DISTORTION OF BOUNDARY SETS UNDER INNER FUNCTIONS AND APPLICATIONS.

by

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(*) Research supported by a grant from CICYT, Ministerio de Educación y Ciencia, Spain.

0. Introduction.

An inner function is a bounded holomorphic function from the unit disk Δ of the complex plane such that the radial boundary values have modulus 1 a. e.. If E is a Borel subset of $\partial\Delta$, we define $f^{-1}(E) = \{e^{i\theta} \mid \lim_{r\to 1} f(re^{i\theta})$ exists and belongs to $E\}$.

In this paper we study the relationship between the metrical sizes of E and $f^{-1}(E)$ and consider some applications. The collection of all Borel subsets of $\partial\Delta$ is denoted $\mathcal{B}$. In this context the classical lemma of Löwner asserts the following:

THEOREM L1. *If f is inner, $f(0) = 0$, and if $E \in \mathcal{B}$, then*

$$L(f^{-1}(E)) = L(E)$$

Here and hereafter L means normalized Lebesgue measure.
There is a companion result about conformal mapping:

THEOREM L2. *If f is univalent, with $f(\Delta) \subset \Delta$, $f(0) = 0$, and if $E \in \mathcal{B}$, with radial limits $f(E) \subset \partial\Delta$, then*

$$L(f(E)) \geq L(E)$$

Both results are easy applications of invariance properties of harmonic measure ([A, p.12], [T p.322]).

Recently, Makarov and Hamilton ([M], [H]; see also [Po 1]) have extended L2 to fractional dimensions. Their results can be summarized as follows:

THEOREM A. *If f is univalent, $f(0) = 0$, and $f(\Delta) \subset \Delta$, then if E is a Borel subset of $\partial f(\Delta) \cap \partial\Delta$, and if $0 < \alpha < 1$, then*

(i) $$M_\alpha(f(E)) \geq C_\alpha M_\alpha(E)$$

and,

(ii) $$cap_\alpha(f(E)) \geq |f'(0)|^{-1/2}\, cap_\alpha(E) \geq cap_\alpha(E)$$

In particular, $Dim(f(E)) \geq Dim(E)$. ∎

Here, M_α, cap_α, and Dim, denote α-dimensional content, α-dimensio- nal capacity and Hausdorff (or capacitary) dimension. We refer to [T] and [K-S] for definitions and basic background. For $\alpha = 0$, cap_0 means logarithmic capacity; (ii) holds and it is due to Pommerenke.

We have

THEOREM 1. *If f is inner, $f(0) = 0$, and if $E \in \mathcal{B}$, we have for $0 < \alpha \leq 1$,*

(i) $$M_\alpha(f^{-1}(E)) \geq C_\alpha M_\alpha(E)$$

and for $0 \leq \alpha < 1$,

(ii) $$cap_\alpha(f^{-1}(E)) \geq C_\alpha cap_\alpha(E)$$

An inmediate consequence is the following:

Corollary. *If f is inner, and $E \in \mathcal{B}$,*

$$Dim(f^{-1}(E)) \geq Dim(E)$$

None of these inequalities can be reversed. See Section 3 for the appropiate examples.

The outline of this paper is as follows: in Section one, we give the proofs of some lemmas needed in the proof of Theorem 1, which is given in Section two; in section 3 we give some examples in order to prove that the inequalities in Theorem 1 cannot be reversed. Finally, section 4 contains the applications to radial boundeness.

We would like to thank J.J. Carmona, J.G. Llorente and Ch. Pommerenke for helpful conversations.

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1. Some lemmas. In what follows p_μ denotes the Poisson extension of a measure μ in $\partial\Delta$.

Lemma 1. *Let $\mu \geq 0$ be a measure in $\partial\Delta$, and let f be an inner function.*
Then, there exist a measure $\nu \geq 0$, such that $(p_\mu)\circ f = p_\nu$, and if ν has singular part σ, and continuous part γ and we denote,

$$A = \{e^{i\theta} \mid p_\sigma(re^{i\theta}) \to \infty, \text{ as } r \to 1\}$$
$$B = \{e^{i\theta} \mid \exists \lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta}),\ |f(e^{i\theta})| = 1 \text{ and } \lim_{r\to 1} p_\gamma(re^{i\theta}) > 0\}$$

then

$$A \cup B \subset f^{-1}(\mathit{support}\ \mu)$$

and so,

$$\nu(f^{-1}(\mathit{support}\ \mu)) = \|\nu\|$$

Proof: Let us denote by E the support of μ; ($p_\mu \circ f$ is harmonic and positive in $\partial\Delta$ and so, as it is well known, there exists ν as above). Now if $e^{i\theta} \in A$, then $|f(re^{i\theta})| \to 1$, as $r \to 1$. The curve $\{f(re^{i\theta}) \mid 0 \leq r < 1\}$ in the w-disk must end on a (unique) point $e^{i\psi}$ of $\partial\Delta$. Indeed, if not, it is easy to see that $p_\mu(re^{i\theta}) \to \infty$, as $r \to 1$, in an open interval of $\partial\Delta$. Now, $e^{i\psi} \in E$, since otherwise p_μ vanishes continuously at $e^{i\psi}$. By this same reason, $B \subset f^{-1}(E)$. Finally, since A has full $\sigma-$measure and B has full $\gamma-$measure, $\nu(A \cup B) = \nu(\partial\Delta)$.

If μ is a probability in $\partial\Delta$, then the α-energy $I_\alpha(\mu)(0 \leq \alpha < 1)$ is defined as

$$I_\alpha(\mu) = \int\int_{\partial\Delta\times\partial\Delta} \phi_\alpha(|x-y|)d\mu(x)d\mu(y)$$

where

$$\phi_\alpha(t) = \begin{cases} \log\frac{1}{t}, \text{ if } \alpha = 0 \\ \\ 1/t^\alpha, \text{ if } 0 < \alpha < 1 \end{cases}$$

Recall that if $E \subset \partial\Delta$ is a closed subset, then

$$\phi_\alpha(cap_\alpha(E)) = \inf\{I_\alpha(\mu);\ \mu \text{ probability supported on } E\}$$

and that the infimum is attained for a probability μ_e which is called the equilibrium distribution on E. Moreover, if $\hat{\mu}(n)$ and γ_n^α denote the Fourier coefficients of the measure μ and the kernel $\varphi_\alpha(t) = \phi_\alpha(|1-e^{it}|)$ respectively, then

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$$I_\alpha(\mu) = 4\pi^2 \sum_{-\infty}^{\infty} |\hat{\mu}(n)|^2 \gamma_n^\alpha = \gamma_0^\alpha + 8\pi^2 \sum_{1}^{\infty} |\hat{\mu}(n)|^2 \gamma_n^\alpha$$

Let us denote, by $J_\alpha(\mu)$ the integral

$$J_\alpha(\mu) = \int\int_\Delta |p_\mu(z) - 1|^2 \frac{dxdy}{|z|^2(\log \frac{1}{|z|})^\alpha}$$

Lemma 2. *There exist a constant $C_\alpha \geq 1$ such that*

$$C_\alpha^{-1} J_\alpha(\mu) \leq I_\alpha(\mu) - \gamma_0^\alpha \leq C_\alpha J_\alpha(\mu)$$

Proof: Notice that

$$J_\alpha(\mu) = 4\pi \sum_{n=1}^{\infty} |\hat{\mu}(n)|^2 \int_0^1 r^{2n-1} \frac{dr}{(\log \frac{1}{r})^\alpha} = 4\pi \frac{\Gamma(1-\alpha)}{2^{1-\alpha}} \sum_{n=1}^{\infty} |\hat{\mu}(n)|^2 n^{\alpha-1}$$

So, since $\gamma_n^\alpha \simeq n^{\alpha-1}$, see [K-S, p.40], the lemma follows.

Lemma 3. *If μ is a probability on $\partial\Delta$, f is an inner function with $f(0) = 0$, and ν is the probability on $\partial\Delta$ such that $p_\nu = (p_\mu) \circ f$, then*

$$I_\alpha(\nu) \leq C_\alpha I_\alpha(\mu)$$

where C_α is a constant ≥ 1.

Proof: The lemma follows from lemma 2 and subordination, since $|p_\mu - 1|$ is subharmonic.

2. Proof of Theorem 1.

We may assume that E is a closed subset of $\partial\Delta$ and $M_\alpha(E) > 0$. Then, see e.g.[T, p.64], there exists a positive mass distribution on E of finite total mass such that: (i) $\mu(E) = M_\alpha(E)$; (ii) $\mu(I) \leq C_\alpha L(I)^\alpha$, for any open interval in $\partial\Delta$, where C_α is a constant independent of E. Given $z = re^{i\theta}(r < 1)$, let us denote by I_z the open interval (in $\partial\Delta$) with center $e^{i\theta}$ and lenght $1 - |z|$.

A standard argument shows that

$$(1) \qquad p_\mu(z) \le \frac{C_\alpha}{(1-|z|)^{1-\alpha}}$$

with C_α a new constant. Let ν be a measure such that $(p_\mu)\circ f = p_\nu$. Schwarz's lemma and (1) give the same inequality for ν. On the other hand, it is well known that

$$p_\nu(z) \ge C\frac{\nu(I_z)}{1-|z|}$$

and so, we obtain that

$$(2) \qquad \nu(I_z) \le C_\alpha L(I_z)^\alpha$$

Now, if σ is the singular part of ν, and we cover the set A in lemma 1 with intervals of radii r_i, we see, by (2), that

$$\sigma(A) \le C_\alpha \sum_i r_i^\alpha$$

and therefore, $\sigma(A) \le C_\alpha M_\alpha(A) \le C_\alpha M_\alpha(f^{-1}(E))$. Since A has full σ-measure we conclude that

$$(3) \qquad \|\sigma\| \le C_\alpha M_\alpha(f^{-1}(E))$$

On the other hand, if γ is the continuous part of ν we obtain from lemma 1 that

$$\gamma(B) \le C_\alpha M_\alpha(B) \le C_\alpha M_\alpha(f^{-1}(E))$$

and since B has full γ-measure we deduce that

$$(4) \qquad \|\gamma\| \le C_\alpha M_\alpha(f^{-1}(E))$$

and so, by (3) and (4), and since $f(0)=0$,

$$M_\alpha(E) = \|\mu\| = \|\nu\| \le C_\alpha M_\alpha(f^{-1}(E))$$

This finishes the proof of (a).

To prove (b), we may assume that E is closed. Let us denote by μ_e the equilibrium distribution of E, and let ν be the positive measure such that $p_\nu = (p_{\mu_e})\circ f$. Since $f(0)=0$, ν is a probability on $\partial\Delta$, and by lemma 3,

$$(5) \qquad I_\alpha(\nu) \le C_\alpha I_\alpha(\mu_e) = C_\alpha \phi_\alpha(cap_\alpha(E))$$

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But, from lemma 1, $\nu(f^{-1}(E)) = 1$, and so

$$I_\alpha(\nu) = \int\int_{f^{-1}(E)\times f^{-1}(E)} \phi_\alpha(|z-w|)d\nu(z)d\nu(w)$$

Now, let $\{K_n\}$ be an increasing sequence of compacts subsets if $\partial\Delta$, $K_n \subset f^{-1}(E)$ such that $\nu(K_n) \nearrow 1$. The monotone convergence theorem gives

$$I_\alpha(\nu) \geq \lim_{n\to\infty} \phi_\alpha(cap_\alpha(K_n)) = \inf_n \phi_\alpha(cap_\alpha(K_n)) \geq \phi_\alpha(cap_\alpha(f^{-1}(E))). \tag{6}$$

(b) is now a consequence of (5) and (6).

3. <u>**Some examples**</u> .

The following examples show that there are no inequalities in the opposite direction.

<u>**EXAMPLE 1**</u>**.** *Let $f_n(z) = z^n$, $z \in \Delta$, $n \in \mathbf{N}$. If E is a small closed interval with center 1, $E = \{e^{i\theta} : \ \theta \in [-\delta, \delta]\}$, and $0 \leq \alpha < 1$, then*

$$cap_\alpha(f_n^{-1}(E)) \to cap_\alpha(\partial\Delta)$$

as n tends to ∞.

<u>Proof</u>: $f_n^{-1}(E)$ consists of n closed intervals of length $\frac{\delta}{n}$ and centered at the points $z_{j,n} = e^{2\pi j i/n}$ $(j = 1, ..., n)$. Let us denote by $\delta_{j,n}$ the measure concentrated in $z_{j,n}$, and write $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{j,n}$.

The α-equilibrium distribution of $\partial\Delta$ is Lebesgue measure. But μ_n tends to L weakly as $n \to \infty$. Consequently,

$$\overline{\lim}_{n\to\infty}\phi_\alpha(cap_\alpha(f_n^{-1}(E)) \leq \overline{\lim}_{n\to\infty} \int\int_{\partial\Delta\times\partial\Delta} \phi_\alpha(|x-y|)d\mu_n(x)d\mu_n(y) =$$

$$= \int\int_{\partial\Delta\times\partial\Delta} \phi_\alpha(|x-y|)dL(x)dL(y) = \phi_\alpha(cap_\alpha(\partial\Delta)) \leq \phi_\alpha(cap_\alpha f_K^{-1}(E))$$

for every K.

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Therefore,

$$\lim_{n\to\infty} \phi_\alpha(cap_\alpha f_n^{-1}(E)) = \phi_\alpha(cap_\alpha(\partial\Delta)).$$

EXAMPLE 2. *Let $F(z) = e^{-\frac{1+z}{1-z}}$ and be $E = F^{-1}(\{\frac{1}{2^n}, n = 1,2,3,...,\}) \cup \{0\}$. If H is a universal covering map from Δ onto $\Delta \setminus E$, then H is inner and*

$$Dim(H^{-1}\{1\}) = 1 > 0 = Dim\{1\}.$$

Proof: E only accumulates at 1. Since E has zero logarithmic capacity we deduce that H is inner, ([CL, p.37]). Let A be the set

$$A = \{e^{i\theta} \mid \lim_{r\to 1}(F \circ H)(re^{i\theta}) = 0\}$$

We shall verify that $Dim(A) = 1$. But notice now that if $F(H(re^{i\theta})) \to 0$, as $r \to 1$, then $H(re^{i\theta}) \to 1$. Thus, $A \subseteq H^{-1}\{1\}$.

Notice that $F \circ H$ is a universal covering map of

$$\Delta \setminus \{\frac{1}{2^n}, n = 1,2,3,...,\} \setminus \{0\}$$

and consequently $F \circ H$ is a singular inner function. Let us denote by μ the corresponding singular measure ($\log|F \circ H| = -p_\mu$). Let g be the reciprocal of $F \circ H$. Then g is a holomorphic mapping in the disk which omits the points $\{2^n; n = 1,2,...\}$. By a theorem of Littlewood, [L, p.228], we conclude that for constants $C > 0, b > 2$.

$$|g(z)| \leq \frac{C}{(1-|z|)^b}, \text{ for each } z \in \Delta$$

Consequently,

$$|p_\mu(z)| \leq b \cdot \log\frac{1}{1-|z|} + \log C$$

One easily concludes that any set of positive μ-measure must have dimension 1. Since $p_\mu(re^{i\theta}) \to \infty$ for μ-a.e. $e^{i\theta}$, we conclude that A has dimension 1, as desired.

4. Radial boundeness of holomorphic functions.

The first application concerns singularities of inner functions. In [F], it is proved that if f is inner, E is the set of points of Δ that f omits and $S(f)$ is the set of singularities of f, then

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$$Dim\, S(f) \geq \alpha(\rho_E)$$

where α is a continuous monotone function in $[0, \infty)$ with $\alpha(x) > \frac{1}{2}$, $\alpha(0) = 1$, and $\rho_E = \inf\{\rho(a,b) \mid a, b \in E,\, a \neq b\}$, where ρ denotes hyperbolic distance in Δ.

The following improvement is useful.

Corollary 1. *Let f be an inner function, and let E be the set of points in Δ that f omits. If B is the set of accumulation points of E in $\partial\Delta$, then*

$$Dim\, S(f) \geq \max\{Dim(B), \alpha(\rho_E)\}$$

Proof: We can assume that $f(0) = 0$. We claim that $\partial\Delta \setminus S(f) \subset f^{-1}(\partial\Delta \setminus B)$. Indeed, if $a \in \partial\Delta \setminus S(f)$ then f is analytic in a neigbourhood U_a of a, so that if $z \in U_a \cap \Delta$ then $|f(z)| < 1$, if $z \in U_a \cap \partial\Delta$ then $|f(z)| = 1$ and if $z \in U_a \setminus \bar{\Delta}$ then $|f(z)| > 1$. Now $f(U_a)$ is a neigbourhood of $f(a)$, and $E \cap f(U_a \cap \Delta) = \emptyset$; therefore $f(a) \notin B$.

So $S(f) \supset f^{-1}(B)$. By theorem 1, we obtain that,

$$Dim\, S(f) \geq Dim\, f^{-1}(B) \geq Dim\,(B)$$

If f is a holomorphic function from Δ into $\mathbf{C}$ we denote by Mf the radial maximal function, i.e.,

$$Mf\left(e^{i\theta}\right) = \sup_{0<r<1} \left|f\left(re^{i\theta}\right)\right|$$

THEOREM 2. *Let f be an inner function, and let E be the set of points in Δ that f omits. Then*

$$Dim\,\{\theta \mid Mf(e^{i\theta}) < 1\} \geq \alpha(\rho_E)$$

Proof: Let $r \in (0,1)$ such that $E \cap \{|z| = r\} = \emptyset$ and $|f(0)| < r$. Let Ω_r be the connected component of $f^{-1}(\Delta_r)$ wich contains zero, where $\Delta_r = \{|w| < r\}$. The domain Ω_r is simply connected. Let $\varphi_r : \Delta \to \Omega_r$ be a conformal mapping onto Ω_r, with $\varphi_r(0) = 0$. Then $h = \frac{1}{r}(\varphi_r \circ f)$ is inner. Indeed, $h \in H^\infty$ and by Fatou's theorem, h has radial boundary values almost everywhere. On the other hand, if we denote by $A_r = \partial\Omega_r \cap \partial\Delta$, and by $B_r = \{e^{i\theta} : \exists \lim_{s\to 1} \varphi_r(se^{i\theta}) \in A_r\}$, we can write $\partial\Delta = B_r \cup N \cup H$ where N has logarithmic capacity zero and H is an open set across which φ_r extends analytically (and $\varphi_r(H) \subset \partial\Omega_r \cap \Delta$).

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So φ_r has finite and non zero angular derivative a.e. in B_r, (a consequence of McMillan's twist point theorem, see e.g. [Po 2, p.326]), and so, in the corresponding points of A_r f has radial boundary values with modulus less than 1. Thus, since f is inner, and because of Löwner's lemma, we deduce that $L(B_r) = 0$. This implies that h is inner. Moreover, we have that

$$S(h) \subseteq B_r \cup N \tag{7}$$

We claim that

$$A_r \subseteq \{e^{i\theta} \mid \sup_{0<s<1} |f(se^{i\theta})| < 1\} \tag{8}$$

This is so, because if $e^{i\theta} \in A_r$, then there exists a curve $\gamma \subseteq \Omega_r$ ending at $e^{i\theta}$ and beginning at zero and since $|f| \leq r$ on γ, and $|f| < 1$ everywhere, an application of Lindelöf's Theorem, gives

$$\sup_{0<s<1} |f(se^{i\theta})| \leq \sqrt{r} < 1$$

The theorem is now a consequence of (7) and (8). For, if it were false we could choose r so close to 1, and β such that

$$Dim\{\theta \mid \sup_s |f(se^{i\theta})| < 1\} < \beta < \alpha(\rho(\frac{E_r}{r})).$$

So by (8), we have $cap_\beta(A_r) = 0$. But, (7) and Theorem A give

$$cap_\beta(S(h)) \leq cap_\beta(B_r) \leq \sqrt{|\varphi_r'(0)|}cap_\beta(A_r) = 0$$

Therefore, we obtain $Dim(S(h)) < \beta < \alpha(\rho(\frac{E_r}{r}))$ which contradicts corollary 1, because h omits $\frac{E_r}{r}$.

Notice the following surprising

Corollary 2. *If f is holomorphic in Δ and omits two values, then*

$$Dim\{\theta \mid Mf(e^{i\theta}) < \infty\} = 1$$

In general (i.e. with no hypothesis on omitted values) f may be radially unbounded everywhere (simply consider $f(z) = \sum_{k=1}^{\infty} \lambda_k z^{a_k}$ which λ_k, and a_k growing very fast). If f is

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simply non-zero then f is radially bounded on a countable dense set. This is the best one can say. Consider, for instance, the function $f = F'$ where F is a universal cover of the plane minus the Gaussian integers. Now, F is in the Bloch class, i.e. for some constant C we have $|f(z)| \leq C \cdot (1 - |z|)^{-1}$, $z \in \Delta$. Also, f never vanishes. If f is radially bounded at $e^{i\theta}$ then $\int_0^1 |F'(re^{i\theta})| dr < \infty$ and, consequently, F has a finite radial limit at $e^{i\theta}$. But F has finite radial limit only at a countable set.

Proof of Corollary 2. Without lost of generality we can assume that $0, 1$ are omitted values of f. Let g be a branch of $\frac{1}{2\pi i} \log f$. Then $|\log |f|| \leq 2\pi |g|$ and so,

$$\{\theta \mid Mg(e^{i\theta}) < \infty\} \subset \{\theta \mid Mf(e^{i\theta}) < \infty\}$$

Notice that g omits $\mathbf{Z}$ and $g = F \circ b$ where F is the universal covering map of $\mathbf{C} \setminus \{0, 1\}$ and $b : \Delta \longrightarrow \Delta$ is holomorphic and omits $F^{-1}(\mathbf{Z})$. Since the hyperbolic distance in $\mathbf{C} \setminus \{0, 1\}$ between k and $k + 1$ ($k \in \mathbf{Z}, k \geq 2$) is at most

$$C \int_k^{k+1} \frac{dx}{x \log x} = C \log \frac{\log(k+1)}{\log k}$$

(see [A, p.17]) we deduce that

$$\rho_{F^{-1}(\mathbf{Z})} = 0$$

Therefore $Dim\{\theta \mid Mb(e^{i\theta}) < 1\} = 1$ by Theorem 2. This implies that $Dim\{\theta \mid Mg(e^{i\theta}) < \infty\} = 1$ and the corollary follows.

Theorem 2 and its corollary could be compared with classical results of Frostman and Nevanlinna which can be stated as

THEOREM B.*If f is holomorphic from Δ into Δ_R ,$0 < R \leq \infty$), and f omits a set E of positive logarithmic capacity then*

(i) If $R < \infty$, $L\left(\{\theta \mid Mf(e^{i\theta}) < R\}\right) > 0$.

(ii) If $R = \infty$, then $L\left(\{\theta \mid Mf(e^{i\theta}) < \infty\}\right) = 1$.

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REFERENCES

[A] AHLFORS, L.V., Conformal invariants, McGraw-Hill, New York, 1973.

[CL] COLLINGWOOD, E.F., LOHWATER, A.J., The theory of cluster sets, Cambridge University Press, London and New York, 1966.

[F] FERNANDEZ, J.L., Singularities of inner functions, Math. Z., 193 (1986), 393-396.

[H] HAMILTON, D., Conformal distortion of boundary sets, Trans. Amer. Math. Soc. 308(1988) 69-81.

[K-S] KAHANE, J.P., SALEM, R., Ensembles Parfaits et Sèries Trigonomètriques, Hermann, Paris, 1963.

[L] LITTLEWOOD, J.E., Lectures on the theory of functions, Oxford University Press, Oxford, 1944.

[M] MAKAROV, N.G., LIL for smooth measures, LOMI Preprint, 1988.

[Po 1] POMMERENKE, CH., On the logarithmic capacity and conformal mapping, Duke Math. J. 35 (1968), 321-326.

[Po 2] POMMERENKE, CH. Univalent functions, Vandenhoek und Ruprecht, Gottingen, 1975.

[T] TSUJI, M., Potential Theory in Modern Function Theory, Chelsea, New York 1959.