On Polar Legendre Polynomials

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Abstract

We introduce a new class of polynomials $\{P_n\}$, that we call polar Legendre polynomials, they appear as solutions of an inverse Gauss problem of equilibrium position of a field of forces with $n+1$ unit masses. We study algebraic, differential and asymptotic properties of this class of polynomials, that are simultaneously orthogonal with respect to a differential operator and a discrete-continuous Sobolev type inner product.

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1 Introduction

Let $\{L_n\}_{\{n\in\mathbb{N}\}}$ be the monic Legendre polynomials. It is well know that L_n satisfies the orthogonality relation

$$\int_{-1}^{1} L_n(x)x^k dx = 0, \quad k = 0, 1, \cdots, n-1\,, \tag{1}$$

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and the differential equation

$$-n(n+1)L_n(z) = \big((1-z^2)L_n'(z)\big)'\,. \tag{2}$$

It can be proved, using integration by parts, that the derivatives of $\{L_n\}$ satisfy the following orthogonality condition

$$\int_{-1}^{1} L_{n+1}'(x)x^k(1-x^2)dx = 0, \quad k = 0, 1, \cdots, n-1\,. \tag{3}$$

For a fixed complex number ζ, that we are going to call the pole, let us define the $P_n = P_{\zeta,n}$ as a monic polynomial, such that

$$(n+1)\,L_n(z) = ((z-\zeta)\,P_n(z))' = P_n(z) + (z-\zeta)P_n'(z)\,, \tag{4}$$

P_n is called the n-th *polar Legendre polynomial.* Obviously, P_n is a monic polynomial of degree n, that by (1) and (4) satisfies the following "orthogonality relation"

$$\int_{-1}^{1} [P_n(x) + (x-\zeta)P_n'(x)]x^k dx = 0, \qquad k = 0, 1, \cdots, n-1. \tag{5}$$

This type of orthogonality relations generated by differential operators was introduced initially in [2], where the existence and uniqueness conditions for more general differential expressions were studied in detail.

Observe that the polynomial

$$\Pi_{\zeta,n+1}(z) = (z-\zeta)\,P_n(z) \tag{6}$$

is the primitive of $(n+1)\,L_n(z)$, such that $\Pi_{\zeta,n+1}(\zeta) = 0$; that is,

$$\Pi_{\zeta,n+1}(z) = (n+1)\int_{\zeta}^{z} L_n(t)dt. \tag{7}$$

$\Pi_{\zeta,n+1}$ will be called the *primitive Legendre polynomial.* Notice that the properties of $P_n = P_{\zeta,n}$ and the properties of $\Pi_{\zeta,n+1}$ are closely related.

It is important to observe that since the functions that we are considering are entire functions, we can assume that the definite integrals considered are line integrals defined on the straight line segment with initial point in the lower limit of integration and end point in the upper limit of integration.

Now, combining (2) and (4) and integrating from ζ to z, we have the fundamental formula

$$n\,(z-\zeta)\,P_n(z) = (1-\zeta^2)L_n'(\zeta) - (1-z^2)L_n'(z)\,. \tag{8}$$

Furthermore, from (4) it is easy to see that $\Pi_{\zeta,n+1}(z)$ is the $(n+1)$-th monic orthogonal polynomial with respect to the Sobolev-type inner product (called "discrete-continuous type", see [1])

$$\langle p, q\rangle = p(\zeta)q(\zeta) + \int_{-1}^{1} p'(x)\, q'(x)\, dx\,.$$

In [4], necessary and sufficient conditions under which such Sobolev-type orthogonal polynomials satisfy a differential equation of spectral type with polynomial coefficients are studied.

The localization of critical points of a given class of polynomials has many physical and geometrical interpretations. Let us consider for instance, a field of forces given by a system of n masses m_j, $(1 \leq j \leq n)$ at the fixed points z_j, $(1 \leq j \leq n)$, that repels a movable unit mass at z according to the inverse distance law.

Let $Q_m(z) = (z-z_1)^{m_1} \cdot (z-z_2)^{m_2} \cdots (z-z_n)^{m_n}$ where $m = m_1 + m_2 + \cdots + m_n$. The logarithmic derivative of $Q_m(z)$ is

$$\frac{d(\log(Q_m(z)))}{dz} = \frac{Q'_m(z)}{Q_m(z)} = \frac{m_1}{(z-z_1)} + \frac{m_2}{(z-z_2)} + \cdots + \frac{m_n}{(z-z_n)}\,. \tag{9}$$

The conjugate of $\dfrac{m_j}{(z-z_j)}$ is a vector directed from z_j to z, so this vector represents the force at the movable unit mass z due to a single fixed particle at z_j. By (9) the positions of equilibrium in the field of force coincide with the zeros of Q'_m, that are not zeros of Q_m. In particular, all multiple zeros of Q_m are equilibrium positions. This result is known as Gauss's theorem ([6, Theorem 1.2.1, Ch.3]).

Now, we consider the following inverse problem: let $z'_1, z'_2, \cdots, z'_n$ be the zeros of the orthogonal polynomial L_n which we assume to be the equilibrium positions of a field of forces with $n+1$ unit masses, one of which is given at the point ζ. What is the location of the remaining masses? Let P_n be the monic polynomial whose zeros are the remaining equlibrium positions. By (4) and (6)

$$\frac{(n+1)\, L_n(z)}{(z-\zeta)P_n(z)} = \frac{1}{z-\zeta} + \frac{P'_n(z)}{P_n(z)} = \frac{\Pi'_{\zeta,n+1}(z)}{\Pi_{\zeta,n+1}(z)}\,. \tag{10}$$

Then, according to (9), (10) and the above interpretation of the logarithmic derivative, the location of the remaining unit masses are the zeros of the polynomial P_n), or equivalently the poles of (10). For this reason we call P_n polar polynomial.

The main purpose of this paper is to study some algebraic, differential and analytic properties of the polar Legendre polynomials, or equivalently of the primitive Legendre polynomials. The paper is organized as follows. In section 2 we study the orthogonality relations and recurrence relations of the polar Legendre polynomials and section 3 is devoted to the study of the location of zeros and the asymptotic behavior of zeros and polynomials.

2 Orthogonality and recurrence relations

Besides the well known results on orthogonality mentioned in the previous section, we have the following additional orthogonality relations between $\{L_n\}$ and $\{P_n\}$,

Theorem 1. *The polar Legendre polynomial P_n with pole $\zeta \in \mathbb{C}$ verifies*

$$\int_{-1}^{1} \left[P_n(x) + (x-\zeta)P_n'(x)\right] L_m(x)dx = \begin{cases} 0 & m \neq n, \\ (n+1)\|L_n\|^2 & m = n, \end{cases} \tag{11}$$

where $\|L_n\|^2 = \int_{-1}^{1} L_n^2(x)dx$.

Additionally if $n > 0$ then

$$\int_{-1}^{1} (x-\zeta)P_n(x)L_m(x)dx = \begin{cases} \frac{2}{n}(1-\zeta^2)L_n'(\zeta) & m = 0, \\ 0 & 0 < m < (n-1), \\ -\frac{n+1}{n}\|L_n\|^2 & m = n-1, \\ 0 & m = n, \\ \|L_{n+1}\|^2 & m = n+1, \\ 0 & (n+1) < m. \end{cases} \tag{12}$$

Proof. Since by (4)

$$\int_{-1}^{1} \left[P_n(x) + (x-\zeta)P_n'(x)\right] L_m(x)dx = (n+1)\int_{-1}^{1} L_n(x)L_m(x)dx,$$

then (11) is a direct consequence of (1). To prove (12), let us denote

$$I_{n,m} = \int_{-1}^{1} (x-\zeta)P_n(x)L_m(x)dx.$$

- The case $m = 0$ is a direct consequence of the fundamental relation (8) and (3).
- If $0 < m < (n-1)$ by (7) and Fubini's theorem

$$\begin{aligned} I_{n,m} &= (n+1)\int_{-1}^{1} \left(\int_{\zeta}^{1} L_n(t)dt + \int_{1}^{x} L_n(t)dt\right) L_m(x)dx \\ &= (n+1)\int_{-1}^{1} \left(\int_{1}^{x} L_n(t)dt\right) L_m(x)dx \\ &= (n+1)\int_{-1}^{1} \left(\int_{t}^{-1} L_m(x)dx\right) L_n(t)dt = 0, \end{aligned} \tag{13}$$

since $\int_t^{-1} L_m(x)dx$ is a polynomial (in t) of degree $m+1 < n$. Notice, that (13) is true if we use any other polynomial Q_m of degree $m < n-1$, instead of L_m.

-If $m = n-1$, observe that as consequence of (2) we have

$$F_n(x) = \int L_{n-1}(x)dx = \frac{(x^2-1)}{n\,(n-1)} L'_{n-1}(x) = \frac{x^n}{n} + \dots\,;$$

therefore, $F_n(-1) = F_n(1) = 0$. Hence, integrating by parts (12), using (3) and (1) we get,

$$I_{n,n-1} = -(n+1)\int_{-1}^{1} L_n(x)F_n(x)dx = -\frac{n+1}{n}\|L_n\|^2.$$

-For the case $m = n$, remember that the Legendre polynomials satisfy the identity $L_n(x) = (-1)^n L_n(-x)$, $x \in \mathbb{R}$, and; therefore, the powers of L_n are either all odd or all even. By (7) it is clear that the powers of $(x-\zeta)P_n(x)$ have opposite parity from the ones of L_n; consequently all the powers of $(x-\zeta)P_n(x)L_n(x)$ are odd and since we are integrating in a symmetric interval,

$$I_{n,n} = \int_{-1}^{1} (x-\zeta)P_n(x)L_n(x)dx = 0,$$

- If $m = n+1$ by (1)

$$I_{n,n+1} = \int_{-1}^{1} (x-\zeta)P_n(x)L_{n+1}(x)dx = \int_{-1}^{1} x^{n+1}L_{n+1}(x)dx = \int_{-1}^{1} L_{n+1}^2(x)dx = \|L_{n+1}\|^2.$$

-Finally, the case $m > (n+1)$ is a direct consequence of (1). □

As a consequence of these orthogonality relations, let us prove now a recurrence relation for the polar Legendre polynomials,

Theorem 2. *The polar Legendre polynomials $\{P_n\}$ with pole $\zeta \in \mathbb{C}$, satisfy the following recurrence relation*

$$P_{n+1}(z) = z\,P_n(z) + a_n P_{n-1}(z) + b_n, \tag{14}$$

for $n > 1$, where $P_0 \equiv 1$, and $P_1(z) = z + \zeta$,

$$a_n = \frac{1-n^2}{n^2}\left(\frac{\|L_n\|}{\|L_{n-1}\|}\right)^2 \quad \text{and} \quad b_n = \frac{2}{n}(\zeta^2-1)L'_n(\zeta). \tag{15}$$

Proof. Let $\{\alpha_{n,0},\, \alpha_{n,1},\, \dots,\, \alpha_{n,n}\}$ be a set of $n+1$ coefficients such that

$$(x-\zeta)P_n(x) = \sum_{k=0}^{n+1} \alpha_{n,k}P_k(x). \tag{16}$$

Observe that $\alpha_{n,n+1} = 1$ since the polar polynomials P_n are monic. Now in order to determine the other coefficients $\alpha_{n,k}$, $k = 0, \cdots, n$, let us consider

$$
\begin{aligned}
(z-\zeta)\left[P_n(z)+(n+1)L_n(z)\right] &= (z-\zeta)\left[P_n(z)+((z-\zeta)P_n(z))'\right] \\
&= \sum_{k=0}^{n+1}\alpha_{n,k}\left[P_k(z)+(z-\zeta)P_k'(z)\right]. \qquad (17)
\end{aligned}
$$

By the orthogonality relation (12), we have,

$$
\sum_{k=0}^{n+1}\alpha_{n,k}\int_{-1}^{1}L_m(x)[P_k(x)+(x-\zeta)P_k'(x)]dx=\alpha_{n,m}\,(m+1)\,\|L_m\|^2, \qquad (18)
$$

for $m=0,\,1,\,\ldots,\,n$. On the other hand, (1) and (12) give us

$$
\int_{-1}^{1}L_m(x)(x-\zeta)\left[P_n(x)+(n+1)L_n(x)\right]dx \;=\; \begin{cases} \dfrac{2}{n}(1-\zeta^2)L_n'(\zeta) & m=0 \\ 0 & 0<m<(n-1) \\ \dfrac{n^2-1}{n}\|L_n\|^2 & m=n-1 \\ -\zeta(n+1)\|L_n\|^2 & m=n. \end{cases} \qquad (19)
$$

Thus multiplying (17) by L_m, integrating over $[-1,1]$ and using (18)–(19) we get,

$$
\alpha_{n,m}=\begin{cases} \dfrac{2}{n}(1-\zeta^2)L_n'(\zeta) & m=0 \\ 0 & 0<m<(n-1) \\ \dfrac{n^2-1}{n^2}\left(\dfrac{\|L_n\|}{\|L_{n-1}\|}\right)^2 & m=n-1 \\ -\zeta & m=n. \end{cases}
$$

Replacing these values in (16) and after cancelations we get (14). □

3 Zeros and asymptotics

Let us now study the distribution of the zeros of the polar Legendre polynomials. The next lemma collects several direct consequences of the formulas contained in the introductory section, in special the fundamental formula (8).

Lemma 1. *The polar Legendre polynomials $\{P_n\}$ with pole $\zeta\in\mathbb{C}$, satisfy*

1. *If n is odd and $\zeta\in\mathbb{R}^*=\mathbb{R}\setminus\{0\}$, then $x=-\zeta$ is a zero of P_n.*

2. *The zeros of the the polar polynomial P_n have multiplicity at most 2 and the multiple zeros are on $[-1,1]$.*

3. *If $\zeta = 1$, $\zeta = -1$ or $L_n'(\zeta) = 0$, then the zeros of $P_n(x)$ are -1 or 1 and the $(n-1)$ critical points of the Legendre polynomial L_n.*

4. *All the zeros of P_n are on the following lemniscate*

$$\Lambda_n(\zeta) := \left\{ z \in \mathbb{C} : \prod_{k=0}^{n} |z - x_{n,k}| = \rho_n(\zeta) \right\}, \tag{20}$$

where $\rho_n(\zeta) = \prod_{k=0}^{n} |\zeta - x_{n,k}|$, $x_{n,0} = -1$, $x_{n,n} = 1$, and $x_{n,1}, x_{n,2}, \cdots, x_{n,n-1}$ are the $(n-1)$ critical points of the Legendre polynomial L_n.

Proof. In order to prove 1., using the fact that if n is odd then all the powers of L_n are odd and (7), one gets $P_n(-\zeta) = 0$.

Now, let us suppose that w is a zero of P_n of multiplicity greater or equal to 3. Notice that by (4) a zero of P_n with multiplicity greater than 2 is a zero of L_n and also a zero of L_n', thus $L_n(w) = L_n'(w) = 0$ which would imply that w is a zero of multiplicity 2 of L_n. This is impossible since the zeros of L_n are all simple, so we have proved 2.

Assertion 3. is a direct consequence of fundamental formula (8) since

$$\left|z_0^2 - 1\right| \left|L_n'(z_0)\right| = \left|\zeta^2 - 1\right| \left|L_n'(\zeta)\right|, \tag{21}$$

and then 4. follows by considering the factorization of $(z^2-1)L_n'(z)$.

□

Remark 1. *The following example shows that the zeros of $P_n(z)$ do not have to be simple. Let $\zeta = \frac{2\sqrt{3}}{3}$ (or $\zeta = -\frac{2\sqrt{3}}{3}$), hence the corresponding polar Legendre polynomial of degree two is*

$$P_2(z) = z^2 + \frac{2\sqrt{3}}{3}z + \frac{1}{3} \; (\text{or } P_2(z) = z^2 - \frac{2\sqrt{3}}{3}z + \frac{1}{3}).$$

Notice that $z = -\frac{\sqrt{3}}{3}$ (or $z = \frac{\sqrt{3}}{3}$) is a zero of multiplicity two of $P_2(z)$.

For the boundedness of the zeros of the polar Legendre polynomials $\{P_n\}$ we have the following result,

Lemma 2. *Given $\zeta \in \mathbb{C}$ let us define $\Delta_\zeta = \sup_{x\in[-1,1]} |\zeta - x|$ and $\delta_\zeta = \inf_{x\in[-1,1]} |\zeta - x|$, then*

1. *All the zeros of the polar Legendre polynomials $\{P_n\}$ with pole $\zeta \in \mathbb{C}$ are contained in $|z| \le \Delta_\zeta + 1$.*

2. *If $\delta_\zeta > 1$, the zeros of the polar Legendre polynomials $\{P_n\}$ with pole $\zeta \in \mathbb{C}$ are simple and contained in the exterior of the ellipse $|z+1| + |z-1| = 2\alpha$, where $1 < \alpha < \delta_\zeta$.*

Proof. 1. We know by(20) that the zeros of $P_n(z)$ are on the lemniscate $\Lambda_n(\zeta)$. Since $\rho_n(\zeta) < \Delta_\zeta^{n+1}$, the zeros of $P_n(z)$ are contained in the interior of the lemniscate $\prod_{k=0}^{n} |z - x_{n,k}| = \Delta_\zeta^{n+1}$, where $x_{n,0} = -1$, $x_{n,n} = 1$, and $x_{n,1}, x_{n,2}, \cdots, x_{n,n-1}$ are the $(n-1)$ critical points of the Legendre polynomial L_n, and therefore $|x_{n,k}| \leq 1$. Now, for any z^*, such that $|z^*| > 1 + \Delta_\zeta$, then

$$\prod_{k=0}^{n} |z^* - x_{n,k}| \geq \prod_{k=0}^{n} ||z^*| - |x_{n,k}|| > \Delta_\zeta^{n+1},$$

and therefore 1. is obtained.

2. Let z be such that $|z+1|+|z-1| = 2\alpha$. From the well known arithmetic-geometric mean inequality we get

$$\prod_{k=0}^{n} |z - x_{n,k}| \leq \left(\frac{1}{n+1} \sum_{k=0}^{n} |z - x_{n,k}| \right)^{n+1} < \alpha^{n+1}.$$

If z is a zero of P_n, from (20) we get that

$$\prod_{k=0}^{n} |z - x_{n,k}| = \prod_{k=0}^{n} |\zeta - x_{n,k}| > \delta_\zeta^{n+1} > \alpha^{n+1}.$$

Thus, by these inequalities and 2. of lemma 1 we obtain 2. of this lemma. □

Finally, let us study the asymptotic behavior of the zeros of the polar Legendre polynomials,

Theorem 3. *Let $\{P_n\}$ be the polar Legendre polynomials with pole $\zeta \in \mathbb{C} \setminus [-1,1]$, such that $\delta_\zeta > 1$. Then the set of accumulation points of zeros of $\{P_n\}$ are on the ellipse*

$$\Lambda(\zeta) := \left\{ z \in \mathbb{C} : z = \frac{\rho^2(\zeta)+1}{2\rho(\zeta)} \cos\theta + i\, \frac{\rho^2(\zeta)-1}{2\rho(\zeta)} \sin\theta, \quad 0 \leq \theta < 2\pi \right\}, \qquad (22)$$

where $\rho(\zeta) := |\zeta + \sqrt{\zeta^2 - 1}|$ and the branch of the square root is chosen so that $|z + \sqrt{z^2-1}| > 1$ for $z \in \mathbb{C} \setminus [-1,1]$.

Proof. By (21) we have that the zeros of the nth polar Legendre polynomial satisfies the equation

$$\left|\frac{z^2-1}{n}\right|^{\frac{1}{n}} |L_n'(z)|^{\frac{1}{n}} = \left|\frac{\zeta^2-1}{n}\right|^{\frac{1}{n}} |L_n'(\zeta)|^{\frac{1}{n}} . \tag{23}$$

On other hand, from the asymptotic properties of the Legendre polynomials it is well known that

$$\lim_{n\to\infty} |L_n'(z)|^{\frac{1}{n}} = \frac{|z+\sqrt{z^2-1}|}{2}, \tag{24}$$

uniformly on compact subset of $\mathbb{C}\setminus[-1,1]$. Taking limit as $n\to\infty$, from 2. of lemma 2, it is possible to use (24) on both sides of (23) and we have that set of accumulation points of the zeros of the sequence of polynomials $\{P_n\}$ are contained in the curve

$$\Lambda(\zeta) = \left\{ z\in\mathbb{C} : |z+\sqrt{z^2-1}| = \rho(\zeta) \right\} .$$

Hence

$$\begin{aligned} z+\sqrt{z^2-1} &= \rho(\zeta)\, e^{i\theta}, \quad 0\le\theta<2\pi, \\ z-\sqrt{z^2-1} &= \rho(\zeta)^{-1}\, e^{-i\theta}, \\ 2z &= \rho(\zeta)e^{i\theta} + \rho(\zeta)^{-1}e^{-i\theta}. \end{aligned}$$

□

Finally, let us study the relative asymptotic of the polar Legendre polynomials $\{P_n\}$ with respect to the Legendre polynomials $\{L_n\}$ and their derivatives $\{L_n'\}$

Theorem 4. *Let ζ be a fixed complex number and L_n, P_n y Δ_ζ be as above. Then*

1. $\dfrac{n\,P_n(z)}{L_n'(z)} \underset{n\to\infty}{\rightrightarrows} \dfrac{z^2-1}{z-\zeta}$, *uniformly on compact subsets of the set* $\{z\in\mathbb{C} : |z| > \Delta_\zeta + 1\}$. (25)
2. $\dfrac{P_n(z)}{L_n(z)} \underset{n\to\infty}{\rightrightarrows} \dfrac{\sqrt{z^2-1}}{z-\zeta}$, *uniformly on compact subsets of the set* $\{z\in\mathbb{C} : |z| > \Delta_\zeta + 1\}$. (26)

Proof. Let K be a compact subset of $\{z\in\mathbb{C} : |z| > \Delta_\zeta + 1\}$. By the fundamental formula (8) we have

$$\frac{n\,P_n(z)}{L_n'(z)} = \frac{1-\zeta^2}{z-\zeta}\frac{L_n'(\zeta)}{L_n'(z)} + \frac{z^2-1}{z-\zeta} .$$

Hence, in order to prove (25) it is sufficient to show that

$$\frac{L_n'(\zeta)}{L_n'(z)} \underset{n\to\infty}{\rightrightarrows} 0 \tag{27}$$

uniformly on compact subset $K \subset \{z \in \mathbb{C} : |z| > \Delta_\zeta + 1\}$. Let $x'_{n,1}, x'_{n,2}, \ldots, x'_{n,n-1}$ be the $n-1$ zeros of $L'_n(z)$ and$d_{\zeta,K} = \inf_{\substack{z \in K \\ |w| = \Delta_\zeta + 1}} |z - w|$, hence

$$\left| \frac{L'_n(\zeta)}{L'_n(z)} \right| = \frac{\prod_{k=1}^{n-1} |\zeta - x'_{n,k}|}{\prod_{k=1}^{n-1} |z - x'_{n,k}|} < \left(\frac{\Delta_\zeta}{d_{\zeta,K} + \Delta_\zeta} \right)^{n-1} < 1, \quad z \in K.$$

which is equivalent to the uniform convergence of (27) to zero on a compact subset K of $\{z \in \mathbb{C} : |z| > \Delta_\zeta + 1\}$.

The statement (26) is a direct consequence of (25) and the well known asymptotic behavior of the Legendre polynomials (see [7, corollary 1.6])

$$\frac{L'_n(z)}{n\, L_n(z)} \underset{n\to\infty}{\rightrightarrows} \frac{1}{\sqrt{z^2 - 1}}, \text{ uniformly on compact subsets of } \mathbb{C} \setminus [-1, 1].$$

□

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