (see [15, pp. 71 and 107]). Let α be a multi-index and $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Then

$$\left. \frac{\partial^{|\alpha|} \Delta}{\partial x^\alpha} \right|_x = (-i)^{\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3} \int_{H_m^+} p^\alpha e^{-ipx} \mu(dp)$$

is an $\begin{cases} \text{even function, if } |\alpha| \text{ is even} \\ \text{odd function, if } |\alpha| \text{ is odd.} \end{cases} \quad (15)$

But from Eqs. (9) it follows that the matrix elements of $\beta_+(p)$ are homogeneous polynomials of degree n in p_μ .³ Equations (12) and (13) are therefore true.

Although in the massless case ($m = 0$) $\beta_+(p)$ has a different form than in Eq. (9) (see Section B.2.1), analogous equations as in (12) and (13) can be proved for $m = 0$ using the same arguments as above, since the corresponding matrix elements are, again, homogeneous polynomials of degree n in p_μ . ■

A.2. CAR- and CCR-algebras

A.2.1. The Fermi case

As the complex Hilbert space we consider $h_F := h_m \oplus h_m$ with the scalar product defined by $\langle \varphi, \psi \rangle_F := \langle \varphi_1, \psi_1 \rangle_{\beta_+} + \langle \varphi_2, \psi_2 \rangle_{\beta_+}$, for all $\varphi = \varphi_1 \oplus \varphi_2$, $\psi = \psi_1 \oplus \psi_2 \in h_F$. Using the expression for Γ_1 given in the proof of Lemma A.1.1, we define on h_F the following involution: $\Gamma(\varphi_1 \oplus \varphi_2) := \Gamma_1^{-1} \varphi_2 \oplus \Gamma_1 \varphi_1$, which satisfies all the required properties (i.e. Γ is anti-linear, $\Gamma^2 = 1$ and $\langle \Gamma\varphi, \Gamma\psi \rangle_F = \langle \psi, \varphi \rangle_F$, for all $\varphi, \psi \in h_F$). $\text{CAR}(h_F, \Gamma)$ is therefore uniquely given.

A.2.2. The Bose case

We interpret $h_B := h_m \oplus h_m$ as a real-linear space with a non-degenerate form defined by

$$\langle \varphi, \psi \rangle_B := \langle \varphi_1, \psi_1 \rangle_{\beta_+} - \langle \varphi_2, \psi_2 \rangle_{\beta_+}. \quad (16)$$

The symplectic form is $\sigma(\cdot, \cdot) := \text{Im}\langle \cdot, \cdot \rangle_B$. $\text{CCR}(h_B, \sigma)$ is then uniquely determined.

³ $(n, 0)$ characterizes the representation D (see the beginning of this section).

A.3. Existence theorems for local algebras

We define an embedding $\mathfrak{J}: C_0^\infty(\mathbb{R}^4, \mathcal{H}) \rightarrow h_{F,B}$ by $\mathfrak{J} := \mathfrak{J}_+ \oplus \mathfrak{J}_-$ (see Remark A.1.3) and the representations of $\widehat{\mathfrak{P}}_+^1$ on $h_{F,B}$ by $V_{F,B}(g) := V_1(g) \oplus V_2(g)$.

A.3.1. The Fermi case

Define a net of linear submanifolds of h_F by

$$\mathbb{R}^4 \supset \mathcal{O} \mapsto M_F(\mathcal{O}) := \text{span}\{\mathfrak{J}f: f \in C_0^\infty(\mathbb{R}^4, \mathcal{H}), \text{supp}f \subset \mathcal{O}\}, \quad (17)$$

for each open and bounded region $\mathcal{O} \subset \mathbb{R}^4$.

LEMMA A.3.1. *The equations $\Gamma V_F(g) = V_F(g)\Gamma$ and $\mathfrak{J}T_1(g) = V_F(g)\mathfrak{J}$ are valid for all $g \in \widehat{\mathfrak{P}}_+^1$.*

Proof: These statements are direct consequences of the definitions of Γ , Γ_1 , V_F and T_1 . \blacksquare

THEOREM A.3.2. *The net $\mathcal{O} \mapsto M_F(\mathcal{O})$ satisfies the properties (0)_F, (I)_F, (II)_F and (III)_F of Section 2.*

Proof: To prove (0)_F, we start with the case $n = 1$. With this assumption and using the definitions of H_p and E (see proof of Lemma A.1.1) we get

$$\Gamma_1 = \frac{1}{m} \Gamma_0 (EP)^t = \frac{1}{m} \Gamma_0 \begin{pmatrix} p_1 + ip_2 & -(p_0 + p_3) \\ p_0 - p_3 & -p_1 + ip_2 \end{pmatrix}, \quad (18)$$

and $\Gamma_1^{-1} = -\Gamma_1$. (Here, the exponent t denotes the transposition of the matrix.) A straightforward calculation shows that for all $\mathfrak{J}f \in M_F(\mathcal{O})$, with $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $\Gamma \mathfrak{J}f = \mathfrak{J}h$, where

$$h = \frac{i}{m} \begin{pmatrix} (\partial_1 - i\partial_2)\bar{f}_1 - (\partial_0 + \partial_3)\bar{f}_2 \\ (\partial_0 - \partial_3)\bar{f}_1 - (\partial_1 + i\partial_2)\bar{f}_2 \end{pmatrix}$$

and therefore $\mathfrak{J}h \in M_F(\mathcal{O})$. This fact, together with the standard properties of the symmetrization operator on $\overset{n}{\otimes} \mathbb{C}^2$ and of the embeddings \mathfrak{J}_+ , \mathfrak{J} , assure the Γ -invariance of $M_F(\mathcal{O})$ for arbitrary n .

The isotony property (I)_F is trivial from the definition of the net. To check the covariance, define for any $\mathfrak{J}f \in M_F(\mathcal{O})$ an element $k := T_1(g)f \in C_0^\infty(\mathbb{R}^4, \mathcal{H})$ such that $\text{supp}k \subset g\mathcal{O}$. We have $V_F(g)\mathfrak{J}f = \mathfrak{J}T_1(g)f = \mathfrak{J}k$. To prove the other inclusion, define for any $\mathfrak{J}k \in M_F(g\mathcal{O})$ an element $f := T_1(g^{-1})k \in C_0^\infty(\mathbb{R}^4, \mathcal{H})$ such that $\text{supp}f \subset \mathcal{O}$. In this case it can easily be verified that $\mathfrak{J}k = V_F(g)\mathfrak{J}f$. Finally, Lemma A.3.1 assures the covariance of the net.

To prove the causality property, choose $\mathfrak{J}f$ and $\mathfrak{J}h$ such that $\text{supp}f \perp \text{supp}h$. We have

$$\langle \mathfrak{J}_-f, \mathfrak{J}_-h \rangle_{\beta_1} = \int_{H_m^+} \left(\widehat{f}(-p), \beta_+(p) \widehat{h}(-p) \right)_{\mathcal{H}} \mu(dp). \quad (19)$$

Therefore

$$\begin{aligned} \langle \mathfrak{J}f, \mathfrak{J}h \rangle_{\mathbb{F}} &= \int_{H_m^+} \left(\int_{\mathbb{R}^4} e^{-ipx} f(x) d^4x, \beta_+(p) \int_{\mathbb{R}^4} e^{-ipy} h(y) d^4y \right)_{\mathcal{H}} \mu(dp) + \\ &\quad + \int_{H_m^+} \left(\int_{\mathbb{R}^4} e^{ipx} f(x) d^4x, \beta_+(p) \int_{\mathbb{R}^4} e^{ipy} h(y) d^4y \right)_{\mathcal{H}} \mu(dp) \\ &= \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left(f(x), \left[\int_{H_m^+} (e^{ip(x-y)} + e^{-ip(x-y)}) \beta_+(p) \mu(dp) \right] h(y) \right)_{\mathcal{H}} d^4x d^4y = 0, \quad (20) \end{aligned}$$

from Eq. (13). ■

Following Section 2 we have

THEOREM A.3.3. *The net of C^* -algebras defined by*

$$\mathbb{R}^4 \supset \mathcal{O} \mapsto \mathcal{A}_{\mathbb{F}}(\mathcal{O}) := C^* (\{A(f): f \in M_{\mathbb{F}}(\mathcal{O})\})^{\overline{\otimes} 2} \quad (21)$$

satisfies the properties (i)_ℱ, (ii)_ℱ and (iii)_ℱ.

A.3.2. The Bose case

In an analogous way we define the net of linear submanifolds of $h_{\mathbb{B}}$ by

$$\mathbb{R}^4 \supset \mathcal{O} \mapsto M_{\mathbb{B}}(\mathcal{O}) := \text{span} \{ \mathfrak{J}f: f \in C_0^\infty(\mathbb{R}^4, \mathcal{H}), \text{supp}f \subset \mathcal{O} \}, \quad (22)$$

for each open and bounded region $\mathcal{O} \subset \mathbb{R}^4$.

LEMMA A.3.4. *The equations $\sigma(V_{\mathbb{B}}(g)\varphi, V_{\mathbb{B}}(g)\psi) = \sigma(\varphi, \psi)$ and $\mathfrak{J}T_1(g) = V_{\mathbb{B}}(g)\mathfrak{J}$ are valid for any $g \in \widehat{\mathfrak{P}}_+^{\uparrow}$ and $\varphi, \psi \in h_{\mathbb{B}}$.*

Proof: These statements are direct consequences of the definitions of σ , $V_{\mathbb{B}}$ and T_1 . ■

THEOREM A.3.5. *The net $\mathcal{O} \mapsto M_{\mathbb{B}}(\mathcal{O})$ satisfies the properties (I)_ℬ, (II)_ℬ and (III)_ℬ of Section 2.*

Proof: The isotony property is again trivial. The covariance (II)_ℬ can be verified using Lemma A.3.4 and analogous arguments as in Theorem A.3.2.

For the causality (III)_B choose $\mathfrak{J}f$ and $\mathfrak{J}h$ such that $\text{supp}f \perp \text{supp}h$. We have

$$\begin{aligned} \langle \mathfrak{J}f, \mathfrak{J}h \rangle_{\mathbb{B}} &= \int_{H_m^+} \left(\int_{\mathbb{R}^4} e^{-ipx} f(x) d^4x, \beta_+(p) \int_{\mathbb{R}^4} e^{-ipy} h(y) d^4y \right)_{\mathcal{H}} \mu(dp) - \\ &\quad - \int_{H_m^+} \left(\int_{\mathbb{R}^4} e^{ipx} f(x) d^4x, \beta_+(p) \int_{\mathbb{R}^4} e^{ipy} h(y) d^4y \right)_{\mathcal{H}} \mu(dp) \\ &= \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left(f(x), \left[\int_{H_m^+} (e^{ip(x-y)} - e^{-ip(x-y)}) \beta_+(p) \mu(dp) \right] h(y) \right)_{\mathcal{H}} d^4x d^4y = 0. \end{aligned} \quad (23)$$

using Eq. (12).⁴ ■

Again following Section 2 we have

THEOREM A.3.6. *The net of C^* -algebras defined by*

$$\mathbb{R}^d \supset \mathcal{O} \mapsto \mathcal{A}_{\mathbb{B}}(\mathcal{O}) := C^* (\{W(f) : f \in M_{\mathbb{B}}(\mathcal{O})\}) \quad (24)$$

satisfies the properties (i)_B, (ii)_B and (iii)_B.

A.4. Fock states and spectrality condition

Since the representations V_1 and V_2 (from which the transformations $V_{\mathbb{F}, \mathbb{B}}$ that generate Bogoliubov automorphisms can be constructed) are anti-unitarily equivalent,⁵ it is natural to ask if there exists a Fock state for which the spectrality condition is satisfied. To answer affirmatively this question, it will be useful in the Fermi case to characterize the Fock states via basis projections (see [4] for a complete exposition of the concept of a basis projection in the context of CAR-algebras). In the Bose case the Fock state will be characterized by an internal complexification J of the real vector space $h_{\mathbb{B}}$ (see e.g. [7, Section 8.2.4]).

A.4.1. The Fermi case

Let α_g , $g \in \widetilde{\mathfrak{P}}_+^1$, denote the Bogoliubov automorphism generated by $V_{\mathbb{F}}(g)$. Then the tuple $(\text{CAR}(h_{\mathbb{F}}, \Gamma), \alpha_{(\cdot)}, \widetilde{\mathfrak{P}}_+^1)$ is a C^* -system [7]. We are interested in a Fock state such that α_g is unitarily implementable and such that the unitarities implementing α_a , $a \in \mathbb{R}^4$, satisfy the spectrality condition (see also [16]).

Define the following operator on $h_{\mathbb{F}}$: $P(\varphi \oplus \psi) := \varphi \oplus 0$ for all $\varphi \oplus \psi \in h_{\mathbb{F}}$.

⁴At this point a simple confirmation of the spin-statistics theorem is possible. Indeed, if one tries to construct CAR-algebras over $h_{\mathbb{B}}$, the corresponding net will not satisfy the causality property since n is even. Analogous problems with the causality appear if one defines CCR-algebras over $h_{\mathbb{F}}$.

⁵This means, for example, that $\text{spec}V_{\mathbb{F}}(g)$ is symmetric.

LEMMA A.4.1. P is a basis projection, i.e. $P + \Gamma P \Gamma = \mathbb{1}$, that commutes with $V_{\mathbb{F}}$.

Proof: The statements are direct consequences of the definitions of P , Γ , Γ_1 and $V_{\mathbb{F}}$. ■

The Hilbert space \mathfrak{H}_P , associated to the representation π_P characterized by the basis projection P , is the antisymmetric part of the Fock space constructed from $P h_{\mathbb{F}} \cong h_m$.

THEOREM A.4.2. The automorphisms α_g , $g \in \widetilde{\mathfrak{P}}_+^1$, are unitarily implementable on \mathfrak{H}_P , i.e. there exists a strongly continuous unitary representation $Q_{(\cdot)}$ of $\widetilde{\mathfrak{P}}_+^1$ on \mathfrak{H}_P such that

$$\pi_P(\alpha_g(X)) = Q_g \pi_P(X) Q_g^{-1} \quad \text{for all } X \in \text{CAR}(h_{\mathbb{F}}, \Gamma). \quad (25)$$

Furthermore, for all $a \in \mathbb{R}^4$, Q_a satisfies the spectrality condition.

Proof: The necessary and sufficient condition for a Bogoliubov automorphism α_g generated by $V_{\mathbb{F}}(g)$ to be unitarily implementable is that $P V_{\mathbb{F}}(g)(\mathbb{1} - P)$ belongs to the Hilbert–Schmidt class on $h_{\mathbb{F}}$ (see [4, p. 77]). But in our case $P V_{\mathbb{F}}(g)(\mathbb{1} - P) = 0$ and 0 is trivially a Hilbert–Schmidt operator.

Since $V_{\mathbb{F}}$ is a unitary representation of $\widetilde{\mathfrak{P}}_+^1$ that commutes with Γ and P , it is clear that the associated Bogoliubov automorphisms α_g leave the Fock state invariant. (For an explicit expression of the Fock state associated to the basis projection P see [4, Eqs. 2.17 and 2.18].) Therefore, the uniqueness of the GNS-construction yields the uniqueness of Q_g . But a direct computation shows that the second quantization of $V_1(g)$ satisfies Eq. (25) on the set of finite particle vectors. Therefore $Q_g = \mathbb{1} \oplus (\bigoplus_{n \geq 1}^n V_1(g))$.

For $a \in \mathbb{R}^4$, we have that $V_1(a)$ satisfies the spectrality condition on the one-particle Hilbert space h_m . This implies that Q_a satisfies the spectrality condition on the Fock space \mathfrak{H}_P (see also [4, Appendix C]). ■

A.4.2. The Bose case

As in the preceding section we have $(\text{CCR}(h_{\mathbb{B}}, \sigma), \alpha_{(\cdot)}, \widetilde{\mathfrak{P}}_+^1)$ as a C^* -system, where α_g , $g \in \widetilde{\mathfrak{P}}_+^1$, denotes the Bogoliubov automorphisms generated by $V_{\mathbb{B}}(g)$. Again, we are interested in a Fock state where the spectrality condition is satisfied.

LEMMA A.4.3. A real-linear mapping defined by $J(\varphi_1 \oplus \varphi_2) := (i\varphi_1) \oplus (-i\varphi_2)$ is an internal complexification on $h_{\mathbb{B}}$ (i.e. J satisfies $J^2 = -\mathbb{1}$, $\sigma(\varphi, J\psi) = -\sigma(J\varphi, \psi)$ and $\sigma(\varphi, J\varphi) \geq 0$ for all $\varphi, \psi \in h_{\mathbb{B}}$). J induces a real-valued scalar product $s(\cdot, \cdot)$ and a complex scalar product $k(\cdot, \cdot)$. Finally, the estimate $|\sigma(\varphi, \psi)| \leq \|\varphi\|_s \cdot \|\psi\|_s$ holds, where $\|\cdot\|_s$ denotes a norm induced by s .

Proof: The equations $J^2 = -\mathbb{1}$ and $\sigma(\varphi, J\psi) = -\sigma(J\varphi, \psi)$ are easy to verify. Furthermore, $s(\varphi, \varphi) := \sigma(\varphi, J\varphi) = \langle \varphi_1, \varphi_1 \rangle_{\beta_+} + \langle \varphi_2, \varphi_2 \rangle_{\beta_+} \geq 0$ for all $\varphi = \varphi_1 \oplus \varphi_2 \in h_{\mathbb{B}}$.

From the polarization formula for real scalar products one has $s(\varphi, \psi) = \sigma(\varphi, J\psi)$ and $k(\varphi, \psi) = s(\varphi, \psi) + i\sigma(\varphi, \psi) = \langle \varphi_1, \psi_1 \rangle_{\beta_+} + \langle \psi_2, \varphi_2 \rangle_{\beta_-}$. (Note that $ik(\varphi, \psi) = k(\varphi, J\psi)$ for all $\varphi, \psi \in h_{\mathbb{B}}$.)

Finally, for any $\varphi, \psi \in h_{\mathbb{B}}$ the estimate

$$|\sigma(\varphi, \psi)| \leq |k(\varphi, \psi)| \leq \|\varphi\|_k \cdot \|\psi\|_k = \|\varphi\|_s \cdot \|\psi\|_s \quad (26)$$

concludes the proof. \blacksquare

REMARK A.4.4: As the complex structure on $h_{\mathbb{B}}$ is completely determined by J , one has to adapt to this case the usual definition of infinitesimal generator of a unitary (with respect to k) group $U(t)$. Concretely, one defines

$$-JA\varphi := \lim_{t \rightarrow 0} \frac{U(t)\varphi - \varphi}{t} \quad \text{for all } \varphi \text{ such that the limit exists.} \quad (27)$$

We extend J to the whole symmetric Fock space $\mathcal{F}_s(h_{\mathbb{B}})$ constructed from $(h_{\mathbb{B}}, k)$ as usual

$$\begin{aligned} J_0 &:= i, \\ J_1 &:= J, \\ J_n(\varphi_1 \otimes \dots \otimes \varphi_n) \\ &:= (J\varphi_1) \otimes \varphi_2 \otimes \dots \otimes \varphi_n = \dots = \varphi_1 \otimes \dots \otimes \varphi_{n-1} \otimes (J\varphi_n), \end{aligned} \quad (28)$$

for all $\varphi_l \in h_{\mathbb{B}}$, $l = 1, \dots, n$. Therefore, $\widehat{J} := \bigoplus_{n \geq 0} J_n$ defines a complexification on $\mathcal{F}_s(h_{\mathbb{B}})$. Note, nevertheless, that in general $(i\varphi) \otimes \psi \neq \varphi \otimes (i\psi)$, $\varphi, \psi \in h_{\mathbb{B}}$, since the construction of the tensor product uses the complex structure given by J .

THEOREM A.4.5. *The automorphisms α_g , $g \in \widetilde{\mathfrak{P}}_+^1$, are unitarily implementable on the Fock space $\mathcal{F}_s(h_{\mathbb{B}})$ and the unitarities implementing the translation group \mathbb{R}^4 satisfy the spectrality condition.*

Proof: The complexification J defined in Lemma A.4.3 characterizes a Fock state of $\text{CCR}(h_{\mathbb{B}}, \sigma)$ through the generating functional $\phi(\varphi) := e^{-\frac{1}{2}\|\varphi\|_s^2}$, $\varphi \in h_{\mathbb{B}}$. But since $V_{\mathbb{B}}(g)$ is unitary (with respect to k) on the one-particle Hilbert space $(h_{\mathbb{B}}, k)$, it is clear that the Bogoliubov automorphisms α_g , $g \in \widetilde{\mathfrak{P}}_+^1$, leave the Fock state invariant. The implementability of α_g is therefore guaranteed by the uniqueness of the GNS-construction. The unitarities that implement α_g are unique.

Denote by Q_g the second quantization of $V_{\mathbb{B}}(g)$ on $\mathcal{F}_s(h_{\mathbb{B}})$. We will prove that

$$\pi_J(\alpha_g(X)) = Q_g \pi_J(X) Q_g^{-1} \quad \text{for all } X \in \text{CCR}(h_{\mathbb{B}}, \sigma), \quad (29)$$

where π_J denotes the Fock representation associated with J . It is sufficient to verify Eq. (29) on a total set of $\mathcal{F}_s(h_B)$ for the Weyl elements of $\text{CCR}(h_B, \sigma)$. But in this case a straightforward calculation shows that Eq. (29) is satisfied on the set of coherent vectors $\left\{ \varphi \in \mathcal{F}_s(h_B) : \varphi = \left(1, \psi, \frac{1}{\sqrt{2!}}(\psi \otimes \psi), \dots \right), \psi \in h_B \right\}$, which is total in $\mathcal{F}_s(h_B)$ (see [17, Chapter 2]).

Finally, taking into account Remark A.4.4, one also has that the infinitesimal generator of the time translation P_0 and the square of the infinitesimal generator of the spacetime translations P^2 are positive operators on the one-particle Hilbert space. Concretely, one has $(P_0\varphi)(p) = p_0\varphi(p)$ and $(P^2\varphi)(p) = m^2\varphi(p)$. This implies that the second quantization of $V_B(g)$ (i.e. Q_g) satisfies the spectrality condition (see [15, p. 63]). ■

Part B: The massless case ($m = 0$)

B.1. Representation theory for massless models

The UIR of $\widetilde{\mathfrak{P}}_+^\uparrow$ can be obtained from the well-known theory of induced representations (see Chapters 2 and 3 of [18] for a neat summary or the standard references [10], [19] and [9]). The strategy is to “induce” the UIR of the whole $\widetilde{\mathfrak{P}}_+^\uparrow$ from a finite-dimensional unitary representation of a certain subgroup of the symmetry group $\widetilde{\mathfrak{P}}_+^\uparrow$. A characteristic feature of massless models, in contrast to massive ones, is that the corresponding little group is non-compact and therefore certain representations of the little group which are physically motivated have to be selected (see [10, p. 202]). Precisely, the little group of the character χ_q with $q = (1, 0, 0, 1)$ is (see [20, p. 162]):⁶

$$\Delta := \left\{ \begin{pmatrix} e^{\frac{1}{2}\theta} & e^{-\frac{1}{2}\theta} z \\ 0 & e^{-\frac{1}{2}\theta} \end{pmatrix} \in \text{SL}(2, \mathbb{C}) : \theta \in [0, 4\pi), z \in \mathbb{C} \right\}. \quad (30)$$

It can also be shown that $\widetilde{\mathfrak{P}}_+^\uparrow/\widetilde{\Sigma}$ is diffeomorphic to the orbit $\mathcal{C}^+ := \{p \in \mathbb{R}^4 : p^2 = 0, p_0 > 0\}$, where $\widetilde{\Sigma} := \Delta \otimes \mathbb{R}^4$. Let $\mathfrak{H} := L^2(\mathcal{C}^+, \mu_0(dp))$, where $\mu_0(dp) = d^3p/p_0$ is the invariant measure on \mathcal{C}^+ . On \mathfrak{H} we have the following irreducible representation of $\widetilde{\mathfrak{P}}_+^\uparrow$ that satisfies the spectrality condition:

$$(U_1(A, a) \varphi)(p) := e^{-ip_a} \left(\left(H_p^{-1} A H_{\Lambda_A^{-1} p} \right)_{22} \right)^n \varphi(\Lambda_A^{-1} p), \quad (31)$$

where $(A, a) \in \widetilde{\mathfrak{P}}_+^\uparrow$ and $n/2$ is the helicity of the model.⁷ The matrices $H_p \in \text{SL}(2, \mathbb{C})$ satisfy the equation:

⁶The corresponding little group associated to the character χ_q with $q = (m, 0, 0, 0)$, $m > 0$, is $\text{SU}(2)$, which is a compact subgroup of $\text{SL}(2, \mathbb{C})$.

⁷The representation U_1 is unitary for the usual L^2 -scalar product $(\varphi, \psi) = \int_{\mathcal{C}^+} \overline{\varphi(p)} \psi(p) \mu_0(dp)$.

$$H_p \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} H_p^* = P, \text{ where } P = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = \sum_{\mu} p_{\mu} \sigma_{\mu}, \quad (32)$$

by convention (10).

REMARK B.1.1: Notice that the H_p matrices are not uniquely determined by Eq. (32), as we can always use $H'_p = H_p L$, with $L \in \Delta$ instead of the original H_p . Although the matrix $H_p^{-1} A H_{\Lambda_A^{-1} p}$ depends on the choice of the H_p matrix, it is easy to see that the matrix elements $(H_p^{-1} A H_{\Lambda_A^{-1} p})_{11}$ and $(H_p^{-1} A H_{\Lambda_A^{-1} p})_{22}$ do not change if we use H'_p instead of H_p . A possible solution of Eq. (32) is

$$H_p = \frac{1}{\sqrt{2p_0(p_0 + p_3)}} \begin{pmatrix} -\sqrt{p_0} (p_0 + p_3) & \frac{p_1 - ip_2}{\sqrt{p_0}} \\ -\sqrt{p_0} (p_1 + ip_2) & -\frac{p_0 + p_3}{\sqrt{p_0}} \end{pmatrix}. \quad (33)$$

Using the polar decomposition and denoting $p = (p_0, \mathbf{p})$ we can write

$$H_p = U^*(\mathbf{p}) H, \quad (34)$$

where

$$H = \begin{pmatrix} p_0^{\frac{1}{2}} & 0 \\ 0 & p_0^{-\frac{1}{2}} \end{pmatrix}, \quad U(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p_3)}} \begin{pmatrix} -(|\mathbf{p}| + p_3) & -p_1 + ip_2 \\ p_1 + ip_2 & -(|\mathbf{p}| + p_3) \end{pmatrix}. \quad (35)$$

H represents a boost in the x_3 -direction with the velocity $v_3 = c(p_0^2 - 1)/(p_0^2 + 1)$, and $U(\mathbf{p})$ represents a rotation of the reference frame that transforms a 4-vector $p = (p_0, p_1, p_2, p_3)$ into a 4-vector $(p_0, 0, 0, p_0)$ (see [19, p. 65]).

REMARK B.1.2: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, we can compute explicitly

$$e^{-\frac{1}{2}\theta(A,p)} = (H_p^{-1} A H_{\Lambda_A^{-1} p})_{22} = \frac{-b(p_1 + ip_2) + d(p_0 + p_3)}{|-b(p_1 + ip_2) + d(p_0 + p_3)|}. \quad (36)$$

Standard properties of the map $\theta: \text{SL}(2, \mathbb{C}) \times \mathbb{C}^+ \rightarrow \mathbb{R}$ are:

- (i) $\theta(\mathbb{I}, p) = 0$.
- (ii) $\theta(A, p) = \theta(A, p)$.
- (iii) $\theta(AB, p) = \theta(A, p) + \theta(B, \Lambda_A^{-1} p)$,

where $A, B \in \text{SL}(2, \mathbb{C})$, $p \in \mathbb{C}^+$. The properties (i)–(iii) can be established without knowing the explicit expression of $e^{-\frac{1}{2}\theta(A,p)}$ (see [21]). Furthermore, the following properties are true:

- (iv) Let $A := \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \in \text{SL}(2, \mathbb{C})$, with $c \in \mathbb{C}$ and $a > 0$. Then $\theta(A, p) = 0$ for all $p \in \mathbb{C}^+$. In general, for $b = b_1 + ib_2$ and $d = d_1 + id_2$,

$$\ker \theta = \left\{ (A, p) \in \mathrm{SL}(2, \mathbb{C}) \times (\mathcal{C}^+ \setminus \{0\}) : A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \right. \\ \left. (p_0 + p_3)d_2 = b_1 p_2 + b_2 p_1, (p_0 + p_3)d_1 > b_1 p_1 - b_2 p_2 \right\}. \quad (37)$$

(v) For all $p_0 > 0$, $e^{i\theta(A, (p_0, 0, 0, \pm p_0))} = \bar{d}/|d|$. Therefore, $\theta(A, (p_0, 0, 0, \pm p_0))$ does not depend on p_0 . Furthermore, let $A := \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$, then $e^{i\theta(A, p)} = |a|/\bar{a}$ for all $p \in \mathcal{C}^+$.

B.2. The factor spaces \mathfrak{H}'_{\pm} and related topics

B.2.1. An isometry between two possible representation spaces

In order to prove the causality and covariance of the net of local C^* -algebras, it is more useful to work with the triple $(\mathfrak{H}'_+, V'_1, \langle \cdot, \cdot \rangle'_{\beta_+})$, instead of the triple $(\mathfrak{H}, U_1, \langle \cdot, \cdot \rangle)$ defined in the preceding section. We show that the spaces $(\mathfrak{H}'_+, \langle \cdot, \cdot \rangle'_{\beta_+})$ and $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ are isometric and that the respective representations are unitarily equivalent. The triple $(\mathfrak{H}'_+, V'_1, \langle \cdot, \cdot \rangle'_{\beta_+})$ will be constructed in this section.

First consider $\mathfrak{H}^{(n)} := \mathbf{L}^2(\mathcal{C}^+, \mathcal{H}, \mu_0(dp))$, where \mathcal{H} is the Hilbert space of dimension $n+1$ associated with the spinor representation of $\mathrm{SL}(2, \mathbb{C})$ of type $(n, 0)$ and define the sesquilinear form

$$\langle \varphi, \psi \rangle_{\beta_+} := \int_{\mathcal{C}^+} (\varphi(p), \beta_+(p) \psi(p))_{\mathcal{H}} \mu_0(dp), \quad \varphi, \psi \in \mathfrak{H}^{(n)}, \quad (38)$$

with the convention⁸

$$\beta_+(p) := D_{P^+} = \bigotimes_{i=1}^n P^+ \quad \text{and} \quad P^+ = \frac{1}{2} \left(p_0 \sigma_0 - \sum_i p_i \sigma_i \right). \quad (39)$$

Define also the representation V_1 on $\mathfrak{H}^{(n)}$ by:

$$(V_1(g)\varphi)(p) := e^{-ip_a} D_A \varphi(\Lambda_A^{-1} p), \quad g = (A, a) \in \widetilde{\mathfrak{P}}_+^1, \quad (40)$$

which leaves the sesquilinear form $\langle \cdot, \cdot \rangle_{\beta_+}$ invariant.

REMARK B.2.1: Notice that $\det P^+ = 0$, so that $\langle \cdot, \cdot \rangle_{\beta_+}$ is semi-definite⁹ for $n \geq 1$. This confirms a general theorem of Barut and Rączka. As a consequence of the fact that

⁸Compare with Eq. (9) for the massive case.

⁹In standard physical literature expressions like “indefinite form” or “indefinite metric” can also be found.

the little group associated to massless models is solvable, connected and non-compact, the sesquilinear form $\langle \cdot, \cdot \rangle_{\beta_+}$ must necessarily be semi-definite. See [22] for details.

B.2.1.1. The neutrino case $n = 1$ ($s = \frac{1}{2}$)

LEMMA B.2.2. For $n = 1$ (i.e. $\mathcal{H} = \mathbb{C}^2$) and denoting $\varphi = (\varphi_1, \varphi_2) \in \mathfrak{H}_+^{(1)}$ we have

- (i) $\|\varphi\|_{\beta_+} = 0$ iff $(p_0 - p_3) \varphi_1(p) = (p_1 - ip_2) \varphi_2(p)$ a.e. for $\mu_0(dp)$.
- (ii) $\mathfrak{H}_+^{(1)} = \mathfrak{H}_+^{(>)} \oplus \mathfrak{H}_+^{(0)}$, where

$$\mathfrak{H}_+^{(>)} = \text{span}\{\varphi = (\varphi_1, \varphi_2) \in \mathfrak{H}_+^{(1)} : (p_0 + p_3) \varphi_1(p) = (-p_1 + ip_2) \varphi_2(p)\}, \quad (41)$$

$$\mathfrak{H}_+^{(0)} = \text{span}\{\varphi = (\varphi_1, \varphi_2) \in \mathfrak{H}_+^{(1)} : (p_0 - p_3) \varphi_1(p) = (p_1 - ip_2) \varphi_2(p)\}, \quad (42)$$

$$\mathfrak{H}'_+ := \mathfrak{H}_+^{(1)} / \mathfrak{H}_+^{(0)}. \quad (43)$$

Proof: (i) A straightforward calculation shows that

$$\|\varphi\|_{\beta_+}^2 = \langle \varphi, \varphi \rangle_{\beta_+} = \int_{\mathcal{C}^+} \left| \sqrt{p_0 - p_3} \varphi_1(p) - \frac{p_1 - ip_2}{\sqrt{p_0 - p_3}} \varphi_2(p) \right|^2 \mu_0(dp). \quad (44)$$

(ii) We have

$$\beta_+(p) \begin{pmatrix} (-p_1 + ip_2) \varphi(p) \\ (p_0 + p_3) \varphi(p) \end{pmatrix} = p_0 \begin{pmatrix} (-p_1 + ip_2) \varphi(p) \\ (p_0 + p_3) \varphi(p) \end{pmatrix}, \quad (45)$$

$$\beta_+(p) \begin{pmatrix} (p_1 - ip_2) \varphi(p) \\ (p_0 - p_3) \varphi(p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (46)$$

Now, for any $\psi \in \mathfrak{H}_+^{(>)}$ and any $p \in \mathcal{C}^+$ the vector $\psi(p) \in \mathcal{H}$ is an eigenvector of $\beta_+(p)$ with eigenvalue p_0 . The elements of $\mathfrak{H}_+^{(0)}$ are eigenvectors with eigenvalue 0. We can decompose a vector of $\mathfrak{H}_+^{(1)}$ in a unique way into a direct sum of vectors of $\mathfrak{H}_+^{(>)}$ and $\mathfrak{H}_+^{(0)}$. ■

LEMMA B.2.3. Using the preceding notation the following statements are true:

- (i) $\mathfrak{H}_+^{(0)}$ is V_1 -invariant.
- (ii) V_1 leaves the $\|\cdot\|_{\beta_+}$ -norm of the elements in $\mathfrak{H}_+^{(>)}$ invariant, but the space $\mathfrak{H}_+^{(>)}$ itself is not V_1 -invariant.
- (iii) In each coset¹⁰ $[\varphi]_+ \in \mathfrak{H}'_+$ there exists a unique representative $\varphi^{(>)} \in [\varphi]_+$ with $\varphi^{(>)} \in \mathfrak{H}_+^{(>)}$.

¹⁰The bracket $[\cdot]_+$ denotes the equivalence class with respect to the semi-definite sesquilinear form $\langle \cdot, \cdot \rangle_{\beta_+}$.

Proof: (i) This follows from the fact that V_1 leaves $\langle \cdot, \cdot \rangle_{\beta_+}$ invariant. This statement can be also confirmed explicitly: if we define $\alpha(p) := (p_1 - ip_2)/(p_0 - p_3)$ and $p' := \Lambda_A^{-1}p$, then for $\varphi(p) = \begin{pmatrix} \alpha(p) \varphi_1(p) \\ \varphi_1(p) \end{pmatrix}$ we have

$$\psi(p) := (V_1(g) \varphi)(p) = \begin{pmatrix} \alpha(p) \psi_1(p) \\ \psi_1(p) \end{pmatrix}, \quad (47)$$

with $\psi_1(p) = e^{-ip_0} (c\alpha(p') + d) \varphi_1(p')$ and where we have used the relation $\alpha(p) = (a\alpha(p') + b)/(c\alpha(p') + d)$.

(ii) Define $\psi := V_1(g) \varphi$, $\varphi \in \mathfrak{H}_+^{(>)}$ and $g \in \widetilde{\mathfrak{P}}_+^{\top}$, and confirm that $\beta(p)\psi(p) \neq p_0\psi(p)$. This shows that, although $\|\psi\|_{\beta_+} = \|\varphi\|_{\beta_+}$, the function ψ must be generally written as $\psi = \psi^{(>)} \oplus \psi^{(0)}$ with $\psi^{(>)} \in \mathfrak{H}_+^{(>)}$ and $\psi^{(0)} \in \mathfrak{H}_+^{(0)}$.

(iii) is a consequence of Lemma B.2.2 (ii). \blacksquare

DEFINITION B.2.4. We define on \mathfrak{H}'_+ a representation V'_1 of $\widetilde{\mathfrak{P}}_+^{\top}$ and the sesquilinear form $\langle \cdot, \cdot \rangle'_{\beta_+}$ by

$$V'_1(g) [\varphi]_+ := [V_1(g) \varphi]_+, \quad (48)$$

$$\langle [\varphi]_+, [\psi]_+ \rangle'_{\beta_+} := \int_{\mathcal{C}^+} (\varphi(p), \beta_+(p)\psi(p))_{\mathcal{H}} \mu_0(dp). \quad (49)$$

REMARK B.2.5: V'_1 and $\langle \cdot, \cdot \rangle'_{\beta_+}$ are well defined (i.e. they do not depend on representatives of the equivalence class). Furthermore, $\langle \cdot, \cdot \rangle'_{\beta_+}$ defines a scalar product on \mathfrak{H}'_+ with respect to which V'_1 is unitary. Indeed, from Lemma B.2.3 (ii) it follows that $[V_1(g) \varphi]_+ = [V_1(g) (\varphi + \varphi^{(0)})]_+$ for any $\varphi^{(0)} \in \mathfrak{H}_+^{(0)}$. The equation

$$\left((\varphi(p) + \varphi^{(0)}(p), \beta_+(p)(\psi(p) + \psi^{(0)}(p)))_{\mathcal{H}} \right) = (\varphi(p), \beta_+(p)\psi(p))_{\mathcal{H}} \quad (50)$$

also holds for any $\psi^{(0)}, \varphi^{(0)} \in \mathfrak{H}_+^{(0)}$, so that $\langle \cdot, \cdot \rangle'_{\beta_+}$ is well defined.

THEOREM B.2.6. For $n = 1$ a linear map $\Phi_+ : (\mathfrak{H}, U_1, (\cdot, \cdot)) \rightarrow (\mathfrak{H}'_+, V'_1, \langle \cdot, \cdot \rangle'_{\beta_+})$, defined by $(\Phi_+ \varphi)(p) := \left[H_p \begin{pmatrix} 0 \\ \varphi(p) \end{pmatrix} \right]_+$, $\varphi \in \mathfrak{H}$, is an isometry that commutes with the representations (i.e. $\Phi_+ U_1 = V'_1 \Phi_+$).

Proof: Notice that

$$H_p \begin{pmatrix} 0 \\ \varphi(p) \end{pmatrix} = -\frac{1}{\sqrt{2p_0^2(p_0 + p_3)}} \begin{pmatrix} (-p_1 + ip_2) \varphi(p) \\ (p_0 + p_3) \varphi(p) \end{pmatrix}$$

is the representative of $\left[H_p \begin{pmatrix} 0 \\ \varphi(p) \end{pmatrix} \right]_+$ in $\mathfrak{H}_+^{(>)}$. It is easy to see that Φ_+ is a bijection between \mathfrak{H} and \mathfrak{H}'_+ . Furthermore,

$$\langle \Phi_+ \varphi_1, \Phi_+ \varphi_2 \rangle'_{\beta_+} = \langle \varphi_1, \varphi_2 \rangle_{L^2}, \quad (51)$$

where we used the relation $\beta_+(p) = (H_p^{-1})^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} H_p^{-1}$.

The proof that Φ_+ intertwines between U_1 and V'_1 is straightforward. Let $g = (A, a) \in \widehat{\mathfrak{P}}_+^1$

$$(\Phi_+ U_1(g) \varphi)(p) = \left[e^{-ipa} \left(H_p^{-1} A H_{\Lambda_A^{-1}p} \right)_{22} H_p \begin{pmatrix} 0 \\ \varphi(\Lambda_A^{-1}p) \end{pmatrix} \right]_+ \quad (52)$$

and further

$$\begin{aligned} (V'_1(g) \Phi_+ \varphi)(p) &= \left[e^{-ipa} H_p \left(H_p^{-1} A H_{\Lambda_A^{-1}p} \right) \begin{pmatrix} 0 \\ \varphi(\Lambda_A^{-1}p) \end{pmatrix} \right]_+ \\ &= \left[e^{-ipa} \left(H_p^{-1} A H_{\Lambda_A^{-1}p} \right)_{22} H_p \begin{pmatrix} 0 \\ \varphi(\Lambda_A^{-1}p) \end{pmatrix} \right]_+, \end{aligned} \quad (53)$$

where we used $H_{(\cdot)} \begin{pmatrix} \psi(\cdot) \\ 0 \end{pmatrix} \in \mathfrak{H}_+^{(0)}$ for all $\psi \in \mathfrak{H}$. ■

COROLLARY B.2.7. V'_1 acts irreducibly on \mathfrak{H}'_+ .

B.2.1.2. The case of arbitrary helicity

The image of $\varphi \in \mathfrak{H}_+^{(n)}$ is of the form $\varphi(p) = \text{Sym}(\varphi_1(p) \otimes \cdots \otimes \varphi_n(p))$, where $\varphi_l \in \mathfrak{H}_+^{(1)}$ for $l = 1, \dots, n$, and Sym is the symmetrization operator on $\overset{n}{\otimes} \mathbb{C}^2$. From Lemma B.2.3 (iii) we can decompose the image of each $\varphi_l \in \mathfrak{H}_+^{(1)}$ in

$$\varphi_l(p) = \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} \alpha_l(p) + \begin{pmatrix} p_1 - ip_2 \\ p_0 - p_3 \end{pmatrix} \beta_l(p), \quad l = 1, \dots, n. \quad (54)$$

Therefore,

$$\varphi(p) = \bigotimes_{l=1}^n \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} \alpha_1(p) \cdots \alpha_n(p) + R(\alpha_k, \beta_l, p), \quad (55)$$

where $\|R(\alpha_k, \beta_l, p)\|_{\beta_+} = 0$ (see Eq. (38)). The preceding lines motivate the following definition:

DEFINITION B.2.8. Let $\langle \cdot, \cdot \rangle_{\beta_+}$ be the *sesquilinear form* defined in Eq. (38)

$$\mathfrak{H}_{+,n}^{(0)} := \text{span}\{\varphi \in \mathfrak{H}_+^{(n)} : \langle \varphi, \varphi \rangle_{\beta_+} = 0\}, \quad (56)$$

$$\mathfrak{H}_{+,n}^{(>)} := \text{span} \left\{ \varphi \in \mathfrak{H}_+^{(n)}: \varphi(p) = \bigotimes^n \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} \chi(p), \text{ for some function } \chi \right\}, \quad (57)$$

$$\mathfrak{H}'_{+,n} := \mathfrak{H}_+^{(n)} / \mathfrak{H}_{+,n}^{(0)}. \quad (58)$$

REMARK B.2.9: Note that:

(i) For $n = 1$ we recover the definitions (41), (42) and (43), respectively.

(ii) As in the case $n = 1$ we can find in each coset $[\varphi]_+ \in \mathfrak{H}'_{+,n}$ a unique representative $\varphi^{(>)} \in [\varphi]_+$ with $\varphi^{(>)} \in \mathfrak{H}_{+,n}^{(>)}$. We have $\mathfrak{H}'_+ = \mathfrak{H}_{+,n}^{(>)} \oplus \mathfrak{H}_{+,n}^{(0)}$.

LEMMA B.2.10. *Let V_1 be defined as in (40). Then $\mathfrak{H}_{+,n}^{(0)}$ is V_1 -invariant, while $\mathfrak{H}_{+,n}^{(>)}$ is not V_1 -invariant.*

Proof: Apply Lemma B.2.3 to each component of the tensor product. ■

DEFINITION B.2.11. We define on $\mathfrak{H}'_{+,n}$ representations V'_1 of $\widetilde{\mathfrak{P}}_+^1$ and a sesquilinear form $\langle \cdot, \cdot \rangle'_{\beta_+}$:

$$V'_1(g)[\varphi]_+ := [V_1(g) \varphi]_+, \quad (59)$$

$$\langle [\varphi]_+, [\psi]_+ \rangle'_{\beta_+} := \int_{\mathcal{C}^+} (\varphi(p), \beta_+(p) \psi(p))_{\mathcal{H}} \mu_0(dp). \quad (60)$$

REMARK B.2.12: Using the same arguments as in Remark B.2.5, for each component of the tensor product it is easy to see that V'_1 and $\langle \cdot, \cdot \rangle'_{\beta_+}$ are well defined.

THEOREM B.2.13. *A linear map $\Phi_+ : (\mathfrak{H}, U_1, (\cdot, \cdot)) \rightarrow (\mathfrak{H}'_{+,n}, V'_1, \langle \cdot, \cdot \rangle'_{\beta_+})$ defined by $(\Phi_+ \varphi)(p) := \left[\bigotimes^n \left(H_p \begin{pmatrix} 0 \\ \varphi^{\frac{1}{n}}(p) \end{pmatrix} \right) \right]_+$ is an isometry that commutes with the representations (i.e. $\Phi_+ U_1 = V'_1 \Phi_+$).*

Proof: Although $\varphi^{\frac{1}{n}}$ is a multi-valued function, any determination of $\varphi^{\frac{1}{n}}$ yields the same value of $(\Phi_+ \varphi)(p)$.

$$\bigotimes^n \left(H_p \begin{pmatrix} 0 \\ \varphi^{\frac{1}{n}}(p) \end{pmatrix} \right) = \left(\bigotimes^n \left(-\frac{1}{\sqrt{2p_0^2(p_0 + p_3)}} \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} \right) \right) \varphi(p) \quad (61)$$

is a representative of $\Phi_+ \varphi \in \mathfrak{H}'_{+,n}$ in $\mathfrak{H}_{+,n}^{(>)}$. Φ_+ is therefore bijective.

Finally, for $g = (A, a) \in \widetilde{\mathfrak{P}}_+^1$ and $\varphi \in \mathfrak{H}$,

$$(\Phi_+ U_1(g) \varphi)(p) = \left[e^{-ipa} \left(H_p^{-1} A H_{\Lambda_A^{-1}p} \right)_{22}^n D_{H_p} \left(\bigotimes^n \left(\begin{array}{c} 0 \\ \varphi^{\frac{1}{n}}(\Lambda_A^{-1}p) \end{array} \right) \right) \right]_+, \quad (62)$$

$$\begin{aligned} (V'_1(g) \Phi_+ \varphi)(p) &= \left[e^{-ipa} D_{H_p} D_{H_p^{-1}} D_A D_{H_{\Lambda_A^{-1}p}} \bigotimes^n \left(\begin{array}{c} 0 \\ \varphi^{\frac{1}{n}}(\Lambda_A^{-1}p) \end{array} \right) \right]_+ \\ &= \left[e^{-ipa} \left(H_p^{-1} A H_{\Lambda_A^{-1}p} \right)_{22}^n D_{H_p} \left(\bigotimes^n \left(\begin{array}{c} 0 \\ \varphi^{\frac{1}{n}}(\Lambda_A^{-1}p) \end{array} \right) \right) \right]_+, \end{aligned} \quad (63)$$

where for the last equation we have used $\bigotimes^n \left(H_{(\cdot)} \left(\begin{array}{c} \psi(\cdot) \\ 0 \end{array} \right) \right) \in \mathfrak{H}_{+,n}^{(0)}$ for all $\psi \in \mathfrak{H}$. ■

COROLLARY B.2.14. V'_1 acts irreducibly on $\mathfrak{H}'_{+,n}$.

B.2.2. Analogous arguments for models with opposite helicity

For the construction of the net of linear submanifolds in the reference space of the CAR- and CCR-algebras, respectively, the representation associated to models with opposite helicity is also necessary. For the convenience of the reader we will list in this case the analogous definitions and results of the preceding section. (The index 1 will be replaced by the index 4.) The choice of the index 4 for this representation will be justified in Section B.3.1.

On the space \mathfrak{H} we define in this section the representation

$$(U_4(g) \varphi)(p) := e^{-ipa} e^{+\frac{i}{2}\theta(A,p)} \varphi(\Lambda_A^{-1}p) \quad (64)$$

(see Eq. (36)). Further, we consider the triple $(\mathfrak{H}_-^{(n)}, V_4, \langle \cdot, \cdot \rangle_{\beta_-})$:

$$\langle \varphi, \psi \rangle_{\beta_-} := \int_{\mathcal{C}^+} (\varphi(p), \beta_-(p) \psi(p))_{\mathcal{H}} \mu_0(dp), \quad (65)$$

where $\varphi, \psi \in \mathfrak{H}_-^{(n)}$, $\beta_-(p) := \Gamma_0 \beta_+(p) \Gamma_0$ and Γ_0 denotes an anti-unitary involution on the space \mathcal{H} . Finally, let V_4 be a representation of $\widehat{\mathfrak{P}}_+^{\uparrow}$ over $\mathfrak{H}_-^{(n)}$ defined by

$$(V_4(g) \varphi)(p) := e^{-ipa} D_{\bar{A}} \varphi(\Lambda_{\bar{A}}^{-1}p). \quad (66)$$

For $n = 1$ we have the following results.

LEMMA B.2.15. *Let $\varphi = (\varphi_1, \varphi_2) \in \mathfrak{H}_-^{(1)}$. Then*

- (i) $\|\varphi\|_{\beta_-} = 0$ iff $(p_0 - p_3) \varphi_1(p) = (p_1 + ip_2) \varphi_2(p)$ a.e. for $\mu_0(dp)$.
- (ii) $\mathfrak{H}_-^{(1)} = \mathfrak{H}_-^{(>)} \oplus \mathfrak{H}_-^{(0)}$, where

$$\mathfrak{H}_-^{(>)} = \text{span}\{\varphi = (\varphi_1, \varphi_2) \in \mathfrak{H}_-^{(1)}: (p_0 + p_3) \varphi_1(p) = -(p_1 + ip_2) \varphi_2(p)\}, \quad (67)$$

$$\mathfrak{H}_-^{(0)} = \text{span}\{\varphi = (\varphi_1, \varphi_2) \in \mathfrak{H}_-^{(1)}: (p_0 - p_3) \varphi_1(p) = (p_1 + ip_2) \varphi_2(p)\}, \quad (68)$$

$$\mathfrak{H}'_- := \mathfrak{H}_-^{(1)} / \mathfrak{H}_-^{(0)}. \quad (69)$$

DEFINITION B.2.16. Using similar arguments as in Definition B.2.4 and Remark B.2.5 we define on \mathfrak{H}'_- a representation V'_4 of $\widetilde{\mathfrak{A}}_+^{\dagger}$ and a sesquilinear form $\langle \cdot, \cdot \rangle'_{\beta_-}$ by¹¹

$$V'_4(g) [\varphi]_- := [V_4(g) \varphi]_-, \quad (70)$$

$$\langle [\varphi]_-, [\psi]_- \rangle'_{\beta_-} := \int_{\mathcal{C}^+} (\varphi(p), \beta_-(p) \psi(p))_{\mathcal{H}} \mu_0(dp). \quad (71)$$

We conclude the case $n = 1$ with the following theorem:

THEOREM B.2.17. For $n = 1$ a linear map $\Phi_-: (\mathfrak{H}, U_4, (\cdot, \cdot)) \rightarrow (\mathfrak{H}'_-, V'_4, \langle \cdot, \cdot \rangle'_{\beta_-})$ defined by $(\Phi_- \varphi)(p) := \left[\Gamma_0 H_p \Gamma_0 \begin{pmatrix} 0 \\ \varphi(p) \end{pmatrix} \right]_-$, $\varphi \in \mathfrak{H}$, is an isometry that commutes with the representations (i.e. $\Phi_- U_4 = V'_4 \Phi_-$).

In the general case we define

DEFINITION B.2.18. (See analogous statements in Definition B.2.8 and Remark B.2.9.)

$$\mathfrak{H}_{-,n}^{(0)} := \text{span}\{\varphi \in \mathfrak{H}_-^{(n)}: \langle \varphi, \varphi \rangle_{\beta_-} = 0\}, \quad (72)$$

$$\mathfrak{H}_{-,n}^{(>)} := \text{span}\left\{ \varphi \in \mathfrak{H}_-^{(n)}: \varphi(p) = \bigotimes^n \begin{pmatrix} p_1 + ip_2 \\ -(p_0 + p_3) \end{pmatrix} \chi(p), \text{ for some function } \chi \right\}, \quad (73)$$

$$\mathfrak{H}'_{-,n} := \mathfrak{H}_-^{(n)} / \mathfrak{H}_{-,n}^{(0)}. \quad (74)$$

DEFINITION B.2.19. On $\mathfrak{H}'_{-,n}$ we define representations V'_4 of $\widetilde{\mathfrak{A}}_+^{\dagger}$ and a sesquilinear form $\langle \cdot, \cdot \rangle'_{\beta_-}$ as

$$V'_4(g) [\varphi]_- := [V_4(g) \varphi]_-, \quad (75)$$

$$\langle [\varphi]_-, [\psi]_- \rangle'_{\beta_-} := \int_{\mathcal{C}^+} (\varphi(p), \beta_-(p) \psi(p))_{\mathcal{H}} \mu_0(dp). \quad (76)$$

¹¹The bracket $[\cdot]_-$ denotes the equivalence class with respect to the semi-definite sesquilinear form $\langle \cdot, \cdot \rangle'_{\beta_-}$.

The following theorem concludes the section.

THEOREM B.2.20. *The linear map $\Phi_-: (\mathfrak{H}, U_4, (\cdot, \cdot)) \rightarrow (\mathfrak{H}'_{-,n}, V'_4, \langle \cdot, \cdot \rangle_{\beta_-})$ defined by $(\Phi_- \varphi)(p) := \left[\otimes \left(\Gamma_0 H_p \Gamma_0 \left(\begin{smallmatrix} 0 \\ \varphi^{\frac{1}{n}}(p) \end{smallmatrix} \right) \right) \right]_-$ is an isometry that commutes with the representations (i.e. $\Phi_- U_4 = V'_4 \Phi_-$).*

B.3. Causal and covariant nets of local C^* -algebras

In this section we will define the reference spaces h_F and h_B . In order to construct causal and covariant nets of local C^* -subalgebras of the algebras $CAR(h_F, \Gamma)$ and $CCR(h_B, \sigma)$, it is sufficient to specify nets of linear submanifolds in the corresponding reference spaces that satisfy the properties (0)_F–(III)_F and (I)_B–(III)_B of Section 2, respectively.

In the following \mathcal{O} , \mathcal{O}_1 and \mathcal{O}_2 denote open and bounded regions of the Minkowski space.

B.3.1. Representations of $\widetilde{\mathfrak{P}}_+^1$

Consider on $C_0^\infty(\mathbb{R}^4, \mathcal{H})$ the following representations of $\widetilde{\mathfrak{P}}_+^1$. We define for $f \in C_0^\infty(\mathbb{R}^4, \mathcal{H})$ and $g = (A, a) \in \widetilde{\mathfrak{P}}_+^1$

$$\begin{aligned} (T_1(g) f)(x) &:= D_A f(A_A^{-1}(x - a)), \\ (T_2(g) f)(x) &:= D_{\bar{A}} f(A_{\bar{A}}^{-1}(x - a)). \end{aligned} \tag{77}$$

Define then $T(g) := T_1(g) \oplus T_2(g)$, which is a representation of $\widetilde{\mathfrak{P}}_+^1$ on $C_0^\infty(\mathbb{R}^4, \mathcal{H}) \oplus C_0^\infty(\mathbb{R}^4, \mathcal{H})$.

Furthermore, define the following unitary strongly continuous representations on $\mathfrak{H}'_{+,n}$ or $\mathfrak{H}'_{-,n}$:

$$\begin{aligned} (V'_1(g) [\varphi]_+)(p) &:= [e^{-ipa} D_A \varphi(A_A^{-1} p)]_+, \\ (V'_2(g) [\varphi]_+)(p) &:= [e^{ipa} D_A \varphi(A_A^{-1} p)]_+, \\ (V'_3(g) [\varphi]_-)(p) &:= [e^{ipa} D_{\bar{A}} \varphi(A_{\bar{A}}^{-1} p)]_-, \\ (V'_4(g) [\varphi]_-)(p) &:= [e^{-ipa} D_{\bar{A}} \varphi(A_{\bar{A}}^{-1} p)]_-, \end{aligned} \tag{78}$$

where $g \in \widetilde{\mathfrak{P}}_+^1$ and $[\varphi]_\pm \in \mathfrak{H}'_{\pm,n}$.

B.3.2. The Fermi case (n odd)

B.3.2.1. CAR-algebras

We consider the complex Hilbert space $h_F := \mathfrak{H}'_{+,n} \oplus \mathfrak{H}'_{+,n} \oplus \mathfrak{H}'_{-,n} \oplus \mathfrak{H}'_{-,n}$ with a scalar product defined by

$$\begin{aligned} \langle [\varphi], [\psi] \rangle_F &:= \langle [\varphi_1]_+, [\psi_1]_+ \rangle'_{\beta_+} + \langle [\varphi_2]_+, [\psi_2]_+ \rangle'_{\beta_+} + \\ &+ \langle [\varphi_3]_-, [\psi_3]_- \rangle'_{\beta_-} + \langle [\varphi_4]_-, [\psi_4]_- \rangle'_{\beta_-} \end{aligned} \quad (79)$$

for all $[\varphi] = [\varphi_1]_+ \oplus [\varphi_2]_+ \oplus [\varphi_3]_- \oplus [\varphi_4]_-$, $[\psi] = [\psi_1]_+ \oplus [\psi_2]_+ \oplus [\psi_3]_- \oplus [\psi_4]_- \in h_F$. As anti-unitary involution we use

$$\Gamma \left([\varphi_1]_+ \oplus [\varphi_2]_+ \oplus [\varphi_3]_- \oplus [\varphi_4]_- \right) := \left[\Gamma_0 \varphi_3 \right]_+ \oplus \left[\Gamma_0 \varphi_4 \right]_+ \oplus \left[\Gamma_0 \varphi_1 \right]_- \oplus \left[\Gamma_0 \varphi_2 \right]_- . \quad (80)$$

Note that the definition of Γ is consistent with the specification of the equivalence classes $[\cdot]_+$ and $[\cdot]_-$. $\text{CAR}(h_F, \Gamma)$ is then given uniquely.

B.3.2.2. Existence theorems for the local algebras

Using the results of Section B.3.1 we have that the definition

$$V_F(g) := V'_1(g) \oplus V'_2(g) \oplus V'_3(g) \oplus V'_4(g) \quad (81)$$

specifies a reducible, unitary, strongly continuous representation of $\widetilde{\mathfrak{P}}_+$ over h_F .

LEMMA B.3.1. Γ and V_F commute (i.e. $\Gamma V_F(g) = V_F(g) \Gamma$ for all $g \in \widetilde{\mathfrak{P}}_+$).

Proof: The statement of the Lemma is a direct consequence of the definitions of V_F and Γ . ■

In order to relate the test functions over “position-space” with elements of the reference space consider the embedding $\mathfrak{J}_F: C_0^\infty(\mathbb{R}^4, \mathcal{H}) \oplus C_0^\infty(\mathbb{R}^4, \mathcal{H}) \rightarrow h_F$, defined by

$$\left(\mathfrak{J}_F(f_1 \oplus f_2) \right)(p) := \left[\left(\mathfrak{J}_+ f_1 \right) \right]_+ \oplus \left[\left(\mathfrak{J}_- f_1 \right) \right]_+ \oplus \left[\left(\mathfrak{J}_- f_2 \right) \right]_- \oplus \left[\left(\mathfrak{J}_+ f_2 \right) \right]_- , \quad (82)$$

for all $f_1 \oplus f_2 \in C_0^\infty(\mathbb{R}^4, \mathcal{H}) \oplus C_0^\infty(\mathbb{R}^4, \mathcal{H})$, and for $\left(\mathfrak{J}_\pm f \right)(p) := \widehat{f}(\pm p)$.¹²

¹²The hat $\widehat{}$ means the Fourier transformation restricted to the forward light cone \mathcal{C}^+ (compare also with Corollary A.1.3 for the massive case).

The covariance of the net will be based on the following result:

LEMMA B.3.2. *The embedding \mathfrak{J}_F intertwines between the representations T and V_F (i.e. $\mathfrak{J}_F T(g) = V_F(g) \mathfrak{J}_F$ for all $g \in \widehat{\mathfrak{A}}_+^1$).*

Proof: Apply the definitions of T , V_F and \mathfrak{J}_F and for all $f \in C_0^\infty(\mathbb{R}^d, \mathcal{H})$ use the relations:

$$\begin{aligned} \left[\mathfrak{J}_+ (T_1(g)f) \right]_+ &= V_1'(g) [\mathfrak{J}_+ f]_+, & \left[\mathfrak{J}_- (T_1(g)f) \right]_+ &= V_2'(g) [\mathfrak{J}_- f]_+, \\ \left[\mathfrak{J}_- (T_2(g)f) \right]_- &= V_3'(g) [\mathfrak{J}_- f]_-, & \left[\mathfrak{J}_+ (T_2(g)f) \right]_- &= V_4'(g) [\mathfrak{J}_+ f]_-. \end{aligned} \quad (83)$$

■

Define the net of linear submanifolds of h_F by

$$\begin{aligned} \mathbb{R}^d \supset \mathcal{O} &\longmapsto M_F(\mathcal{O}) \\ &:= \text{span} \{ \mathfrak{J}_F (f_1 \oplus f_2) : f_i \in C_0^\infty(\mathbb{R}^d, \mathcal{H}), \text{supp} f_i \subset \mathcal{O}, i = 1, 2 \}. \end{aligned} \quad (84)$$

THEOREM B.3.3. *The net $\mathcal{O} \mapsto M_F(\mathcal{O})$ satisfies the properties (0)_F, (I)_F, (II)_F and (III)_F of Section 2.*

Proof: The Γ -invariance of $M_F(\mathcal{O})$ is trivial from the definition of Γ and the isotony property is also clear.

To prove the covariance property of the net use Lemma B.3.2 and similar arguments as in the proof of Theorem A.3.2.

For the causality choose $f = f_1 \oplus f_2$, $h = h_1 \oplus h_2 \in h_F$ such that $\text{supp} f_i \perp \text{supp} h_i$, $i = 1, 2$. Then

$$\begin{aligned} \langle \mathfrak{J}_F (f_1 \oplus f_2), \mathfrak{J}_F (h_1 \oplus h_2) \rangle_F &= \int_{c^-} \left(\widehat{f}_1(p), \beta_+(p) \widehat{h}_1(p) \right)_{\mathcal{H}} \mu_0(dp) + \\ &+ \int_{c^+} \left(\widehat{f}_1(-p), \beta_-(-p) \widehat{h}_1(-p) \right)_{\mathcal{H}} \mu_0(dp) + \int_{c^+} \left(\widehat{f}_2(-p), \beta_-(-p) \widehat{h}_2(-p) \right)_{\mathcal{H}} \mu_0(dp) + \\ &+ \int_{c^+} \left(\widehat{f}_2(p), \beta_-(p) \widehat{h}_2(p) \right)_{\mathcal{H}} \mu_0(dp) = 0, \end{aligned} \quad (85)$$

again by Eq. (13). ■

As in the massive case, using the results of Section 2, we have the following

THEOREM B.3.4. *The net of C^* -algebras defined by*

$$\mathbb{R}^d \supset \mathcal{O} \longmapsto \mathcal{A}_F(\mathcal{O}) := C^* (\{ A(f) : f \in M_F(\mathcal{O}) \})^{\overline{\otimes}} \quad (86)$$

satisfies the properties (i)_F, (ii)_F and (iii)_F.

REMARK B.3.5: Notice that a “smaller” causal and covariant net of local C^* -algebras can be constructed from a certain local linear submanifold of $M_F(\mathcal{O})$:

$$\begin{aligned} \mathbb{R}^4 \supset \mathcal{O} &\longmapsto M_{F_1}(\mathcal{O}) \\ &:= \text{span} \{ \mathfrak{J}_F(f \oplus I_0 f) : f \in C_0^\infty(\mathbb{R}^4, \mathcal{H}), \text{supp} f \subset \mathcal{O} \}. \end{aligned} \quad (87)$$

Note that Γ acts on elements of $M_{F_1}(\mathcal{O})$ as the identity (i.e. we have that $\Gamma(\mathfrak{J}_F(f \oplus I_0 f)) = \mathfrak{J}_F(f \oplus I_0 f)$ for all $f \in C_0^\infty(\mathbb{R}^4, \mathcal{H})$), so the corresponding local algebra $\mathcal{A}_{F_1}(\mathcal{O})$ will be generated by self-adjoint elements.

B.3.3. The Bose case (n even)

B.3.3.1. CCR-algebra

Define $h_B := \mathfrak{H}'_{+,n} \oplus \mathfrak{H}'_{+,n}$ and interpret it as a real space. Consider the following real-bilinear form: for $\varphi = [\varphi_1]_+ \oplus [\varphi_2]_+$, $\psi = [\psi_1]_+ \oplus [\psi_2]_+ \in h_B$ let

$$\langle \varphi, \psi \rangle_B := \langle [\varphi_1]_+, [\psi_1]_+ \rangle'_{\beta_+} - \langle [\varphi_2]_+, [\psi_2]_+ \rangle'_{\beta_+}, \quad (88)$$

and define a non-degenerate symplectic form

$$\sigma(\cdot, \cdot) := \text{Im} \langle \cdot, \cdot \rangle_B. \quad (89)$$

LEMMA B.3.6. *The representation V_B on h_B defined by $V_B := V'_1 \oplus V'_2$ leaves the symplectic form σ invariant.*

Proof: For $[\varphi] = [\varphi_1]_+ \oplus [\varphi_2]_+$, $[\psi] = [\psi_1]_+ \oplus [\psi_2]_+ \in h_B$ we have

$$\begin{aligned} \langle V_B(g)[\varphi], V_B(g)[\psi] \rangle'_B &= \left\langle V'_1(g)[\varphi_1]_+, V'_1(g)[\psi_1]_+ \right\rangle'_{\beta_+} - \\ &\quad - \left\langle V'_2(g)[\varphi_2]_+, V'_2(g)[\psi_2]_+ \right\rangle'_{\beta_+} = \langle [\varphi], [\psi] \rangle'_B. \end{aligned} \quad (90)$$

Therefore, V_B leaves also the symplectic form σ invariant. \blacksquare

B.3.3.2. Existence theorem for the local algebras

Let T_1 and \mathfrak{J}_\pm be defined as in Sections B.3.1 and B.3.2. Consider an embedding $\mathfrak{J}_B: C_0^\infty(\mathbb{R}^4, \mathcal{H}) \rightarrow h_B$ defined by the equation

$$\mathfrak{J}_B f := [\mathfrak{J}_+ f]_+ \oplus [\mathfrak{J}_- f]_+ \quad (91)$$

for all $f \in C_0^\infty(\mathbb{R}^4, \mathcal{H})$.

The covariance will be based on the following

LEMMA B.3.7. *The embedding \mathfrak{J}_B intertwines between T_1 and V_B (i.e. $\mathfrak{J}_B T_1 = V_B \mathfrak{J}_B$).*

Proof: Use relations (83). ■

Define the net of linear submanifolds of h_B by

$$\mathbb{R}^4 \supset \mathcal{O} \longmapsto M_B(\mathcal{O}) := \text{span} \{ \mathcal{J}_B f : f \in C_0^\infty(\mathbb{R}^4, \mathcal{H}), \text{supp} f \subset \mathcal{O} \}. \quad (92)$$

THEOREM B.3.8. *The net $\mathcal{O} \mapsto M_B(\mathcal{O})$ satisfies the properties (I)_B, (II)_B and (III)_B of Section 2.*

Proof: Isotony is trivially satisfied and for the covariance use Lemma B.3.7 and analogous arguments as in Theorem A.3.2.

To prove the causality property, choose the test functions $f, h \in C_0^\infty(\mathbb{R}^4, \mathcal{H})$ such that $\text{supp} f \perp \text{supp} h$. Then

$$\begin{aligned} \langle \mathcal{J}_B f, \mathcal{J}_B h \rangle'_B &= \int_{\mathcal{C}^+} \left(\widehat{f}(p), \beta_+(p) \widehat{h}(p) \right)_{\mathcal{H}} \mu_0(dp) - \\ &\quad - \int_{\mathcal{C}^+} \left(\widehat{f}(-p), \beta_+(p) \widehat{h}(-p) \right)_{\mathcal{H}} \mu_0(dp) = 0, \end{aligned} \quad (93)$$

again by Eq. (12). Therefore, $\sigma(\mathcal{J}_B f, \mathcal{J}_B h) = 0$ and the net is causal. ■

Again, following Section 2 one has

THEOREM B.3.9. *The net of C^* -algebras defined by*

$$\mathbb{R}^4 \supset \mathcal{O} \longmapsto \mathcal{A}_B(\mathcal{O}) := C^* (\{ W(\varphi) : \varphi \in M_B(\mathcal{O}) \}) \quad (94)$$

satisfies the properties (i)_B, (ii)_B and (iii)_B.

B.4. Fock states and spectrality condition

As in Part A the Fock states for the fermionic models will be characterized by basis projections. In the Bose case we will specify an internal complexification, which characterizes the Fock state with spectrality condition.

B.4.1. The Fermi case

The statement of the problem is analogous as in Section A.4.1. Define the following operator on h_F :

$$P([\varphi_1]_+ \oplus [\varphi_2]_+ \oplus [\varphi_3]_- \oplus [\varphi_4]_-) := [\varphi_1]_+ \oplus 0 \oplus 0 \oplus [\varphi_4]_-$$

for all $[\varphi_1]_+ \oplus [\varphi_2]_+ \oplus [\varphi_3]_- \oplus [\varphi_4]_- \in h_F$.

LEMMA B.4.1. *P is a basis projection, i.e. $P + \Gamma P \Gamma = \mathbf{1}$, that commutes with V_F .*

Proof: This is a direct consequence of the definitions of P , Γ and V_F . ■

The Hilbert space \mathfrak{H}_P , associated to the representation π_P characterized by the basis projection P , is the antisymmetric part of the Fock space constructed from $P h_F \cong \mathfrak{H}_{n,+} \oplus \mathfrak{H}_{n,-}$.

THEOREM B.4.2. *The automorphisms α_g , $g \in \widetilde{\mathfrak{P}}_+^1$, are unitarily implementable on \mathfrak{H}_P , i.e. there exists a strongly continuous unitary representation $Q_{(\cdot)}$ of $\widetilde{\mathfrak{P}}_+^1$ on \mathfrak{H}_P such that*

$$\pi_P(\alpha_g(X)) = Q_g \pi_P(X) Q_g^{-1} \quad \text{for all } X \in \text{CAR}(h_F, \Gamma). \quad (95)$$

Furthermore, for all $a \in \mathbb{R}^4$, Q_a satisfies the spectrality condition.

Proof: The proof uses the same arguments as in the proof of Theorem A.4.2. The unitarities implementing the Bogoliubov automorphisms are now the second quantization of $V_1'(g) \oplus V_4(g)$ on the antisymmetric part of the Fock space over \mathfrak{H}_P . ■

B.4.2. The Bose case

One may choose the internal complexification J defined by $J([\varphi_1]_+ \oplus [\varphi_2]_+) := [i\varphi_1]_+ \oplus [-i\varphi_2]_+$ for all $[\varphi_1]_+ \oplus [\varphi_2]_+ \in h_B$. Similar arguments as in Section A.4.2 show that this choice specifies a Fock state where the spectrality condition is satisfied. The Bogoliubov automorphisms α_g generated by $V_B(g)$ are implemented by the second quantization of $V_B(g)$ on the symmetric Fock space over h_B , considered as a Hilbert space with the scalar product characterized by J .

Acknowledgements

We are indebted to Prof. H. J. Borchers for valuable remarks regarding a preliminary version of this paper. One of the authors (M. J.) would like to thank Prof. K. Fredenhagen for some useful advices concerning specification of the basis projection associated to the fermionic models.

REFERENCES

- [1] R. Haag and D. Kastler: *J. Math. Phys.* **5** (1964), 848.
- [2] R. Haag: *Local Quantum Physics*, Springer, Berlin 1992.
- [3] G. F. dell'Antonio: *Commun. Math. Phys.* **9** (1968), 81.
- [4] H. Araki: in: *Operator Algebras and Mathematical Physics*, Vol. 62 of *Contemp. Math.* (AMS) (1987), 23.
- [5] H. Araki and M. Shiraishi: *Publ. RIMS Kyoto Univ.* **7** (1971/72), 105.
- [6] H. Araki: *Publ. RIMS Kyoto Univ.* **7** (1971/72), 121.
- [7] H. Baumgärtel and M. Wollenberg: *Causal Nets of Operator Algebras, Mathematical Aspects of Algebraic Quantum Field Theory*, Akademie Verlag, Berlin 1992.
- [8] O. Bratteli and D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics*, Vol. II, Springer, Berlin 1981.
- [9] E. P. Wigner: *Ann. Math.* **40** (1939), 149.
- [10] A. O. Barut and R. Rączka: *Theory of Group Representations and Applications*. Polish Scientific Publishers, Warsaw 1980.

- [11] H. Baumgärtel and M. Jurke: Preprint No. 60, SFB 288, TU Berlin, 1993.
- [12] H. Baumgärtel, M. Jurke and F. Lledó: Preprint No. 120, SFB 288, TU Berlin, 1994.
- [13] M. Jurke and F. Lledó: Preprint No. 137, SFB 288, TU Berlin, 1994.
- [14] M. A. Neumark: *Normierte Algebren*. DVW, Berlin 1990.
- [15] M. Reed and B. Simon: *Methods of Modern Mathematical Physics*, Vol. II. AP, San Diego 1975.
- [16] H. J. Borchers: in: *Operator Algebras and Mathematical Physics*, Vol. 62 of *Contemp. Math.* (AMS) (1987), 143.
- [17] A. Guichardet: *Symmetric Hilbert Spaces and Related Topics*, Springer, Berlin 1972.
- [18] B. Thaller: *The Dirac Equation*, Springer, Berlin 1992.
- [19] D. J. Simms: *Lie Groups and Quantum Mechanics*, Springer, Berlin 1968.
- [20] N. N. Bogoliubov, A. A. Logunov and I. T. Todorov: *Introduction to Axiomatic Quantum Field Theory*, W. A. Benjamin Inc., Reading (Massachusetts) 1975.
- [21] P. D. Hislop: *Commun. Math. Phys.* **185** (1988), 193.
- [22] A. O. Barut and R. Rączka: *Ann. Inst. Henri Poincaré* **2** (1972), 111.