Asymptotically extremal polynomials with respect to varying weights and application to Sobolev orthogonality

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Abstract

We study the asymptotic behavior of the zeros of a sequence of polynomials whose weighted norms, with respect to a sequence of weight functions, have the same nth root asymptotic behavior as the weighted norms of certain extremal polynomials. This result is applied to obtain the (contracted) weak zero distribution for orthogonal polynomials with respect to a Sobolev inner product with exponential weights of the form $e^{-\varphi(x)}$, giving a unified treatment for the so-called Freud (i.e., when φ has polynomial growth at infinity) and Erdös (when φ grows faster than any polynomial at infinity) cases. In addition, we provide a new proof for the bound of the distance of the zeros to the convex hull of the support for these Sobolev orthogonal polynomials.

Key words: Logarithmic potential theory, orthogonal polynomials, zero location, asymptotic behavior.

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1 Introduction and main results

In this paper, we deal with the asymptotic behavior of asymptotically extremal polynomials with respect to varying weights. In this sense, our main result is related to the weak zero asymptotics for such sequences of polynomials. Then, this result is applied to study the asymptotic behavior of polynomials orthogonal with respect to Sobolev inner products with exponential weights.

In order to state our main results it is convenient to recall some basic topics in potential theory. First, recall the notion of admissible weights (see [10, Def.I.1.1]).

Definition 1.1 *Given a closed set $\Sigma \subset \mathbb{C}$, we say that a function $\omega : \Sigma \to [0,\infty)$ is an admissible weight on Σ if the following conditions are satisfied:*

i) ω is upper semi-continuous;
ii) the set $\{z \in \Sigma : \omega(z) > 0\}$ has positive (logarithmic) capacity;
iii) if Σ is unbounded, then $\lim\limits_{\substack{|z|\to\infty \\ z\in\Sigma}} |z|\omega(z) = 0$.

Given such an admissible weight ω in the closed set Σ and setting $\phi(z) = -\log\omega(z)$, we know (see e.g. [10, Ch.I]) that there exists a unique measure μ_ω, with (compact) support in Σ, for which the infimum of the weighted (logarithmic) energy

$$I_\omega(\mu) \;=\; -\int\!\!\int \log|z-x|d\mu(z)d\mu(x) \;+\; 2\int \phi(x)d\mu(x)\,,\ \mu\in M(\Sigma)\,,$$

is attained, where, as usual, $M(\Sigma)$ denotes the set of probability measures supported in Σ. Moreover, setting $F_\omega = I_\omega(\mu_\omega) - \int \phi d\mu_\omega$, we have the following property, which uniquely characterizes both the equilibrium measure μ_ω and its support $\operatorname{supp}\mu_\omega$:

$$V^{\mu_\omega}(z) + \phi(z) \begin{cases} \leq\ F_\omega\,, & \text{for } z \in \operatorname{supp}\mu_\omega\,, \\ \geq\ F_\omega\,, & \text{for quasi-every } z \in \Sigma\,, \end{cases} \qquad (1)$$

where for a measure σ, V^σ denotes its logarithmic potential, that is,

$$V^\sigma(z) \;=\; -\int \log|z-x|\,d\sigma(x)\,,$$

and a property is said to be satisfied for "quasi-every" z in a certain set, if it holds except in a possible subset of zero capacity (see e.g. [10] for details).

Indeed, let Σ be a closed set and ω an admissible weight on Σ. Then, a sequence of monic polynomials $\{p_n\}_{n\in\mathbb{N}}$ is said to be asymptotically extremal with respect to the weight ω if it holds (see [10]):

$$\lim_{n\to\infty} \|\omega^n p_n\|_{L^\infty(\Sigma)}^{1/n} = \exp(-F_\omega)\,, \tag{2}$$

where, as usual, the symbol $\|\cdot\|_{L^\infty(\Sigma)}$ denotes the sup-norm in the set Σ.

The study of weighted polynomials of the form $\omega(z)^n p_n(z)$ has applications to many problems in approximation theory (see e.g. the monographes [10] and [11]). It is well known that if for each $n \in \mathbb{N}$ T_n^ω is the n-th (weighted) Chebyshev polynomial with respect to the weight ω^n, that is, if it is the unique monic polynomial of degree n such that

$$\|\omega^n T_n^\omega\|_{L^\infty(\Sigma)} = \inf\{\|\omega^n p\|_{L^\infty(\Sigma)}\,,\, p(z) = z^n + \ldots\}\,,$$

then the sequence $\{T_n^\omega\}$ has the asymptotic behavior given in (2) (see [10, Ch.III]).

Under mild conditions on the weight ω, in [10, Ch.III] it is shown that the zeros of such a sequence of polynomials asymptotically follow the equilibrium measure μ_ω, in the sense of the weak-* convergence. To be precise, denoting by $x_{n,k}, k = 1, \ldots, n$, the zeros of p_n and by $\nu_n = \dfrac{1}{n}\sum_{k=1}^{n} \delta_{x_{n,k}}$ the corresponding unit zero counting measure, where as usual the symbol δ_a stands for the Dirac delta with mass concentrated at a, we have that if $\operatorname{supp}\mu_\omega$ has empty interior and connected complement, then (see [10, Th.III.4.2] or the previous paper [8]):

$$\nu_n \longrightarrow \mu_\omega\;,\; n\to\infty\,, \tag{3}$$

where the convergence holds in the weak-* sense.

In this paper we extend this result by considering asymptotically extremal monic polynomials $\{p_n\}$ with respect to varying weights $\{\omega_n^{1/n}\}_{n\in\mathbb{N}}$, that is, when it holds

$$\lim_{n\to\infty} \|\omega_n p_n\|_{L^\infty(\Sigma)}^{1/n} = \exp\left(-F_\omega\right)\,, \tag{4}$$

where $\omega_n^{1/n}$ tends to ω in some sense, which will be indicated below. For simplicity, we deal with the case where Σ is a closed interval, but the extension for general closed sets seems feasible. Our result also extends previous theorems by Gautschi and Kuijlaars [3, Theorem 5] and Gonchar and Rakhmanov [4].

Now, we are going to state our main result.

Theorem 1.1 *Let Σ be a closed interval in $\mathbb{R}$ and ω an admissible weight on Σ. If $\{\omega_n^{1/n}\}$ is a sequence of admissible weights on Σ such that*

$$\liminf_{n\to\infty} \omega_n(x)^{1/n} \ge \omega(x)\,,\ \textit{for qu.e. } x \in \Sigma\,, \tag{5}$$

and $\{p_n\}$ *a sequence of monic polynomials satisfying condition* (4)*, then* (3) *holds.*

Next, this general result will be applied to obtain the (contracted) weak zero asymptotics for orthogonal polynomials with respect to a Sobolev inner product with exponential weights. These exponential weights, of the form $e^{-\varphi(x)}$, include as particular cases the so-called Freud (i.e., when φ has a polynomial growth at infinity) and Erdös (when φ grows faster than any polynomial at infinity) weights.

Indeed, let $\{\mu_i\}_{i=0}^M$ be a set of $M+1$ positive Borel measures supported on $\Delta \subset \mathbb{R}$. In the linear space $\mathbb{P}$ of polynomials with complex coefficients we introduce the inner product

$$\langle p, q\rangle_S = \sum_{i=0}^{M} \int_\Delta p^{(i)}(x)\, \overline{q}^{(i)}(x)\, d\mu_i(x)\,, \tag{6}$$

assuming all the integrals are convergent and $p^{(i)}$ denotes the i-th derivative of p. For a polynomial p, the corresponding Sobolev norm is

$$\|p\|_S^2 \;=\; \sum_{i=0}^{M} \|p^{(i)}\|_{L^2(d\mu_i)}^2\,, \tag{7}$$

When Δ is a compact subset of the complex plane, the algebraic properties and analytic/asymptotic behavior of the system of orthogonal polynomials with respect to the above inner product have been extensively studied in the last fifteen years. Notwithstanding, the case where Δ is unbounded has only been considered recently. As it is well known, one of the main facts in the theory of standard orthogonality (i.e., when $M=0$) is that the zeros of the orthogonal polynomials lie in $\mathrm{Co}(\operatorname{supp}\mu_0)$ (the convex hull of the support of the measure μ_0); but in the Sobolev case ($M>0$) this is no longer true. In this sense, in [1] it is obtained an upper bound for the distance of the zeros to the convex hull of the support, under certain conditions on the measures; in section 3 we give an alternative and very simple proof of that result. For an updated summary on the analytic properties of Sobolev polynomials orthogonal with respect to exponential type weights supported on unbounded sets of the real line we recommend the introduction of [7].

In the sequel, we consider that $\Delta=\mathbb{R}$ and that each measure μ_k, $k=0,\ldots,M$, can be written as $d\mu_k(x)=\rho_k^2(x)\,dx$, where

$$\rho_k(x) \;=\; \exp(-\varphi_k(x))\,, \tag{8}$$

with

$$\varphi_k(x) = \exp_{\eta_k}(|x|^{\alpha_k}) - \exp_{\eta_k}(0)\,,\; \eta_k \in \mathbf{Z}_+\,,\; \alpha_k > \begin{cases} 0\,, & \text{if } \eta_k > 0\,, \\ 1\,, & \text{if } \eta_k = 0\,. \end{cases} \tag{9}$$

Here, $\exp_l(x)$ denotes the *l-th iterated exponential*, defined as

$$\exp_l(x) = \begin{cases} \underbrace{\exp(\exp(\exp(\ldots\exp(x)))}_{l \text{ times}} & \text{if } l > 0\,, \\ x & \text{if } l = 0\,. \end{cases}$$

For these weights, it is well known that there exists a constant $a_{k,n}$, usually called the Mhaskar-Rakhmanov-Saff constant, such that (see [6, formula 1.1.17]),

$$\|P\,\rho_k\|_{L^\infty(\mathbb{R})} \;=\; \|P\,\rho_k\|_{L^\infty[-a_{k,n},a_{k,n}]}\,, \tag{10}$$

for any polynomial P of degree at most n. These constants (see [6, §1.6]), satisfy:

$$a_{k,n} \;\sim\; \left(\log_{\eta_k}(n)\right)^{(1/\alpha_k)}\,, \tag{11}$$

where $\log_l$ denotes the l-th iterated logarithm:

$$\log_l(x) = \begin{cases} \underbrace{\log(\log(\log(\ldots\log(x)))}_{l \text{ times}} & \text{if } l > 0\,, \\ x & \text{if } l = 0\,, \end{cases}$$

and the notation "$c_n \sim d_n$" means that there exist two positive constants A, B such that $Ad_n \le c_n \le Bd_n$. In the particular case where $\eta_k = 0$ (that is, the Freud case), we have the closed expression (see [6] or [10]):

$$a_{k,n} \;=\; (\gamma_{\alpha_k} n)^{1/\alpha_k}\;,\; \gamma_{\alpha_k} \;=\; \frac{\Gamma\left(\frac{\alpha_k}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{\alpha_k+1}{2}\right)}\,. \tag{12}$$

Notice that there exist $\overline{k} \in \{0, 1, \ldots, M\}$ and a constant $C > 0$ such that for each $0 \le k \le M$ the following inequality holds

$$e^{-\varphi_k(x)} \le Ce^{-\varphi_{\overline{k}}(x)}\,,\; x \in \mathbb{R}\,, \tag{13}$$

where $\varphi_{\overline{k}}(x) = \exp_{\overline{\eta}}(|x|^{\overline{\alpha}}) - \exp_{\overline{\eta}}(0)$, $\overline{\eta} = \min\limits_{0\le k\le M} \eta_k$, $\overline{\alpha} = \min\limits_{\{k:\,\eta_k=\overline{\eta}\}} \alpha_k$ and $\overline{k} = \min\limits_{\substack{\eta_k=\overline{\eta}\\ \alpha_k=\overline{\alpha}}} k$.

In the sequel, we shall refer to $\rho_{\overline{k}} = \exp(-\varphi_{\overline{k}})$ as the "main" weight.

Denote by S_n the n-th monic orthogonal polynomial with respect to the inner product (6) with weights given by (8)-(9), and by μ_{eq} the "arcsine" measure in $[-1, 1]$:

$$d\mu_{eq}(x) = \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}} \,,$$

or, what is the same, the equilibrium measure for $\Sigma = [-1, 1]$ when $\omega \equiv 1$ ($F_\omega = \log 2$), and by μ_α the so-called Ullman distribution (see e.g. [6] and [10]):

$$d\mu_\alpha(x) = \left(\frac{\alpha}{\pi} \int_{|x|}^{1} \frac{t^{\alpha-1}}{\sqrt{t^2 - x^2}} \, dt \right) dx \,,$$

that is, the equilibrium measure for $\Sigma = [-1, 1]$ when $\omega(x) = \gamma_\alpha |x|^\alpha$, with γ_α as in (12) ($F_\omega = \log 2 + \frac{1}{\alpha}$).

Under these conditions, and as an application of Theorem 1.1, we have the following

Theorem 1.2 *Under the conditions above, if we denote by $x_{n,j}^{(k)}$, $j = 1, \ldots, n-k$, the zeros of the k-th derivative $S_n^{(k)}$ of the polynomial S_n, $k = 0, 1, \ldots, M$, we have for $k \geq \overline{k}$:*

$$\frac{1}{n-k} \sum_{j=1}^{n-k} \delta_{x_{n,j}^{(k)}/a_n} \longrightarrow \begin{cases} d\mu_{eq} \,, & \text{if } \eta_{\overline{k}} > 0 \,, \\ d\mu_{\alpha_{\overline{k}}} \,, & \text{if } \eta_{\overline{k}} = 0 \,, \end{cases} \tag{14}$$

where the convergence holds in the weak- sense, and $a_n = a_{\overline{k},n}$ is the Mhaskar-Rakhmanov-Saff constant associated to the main weight.*

Remark 1.1 *In the particular case when $\overline{k} = 0$, that is, when ρ_0 is the main weight,* (14) *yields:*

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{x_{n,j}/a_n} \longrightarrow \begin{cases} d\mu_{eq} \,, & \text{if } \eta_0 > 0 \,, \\ d\mu_{\alpha_0} \,, & \text{if } \eta_0 = 0 \,, \end{cases}$$

where $x_{n,j} = x_{n,j}^{(0)}$, $j = 1, \ldots, n$, are the zeros of S_n.

That is, when ρ_0 is the main weight, the contracted zeros of the Sobolev orthogonal polynomials S_n asymptotically follow the equilibrium distribution of the interval $[-1, 1]$, if $\eta_0 > 0$ (that is, if the main weight is of Erdös type). If ρ_0 is of Freud type, i.e. $\eta_0 = 0$, the zeros of the rescaled Sobolev orthogonal polynomials are distribute according to the Ullman distribution.

Notice that Theorem 1.1 enables us to give in Theorem 1.2 the weak zero asymptotics for Sobolev orthogonal polynomials with exponential weights both of Freud and Erdös type, with a unified treatment. In this sense, this result extends the previous analysis carried out in [7], which was restricted to the Freud-type cases.

Finally, we show a result on the n-th root asymptotics of the derivatives of the rescaled Sobolev orthogonal polynomials. To this end, we need a result about the location of zeros of Sobolev orthogonal polynomials in the unbounded case. Up to now, the only known result in this direction was obtained by Durán and Saff in [1, Theorem 1.1]. By applying that result to our case, we get the following:

Theorem 1.3 *Suppose that we have a Sobolev inner product* (6) *with* $d\mu_k(x) = w_k(x)dx, k = 0, \ldots, M,$ *and there exist positive constants* $\{C_i\}_{i=1}^M$ *such that the weights* w_k *satisfy*

$$C_k = \left\| \frac{w_k}{w_{k-1}} \right\|_{L^\infty(\mathbb{R})} < \infty, \quad k = 1, \ldots, M. \tag{15}$$

Then, there exists another positive constant C*, only depending on* $\{C_k\}_{k=1}^M$*, such that if* z_0 *is a zero of the Sobolev orthogonal polynomial* S_n*, we have*

$$|\Im(z_0)| \leq C.$$

A set of weights satisfying condition (15) is usually called a sequentially dominated family of weights. Durán and Saff [1] proved that $C = \frac{1}{2}\sqrt{\sum_{k=1}^{M} k^2 C_k}$. In section 3 we shall give a new proof of Theorem 1.3, which is much simpler than that given in [1].

For each $n \in \mathbb{N}$ let us denote the n-th monic rescaled Sobolev orthogonal polynomial by $R_n(x)$, that is, $R_n(x) = a_n^{-n} S_n(a_n x)$. Then, as a consequence, we have:

Corollary 1.1 *If the weights* $w_k(x) = \rho_k^2(x), k = 0, \ldots, M,$ *satisfy condition* (15)*, then the monic k-th derivatives of the rescaled Sobolev orthogonal polynomials,* $R_{n,k}(x) = \frac{(n-k)!}{n!} R_n^{(k)}(x), k = 0, \ldots, M,$ *satisfy the following asymptotic behavior, uniformly in compact subsets of* $\mathbb{C} \setminus \mathbb{R}$*:*

$$\lim_{n\to\infty} |R_{n,k}(x)|^{1/n} = \begin{cases} \frac{1}{2}\left|x + \sqrt{x^2-1}\right|, & \text{if } \eta_0 > 0, \\ \frac{1}{2e^{1/\alpha_0}}\left|x + \sqrt{x^2-1}\right| e^{\zeta_{\alpha_0}(x)}, & \text{if } \eta_0 = 0, \end{cases}$$

where $\zeta_{\alpha_0}(x) = \Re \int_0^1 \frac{x t^{\alpha_0 - 1}}{\sqrt{x^2 - t^2}} dt.$

The rest of the paper is organized as follows. In section 2 we prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2, the new proof of Theorem 1.3 and the proof of Corollary 1.1.

2 Proof of Theorem 1.1

The proof of this result is inspired in the method used in the proof of Theorem 5 in [3].

For each $n \in \mathbb{N}$, denote by $x_{n,k}, k = 1, \ldots, n$, the zeros of p_n and by $\nu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{x_{n,k}}$, its unit zero counting measure.

Since ω is an admissible weight on Σ, we know by (1) that there exists a unique measure μ_ω, supported on Σ, such that:

$$V^{\mu_\omega}(z) + \phi(z) \begin{cases} \leq F_\omega\,, & \text{for } z \in \operatorname{supp}\mu_\omega \subseteq \Sigma\,, \\ \geq F_\omega\,, & \text{for quasi-every } z \in \Sigma\,, \end{cases}$$

where $\operatorname{supp}\mu_\omega$ is a compact subset of Σ of positive capacity.

Condition (4) implies that for any $z \in \Sigma$, we have:

$$\limsup_{n\to\infty} |\omega_n(z) p_n(z)|^{1/n} \,\leq\, \exp(-F_\omega)\,,$$

and thus

$$\liminf_{n\to\infty} V^{\nu_n}(z) \,\geq\, F_\omega + \liminf_{n\to\infty} \frac{1}{n}\log\omega_n(z)\;,\; z \in \Sigma\,. \tag{16}$$

Now, let $\widetilde{\nu_n}$ be the restriction of ν_n to $\overline{\mathbb{C}} \setminus \operatorname{supp}\mu_\omega$ and denote by $\nu_n^{'}$ the balayage (or sweeping out, see e.g. [10]) of $\widetilde{\nu_n}$ from $\overline{\mathbb{C}} \setminus \operatorname{supp}\mu_\omega$ onto $\operatorname{supp}\mu_\omega$. Therefore, $\nu_n^{'}$ is supported on $\operatorname{supp}\mu_\omega$ and we have:

$$V^{\nu_n^{'}}(z) - V^{\widetilde{\nu_n}}(z) \,=\, c_n\,,\ \text{qu.e. } z \in \operatorname{supp}\mu_\omega\,,$$

with the constant c_n given by:

$$c_n \,=\, \int g(x;\infty)\, d\widetilde{\nu_n}(x) \,\geq 0\,,$$

where $g(x;\infty)$ denotes the Green function of $\overline{\mathbb{C}} \setminus \operatorname{supp}\mu_\omega$ with a singularity at infinity. Thus,

$$V^{\widehat{\nu_n}}(z) - V^{\nu_n}(z) \;=\; c_n\,, \text{ qu.e. } z \in \operatorname{supp}\mu_\omega\,, \tag{17}$$

where $\widehat{\nu_n} \;=\; \nu_n^{'} \,+\, (\nu_n \,-\, \widetilde{\nu_n})$ and

$$c_n \;=\; \int g(x;\infty)\, d\nu_n(x) \;\geq 0\,. \tag{18}$$

Suppose that $\{\widehat{\nu_n}\}_{n\in\Lambda}$, $\Lambda \subset \mathbb{N}$, is a subsequence weakly convergent to a measure $\widehat{\nu}$ supported in $\operatorname{supp}\mu_\omega$. Applying the lower envelope theorem [10, Th.I.6.9], (5) and (16)-(18), for quasi-every $z \in \operatorname{supp}\mu_\omega$ we have:

$$\begin{aligned} V^{\widehat{\nu}}(z) &= \liminf_{n\in\Lambda} V^{\widehat{\nu_n}}(z) = \liminf_{n\in\Lambda} \left(V^{\nu_n}(z) + c_n\right) \\ &\geq \liminf_{n\in\Lambda} V^{\nu_n}(z) \,+\, \liminf_{n\in\Lambda} c_n \geq\, \liminf_{n\in\Lambda} \frac{1}{n}\log\omega_n(z) + F_\omega \\ &\geq \log\omega(z) + F_\omega \geq F_\omega -\, \phi(z) \;=\; V^{\mu_\omega}(z)\,. \end{aligned} \tag{19}$$

The function $\left(V^{\widehat{\nu}} - V^{\mu_\omega}\right)$ is harmonic in $\widehat{\mathbb{C}} \setminus \operatorname{supp}\mu_\omega$ and equal to zero at infinity. Thus, by the minimum principle and (19), we obtain that

$$V^{\widehat{\nu}}(z) - V^{\mu_\omega}(z) \;=\; 0\,,\; z \in \overline{\mathbb{C}} \setminus \operatorname{supp}\mu_\omega$$

and by the unicity theorem [10, Th.II.4.11], we have that $\widehat{\nu} = \mu_\omega$. Therefore, $\{\widehat{\nu_n}\}$ is weakly convergent to μ_ω.

Now, we are going to show that the same conclusion holds for $\{\nu_n\}$. From the chain of inequalities in (19), then

$$\liminf_{n\to\infty, n\in\Lambda} c_n \;=\; 0\,, \text{ for every } \Lambda \subseteq \mathbb{N}\,,$$

and thus,

$$\lim_{n\to\infty} c_n \;=\; 0. \tag{20}$$

Now, if F is a closed subset of $\overline{\mathbb{C}}\setminus\operatorname{supp}\mu_\omega$, we have that there exists a positive constant $M = M(F) > 0$, such that $g(z;\infty) \geq M$, $z \in F$, and so, (18) and (20) imply that

$$\lim_{n\to\infty} \nu_n(F) \;=\; 0.$$

This shows that if ν is a weak limit of an infinite subsequence $\{\nu_n\}_{n\in\Lambda}$, then $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu_\omega$. Applying again the lower envelope theorem [10, Th.I.6.9], (17) and (20), we have for qu.e. $z \in \operatorname{supp}\mu_\omega$,

$$V^{\mu_\omega}(z) = \liminf_{n\to\infty, n\in\Lambda} V^{\widehat{\nu_n}}(z) = \liminf_{n\to\infty, n\in\Lambda} V^{\nu_n}(z) = V^{\nu}(z)\,.$$

Finally, using the same arguments as above, we conclude $\nu = \mu_\omega$. $\square$

Remark 2.1 *Instead of* (4)*, in the proof we actually use the weakest condition*

$$\limsup_{n\to\infty} [\omega_n(z)|p_n(z)|]^{1/n} \leq \exp(-F_\omega)\,,\ qu.e.\ z \in \operatorname{supp}\mu_\omega\,.$$

However, we prefer to state the result using (4)*, because it is a more natural extension of classical condition* (2)*.*

3 Sobolev orthogonality with exponential weights

The main goal of this section is to prove Theorem 1.2 about the weak asymptotics for the (rescaled) Sobolev orthogonal polynomials with exponential weights. In this sense, we will show that the rescaled monic Sobolev orthogonal polynomials with exponential weights are asymptotically extremal. For it, it plays a key role the asymptotic behavior of the related standard orthogonal polynomials.

First, we remind some basic facts concerning exponential weights. For the proof of our result, we need to make use of the so-called Markov and Nikolskii type inequalities for exponential weights (see [6, Example 1 (IV), page 28], [6, Example 2 (III), page 30] and [6, Theorem 10.3]).

Lemma 3.1 *Let* $\omega_{\eta,\alpha}(x) = \exp\Big(-\exp_\eta(|x|^\alpha)\Big)$*, and let* P *be a polynomial of degree at most* n*,* $n = 1, 2, \cdots$*. Then*

a) $\|P^{(k)}\,\omega_{\eta,\alpha}\|_{L_2(\mathbb{R})} \leq A(n,k,\eta,\alpha)\,\|P\,\omega_{\eta,\alpha}\|_{L_2(\mathbb{R})}$*, where* $A_n = A(n,k,\eta,\alpha)$ *is such that* $A_n^{1/n} \underset{n}{\longrightarrow} 1$*.*

b) $B(n,\eta,\alpha)\,\|P\,\omega_{\eta,\alpha}\|_{L_\infty(\mathbb{R})} \leq \|P\,\omega_{\eta,\alpha}\|_{L_2(\mathbb{R})} \leq C(n,\eta,\alpha)\,\|P\,\omega_{\eta,\alpha}\|_{L_\infty(\mathbb{R})}$*, where the constants* $B_n = B(n,\eta,\alpha)$ *and* $C_n = C(n,\eta,\alpha)$ *are such that* $B_n^{1/n} \underset{n}{\longrightarrow} 1$ *and* $C_n^{1/n} \underset{n}{\longrightarrow} 1$*.*

Finally, from [6, Th. 1.22], under the same conditions as in Lemma 3.1, the following result on the asymptotics of extremal polynomials with respect to such weights easily follows. See also [5], and [9] for the Freud case, i.e., when $\eta = 0$.

Lemma 3.2 *Let us denote*

$$\varepsilon_n(\omega_{\eta,\alpha}) = \inf_{P\in\mathbb{P}_{n-1}} \|\,(x^n - P(x))\,\omega_{\eta,\alpha}\|_{L^2(\mathbb{R})}.$$

Then,

$$\lim_{n\to\infty} \frac{1}{a_n}\, \varepsilon_n(\omega_{\eta,\alpha})^{1/n} \;=\; \begin{cases} \frac{1}{2}\,, & \text{if } \eta > 0\,, \\ \frac{1}{2}\, e^{-1/\alpha}\,, & \text{if } \eta = 0\,, \end{cases} \tag{21}$$

where a_n is the Mhaskar-Rakhmanov-Saff constant for the weight $\omega_{\eta,\alpha}$.

Now, we are in a position to prove our main result about Sobolev orthogonal polynomials with respect to exponential weights.

3.1 Proof of Theorem 1.2

Hereafter, let us denote by S_n the n-th monic orthogonal polynomial with respect to the Sobolev inner product (6) with exponential weights $\rho_k^2(x) = \exp(-2\varphi_k(x))$, where φ_k , $k = 0, 1, \ldots, M$, are given by (9). Let $E_{k,n}$ be the n-th monic orthogonal polynomial with respect to the exponential weight ρ_k^2.

By (21), for $k = 0, 1, \ldots, M$, we know that

$$\lim_{n\to\infty} a_{k,n}^{-1}\, \|E_{k,n}\rho_k\|_{L^2(\mathbb{R})}^{1/n} \;=\; \begin{cases} \frac{1}{2}\,, & \text{if } \eta_k > 0\,, \\ \frac{1}{2}\, e^{-1/\alpha_k}\,, & \text{if } \eta_k = 0\,. \end{cases} \tag{22}$$

On the other hand, taking into account the extremality of $E_{k,n}$ and S_n with respect to $\|\cdot\|_{L^2(\rho_k^2)}$ and the Sobolev norm $\|\cdot\|_S$ (7), respectively, as well as the Markov inequalities (Lemma 3.1 a) and (13), we can write, using the fact that $\rho_{\overline{k}}$ is the main weight,

$$\begin{aligned} \frac{n!}{(n-\overline{k})!}\|E_{\overline{k},n-\overline{k}}\,\rho_{\overline{k}}\|_{L^2(\mathbb{R})} &\le \|S_n^{(\overline{k})}\rho_{\overline{k}}\|_{L^2(\mathbb{R})} \le \|S_n\|_S \\ &\le \|E_{\overline{k},n}\|_S \le C(n)\|E_{\overline{k},n}\,\rho_{\overline{k}}\|_{L^2(\mathbb{R})}\,, \end{aligned} \tag{23}$$

with $\lim\limits_{n\to\infty} C(n)^{1/n} = 1$. So, taking into account that $\overline{k}$ is a fixed nonnegative integer and $\lim\limits_{n\to\infty}\left(\dfrac{n!}{(n-\overline{k})!}\right)^{1/n} = 1$, (22) and (23) yield:

$$\lim_{n\to\infty} a_{\overline{k},n}^{-1}\, \|S_n\|_S^{1/n} \;=\; \begin{cases} \frac{1}{2}\,, & \text{if } \eta_{\overline{k}} > 0\,, \\ \frac{1}{2}\, e^{-1/\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,, \end{cases}$$

and

$$\lim_{n\to\infty} a_{\overline{k},n}^{-1}\, \|S_n^{(\overline{k})}\,\rho_{\overline{k}}\|_{L^2(\mathbb{R})}^{1/n} \;=\; \begin{cases} \frac{1}{2}\,, & \text{if } \eta_{\overline{k}} > 0\,, \\ \frac{1}{2}\, e^{-1/\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,. \end{cases}$$

Now, applying the Nikolskii inequalities for exponential weights (Lemma 3.1 b), we have

$$\lim_{n\to\infty} a_{\overline{k},n}^{-1}\,\|S_n^{(\overline{k})}\rho_{\overline{k}}\|_{L^\infty(\mathbb{R})}^{1/n} \;=\; \begin{cases} \frac{1}{2}\,, & \text{if } \eta_{\overline{k}} > 0\,, \\ \frac{1}{2}\,e^{-1/\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,. \end{cases} \tag{24}$$

Thus, rescaling the polynomial S_n, that is, setting

$$R_n(x) \;=\; a_{\overline{k},n}^{-n} S_n(a_{\overline{k},n}x)\,,$$

taking $\omega_{\overline{k},n}(x) = \rho_{\overline{k}}(a_{\overline{k},n}x)$ and applying the infinite-finite range inequalities for exponential weights (10), (24) yields

$$\lim_{n\to\infty} \|R_{n,\overline{k}}\omega_{\overline{k},n}\|_{L^\infty[-1,1]}^{1/n} \;=\; \begin{cases} \frac{1}{2}\,, & \text{if } \eta_{\overline{k}} > 0\,, \\ \frac{1}{2}\,e^{-1/\alpha_{\overline{k}}}\,, & \text{if } \eta_{\overline{k}} = 0\,, \end{cases} \tag{25}$$

where $R_{n,k}(x) \;=\; \dfrac{(n-k)!}{n!} R_n^{(k)}(x), k \;=\; 0,\ldots,M,$ and thus, by the Markov inequalities (Lemma 3.1 a), we have that (25) also holds replacing $R_{n,\overline{k}}$ by any derivative $R_{n,k}$, with $k \geq \overline{k}$.

Observe that if $\eta_{\overline{k}} > 0$ (that is when the main weight is of Erdös type), then

$$\lim_{n\to\infty} \omega_{\overline{k},n}(x)^{1/n} \;=\; 1\,,\;\; \text{qu.e. } x \in [-1,1]\,,$$

(in fact, the convergence does not hold only at the endpoints). Therefore, applying Theorem 1.1, we conclude that the contracted zeros of $R_n^{(k)}$, with $k \geq \overline{k}$, asymptotically follow the arcsine measure in $[-1,1]$. If $\eta_k = 0$, i.e. the main weight is of Freud type, we have that $\omega_{\overline{k},n}(x)^{1/n} \;=\; \gamma_{\alpha_{\overline{k}}}|x|^{\alpha_{\overline{k}}}$, for each $n \in \mathbb{N}$, and so, application of Theorem 1.1 shows that the contracted zeros of $R_n^{(k)}$, with $k \geq \overline{k}$, distribute according to the Ullman distribution (see [10, Th. IV.5.1]). □

Remark 3.1 *In fact, we have proved that the zeros of some derivatives of the rescaled Sobolev orthogonal polynomials have the same weak asymptotics as the standard orthogonal polynomials with respect to the main weight. Thus, following the same method as above and applying Theorem 1.1 we can prove that if E_n is the n-th orthogonal polynomial with respect to an exponential weight of Erdös type, then its contracted zeros asymptotically follow the arcsine (equilibrium) distribution on the interval $[-1,1]$. As far as we know, this result was established in a previous work by Erdös [2], using a different method.*

On the other hand, if E_n is orthogonal with respect to an exponential weight of Freud type, application of Theorem 1.1 provides a well-known result, namely, *that its contracted zeros asymptotically follow the Ullman distribution on the interval $[-1,1]$ (see [6] and [10]).*

3.2 New proof of Theorem 1.3

Now, we give an alternative proof of Theorem 1.3, which is much simpler than that given in [1, Th.1.1]. To this end, we need a technical lemma.

Lemma 3.3 *If $a_k > 0$, $k = 0, \dots, m$, then*

$$\sum_{k=1}^{m} a_{k-1}a_k \,\leq\, \frac{1}{4}\left(\sum_{k=0}^{m} a_k\right)^2 . \tag{26}$$

Proof: We have that

$$0 \,\leq\, \left(\sum_{k=0}^{m} a_k\right)^2 \,=\, \sum_{k=0}^{m} a_k^2 \,+\, 2\sum_{k=1}^{m} a_{k-1}a_k \,+\, 2\sum_{l-k>1} a_k a_l. \tag{27}$$

In the same way,

$$0 \,\leq\, \left(\sum_{k=0}^{m}(-1)^k a_k\right)^2 \,=\, \sum_{k=0}^{m} a_k^2 \,-\, 2\sum_{k=1}^{m} a_{k-1}a_k \,+\, 2\sum_{l-k>1}(-1)^{k+l} a_k a_l \,,$$

and so,

$$2\sum_{k=1}^{m} a_{k-1}a_k \,\leq\, \sum_{k=0}^{m} a_k^2 \,+\, 2\sum_{l-k>1}(-1)^{k+l} a_k a_l \,.$$

Thus, since $a_k > 0$, $k = 0, \dots, m$, we have:

$$2\sum_{k=1}^{m} a_{k-1}a_k \,\leq\, \sum_{k=0}^{m} a_k^2 \,+\, 2\sum_{l-k>1} a_k a_l \,. \tag{28}$$

Finally, (27) and (28) yield (26). □

Now, we deal with our proof of Theorem 1.3. Indeed, let $z_0 = x_0 + iy_0$ be a zero of a n-th monic Sobolev orthogonal polynomial S_n with respect to the inner product:

$$\langle p, q\rangle_S := \sum_{i=0}^{M} \int_{\mathbb{R}} p^{(i)}(x)\, \overline{q}^{(i)}(x)\, w_i(x) dx \,.$$

Thus, $S_n(z) = (z - z_0)q(z)$, for some polynomial q of degree $n-1$. Then, by orthogonality we have $\langle (z-z_0)q(z), q(z)\rangle_S = 0$, and so,

$$\sum_{k=0}^{M} \int_{\mathbb{R}} (x - z_0) \left| q^{(k)}(x) \right|^2 w_k(x)dx + \sum_{k=1}^{M} k \int_{\mathbb{R}} q^{(k-1)}(x)\overline{q}^{(k)}(x) w_k(x)dx = 0. \quad (29)$$

Therefore, taking imaginary parts in (29), we have:

$$\Im\left(\sum_{k=0}^{M} \int_{\mathbb{R}} (z_0 - x) \left| q^{(k)}(x) \right|^2 w_k(x)dx \right) = \Im\left(\sum_{k=1}^{M} k \int_{\mathbb{R}} q^{(k-1)}(x)\overline{q^{(k)}(x)} w_k(x)dx \right),$$

and thus,

$$y_0 \|q\|_S^2 = \sum_{k=1}^{M} k \int_{\mathbb{R}} \Im\left(q^{(k-1)}(x)\overline{q}^{(k)}(x) \right) w_k(x)dx\,.$$

Thus, applying the Cauchy-Schwarz inequality and condition (15), we get:

$$\begin{aligned} |y_0| \|q\|_S^2 &\leq \sum_{k=1}^{M} k \int_{\mathbb{R}} \left| q^{(k-1)}(x)\overline{q}^{(k)}(x) \right| w_k(x)dx \\ &\leq \sum_{k=1}^{M} k \|q^{(k-1)}\|_{L^2(w_k)} \|q^{(k)}\|_{L^2(w_k)} \\ &\leq \sum_{k=1}^{M} k\sqrt{C_k} \|q^{(k-1)}\|_{L^2(w_{k-1})} \|q^{(k)}\|_{L^2(w_k)}. \end{aligned}$$

Now, applying again the Cauchy-Schwarz inequality, we have:

$$|y_0| \|q\|_S^2 \leq \sqrt{\sum_{k=1}^{M} k^2 C_k} \sqrt{\sum_{k=1}^{M} \|q^{(k-1)}\|_{L^2(w_{k-1})}^2 \|q^{(k)}\|_{L^2(w_k)}^2}.$$

Thus, applying Lemma 3.3 with $a_k = \|q^{(k)}\|_{L^2(w_k)}^2$, we conclude that:

$$|y_0| \|q\|_S^2 \leq \frac{1}{2} \sqrt{\sum_{k=1}^{M} k^2 C_k} \; \|q\|_S^2$$

and this completes the proof. □

3.3 Proof of Corollary 1.1

Take into account that if the unit zero counting measures of a sequence of polynomials $\{p_n\}$ converge (in the weak-* topology) to a certain measure μ with compact support, and K is a compact subset of $\mathbb{C} \setminus \operatorname{Co}(\operatorname{supp} \mu)$ not intersecting the set of limit points of zeros of $\{p_n\}$, then:

$$\lim_{n\to\infty} |p_n(x)|^{1/n} = \exp(-V^{\mu}(x)) \,, \tag{30}$$

uniformly on K. On the other hand, Theorem 1.3 yields that for each n, the zeros of the monic rescaled Sobolev orthogonal polynomial $R_{n,0}(x) = R_n(x) = a_{0,n}^{-n} S_n(a_{0,n}x)$ lie in the strip

$$\left\{ z \in \mathbb{C} / -\frac{C}{a_{0,n}} \leq \Im(z) \leq \frac{C}{a_{0,n}} \right\} ,$$

with the constant C given in Theorem 1.3, and therefore the contracted zeros may only have accumulation points on the real line. Since Theorem 1.2 implies that these zeros asymptotically distribute as μ_{eq}, when $\eta_k > 0$, and as μ_{α_0}, when $\eta_k = 0$, then the expression of their respective potentials (see [10, Th. IV.5.1]) and the asymptotic behavior of $a_{0,n}$ (see (11)-(12)) render the proof. $\square$

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