Geodesic Excursions into Cusps in Finite-Volume Hyperbolic Manifolds

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0. Introduction

Throughout, $\mathfrak{M}^{d+1}$ will be a fixed, complete, noncompact Riemannian manifold of constant negative sectional curvature and finite volume. Given a point p on $\mathfrak{M}$, we denote by $S(p)$ the unit ball of the tangent space of $\mathfrak{M}$ at p, and for every $v \in S(p)$ let $\gamma_v(t)$ be the geodesic emanating from p in the direction v. In this paper, we study the long time behaviour of $\gamma_v(t)$.

Sullivan proved in [S] that for almost every direction $v \in S(p)$, one has

$$\limsup_{t\to\infty} \frac{\operatorname{dist}(\gamma_v(t), p)}{\log t} = \frac{1}{d},$$

where dist is the distance in $\mathfrak{M}$. On the other hand, for just a countable number of directions $v \in S(p)$,

$$\limsup_{t\to\infty} \frac{\operatorname{dist}(\gamma_v(t), p)}{t} = 1.$$

We give a result interpolating between these two.

THEOREM 1. *For* $0 \le \alpha \le 1$,

$$\operatorname{Dim}\left\{v : \limsup_{t\to\infty} \frac{\operatorname{dist}(\gamma_v(t), p)}{t} \ge \alpha\right\} = d(1-\alpha).$$

Here and hereafter, Dim denotes Hausdorff dimension. Dimension refers here to the induced distance in $S(p)$. Also, we will use the notation M_α for α-dimensional content. We refer to [C] or [R] for definitions and background on these metrical notions.

Let $\mathbf{H}^{d+1}$ be the upper half plane of $\mathbf{R}^{d+1}$,

$$\mathbf{H}^{d+1} = \{(x_1, \dots, x_{d+1}) \in \mathbf{R}^{d+1} : x_{d+1} > 0\},$$

and let λ be the hyperbolic metric in $\mathbf{H}^{d+1}$,

$$d\lambda = \frac{|dx|}{x_{d+1}}.$$

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We will denote by Möb($\mathbf{H}^{d+1}$) the group of orientation-preserving Möbius transformations which map $\mathbf{H}^{d+1}$ on itself. It is well known that $\mathbf{H}^{d+1}$ is the unique (up to isometries and a constant conformal factor) simply connected complete Riemannian manifold of constant negative sectional curvature and $\mathfrak{M}^{d+1}=\mathbf{H}^{d+1}/\Gamma$, where Γ is a discrete subgroup of Möb($\mathbf{H}^{d+1}$) with parabolic elements (since $\mathfrak{M}^{d+1}$ is noncompact) and finite covolume; that is, the hyperbolic volume of a Dirichlet region D_a of Γ is finite. We recall that

$$D_a=\{x\in\mathbf{H}^{d+1}: \rho_{\mathbf{H}^{d+1}}(x,a)\le\rho_{\mathbf{H}^{d+1}}(\gamma(x),a) \text{ for all } \gamma\in\Gamma\},$$

where $a\in\mathbf{H}^{d+1}$ is a non-fixed point of Γ and $\rho_{\mathbf{H}^{d+1}}$ is the hyperbolic distance in $\mathbf{H}^{d+1}$.

We remark that for the cases $d=1,2$, if Γ is any discrete subgroup of Möb($\mathbf{H}^{d+1}$) then we can ensure that $\mathbf{H}^{d+1}/\Gamma$ is a Riemannian manifold. We refer to [A] and [B] for general background on Möbius Transformations.

Here is a brief description of the geometry at infinity of $\mathfrak{M}^2=\mathbf{H}^2/\Gamma$. It can be shown that $\mathfrak{M}^2=X_0\bigcup_{i=1}^k Y_i$, where X_0 is compact and Y_i is isometric to $S^1\times[a,+\infty)$ with the metric $dr^2+e^{-2r}\,d\theta^2$ [P]. The Y_i's are usually called *cusps*. Notice that the infimum of the lengths of curves in nontrivial free homotopy classes on each cusp is zero.

Moreover, given a fixed cusp $\mathcal{E}$ there exists a conjugacy class of maximal cyclic parabolic subgroups of Γ, usually also called a cusp, which contains a subgroup of Γ generated by a parabolic element γ with fixed point ξ in the limit set of Γ. Besides, there exists a Möbius transformation A such that $A(\infty)=\xi$ and $A^{-1}\circ\gamma\circ A$ is the translation $z\mapsto z+1$. Also, there exists a half-plane

$$U_c=\{z\in\mathbf{C}: \operatorname{Im} z>c\},$$

verifying that the image of $A(U_c)$ under $\pi:\mathbf{H}^2\to\mathbf{H}^2/\Gamma$, the canonical projection, is homeomorphic to $\mathcal{E}$ [K, p. 52].

By a theorem of H. Shimizu [K, p. 60] we have that the set

$$\bigcup\{g(U_c): g\in A^{-1}\circ\Gamma\circ A\backslash\{\text{identity}\}\}$$

consists of a pairwise disjoint and countable union of balls in $\mathbf{H}^2$ with diameter at most c. These balls are tangent to $\mathbf{R}$ in certain base-points a_i which are the parabolic fixed points fixed by the elements belonging to the conjugacy class in $A^{-1}\circ\Gamma\circ A$ of the translation $z\mapsto z+1$. Also, notice that

$$a_i=A^{-1}\circ\gamma_i\circ A(\infty) \quad \text{with } \gamma_i\in\Gamma\backslash\Gamma_\xi,$$

where $\Gamma_\xi=\{\gamma\in\Gamma:\gamma(\xi)=\xi\}$.

This description holds in higher dimensions. We have that a cusp $\mathcal{E}$ in $\mathbf{H}^{d+1}/\Gamma$ is isometric to $(S^1)^d\times[a,+\infty)$, and there exists a conjugacy class of infinite maximal parabolic subgroups of Γ associated to the cusp. Since Γ has finite covolume, each parabolic subgroup in the cusp is an abelian group with rank d. Besides, there exists a conjugate group $\bar{\Gamma}$ of Γ such that the inverse image of $\mathcal{E}$ by the canonical projection consists of a semispace above a hyperplane parallel to $\mathbf{R}^d$, at height c, and a pairwise disjoint and countable union of $(d+1)$-balls in $\mathbf{H}^{d+1}$ resting on $\mathbf{R}^d$ with base-points

$$a_i = \bar{\gamma}_i(\infty) \quad \text{where } \bar{\gamma}_i \in \bar{\Gamma} \backslash \bar{\Gamma}_\infty$$

and radii $R(a_i) \leq c/2$.

Henceforth we will refer to these $(d+1)$-balls as the *horoballs* corresponding to the cusp $\mathcal{E}$. The boundary of a horoball is called a *horosphere*.

Following [S], we will study the excursions of geodesics into the cusps of $\mathbf{H}^{d+1}/\Gamma$ by translating this problem to $\mathbf{H}^{d+1}$ and considering there the corresponding geodesics and the set of horoballs associated to each cusp. Thus, the proof of Theorem 1 is reduced to the following theorem.

THEOREM 2. *Let $\{\mathcal{E}_l\}_{l=1}^n$ be the set of all cusps of $\mathfrak{M}$. Then, for $0<\tau<1$, the Hausdorff dimension of the set of $\xi \in \mathbf{R}^d$ such that $\|\xi - a_i\| < C(\xi)(R(a_i))^{1/\tau}$ for infinitely many a_i is τd. Here each a_i is a base-point of a horosphere corresponding to some cusp $\mathcal{E} \in \{\mathcal{E}_l\}_{l=1}^n$ and $R(a_i)$ is the radius of the horosphere.*

In fact, we can also prove the following improvement.

THEOREM 3. *Let $\{\mathcal{E}_l\}_{l=1}^n$ be the set of all cusps of $\mathfrak{M}$. Then, for $0<\tau<1$, the Hausdorff dimension of the set of $\xi \in \mathbf{R}^d$ such that*

$$\|\xi - a_{l,i}\| < C(\xi)(R(a_{l,i}))^{1/\tau}$$

for infinitely many i and for all $l \in \mathcal{L}$, where $\mathcal{L}$ is a subset of $\{1, 2, \ldots, n\}$, is τd.

Here each $a_{l,i}$ and $R(a_{l,i})$ are respectively the base-points and the radii of the horospheres corresponding to the cusp $\mathcal{E}_l$.

In particular, when $\Gamma = SL(2, \mathbf{Z})$ we have that the base-points a_i run over all nonzero rationals p/q, with g.c.d.$(p,q)=1$ and $R(p/q) = 1/q^2$. So, one obtains the following classical theorem on metrical diophantine approximation [Be; J; Ka].

COROLLARY 1 (Jarník–Besicovitch theorem). *For $\lambda \geq 1$, the Hausdorff dimension of the set of the points $\xi \in \mathbf{R}$ such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{C(\xi)}{|q|^{2\lambda}}$$

for infinitely many relatively prime integers p, q is $1/\lambda$.

If $\Gamma = SL(2, \mathbf{Z}[i])$ or, more generally, if $\Gamma = SL(2, \mathfrak{R})$ where $\mathfrak{R}$ is the ring of integers of $\mathbf{Q}(\sqrt{-n})$ and n is a positive integer which is not a perfect square (see e.g. [PD, p. 77]), we obtain, as in [S], that the base-points a_i run over all the nonzero fractions p/q with p, q relatively prime integers in $\mathfrak{R}$, and

$$R\left(\frac{p}{q}\right) = \frac{1}{|q|^2}.$$

Hence, we obtain the next corollary.

COROLLARY 2. *For $\lambda \geq 1$, the Hausdorff dimension of the set of the points $\xi \in \mathbf{C}$ such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{C(\xi)}{|q|^{2\lambda}}$$

for infinitely many p, q relatively prime integers in $\mathfrak{R}$ is $2/\lambda$.

The outline of this paper is as follows: In Section 1, we give the proofs of some lemmas on orbit distribution needed in the proof of theorems. In Section 2 we use the concept of regular system of Baker–Schmidt in order to prove some approximation results. In Section 3 we prove the theorems.

NOTATION. We will use $\|\cdot\|$, m, and Vol to denote Euclidean norm, Lebesgue measure, and hyperbolic volume, respectively. The notation $|z|$ will denote the absolute value of the complex number z. Ω_d will mean the Lebesgue measure of the unit ball of $\mathbf{R}^d$, and ∂A will be the boundary of the set A. We will denote by $B(a, r)$ the Euclidean open ball of center a and radius r; $\bar{B}(a, r)$ will be the corresponding closed ball. By $\#A$ we will denote the cardinality of the set A.

As usual, $C(a, b, \dots)$ will denote a variable constant whose value depends only on the arguments shown. Thus its value may vary from line to line and even in the same line.

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1. Distribution of Orbits

In this section we collect some known results on distribution of orbits. The first one is an asymptotic result due to Nicholls [N1; N2, p. 204] concerning the distribution of orbits under a discrete group $\tilde{\Gamma}$ of hyperbolic isometries of B^d—the unit ball of $\mathbf{R}^d$ with the Euclidean metric—with finite hyperbolic covolume. This result is an improvement of a theorem of Tsuji [T, p. 518].

Given $\xi \in \partial B^d$ and α an angle satisfying $0 < \alpha < \pi/2$, consider the set $\Omega(\xi, \alpha)$ defined as

$$\Omega(\xi, \alpha) = \{\eta \in B^d : |\langle \eta, \xi \rangle| \geq \|\eta\| \cos\alpha\}.$$

Thus, $\Omega(\xi, \alpha)$ is the portion in B^d of the solid cone of axis $O\xi$ and aperture angle α.

For $\eta \in B^d$ we define $N(s, \eta, \xi, \alpha)$ as the number of elements $\gamma \in \tilde{\Gamma}$ such that

$$\gamma(\eta) \in \Omega(\xi, \alpha) \cap \{x : \rho_{B^d}(0, x) \leq s\},$$

where ρ_{B^d} denotes the hyperbolic distance in B^d associated to the metric

$$d\lambda = \frac{2|dx|}{1 - |x|^2}.$$

LEMMA 1.1 [N1].

$$\lim_{s\to\infty} \frac{N(s,\eta,\xi,\alpha)}{\mathrm{Vol}\{x\colon \rho(x,0)<s\}} = C(\Gamma)\alpha^{d-1},$$

and the convergence is uniform in ξ.

In the next lemma we make precise an idea of Sullivan.

LEMMA 1.2. *Let H be any horoball and Γ be a discrete subgroup of* $\mathrm{Möb}(\mathbf{H}^{d+1})$. *Consider the following sum with* $p_0, q_0 \in \mathbf{H}^{d+1}$:

$$\mathcal{S} = \sum_{\substack{\gamma\in\Gamma \\ \gamma(q_0)\in\partial H}} e^{-\delta\rho(p_0,\gamma(q_0))},$$

where $\rho = \rho_{\mathbf{H}^{d+1}}$. *If* $p_0 \notin H$ *then there exists a constant* $C_1 = C_1(q_0, \Gamma)$ *such that, for* $\delta > d/2$,

$$\mathcal{S} \le C_1 e^{-\delta\rho(p_0,\partial H)}.$$

As a matter of fact, C_1 *depends only on*

$$\omega = \min\{\rho(q_0, \eta(q_0)),\ \eta\in\Gamma\backslash\{\text{identity}\}\}$$

REMARK. If $p_0 \in H$ then there exists a constant $C_2 = C_2(\omega)$ such that, for $\delta > d/2$,

$$\mathcal{S} \le C_2 e^{-(\delta-d)\rho(p_0,\partial H)}.$$

Proof. We may assume by conjugation that ∂H is the hyperplane of equation $x_{d+1} = 1$, $p_0 = \lambda e_{d+1}$, where $e_{d+1} = (0,0,\ldots,0,1)$ and $\lambda \le 1$.

There exists $a = a(\omega) > 0$ such that if $P, Q \in \partial H$ and $\rho(P,Q) \ge \omega$ then $\|P-Q\| \ge a$. On $\Omega_k = \{P\in\partial H\colon \|P-e_{d+1}\| \in [k-1,k)\}$ there are at most $C(\omega)\cdot k^{d-1}$ points of $\Gamma(q_0)$ $(k=1,2,\ldots)$, and if $P\in\Omega_k$ then

$$\begin{aligned}\rho(p_0,P) &\ge \rho(p_0,(k-1,0,\ldots,0,1))\\ &= \rho_{\mathbf{H}^2}(i\lambda,(k-1)+i) \ge \log\frac{(k-1)^2+(\lambda+1)^2}{4\lambda}.\end{aligned}$$

Therefore, if $P\in\Omega_k$,

$$e^{-\delta\rho(P,p_0)} \le C\frac{\lambda^\delta}{((k-1)^2+(\lambda+1)^2)^\delta} \le C\left(\frac{\lambda}{k^2}\right)^\delta.$$

Hence

$$\mathcal{S} = \sum_{k=1}^{\infty} \sum_{\substack{\gamma\in\Gamma \\ \gamma(q_0)\in\Omega_k}} e^{-\delta\rho(p_0,\gamma(q_0))} \le C(\omega)\lambda^\delta \sum_{k=1}^{\infty} \frac{1}{k^{2\delta-d+1}} = C_1(\omega)e^{-\delta\rho(p_0,\partial H)},$$

since $\log(1/\lambda) = \rho(p_0,\partial H)$. □

Next, using these two lemmas, we obtain a local version of an estimate of Sullivan [S, p. 227].

LEMMA 1.3. *There exists* $\mu\in(0,1)$ *such that the number* $\nu_n(\mathcal{E},\bar{\mathcal{B}})$ *of horoballs corresponding to a cusp* $\mathcal{E}$ *of* $\mathbf{H}^{d+1}/\Gamma$ *with base-points in a closed ball* $\bar{\mathcal{B}}$ *of* $\mathbf{R}^d$ *and radii* $R\in(\mu^{n+1},\mu^n]$ *satisfies, for all* $n \ge n_0(\Gamma,\mathcal{E},\bar{\mathcal{B}})$,

$$C_1\left(\frac{1}{\mu^n}\right)^d m(\bar{\mathcal{B}}) \le \nu_n(\mathcal{E},\bar{\mathcal{B}}) \le C_2\left(\frac{1}{\mu^n}\right)^d m(\bar{\mathcal{B}})$$

with constants $C_1 = C_1(\Gamma,\mathcal{E})$ and $C_2 = C_2(\Gamma,\mathcal{E})$.

Proof. We may assume without loss of generality that $\bar{\mathcal{B}}$ is contained in the unit ball of $\mathbf{R}^d$ and that $m(\bar{\mathcal{B}})$ is small. Let T be a Möbius transformation such that $T(\mathbf{H}^{d+1}) = B^{d+1}$ and let $\{H_i\}_{i=1}^{\infty}$ be the collection of horoballs in $\mathbf{H}^{d+1}$ corresponding to $\mathcal{E}$ with base-points in $\bar{\mathcal{B}}$ and radii $R_i \le 1$, say. Then $\{T(H_i)\}_{i=1}^{\infty}$ is a new collection of horoballs in B^{d+1}. For all i, the radii R_i and R_i' of H_i and $T(H_i)$ respectively satisfy

$$C_1(\bar{\mathcal{B}})R_i \le R_i' \le C_2(\bar{\mathcal{B}})R_i.$$

So, by conjugation, we can work in B^{d+1}. Also we can assume that the image of the origin, by the canonical projection, does not belong to $\mathcal{E}$ and therefore $R_i' < 1/2$. To simplify notation we still denote by $\bar{\mathcal{B}}$ a closed ball in ∂B^{d+1}, by $\{H_i\}_{i=1}^{\infty}$ the collection of horospheres in B^{d+1} corresponding to $\mathcal{E}$, and by R_i the radius of H_i. In this proof ρ means $\rho_{\mathbf{H}^{d+1}}$.

Take one of these horoballs, H_0, say, and let q be a point in ∂H_0. Let $\xi \in \partial B^{d+1}$ be the center of $\bar{\mathcal{B}}$ and α be the aperture of the cone with vertex at the origin whose intersection with ∂B^{d+1} is equal to $\bar{\mathcal{B}}$. Given $a, b \in \mathbf{R}$ with $a < b$, we will use the following notation:

$$L(a,b) = \{x \in B^{d+1} : \log(e^a - 1) \le \rho(0,x) < \log(e^b - 1)\}$$

$$N(a) = N(a,q,\xi,\alpha)$$

$$\mathfrak{N}(a,b) = \#\{H_i : e^{-b} \le R_i < e^{-a}\}$$

We recall that $N(a,q,\xi,\alpha)$ is the number of elements $\gamma \in \Gamma$ such that $\rho(0,\gamma(q)) \le a$ and $\gamma(q)$ belongs to the portion in B^{d+1} of the solid cone of axis $O\xi$ and aperture angle α. $\#A$ means the cardinality of the set A.

Notice that the orbit of q consists of points equally spaced on each of the horospheres ∂H_i, and therefore there exists a constant $k_0 = k_0(\Gamma,\mathcal{E})$ such that if H_i is a horoball of radius $R_i \ge e^{-b}$ then $L(b, b+k_0)$ contains at least a point $\gamma(q) \in \partial H_i$. So, for T, K real positive numbers

$$\mathfrak{N}(T, T+K) \le N(\log(e^{T+K+k_0} - 1))$$

and for $T \ge T_0$, using that

$$\text{(1.1)} \qquad \lim_{T\to\infty} \frac{\log(e^T - 1)}{T} = 1 \quad \text{and} \quad \lim_{T\to\infty} \frac{\mathrm{Vol}\{x : \rho(0,x) < T\}}{e^{dT}} = C(d),$$

we have by Lemma 1.1 that

$$\text{(1.2)} \qquad \mathfrak{N}(T, T+K) \le C(\Gamma, K)\alpha^d e^{d(T+K)}.$$

Next we will obtain an opposite inequality for some large enough K,

$$\text{(1.3)} \qquad C'(\Gamma, K)\alpha^d e^{dT} \le \mathfrak{N}(T, T+K),$$

and since the constants in (1.2) and (1.3) are independent of T we can conclude that, for $n=0,1,2,\dots,$

$$C'(\Gamma,K)\alpha^d e^{d(T+nK)} \le \mathfrak{N}(T+nK, T+(n+1)K) \le C(\Gamma,K)\alpha^d e^{d(T+(n+1)K)}.$$

Let n_o be a positive integer such that $n_0K \ge T_0$. Now, let T be such that $T=n_0K$. Then for $n \ge n_0$,

$$C'(\Gamma,K)\alpha^d e^{dnK} \le \mathfrak{N}(nK,(n+1)K) \le C(\Gamma,K)\alpha^d e^{d(n+1)K};$$

choosing $\mu=e^{-K}$ and $\nu_n(\Gamma,\mathcal{E})=\mathfrak{N}(nK,(n+1)K)$, the lemma follows.

Now, we prove (1.3). Consider the following sum:

$$S(T,K)=\sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K) \\ e^{-(T+K)}\le R_i<e^{-T}}} e^{-\delta\rho(0,\gamma(q))},$$

where δ is a real number such that $d/2<\delta<d$. Notice that

$$S(T,K)=\sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i \\ e^{-(T+K)}\le R_i<e^{-T}}} e^{-\delta\rho(0,\gamma(q))}$$

and, by Lemma 1.2,

$$S(T,K)\le A\mathfrak{N}(T,T+K)e^{-\delta T}. \tag{1.4}$$

So, in order to prove (1.3), it is enough to obtain a lower bound for $S(T,K)$. If we consider the sums

$$S_1(T,K)=\sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K) \\ R_i\ge e^{-T}}} e^{-\delta\rho(0,\gamma(q))}$$

and

$$S_2(T,K)=\sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K)}} e^{-\delta\rho(0,\gamma(q))}$$

then, since $\partial H_i\cap L(T,T+K)\ne\emptyset$ only if $R_i\ge e^{-(T+K)}$, we have that

$$S_2(T,K)-S_1(T,K)=S(T,K). \tag{1.5}$$

On the other hand,

$$\begin{aligned} S_1(T,K) &\le \sum_{j=2}^{[T+1]} \sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K) \\ e^{-j}\le R_i<e^{-(j-1)}}} e^{-\delta\rho(0,\gamma(q))} + \sum_{\substack{\gamma\in\Gamma \\ \gamma(q)\in\partial H_i\cap L(T,T+K) \\ e^{-1}\le R_i<1/2}} e^{-\delta\rho(0,\gamma(q))} \\ &\le \sum_{j=2}^{[T+1]} (e^{j-1}-1)^{-\delta} N(\log(e^j-1)) + N(\log(e-1)) \\ &\le 2^\delta \sum_{j=1}^{[T+1]} e^{-\delta(j-1)} N(\log(e^j-1)) \end{aligned}$$

and

$$S_2(T,K) \geq e^{-\delta(T+K)}(N(\log(e^{T+K}-1)) - N(\log(e^T-1))),$$

where $[x]$ denotes the integer part of the real number x.

Using Lemma 1.1 and (1.1), we obtain

$$N(\log(e^j-1)) \leq C\alpha^d e^{d(j-1)} \quad \text{for all } j.$$

Therefore,

$$S_1(T,K) \leq C(\Gamma)\alpha^d \sum_{j=1}^{[T+1]} e^{(d-\delta)(j-1)} = C(\Gamma)\alpha^d e^{(d-\delta)T} \tag{1.6}$$

and for T large enough, again using Lemma 1.1 and (1.1),

$$S_2(T,K) \geq C(\Gamma,K)\alpha^d e^{(d-\delta)(T+K)} \quad \text{with } C(\Gamma,K) = C(\Gamma)(1-e^{-dK}). \tag{1.7}$$

Thus, by (1.5), (1.6), and (1.7),

$$S(T,K) \geq \alpha^d e^{(d-\delta)T}(C(\Gamma,K)e^{(d-\delta)K} - C(\Gamma)).$$

Finally, since we can choose K large enough so that

$$C(\Gamma,K)e^{(d-\delta)K} - C(\Gamma) > C > 0,$$

we obtain

$$S(T,K) \geq C\alpha^d e^{(d-\delta)T}, \tag{1.8}$$

and (1.3) is now a consequence of (1.4) and (1.8). □

2. Well-Distributed Systems of Balls

Baker and Schmidt introduced in [BS] the concept of regular system of intervals in order to get some results on diophantine approximation of algebraic numbers. We will extend their definition to systems of balls in $\mathbf{R}^d$ to obtain results of the same kind in any dimension.

DEFINITION. Let $\mathcal{W}$ be a countable collection of Euclidean balls $B_i = B(a_i, R_i)$ in $\mathbf{R}^d$. We will say that $\mathcal{W}$ is a *well-distributed system of balls* with constant Θ if, for every ball $\mathcal{B}$ in $\mathbf{R}^d$, there exists a positive number $K(\mathcal{B})$ such that for every K with $K \geq K(\mathcal{B})$ we have a subcollection $\mathcal{W}(K,\mathcal{B}) \subseteq \mathcal{W}$ satisfying:

(W1) $a_i \in \mathcal{B}$ and $R_i \geq 1/K$ for all $B_i \in (K,\mathcal{B})$;
(W2) For all $B_i, B_j \in \mathcal{W}(K,\mathcal{B})$ with $i \neq j$, $\|a_i - a_j\| > \min\{R_i, R_j\}$;
(W3) $\#\mathcal{W}(K,\mathcal{B}) \geq \Theta K^d m(\mathcal{B})$.

A simple example of a well-distributed system in $\mathbf{R}$ is the collection $\mathcal{W}$ of intervals with center a nonzero rational p/q, g.c.d.$(p, q) = 1$, and radius $1/q^2$. Another example is given, in $\mathbf{R}^2$, by the balls of center z/w and radius $1/|w|^2$, where z and w are Gaussian integers and $w \neq 0$. However, the collection of intervals in $\mathbf{R}$ with center a dyadic number $r + p/2^n$ (with $n \in \mathbf{N}$, $r \in \mathbf{Z}$, and p an odd integer) and radius $1/2^{2n}$ is not a well-distributed system in $\mathbf{R}$.

Using the notion of well-distributed system we obtain the following results.

THEOREM 2.1. *Let $\{\mathcal{W}^l\}_{l=1}^n$ be a collection of well-distributed systems of balls, $\mathcal{W}^l = \{B(a_{l,i}, R_{l,i})\}_{i=1}^{\infty}$, in $\mathbf{R}^d$ with constants Θ_l. Let*

$$\Theta = \min\{\Theta_1, \Theta_2, \ldots, \Theta_n\}.$$

Then, for $0<\alpha<\tau<1$ and $\mathcal{B}$ a ball in $\mathbf{R}^d$, the $(d\alpha)$-dimensional content of the set

$$H = \{\xi \in \mathcal{B} : \|\xi - a_{l,i}\| < C(\xi) R_{l,i}^{1/\tau} \text{ for infinitely many } i \text{ and for all } l \in \mathcal{L}\},$$

where $\mathcal{L} \subset \{1, 2, \ldots, n\}$, is at least $C(\Theta, \alpha)(m(\mathcal{B}))^{\alpha}$.

COROLLARY 2.2. *If $\{\mathcal{W}^l\}_{l=1}^n$, $\mathcal{L}$ and $\mathcal{B}$ are as above, and if $0<\tau<1$, then the Hausdorff dimension of the set of points $\xi \in \mathcal{B}$ such that*

$$\|\xi - a_{l,i}\| < C(\xi) R_{l,i}^{1/\tau} \text{ for infinitely many } i \text{ and for all } l \in \mathcal{L}$$

is at least τd.

In [BS] Baker and Schmidt proved Corollary 2.2 in the case $d=1$, refining some ideas of Besicovitch [Be]. Our argument is an extension of theirs.

In the proof of Theorem 2.1 we will need the following lemma.

LEMMA 2.3. *Let ϵ, R be positive numbers such that $\epsilon \geq 2R$, and let $\mathcal{F}$ be a family of balls in $\mathbf{R}^d$ of radius R such that, for all $B(a_i, R), B(a_j, R) \in \mathcal{F}$ $(i \neq j)$, we have that $\|a_i - a_j\| > \epsilon$. Let $\mathcal{S} = \{S_j\}$ be a countable family of balls in $\mathbf{R}^d$ such that*

(i) *$\sum_j (\text{diam}(S_j))^{\alpha d} < \delta$, and*
(ii) *$\text{diam}(S_j) < \omega$ for all $S_j \in \mathcal{S}$,*

where α, δ, ω are positive numbers and diam(A) *denotes the diameter of the ball A.*

If $\mathcal{F}' \subseteq \mathcal{F}$ denotes the set of balls B in $\mathcal{F}$ such that there exists a ball $S_j \in \mathcal{S}$ whose intersection with B contains a ball of diameter at least $R/2$, then

$$\#\mathcal{F}' \leq \frac{6^d \delta \omega^{d(1-\alpha)}}{\epsilon^d}.$$

Proof. Let $\mathcal{D}$ be the collection of balls $S_j \in \mathcal{S}$ whose intersection with some $B \in \mathcal{F}$ contains a ball of diameter at least $R/2$. For all $D \in \mathcal{D}$, we denote by $\mathcal{G}_D$ the collection of balls of $\mathcal{F}$ which intersect D as we have just described.

We will obtain an upper bound of $\#\mathcal{G}_D$, and since

$$\#\mathcal{F}' \leq \sum_{D \in \mathcal{D}} \#\mathcal{G}_D \tag{2.1}$$

we will get an upper bound of $\#\mathcal{F}'$.

Let r_D be the radius of a ball $D \in \mathcal{D}$ and let $\tilde{D}$ be the ball with the same center as D and radius $r_D + R/2$. It is clear that the centers c_G of the balls G in $\mathcal{G}_D$ belong to $\tilde{D}$ and, since the distance between them is at least ϵ, we have that there exists a constant $C > 1/2^d$ such that

$$m(\tilde{D}) \geq C \sum_{G \in \mathcal{G}_D} m(B(c_G, \epsilon/2)) \geq \frac{\epsilon^d \Omega_d}{2^{2d}} \# \mathcal{G}_D.$$

Hence,

$$\# \mathcal{G}_D \leq \frac{2^{2d}}{\epsilon^d \Omega_d} m(\tilde{D}) = \frac{2^{2d}}{\epsilon^d} \left(r_D + \frac{R}{2} \right)^d.$$

But $R/2 \leq 2r_D$ and so we have that

$$\# \mathcal{G}_D \leq \frac{6^d}{\epsilon^d} (\operatorname{diam}(D))^d.$$

Therefore, by (2.1),

$$\# \mathcal{F}' \leq \frac{6^d}{\epsilon^d} \sum_{D \in \mathfrak{D}} (\operatorname{diam}(D))^d.$$

But, by (i) and (ii),

$$\sum_{D \in \mathfrak{D}} (\operatorname{diam}(D))^{d(1-\alpha)} (\operatorname{diam}(D))^{d\alpha} < \omega^{d(1-\alpha)} \delta,$$

and so we conclude that

$$\# \mathcal{F}' \leq 6^d \frac{\omega^{d(1-\alpha)} \delta}{\epsilon^d}. \qquad \square$$

Proof of Theorem 2.1. We can suppose, by rearrangement, that $\mathcal{L} = \{1, 2, \ldots, p\}$ $(p \leq n)$. If $\mathcal{B}$ is a ball of radius 1 in $\mathbf{R}^d$, we let $\tilde{H}$ denote the set of $\xi \in \mathcal{B}$ such that there exists a sequence $K_j(\xi)$ tending to infinity and a subsequence $\{B_{i(j)}\}$ of $\bigcup_{l \in \mathcal{L}} \mathcal{W}^l$, which also depends on ξ, such that for all j there exists a ball $B(a_{t(j), i(j)}, R_{t(j), i(j)})$ in $\mathcal{W}^{t(j)}$, where $t(j) \in \mathcal{L}$ and $t(j) \equiv j \pmod p$, satisfying

$$\|\xi - a_{t(j), i(j)}\| < \frac{1}{K_j^{1/\tau}} \quad \text{and} \quad R_{t(j), i(j)} \geq \frac{1}{K_j}.$$

Then, we will see that $M_{d\alpha}(\tilde{H}) \geq C(\Theta, \alpha)$, and since

$$\tilde{H} = \bigcap_{l \in \mathcal{L}} \left\{ \xi \in \mathcal{B} : \|\xi - a_{l, i(pk+l)}\| < \frac{1}{K_{pk+l}^{1/\tau}} \right.$$
$$\left. \text{and } R_{l, i(pk+l)} \geq \frac{1}{K_{pk+l}} \text{ for } k = 0, 1, \ldots \right\} \subset H,$$

the theorem follows for balls of radius 1.

In the general case, with $\mathcal{B}$ a ball in $\mathbf{R}^d$ with center h and radius r, we have that

$$M_{d\alpha}\left(\left\{ \xi \in \mathcal{B} : \|\xi - a_{l,i}\| < r\left(\frac{R_{l,i}}{r}\right)^{1/\tau} \text{ for infinitely many } i, \text{ for all } l \in \mathcal{L} \right\}\right)$$
$$= r^{d\alpha} M_{d\alpha}\left(\left\{ \eta \in B\left(\frac{h}{r}, 1\right) : \left\| \eta - \frac{a_{l,i}}{r} \right\| < \left(\frac{R_{l,i}}{r}\right)^{1/\tau} \right.\right.$$
$$\left.\left. \text{for infinitely many } i, \text{ for all } l \in \mathcal{L} \right\}\right)$$

It is easy to see that the families $\{B(a_{l,i}/r, R_{l,i}/r)\}_{i=1}^{\infty}$ $(l \in \mathcal{L})$ are also well-distributed systems, with constants Θ_l respectively, and so the theorem follows.

Let δ be a real number such that

$$\delta < \left(\frac{\Theta\, m(\mathcal{B})}{2.12^d}\right)^{\alpha}, \tag{2.2}$$

and let $\mathcal{U} = \{U_j\}$ be a countable family of balls in $\mathbf{R}^d$ such that

$$\sum_j (\operatorname{diam}(U_j))^{d\alpha} < \delta. \tag{2.3}$$

We will now prove that $\mathcal{U}$ cannot be a covering of $\tilde{H}$ and, consequently, that $M_{\alpha d}(\tilde{H}) \geq \delta$. In order to see this, we will construct by induction a sequence $\{K_j\}_{j=1}^{\infty}$ of positive numbers tending to infinity and a sequence $\mathcal{V} = \{V_j\}_{j=1}^{\infty}$ of finite unions of nonempty and disjoint closed balls, $V_j = \bigcup_{s \in I_j} V_{j,s}$, contained in $\mathcal{B}$. We will have the following conditions on K_j, $V_j = \bigcup_{s \in I_j} V_{j,s}$, and the positive number λ_j defined as

$$\lambda_j = \frac{C}{K_j^{1/\alpha}(m(\frac{1}{2}V_{j-1}))^{1/d\alpha}} \quad \text{with } C = \left(\frac{2^{2d+2}3^d\delta}{\Theta^2\Omega_d}\right)^{1/d\alpha}$$

(in this proof, if A is a set which is a union of balls, $A = \bigcup_k B(p_k, r_k)$, then we will denote the set $\bigcup_k B(p_k, r_k/2)$ by $\frac{1}{2}A$):

- (I.1) $V_j \subseteq V_{j-1}$;
- (I.2) for each $V_{j,s}$, there exists a ball $B(a, R)$ belonging to $\mathcal{W}^{t(j)}$ with $R \geq 1/K_j$ such that $V_{j,s} = \bar{B}(a, \lambda_j)$;
- (I.3) $V_j \cap U_k = \emptyset$ for all $U_k \in \mathcal{U}$, with $\operatorname{diam}(U_k) > \lambda_j$;
- (I.4) $\lambda_j < \min\{1/(4K_j), \lambda_{j-1}/4, 1/K_j^{1/\tau}\}$;
- (I.5) for all $V_{j,s}, V_{j,s'}$ with $s, s' \in I_j$ $(s \neq s')$, the distance between them is at least $3/(4K_j)$;
- (I.6) $m(\frac{1}{2}V_j) \geq (1/2^{d+1})\Theta\Omega_d\lambda_j^d K_j^d m(\frac{1}{2}V_{j-1})$.

Since the balls in V_j are disjoint and with radii λ_j (by (I.2) and (I.5)), condition (I.6) simply means that the number of balls in V_j is at least

$$\frac{1}{2}\Theta K_j^d m\left(\frac{1}{2}V_{j-1}\right).$$

Notice that by (I.1), (I.2), and (I.4) we get that $\emptyset \neq \bigcap_{j=0}^{\infty} V_j \subset \tilde{H}$ and, since by (I.4) the sequence $\{\lambda_j\}_{j=0}^{\infty}$ tends to zero as $j \to \infty$, we have by (I.3) that $(\bigcap_{j=0}^{\infty} V_j) \cap U_k = \emptyset$ for all $U_k \in \mathcal{U}$.

Here is the inductive construction of $\mathcal{V}$.

Initial step: We take $V_0 = \mathcal{B}$. Notice that, by (2.2), there exists a number β such that

$$\delta < \beta \leq \left(\frac{\Theta\, m(\mathcal{B})}{2.12^d}\right)^{\alpha}.$$

We define λ_0 by the condition $\lambda_0^{d\alpha(1-\alpha)}\delta^{\alpha} = \beta$. Then, it is easy to see that

(2.4) $$\lambda_0^{d(1-\alpha)} \le \frac{\Theta}{2.12^d \delta} m(\tfrac{1}{2}V_0);$$

(2.5) $$\delta < \lambda_0^{d\alpha}.$$

Now, by (2.3) and (2.5), it is clear that

(2.6) $$\operatorname{diam}(U_k) < \lambda_0 \quad \text{for all } k.$$

Inductive step: We now fix j in the rest of the argument. If $K_1, \dots, K_{j-1}$ and $V_0, V_1, \dots, V_{j-1}$ have already been constructed, then we take K_j large enough so that (I.4) is verified and K_j also satisfies the following two conditions:

(2.7) $$K_j \ge K^{t(j)}(V_{j-1,s}) \quad \text{for all } s \in I_{j-1},$$

where $K^{t(j)}(V_{j-1,s})$ is the constant given for the ball $V_{j-1,s}$ in the definition of the well-distributed system $\mathcal{W}^{t(j)}$; and

(2.8) $$\frac{3}{4K_{j-1}} \ge \frac{1}{K_j}.$$

Notice that (I.4) can be satisfied since $\alpha < \tau < 1$.

Now, let $\mathfrak{I}_j$ be the finite collection given by

$$\mathfrak{I}_j = \bigcup_{s \in I_{j-1}} \mathcal{W}^{t(j)}(K_j, \tfrac{1}{2}V_{j-1,s}).$$

We recall that $\mathcal{W}^{t(j)}(K_j, \frac{1}{2}V_{j-1,s})$ is the subset of the well-distributed system $\mathcal{W}^{t(j)}$ obtained by applying the definition to each $\frac{1}{2}V_{j-1,s}$ and the number K_j.

Let $a_1, \dots, a_m$ be the centers of the balls in $\mathfrak{I}_j$, and let $\mathfrak{F}_j$ be the collection of closed balls $\bar{B}(a_i, 2\lambda_j)$ $(i = 1, 2, \dots, m)$. Let us observe that

$$m = \#\mathfrak{I}_j = \#\mathfrak{F}_j = \sum_{s \in I_{j-1}} \#\mathcal{W}^{t(j)}(K_j, \tfrac{1}{2}V_{j-1,s});$$

using (W3) (for the well-distributed system $\mathcal{W}^{t(j)}$) and the fact that, by induction, V_{j-1} is a union of disjoint balls, we obtain

(2.9) $$\#\mathfrak{F}_j \ge \sum_{s \in I_{j-1}} \Theta K_j^d m(\tfrac{1}{2}V_{j-1,s}) = \Theta K_j^d m(\tfrac{1}{2}V_{j-1}).$$

We note that if two balls in the collection $\mathfrak{F}_j$ have their centers in different balls $\frac{1}{2}V_{j-1,s}$, then, by (I.5) for $j-1$ and (2.8), the distance between them is at least $1/K_j$. On the other hand, if the centers belong to the same ball $\frac{1}{2}V_{j-1,s}$, then applying (W1) and (W2) (for the well-distributed system $\mathcal{W}^{t(j)}$) we get the same conclusion. So, in any case, by (I.4) the balls in $\mathfrak{F}_j$ are disjoint. Also it is clear, from (I.2) for $j-1$ and (I.4) for j, that the balls in $\mathfrak{F}_j$ are contained in V_{j-1}. Hence if $j > 1$ then, by (I.3) (which holds for $j-1$ by induction), for all $\bar{B}(a_i, 2\lambda_j) \in \mathfrak{F}_j$ we have that

(2.10) $$\bar{B}(a_i, 2\lambda_j) \cap U_k = \emptyset \quad \text{for all } U_k \in \mathcal{U} \text{ with } \operatorname{diam}(U_k) > \lambda_{j-1}.$$

Next we split $\mathfrak{F}_j$ into two disjoint families $\mathfrak{F}_j'$ and $\mathfrak{F}_j''$. $\mathfrak{F}_j'$ consists of those balls Q of $\mathfrak{F}_j$ such that there exists a ball $U_k \in \mathfrak{U}$ whose intersection with Q contains a ball of diameter at least λ_j. By Lemma 2.3 with $\mathfrak{F} = \mathfrak{F}_j$, $R = 2\lambda_j$, $\epsilon = 1/K_j$, $\omega = \lambda_{j-1}$, and $\mathcal{S} = \{U \in \mathfrak{U} : \operatorname{diam}(U) \le \lambda_{j-1}\}$, we get that

$$\#\mathfrak{F}_j' < 6^d K_j^d \lambda_{j-1}^{d(1-\alpha)} \delta.$$

So, for case $j=1$, using (2.4), we obtain

$$\#\mathfrak{F}_1' < \tfrac{1}{2}\Theta K_1^d m(\tfrac{1}{2}V_0);$$

for case $j>1$, using (I.6) (which holds for $j-1$ by induction), we have

$$\#\mathfrak{F}_j' < \frac{6^d \delta K_j^d}{\lambda_{j-1}^{d\alpha}} \frac{2^{d+1} m(\frac{1}{2}V_{j-1})}{\Theta \Omega_d K_{j-1}^d m(\frac{1}{2}V_{j-2})}.$$

By the definition of λ_{j-1} we obtain that

$$\#\mathfrak{F}_j' < \tfrac{1}{2}\Theta K_j^d m(\tfrac{1}{2}V_{j-1}).$$

Hence, using (2.9),

$$\#\mathfrak{F}_j' < \tfrac{1}{2}\#\mathfrak{F}_j,$$

and so

$$\#\mathfrak{F}_j'' \ge \tfrac{1}{2}\#\mathfrak{F}_j \ge \tfrac{1}{2}\Theta K_j^d m(\tfrac{1}{2}V_{j-1}) > 0. \tag{2.11}$$

If $\mathfrak{F}_j'' = \{Q_s : s \in I_j\}$, then we define $V_{j,s} = \frac{1}{2}Q_s$ and $V_j = \bigcup_{s \in I_j} V_{j,s}$.

We need to check that the conditions (I.1)–(I.6) hold for K_j and V_j: (I.1)–(I.4) follow by construction; (I.5) follows from (I.4) because the distance between the centers of the balls $V_{j,s}$ is at least $1/K_j$ and the radii are λ_j. Finally, since

$$m(\tfrac{1}{2}V_j) = \#\mathfrak{F}_j'' m(\tfrac{1}{2}V_{j,s}) = \#\mathfrak{F}_j'' \left(\frac{\lambda_j}{2}\right)^d \Omega_d,$$

using (2.11) we get

$$m(\tfrac{1}{2}V_j) \ge \frac{1}{2^{d+1}} \Theta \Omega_d \lambda_j^d K_j^d m(\tfrac{1}{2}V_{j-1}),$$

and so (I.6) holds too. □

3. Proof of Theorems

LEMMA 3.1. *Let $\mathcal{S}$ be a countable collection of balls $B_j = B(c_j, r_j)$ (with $r_j \le 1$) in $\mathbf{R}^d$ such that for all i, j with $i \ne j$,*

$$\|c_i - c_j\| > \min\{r_i, r_j\} \tag{3.1}$$

Then, given a number τ, $0<\tau<1$, the Hausdorff dimension of the set of points ξ such that

$$\|\xi - c_j\| < C(\xi) r_j^{1/\tau} \quad \textit{for infinitely many } c_j$$

is at most τd.

Proof. Let $\mathcal{B}$ be a ball in $\mathbf{R}^d$ of radius r, and let M be a positive real number. Consider the set $\mathcal{H}$ defined as

$$\mathcal{H}=\{\xi\in\mathcal{B}: \|\xi-c_i\|<Mr_j^{1/\tau} \text{ for infinitely many } B_j \text{ with } c_j\in\mathcal{B}\}.$$

To prove the lemma it is enough to show that $\mathrm{Dim}(\mathcal{H})$ is at most τd.

Given a number $\mu\in(0,1)$, let $\mathcal{A}_n$ denote the set

$$\{B_j\in\mathcal{S}\mid c_j\in\mathcal{B} \text{ and } r_j\in(\mu^{n+1},\mu^n]\}$$

It is clear that for every $B_i, B_j\in\mathcal{A}_n$, $i\neq j$,

$$B\left(a_i,\frac{\mu^{n+1}}{2}\right)\cap B\left(a_j,\frac{\mu^{n+1}}{2}\right)=\emptyset.$$

Comparing volumes, we have that

$$\sum_{i\in I} m\left(B\left(a_i,\frac{\mu^{n+1}}{2}\right)\right)\leq m(B'),$$

where $I=\{i: B_i\in\mathcal{A}_n\}$ and B' is the ball with the same center as $\mathcal{B}$ and radius $r+\mu^{n+1}/2$. Thus, we get

$$\begin{aligned}(3.2)\qquad \#\mathcal{A}_n&\leq\frac{2^d}{\Omega_d\mu^d}\left(\frac{1}{\mu^n}\right)^d m(B')\\ &=\frac{2^d}{\Omega_d\mu^d}\left(1+\frac{\mu^{n+1}}{2r}\right)^d\left(\frac{1}{\mu^n}\right)^d m(B).\end{aligned}$$

If $2r\geq 1$, then using (3.2) we obtain

$$(3.3)\qquad \#\mathcal{A}_n\leq\frac{2^{2d}}{\Omega_d\mu^d}\left(\frac{1}{\mu^n}\right)^d m(B)\quad\text{for all } n\in\mathbf{N}.$$

If $2r\in(\mu^{n_0+1},\mu^{n_0}]$ with $n_0\in\mathbf{N}$, then we also obtain (3.3) for $n\geq n_0$. Furthermore, if there exist $a_l\in\mathcal{B}$ such that $B(a_l,r_l)\in\mathcal{S}$ and $r_l>\mu^{n_0}$, then for all a_j such that $r_j>\mu^{n_0}$ we have that

$$\|a_l-a_j\|>\min\{r_l,r_j\}>\mu^{n_0},$$

and since $2r\leq\mu^{n_0}$ we conclude that $a_j\notin\mathcal{B}$. Hence, if $2r\in(\mu^{n_0+1},\mu^{n_0}]$ then

$$(3.4)\qquad \sum_{t=0}^{n_0+1}\#\mathcal{A}_t\leq 1.$$

Notice that, since $\#A_n<\infty$ for all $n\in\mathbf{N}$, we have that for all ξ in $\mathcal{H}$ there exists a sequence $\{r_j(\xi)\}$ such that r_j tends to zero as $j\to\infty$ and $\|\xi-c_j\|<Mr_j^{1/\tau}$. Hence we get that $\mathcal{H}$ is covered by the collection of balls

$$\tilde{\mathcal{S}}_k=\{\tilde{B}_j=B(c_j,\tilde{r}_j)\mid\tilde{r}_j=Mr_j^{1/\tau},\ c_j\in\mathcal{B},\ r_j\leq\mu^k\}$$

for each positive integer k. Since

$$\sum_{\substack{j\\ \tilde{B}_j\in\tilde{\mathcal{S}}_k}}\tilde{r}_j^{\beta}=M^\beta\sum_{\substack{j\\ c_j\in\mathcal{B}\\ r_j\leq\mu^k}}r_j^{\beta/\tau}\leq M^\beta\sum_{n=k}^{\infty}\sum_{\substack{r_j\in(\mu^{n+1},\mu^n]\\ c_j\in\mathcal{B}}}r_j^{\beta/\tau},$$

using (3.3) and (3.4) we have that, for all $k \geq n_0$,

$$\sum_{\substack{j \\ \tilde{B}_j \in \tilde{\mathcal{S}}_k}} \tilde{r}_j^{\beta} \leq C(M) \sum_{n=k}^{\infty} \frac{\mu^{n\beta/\tau}}{\mu^{nd}}.$$

So, if $\beta/\tau > d$ then $\sum_{j, \tilde{B}_j \in \tilde{\mathcal{S}}_k} \tilde{r}_j^{\beta}$ tends to zero as $k \to \infty$, because $\sum_n \mu^{n(\beta/\tau - d)}$ is convergent. Hence $M_{\beta}(\mathcal{K}) = 0$ and consequently Dim $\mathcal{K} \leq \tau d$. □

Proof of Theorem 1. Let $\{\mathcal{E}_l\}_{l=1}^{n}$ be the set of all cusps of $\mathfrak{M} = \mathbf{H}^{d+1}/\Gamma$. For each l, let $\{H_i^l\}_{i=1}^{\infty}$ denote the set of horoballs corresponding to the cusp $\mathcal{E}_l$.

Let $\gamma_v(t)$ be a geodesic in $\mathfrak{M}$ emanating from p with direction v and such that

$$\limsup_{t \to \infty} \frac{\operatorname{dist}(\gamma_v(t), p)}{t} \geq \alpha. \tag{3.5}$$

Then we have a sequence t_i tending to infinity such that $\gamma_v(t_i)$ is inside some cusp $\mathcal{E}_{l(i)}$ of $\mathfrak{M}$ ($l(i) \in \{1, 2, \ldots, n\}$) and $d_i \geq \alpha t_i$, where

$$d_i = \max\{\operatorname{dist}(\gamma_v(t), p) : t \in [0, t_i]\}.$$

Now, let $\bar{\gamma}_v$ be a lifting to $\mathbf{H}^{d+1}$ of γ_v. Without loss of generality we can suppose that $\bar{\gamma}_v$ is a vertical ray ending at a point $\xi \in \mathbf{R}^d$. We have that

$$d_i = C_{l(i)} + \log \frac{R_{k(i)}}{r_{k(i)}} \quad (k(i) \in \mathbf{N}),$$

where $R_{k(i)}$ is the radius of the horoball $H_{k(i)}^{l(i)}$ corresponding to the cusp $\mathcal{E}_{l(i)}$ which contains $\bar{\gamma}_v(t_i)$, and $r_{k(i)}$ is the radius of the horoball, with the same base-point $a_{k(i)}$ as $H_{k(i)}^{l(i)}$, whose projection on $\mathfrak{M}$ is the region of $\mathcal{E}_{l(i)}$ not attained by γ_v before the time t_i. $C_{l(i)}$ denotes a constant which depends only on the cusp $\mathcal{E}_{l(i)}$. For the sake of simplicity, hereafter we will write r_i and R_i instead of $r_{k(i)}$ and $R_{k(i)}$.

It is clear that $r_i = Ce^{-t_i}$, and so

$$\frac{R_i}{r_i} \geq C_{l(i)} \left(\frac{1}{r_i}\right)^{\alpha}.$$

Therefore

$$\|\xi - a_i\| = r_i \leq C(\xi) R_i^{1/(1-\alpha)}, \tag{3.6}$$

where $C = \max\{C_1, \ldots, C_n\}$.

Thus, if ξ is not a base-point of a horoball corresponding to some cusp $\mathcal{E}_l$, then there are infinitely many solutions a_i of the inequality (3.6). On the other hand, if (3.6) has infinitely many solutions a_i, where each a_i is the base-point of a horoball corresponding to some cusp $\mathcal{E}_{l(i)}$, then the geodesic $\bar{\gamma}_v$ in $\mathbf{H}^{d+1}$ with endpoint $\xi \in \mathbf{R}^d$ projects on a geodesic γ_v in $\mathfrak{M}$ which satisfies (3.5).

Hence, the set appearing in Theorem 1 has the same Hausdorff dimension as the set of points $\xi \in \mathbf{R}^d$ such that the inequality (3.6) holds for infinitely many a_i's. Thus, Theorem 1 follows from Theorem 2. □

REMARK. We can prove more than stated in Theorem 1 by using a similar argument and Theorem 3 instead of Theorem 2.

Given a cusp $\mathcal{E}_l$, let T_l be the set of times t such that $\gamma_v(t) \in \mathcal{E}_l$. Then the Hausdorff dimension of the set of $v \in S(p)$ such that

$$\limsup_{\substack{t \to \infty \\ t \in T_l}} \frac{\operatorname{dist}(\gamma_v(t), p)}{t} \geq \alpha \quad \text{for all } l \in \mathcal{L} \subset \{1, 2, \ldots, n\}$$

is $d(1-\alpha)$.

PROOF OF THEOREM 3. We will prove that the system $\mathcal{W}$ of balls $B(a_i, R(a_i))$ in $\mathbf{R}^d$, where a_i and $R(a_i)$ are respectively the base-points and the radii of the horoballs corresponding to a fixed cusp $\mathcal{E}$ of $\mathbf{H}^{d+1}/\Gamma$, is a well-distributed system. Thus the inequality $\operatorname{Dim} \geq \tau d$ follows from Corollary 2.2, and the opposite inequality is a consequence of Lemma 3.1.

Given a ball $\mathcal{B}$ in $\mathbf{R}^d$, let $\mu \in (0,1)$ and $n_0 \in \mathbf{N}$ be the numbers in Lemma 1.3, and let $K(\mathcal{B}) = 1/\mu^{n_0}$. Then, for $K \geq K(\mathcal{B})$, consider the subcollection

$$\mathcal{W}(K, \mathcal{B}) = \{B(a_i, R(a_i)) \mid a_i \in \mathcal{B} \text{ and } R(a_i) \geq 1/K\}$$

By definition, $\mathcal{W}(K, \mathcal{B})$ satisfies (W1). (W2) follows immediately from the fact that the horoballs in $\mathbf{H}^{d+1}$ with base-points a_i and radii $R(a_i)$ come from a cusp of $\mathbf{H}^{d+1}/\Gamma$ and hence are disjoint. Finally, if $1/K \in (\mu^{n+1}, \mu^n]$ (and so $n \geq n_0$), then $\#\mathcal{W}(K, \mathcal{B})$ is at least the number $\nu_n(\mathcal{E}, \bar{\mathcal{B}})$ appearing in Lemma 1.3 and so (W3) follows from that lemma. □

PROOF OF THEOREM 2. Obviously, any collection of balls which contains a well-distributed system of balls is also a well-distributed system. Therefore, since the family $\mathcal{W}'$ of balls in $\mathbf{R}^d$, $\{B(a_i, R(a_i))\}$ (where a_i and $R(a_i)$ are respectively the base-points and the radii of the horoballs corresponding to any cusp of $\mathfrak{M}$) contains the family $\mathcal{W}$ appearing in the proof of Theorem 3, $\mathcal{W}'$ is a well-distributed system. Hence, the inequality $\operatorname{Dim} \geq \tau d$ follows from Corollary 2.2.

On the other hand, we can get that the horoballs corresponding to different cusps of $\mathfrak{M}$ are disjoint (if they correspond to the same cusp then by construction they are also disjoint), and therefore the balls in $\mathcal{W}'$ satisfy the condition in Lemma 3.1. Thus we obtain the inequality $\operatorname{Dim} \leq \tau d$. □

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