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HOW DOES FINANCIAL THEORY APPLY TO CATASTROPHE-LINKED DERIVATIVES? AN EMPIRICAL TEST OF SEVERAL PRICING MODELS [1]

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Abstract

The paper focuses on the PCS Catastrophe Insurance Option Contracts and empirically tests the degree of agreement between their real quotes and the standard financial theory. The highest possible precision is incorporated since the real quotes are perfectly synchronized and the bid-ask spread is always considered. A static setting is assumed and the main topics of arbitrage, hedging and portfolio choice are involved in the analysis. Three significant conclusions are reached. First, the catastrophe derivatives may be very often priced by arbitrage methods, and the paper provides some examples of practical strategies that were available in the market. Second, hedging arguments also yield adequate criteria to price the derivatives and some real examples are provided as well. Third, in a variance aversion context many agents could be interested in selling derivatives to invest the money in stocks and bonds. These strategies show a suitable level in the variance for any desired expected return. Furthermore, the methodology here applied seems to be quite general and may be useful to price other derivative securities. Simple assumptions on the underlying asset behavior are the only required conditions.

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1. Introduction

New investment and financing opportunities, and innovative risk management techniques involving derivatives have been developed to allow individuals and corporations to cost-effectively reallocate funds and transfer risks to other parties. A growing concern on catastrophe losses has particularly brought attention to catastrophe derivatives and their potential financing and risk sharing benefits for the insurance industry.

The PCS (Property Claim Services) Catastrophe Insurance Options Contracts launched by the Chicago Board of Trade (CBOT) on September 29, 1995 are among the most significant catastrophe derivatives. These are standardized option contracts based on indices that track the insured losses, as estimated by PCS, resulting from catastrophic events that occur in a given area and period. Previously, the CBOT had traded catastrophe futures and options contracts on an index provided by the Insurance Services Office (ISO). Moreover, the CBOT planned to list PCS Single-Event Catastrophe options in 1998 to broad its product offering. In 1997 the Bermuda Commodities Exchange (BCOE) also began trading derivative securities based on the Guy Carpenter Catastrophe Index, an index of losses from climate events in US.

In this paper we focus on the CBOT's PCS options. Previous literature on these particular contracts and other related catastrophe derivatives can be roughly divided into two major categories, according to their main objective. The first group of papers concentrates on pricing issues. They view catastrophe derivatives as financial instruments and, accordingly, they take a financial approach to valuing (see Cummins and Geman (1995), Geman and Yor (1997), among others; Tomas (1998) suggests an actuarial approach). They theorize on the dynamic stochastic behavior of the relevant underlying variables in order to obtain the desired pricing result. From a theoretical point of view this line of research is really important and very promising. From a practical point of view there are some difficulties due to market imperfections (bid-ask spread, other transaction costs, short-selling restrictions, illiquidity that makes difficult a continuous trading, etc.) and some specific properties shown by the underlying indices (their stochastic behavior, the absence of any underlying security available for trade, etc.).This motivates the existence of a second group of papers devoted to describe the contracts and illustrate their most significant applications to both, insurance and capital markets (*e.g.* see D'Arcy and France (1992), Canter *et al.* (1996), Litzenberger *et al.* (1996), O'Brien (1997) and Jaffee and Russell (1997)). They explore the potential benefits of using catastrophe derivatives for the insurance industry, and they compare these securities to other competitive alternatives such as reinsurance and catastrophe-linked bonds. Some papers analyze the potentially attractive new investment opportunities provided by the catastrophe-linked assets from any investor's perspective. Most of all these papers stress the traders' need for an understandable and reliable complete pricing methodology for these innovative securities.

The present paper may be included in the second group, but the standard static asset pricing models are applied. Real bid and ask prices of catastrophe-linked derivatives are considered, and their level of adequation to the static financial theory is empirically tested. Examining static valuation minimizes the impact of real market imperfections, and problems implied by the nature of the underlying variables are avoided if one prices an arbitrary derivative by only bearing in mind the interest rates and the prices of other derivative securities. Thus, we can apply the main topics of asset pricing, arbitrage, hedging and portfolio choice, in a model where bonds and derivatives are the only marketed assets.[1]

In order to apply the static theory, we will consider a two period model characterized by the current date, the derivatives expiration date, their current bid and ask prices and their final payoffs. The analysis is independently implemented once a day.

Once the context has been fixed, we start by analyzing the existence of arbitrage portfolios. We take two different perspectives. First, we test the situation of an investor who incurs in the cost of the bid-ask spread, *i.e.* he/she sells at the bid and buys at the ask price. As expected, we found that it was not possible to form any arbitrage portfolio in that case. Second, we explore the position of any agent who posted one of the available prices (*i.e.* either he/she buys at the bid or sells at the ask for a given asset and incurs on the cost of the bid-ask spread for the rest of assets). If an arbitrage portfolio were available in this context, any other agent could offer a better price and still retain some of the arbitrage gains. Competition among traders willing to earn money without any risk of losing money should lead to a more reduced spread in this situation. We found some relative misspricings in this market what suggests that a narrower spread could have been possible. Consequently, the analysis reveals that traders can sometimes improve real bid or ask quotes without any type of risk or, equivalently, they can price by arbitrage methods.

When a concrete derivative can not be priced by arbitrage methods, we explore the existence of hedged portfolios that contain this derivative. In particular, there may exist some hedged portfolios with a guaranteed positive return slightly lower than the risk-free return (or equal to it, if there exists an arbitrage portfolio of the first type) but with a possible return far larger than it, if some facts take place. Again, competition among traders trying to exploit the attractive benefits of this portfolios should lead to reductions of the spread. Our results show that interesting portfolios of this type could have been formed in some cases.

Previous work on individual portfolio selection and the potential benefits to investors from participating in the (re)insurance market through catastrophe insurance-linked securities includes the papers by Canter *et al.* (1996) and Litzenberger *et al.* (1996). Based on some empirical evidence on the insignificant correlation of the PCS national index with the S&P 500 index (see also Litzenberger *et al.* (1996) and the references contained therein for more evidence on this regard), the first article stresses the diversification benefits that participating in the new securitized insurance risk opens to investors. Using an approach suggested by Fisher Black and Robert Litterman based on the Capital Asset

[1]Stocks, whose returns show an insignificant correlation with the PCS indices (see Canter *et al.* (1996)), can be included too.

Pricing Model (C.A.P.M.), and after calculating some necessary parameters based on historical data of insured losses and premiums, the second paper finds some evidence on the attractiveness of including some hypothetical catastrophe bonds in diversified portfolios of stocks or bonds in terms of the new offered risk/return opportunities (see also more references about other related evidence in Canter *et al.* (1996)).

Our study on portfolio choice in a mean-variance context is closely related to this previous work, but takes a different point of view. It concentrates on the investment in catastrophe insurance options market. Suppose that an investor, possibly attracted by the accompanied diversification benefits, add insurance risk to his/her traditional portfolio of stocks, bonds and real estate. How should he/she efficiently combine PCS options in this insurance portfolio with the risk-less asset in order to obtain the desired expected return with a minimum variance? We try to answer this question armed with two important results from static pricing theory.

We also need the real probability distribution for the underlying insured loss index, and a linear pricing rule compatible with the real quotes. The probability distribution is obtained via simulation and historical catastrophe data. The linear pricing rule is generated from a risk-neutral probability measure attained by applying a methodology proposed by Rubinstein (1994) and a number of others.

Once we have the probability measure and the pricing rule, we look for minimum variance portfolios. An interesting result seems to hold. For investors whose risk is not correlated with the PCS indices (*i.e.*, investors that are not insurers) it may be very useful to sell catastrophe derivatives and to invest the money in other kind of assets, like bonds or stocks.

Summarizing, our empirical results confirm the potential interest of catastrophe-linked derivatives. They are useful for insurers because, in some sense, they can be considered like a special type of reinsurance. Besides, they may be also interesting for another Financial Institutions (banks, for instance) because, if arbitrage and hedging arguments lead to low bid-ask spreads, these Institutions can adequately diversify their portfolios by selling derivatives. Consequently, the high level of risk due to catastrophic events may be appropriately diversified among large numbers of investors who trade "reinsurances" in a financial market.

Let us finally remark that the methodology here applied seems to reveal two interesting properties. First, it is useful for traders because practical criteria and strategies to invest are provided. Second, it may be quite general and can be implemented to analyze other kind of securities. Very weak assumptions are required. Arbitrage and hedging arguments will hold if one is able to identify the underlying uncertainty, *i.e.*, the underlying variables if we are working with derivatives. Variance aversion and C.A.P.M.-type arguments will work well when the probability measure, affecting the underlying uncertainty, can be determined with precision. This has been the case in this paper.[2]

[2]When the usual C.A.P.M. is tested, it is not possible to describe the underlying probability space, and researchers have to obtain information about it by studying some relations involving the returns associated with the different securities. Nevertheless, in this case, the underlying PCS indices behavior has been directly analyzed and the derivatives quotes were not used for this purpose.

The remainder of the paper is organized as follows. Section 2 briefly reviews the main theoretical results we rely on to carry out our empirical analysis of PCS options quotes. Section 3 summarizes the foremost characteristics of PCS options and all our data. In Section 4 we detail the concrete methodology we adopt in our empirical research of PCS options quotes and provide our results. The paper ends with some concluding remarks in Section 5.

2. Theoretical Background

Throughout the paper we will consider a static setting to analyze how the theory of portfolio selection and different asset pricing models may be applied to PCS option contracts. Thus, first of all, we must summarize the general framework and the basic assumptions that lead to the most important theoretical results on asset pricing. A brief review of these topics is the main purpose of the present section. Later, we will provide the way the theory applies in this article to study the market of PCS option contracts.

We focus on the two periods approach characterized by the present date t_0, a future date t_1, n securities denoted by $S_1, S_2, ..., S_n$, their bid prices at t_0 denoted by $v_1, v_2, ..., v_n$, their ask prices $c_1, c_2, ..., c_n$, and the future prices (or final payoffs) at t_1 which depend on a finite number of states of the world $W_1, W_2, ..., W_k$ and are given by the matrix $A = (a_{ij})$, $i = 1, 2, ..., k$, $j = 1, 2, ..., n$, being $a_{ij} \geq 0$ the price of S_j if the state W_i takes place.[3] $\mu_i > 0$ will denote the probability of W_i, $i = 1, 2, ..., k$, and μ will denote the whole probability measure. The inequalities $c_j \geq v_j$ for $j = 1, 2, ..., n$ are clear, and we will accept the convention $v_j = 0$ ($c_j = \infty$) if there is no bid (ask) price available for S_j. The first security S_1 will be a riskless asset (its final payoff is 1 and does not depend on the state of the world) and $c_1 = v_1 > 0$.[4] As usual, the riskless return is given by $R = \frac{1}{v_1}$.

The row matrix $x = (x_1, x_2, ..., x_n)$ will represent the portfolio composed by x_j units of S_j, $j = 1, 2, ..., n$, and $x_j \geq 0$ ($x_j \leq 0$) must hold if $v_j = 0$ ($c_j = \infty$). Its current (at t_0) price will be $P(x) = \sum_{j=1}^{n} p_j x_j$ being $p_j = c_j$ ($p_j = v_j$) if $x_j \geq 0$ ($x_j \leq 0$).[5] Its price at t_1 depends on the state of the world and is given by the column matrix Ax^t where x^t is the transpose of x.

For an arbitrary portfolio x, we will consider the portfolios $x^+ = (x_1^+, ..., x_n^+)$ and $x^- = (x_1^-, ..., x_n^-)$ composed by the purchased and sold securities respectively. To be precise, $x_j^+ = Max\{x_j, 0\}$ and $x_j^- = Max\{-x_j, 0\}$ for $j = 1, 2, ..., n$.

We will follow the approach by Prisman (1986) or Ingersoll (1987) to introduce the concept of arbitrage.[6]

[3] Almost all the results still hold for a matrix A whose elements are also negative, but the constraint $a_{ij} \geq 0$ makes things a little easier and is always fulfilled in our empirical analysis.

[4] Once again, this assumption can be avoided in a general framework, but it is useful and fulfilled in this article.

[5] *i.e.* agents can buy or sell any security but prices are larger if they buy.

[6] In what follows, $Ax^t \geq 0$ denotes that all the elements in this matrix are larger than or equal to 0. Analogously, $Ax^t >> 0$ denotes that all the elements are larger than 0, and $Ax^t > 0$ denotes that elements are larger than or equal to 0, but at least one element is strictly positive. Similar notations will

Definition 1. *The portfolio x is said to be an arbitrage portfolio of the second type, or a strong arbitrage portfolio, if $P(x) < 0$ and $Ax^t \geq 0$, or $P(x) = 0$ and $Ax^t >> 0$.*

The portfolio x is said to be an arbitrage portfolio of the first type, or a weak arbitrage portfolio, if $P(x) = 0$ and $Ax^t > 0$.

Previous literature has characterized the absence of arbitrage by the existence of state prices or discount factors (see for instance Chamberlain and Rosthchild (1983), Ingersoll (1987) or Hansen and Richard (1987)). The following result is a minor extension that incorporates the bid-ask spread and may be easily proved by readapting classical proofs (see also Jouini and Kallal (1995)).

Theorem 1. *There are no arbitrage opportunities if and only if there exists a vector $d = (d_1, d_2, ..., d_k)$ of discount factors such that $d_i > 0$, $i = 1, 2, ..., k$ and*

$$v_j \leq \sum_{i=1}^{k} a_{ij} d_i \mu_i \leq c_j \tag{2.1}$$

for $j = 1, 2, ..., n$.

There are no arbitrage opportunities of the second type if and only if there exists a vector $d = (d_1, d_2, ..., d_k)$ of discount factors such that $d_i \geq 0$, $i = 1, 2, ..., k$, and (2.1) holds.[7] ■

Let us remark that (2.1) leads to $\frac{1}{R} = \sum_{i=1}^{k} d_i \mu_i$. If we set

$$\lambda_i = R d_i \mu_i \tag{2.2}$$

$i = 1, 2, ..., k$, then $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$, and therefore, $\lambda = (\lambda_1, \lambda_2, ...\lambda_k)$ can be considered a probability measure. Furthermore, (2.1) leads to

$$v_j \leq \frac{1}{R} E^{\lambda}(S_j) \leq c_j \tag{2.3}$$

for $j = 1, 2, ..., n$, being $E^{\lambda}(S_j)$ the expected value of S_j at t_1 computed with the probability measure λ instead of μ. This is the reason why λ is called a *Risk Neutral Probability Measure,* and Theorem 1 shows that its existence (and positiveness) is the necessary and sufficient condition to guarantee the absence of arbitrage of the second type (of any kind).

The latter Theorem provides a very well known and important condition to ensure the absence of arbitrage, but we will need to know what happens when this absence fails. In order to improve bid or ask real prices for PCS option contracts, it is interesting to measure, in monetary terms, the degree of arbitrage. It is also useful to analyze market imperfections. For instance, due to transaction costs, the presence of arbitrage could be only apparent but not real.

The following result summarizes some properties of the measures developed by Balbás and Muñoz (1998).

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appear in similar cases.

[7] An analogous result holds if probabilities μ_i are not specified. In such a case, the discount factors $d'_i > 0$ (≥ 0) must verifiy $v_j \leq \sum_{i=1}^{k} a_{ij} d'_i \leq c_j$. The proof is trivial since one can define $d'_i = d_i \mu_i$.

Theorem 2. *Suppose that the set X of arbitrage strategies of the second type is non void. Then, problems*

$$Max\ \frac{P(x)}{P(-x^-)} \quad \{x \in X$$

and

$$Max\ \frac{-P(x)}{P(x^+) - P(-x^-)} \quad \{x \in X$$

achieve an optimal value at the same portfolio x^*. ∎

The disagreement measures m and l are defined by $m = \frac{P(x^*)}{P(-x^{*-})}$, $l = \frac{-P(x^*)}{P(x^{*+})-P(-x^{*-})}$ or zero if no arbitrage opportunities of the second type do exist. Measures m and l vanish if and only if there are no arbitrage opportunities of the second type. When the arbitrage appears m (l) yields available relative arbitrage earns with respect to the value of the sold (interchanged) assets. The inequalities $0 \leq l \leq m \leq 1$ may be proved, and the level of violation of the arbitrage absence grows up as the measures move from 0 to 1. The relationship $l = \frac{m}{2-m}$ holds, and thus since $[0,1] \ni z \longrightarrow \frac{z}{2-z} \in [0,1]$ is an increasing one to one function, both measures provide equivalent information. Further details may be found in Balbás and Muñoz (1998) or Balbás *et al.* (1998).[8] [9]

Let us turn now to hedging strategies and arbitrage portfolios of the first type. If the model does not allow arbitrage portfolios of the second type, R is the highest return than can be guaranteed. However, an investor may be interested in a hedging portfolio whose guaranteed return is very close to (and lower than[10]) R but provides larger returns in some states of the world.

Let us fix a concrete security S_{j_0}, and consider the usual way to hedge the purchase of this security, *i.e.* solve the problem[11]

$$Min\ P(x) \quad \begin{cases} x_{j_0} = 1 \\ Ax^t \geq 1 \end{cases} \tag{2.4}$$

If the solution is attained at $\tilde{x}$, it is clear that $P(\tilde{x}) > 0$ and $\frac{1}{P(\tilde{x})}$ is the optimal guaranteed return if a unit of S_{j_0} is bought. Moreover, there are arbitrage portfolios of the first type such that $x_{j_0} = 1$ if and only if $R = \frac{1}{P(\tilde{x})}$ and $Ax^t > 1$.

An analogous analysis may be done to hedge the sale of S_{j_0}. Just write $x_{j_0} = -1$ in (2.4) instead of $x_{j_0} = 1$. Obviously, not only hedging portfolios, but also arbitrage of the first type can be detected by computing all the hedging portfolios when j_0 moves from 1 to n.

The last part of this synopsis focuses on individual portfolio selection and variance aversion. Assume that there are no arbitrage opportunities (of any sort) in the model.

[8] Once again, these results are also verified in a model where the probabilities μ_i are not specified.

[9] Another procedure, useful to detect arbitrage portfolios, may be found in Garman (1976).

[10] Or equal to R if there are arbitrage strategies of the first type.

[11] Recall that $Ax^t \geq 1$ means that all the elements in the column matrix Ax^t are larger than or equal to 1.

Then, the arbitrage absence still holds for some concrete linear pricing rule π such that $v_j \leq \pi_j \leq c_j$, $j = 1, 2, ..., n$, being π_j the price of S_j provided by π. Besides, π may be considered as a positive real valued linear operator over the space $Span(A)$, span of the columns of A.[12] Then, the *Riesz Representation Theorem* of linear operators in Hilbert spaces allows to establish the following result (see Chamberlain and Rosthchild (1983)).

Theorem 3. *There exists a unique discount factor $d >> 0$ such that d^t belongs to $span(A)$.* ■

We will assume that d^t is not the payoff of a riskless asset. This hypothesis is not restrictive (it only affirms that the market is not risk-neutral and, consequently, $\lambda \neq \mu$) and will always hold in our empirical test.

In order to achieve an easier notation, denote by S_j the j^{th}-column of A, $j = 1, 2, ..., n$, and let us identify each feasible portfolio x with its final payoff $Ax^t = y \in Span(A)$. Denote by $\pi(Ax^t) = \pi(y) = \sum_{i=1}^{k} y_i d_i \mu_i$ its current price provided by π. In particular, $\pi(d^t) = \sum_{i=1}^{k} d_i^2 \mu_i > 0$. Define the return (provided by π) of any $y \in Span(A)$ such that $\pi(y) > 0$ by $R(y) = \frac{y}{\pi(y)}$, and consider its expected value $E^{\mu}(R(y))$ and standard deviation $\sigma^{\mu}(R(y))$. [13] Then, the statement below, whose proof is a consequence of the *Projection Lemma* of Hilbert spaces (see for instance Duffie (1988)), provides the optimal portfolios in a variance-averse model

Theorem 4. *For any $y \in span(A)$ such that $\pi(y) > 0$, there exists a linear combination of d^t and the riskless asset, $\varphi S_1 + \psi d^t$, such that*

i) $\psi \leq 0$
ii) $\pi(y) = \pi(\varphi S_1 + \psi d^t)$
iii) $E^{\mu}(R(y)) = E^{\mu}(R(\varphi S_1 + \psi d^t))$
iv) $\sigma^{\mu}(R(y)) \geq \sigma^{\mu}(R(\varphi S_1 + \psi d^t))$ ■

Hence, for a desired expected return, the minimum variance is attained by selling the portfolio $\tilde{x}$ such that $A\tilde{x}^t = d^t$ and investing the price of $\tilde{x}$, jointly with the own capital, in the riskless asset.[14]

3. Markets and Data

CBOT's PCS Catastrophe options are standardized contracts based on PCS indices that track the insured losses resulting from catastrophic events that occur in a given area and risk period, as estimated by PCS.

[12] *i.e.* the space of $kx1$ column matrixes that can be obtained by linear combinations of the columns of A.

[13] Recall that y and $R(y)$ may be considered random variables and, therefore, $E^{\mu}(R(y)) = \frac{1}{\pi(y)} \sum_{i=1}^{k} y_i \mu_i$ and $\sigma^{\mu}(R(y)) = \frac{1}{\pi(y)} \left(\sum_{i=1}^{k} [y_i - \pi(y) E^{\mu}(R(y))]^2 \mu_i \right)^{1/2}$

[14] Under the usual C.A.P.M. assumptions, the considered securities are stocks and, under the suitable hypotheses on their stochastic behavior, the portfolio $\tilde{x}$ is composed by a long position in the riskless asset and short positions in the stocks. Since the coefficient ψ must be negative, the *Market Portfolio* is composed by the stocks in a long position.

When PCS estimates that a natural or man-made event within the US is likely to cause more than $25 million in total insured property losses and determines that such effect is likely to affect a significant number of policyholders and property/casualty insurance companies, PCS identifies the event as a catastrophe and assigns it a catastrophe serial number. PCS provides nine loss indices daily to the CBOT: a national index, five regional indices, and three state indices (National, Eastern, Northeastern, Southeastern, Midwestern, Western, Florida, Texas and California loss indices). Each PCS loss index represents the sum of current PCS estimates for insured catastrophic losses in the area and loss period covered divided by $100 million and rounded to the nearest first decimal point.

The loss period is the time during which a catastrophic event must occur in order for the resulting losses to be included in a particular index. Most PCS indices have quarterly loss periods, some of them (California and Western) have annual loss periods, and one of them (National) has both quarterly and annual risk periods. Following the loss period, there exists a development period (twelve months) during which PCS continues estimating and reestimating losses for catastrophes that occurred during the loss period. The development period estimates affect PCS indices and determine the final settlement value of the indices.

Catastrophe options are available for trading till the end of the development period. They are European and cash-settled (each point equals $200 cash value). They can be traded as either "small-cap" or "large-cap" contracts. These caps limit the amount of losses that are included under each contract: insured losses from $0 to $20 billion for the small contracts and losses from $20 to $50 billion for the large contracts. In practice, traders prefer negotiating call spreads and so further limiting their associated payoffs. Sophisticated combinations traded as a package, that include several expirations dates and indices, are also available.[15] Catastrophe options bid and ask quotes and the current value of the indices are daily provided by the CBOT. Premiums are quoted in (index) points and tenths of a point (each point equals $200). Strikes values are listed in integral multiples of five points.

Our empirical work studies two periods: from February 25 to April 20, 1998 and from June 23 to July 30, 1998. The quotes used in the empirical analysis were provided by the CBOT and correspond to synchronized bid and ask quotes posted at the end of each day. For each of the days considered, we have also included a risk-free asset. Its prices were obtained from the coupon-only strips quotes reported by The Wall Street Journal.[16] Treasury-bills could have been used instead but strips maturities were much closer to the options expiration dates.[17]

In order to learn about the distributional properties of the catastrophe waiting times

[15] Henceforth, for short, we will merely say 'derivative' or 'option' to refer a single option or a package of ones.

[16] Exact values were obtained through linear interpolation of midpoints of the bid-ask prices by using the closest maturities to the option expiration dates.

[17] Some of our computations were also implemented with T-bills returns. Our main results remained unchanged.

and their associated amount of insured losses, our data also includes a 25-year (1973-1997) catastrophe record provided by PCS. This record included all catastrophes occurred in each state with indication of its serial number, begin and end dates, causes and the PCS's estimates of insured losses. The monetary value of losses was converted to 1997 dollars by using the Producer Price Index reported by the Bureau of Labor Statistics (US Department of Labor).

4. Empirical Research: Methodology and Results

From now onwards, we will first make concrete the methodology employed in the empirical analysis and then we will present the obtained results.

Our analysis only targets those derivatives with a single underlying index and a unique expiration date. We group those derivatives with the same expiration date and the same underlying index that are available for trading. We require for a given day a minimum of four assets in each set. For the first period this filtering left us with derivatives associated to the following indices: National Annual-98 (36 valid days), California Annual-98 (36 days), Eastern September-98 (12 days) and Southeastern September-98 (10 days). For the second period we have the National Annual-98 (27 valid days), Eastern September-98 (10) and Southeastern September-98 (10).[18] Table 1 summarizes the number of relevant derivatives satisfying the above criteria and the number of their quotes available in the two periods.[19] This constitutes our whole sample for Subsections 4.1 and 4.2.

Table 1
Overview of the PCS Catastrophe Options Sample

This table describes our sample of derivatives. For each specific index, we require a minimum of four tradable securities in order to includ a given day in the analysis. The first column gives cumulated figures corresponding to the entire periods, and the subsequent one summarize the daily number of derivatives and quotes.

	NAT ANN 98					EST SEP 98					SE SEP 98					CAL ANN 98				
	Total	Daily Statistics				Total	Daily Statistics				Total	Daily Statistics				Total	Daily Statistics			
Number of		Mean	Min.	Median	Max.		Mean	Min.	Median	Max.		Mean	Min.	Median	Max.		Mean	Min.	Median	Max.
Panel A: First Period																				
Days	36					12					10					36				
Derivatives	316	8.77	8	9	9	53	4.42	4	4	5	40	4	4	4	4	144	4	4	4	4
Bid Quotes	253	7.03	7	7	8	48	4	4	4	4	35	3.5	3	3.5	4	112	3.11	1	4	4
Ask Quotes	260	7.22	6	7	9	53	4.42	4	4	5	40	4	4	4	4	120	3.33	3	3	4
Panel B: Second Period																				
Days	27					10					10									
Derivatives	283	10.48	8	11	13	40	4	4	4	4	40	4	4	4	4					
Bid Quotes	187	6.93	5	7	8	40	4	4	4	4	18	1.8	1	2	2					
Ask Quotes	232	8.59	5	10	11	27	2.7	2	2	4	36	3.6	2	4	4					

[18]The expiration dates for the 98 annual contracts and the September 98 contracts are 12/31/99 and 09/30/99, respectively.

[19]We also detected that it was possible to synthetically produce some other Eastern September-98 options based on their corresponding Northeastern and Southeastern options for some days in our sample. To be consistent, as the latter two negotiated independently, we decided not to include the synthesized Eastern options as other derivatives available for trading in our sample.

In Subsections 4.3 and 4.4, a large amount of data on waiting times and their associated amount of losses is required in order to perform reliable simulations and therefore, the characteristics of our historical data set forces us to exclusively concentrate on the National Annual-98 Index.

4.1. Pricing by strong arbitrage methods

The price of the PCS derivatives will be analyzed once a day along each tested period. Hence, under the notations of the second section, the date t_0 will always be the corresponding day, while securities $S_2, S_3, ..., S_n$ will be PCS option contracts (call or put spreads, butterflies etc.), available this day, and with the same underlying index W and expiration date t_1.[20] Their bid and ask prices are perfectly synchronized and provided by CBOT. S_1 will be a pure discount bond available at t_0 and such that its maturity is as close to t_1 as possible. Of course, all the data and parameters (dates t_0 or t_1, securities, prices etc.) depend on the concrete day under revision.

Let E_1 be the current value of the index and denote by $E_2, ..., E_r$ the striking prices, corresponding to S_i, $i = 2, 3, ..., n$. The future state of the world will be determined by the final (at t_1) value of W, and the matrix A of final payoffs may be easily computed. In fact, all the elements in its first column (payoffs of the riskless asset) are equal to 1, and the rest of the columns are given by the usual differences between W and E_j, $j = 1, 2, \ldots, r$. It is obvious that, for an arbitrary strategy x, its final payoffs verify the constraints $Ax^t \geq 0$, $Ax^t > 0$ or $Ax^t >> 0$ if and only if these constraints are fulfilled when the settlement value of W belongs to the set $\{E_1, E_2, ..., E_r\}$.[21] So, the absence or existence of arbitrage may be tested under the assumption that these elements are the only possible states of the world . Furthermore, this simplification neither modifies the value of the disagreement measures m and l, nor affects the results when hedging or weak arbitrage portfolios are being computed. Consequently, we will permit W to attain all the feasible values only when testing portfolio choice models.

Once the available derivatives, their real bid-ask prices provided by CBOT, the r states of the world and the matrix A are fixed, we can compute the measure m and the portfolio x^* introduced in Theorem 2. If $m \neq 0$, there are arbitrage opportunities. This case has never appeared along the tested periods.

Next, we fix an arbitrary option S_{j_0}, $j_0 = 2, 3, ..., n$, and consider an agent who can buy this derivative by paying the price v_{j_0}.[22] If the new values for m and x^* show the presence of arbitrage and the profits represented by m are high enough to overcome the market frictions, it may be concluded that the market allows to price S_{j_0} by arbitrage methods. An agent can offer a new bid price v'_{j_0} (such that $v_{j_0} \leq v'_{j_0} \leq c_{j_0}$ and, therefore, better than the current bid price v_{j_0}) without any kind of risk. The position will be hedged by

[20] *i.e.* the underlying index and the loss and development periods coincide for all the considered derivatives.

[21] And the same property holds if one writes 1 instead of 0 in the right side of above inequalities.

[22] *i.e.* the bid-ask spread vanishes for the j_0^{th}–security. The rest of the prices are not modified. Of course, this analysis has not been implemented in cases where $v_{j_0} = 0$.

implementing the arbitrage portfolio x^* if a new investor accepts the new bid price.

Analogously, one can analyze if the ask price c_{j_0} may be improved. Just consider that c_{j_0} equals both, the bid and the ask price, and compute the new solutions for m and x^*.[23]

The above procedure can be applied for all the available securities (*i.e.* for $j_0 = 2, 3, ..., n$) in order to test how often the market allows to price by strong arbitrage methods. The empirical results are confined to Table 2.

Table 2
Second Type Arbitrage Opportunities

The bid-ask spread has been removed for each option at a time and its price was set equal to alternatevely the bid quote and the ask quote when possible. The table summarizes the resulting arbitrage opportunities of the second type and their associated optimal gains as quantified by the m measure. The first two columns show the number of days for which there are some arbitrage opportunities. Subsequent columns give statistics computed over those days with arbitrage opportunities ($m \neq 0$).

	Detected Second Type Arbitrage Opport.						Daily Maximum m (per cent)				
	Days		Daily Number of Opport.				Mean	Min.	Median	Max.	Mode
Index	No.	Per cent	Mean	Min.	Median	Max.					
Panel A: First Period											
NAT ANN 98	7	19.44	2	1	2	3	10.23	9.09	9.09	13.98	9.09
EST SEP 98	5	41.67	1	1	1	1	6.82	6.67	6.82	6.98	6.70
SE SEP 98	0	0.00									
CAL ANN 98	26	72.22	1	1	1	1	15.46	11.76	16.67	21.05	16.67
Panel B: Second Period											
NAT ANN 98	27	100.00	2.22	1	2	3	19.05	2.50	9.09	37.50	37.50
EST SEP 98	9	90.00	1	1	1	1	10.13	9.96	10.17	10.23	10.17
SE SEP 98	0	0.00									

This particular type of arbitrage is detected quite often. It should be noted that these results seem to reveal that the price setting process might be improved. Hedging (with arbitrage portfolios) would be feasible. The arbitrage profits are quite large and this should be used by investors to offer new prices. For the National Annual-98 index, arbitrage opportunities appear in seven out of 36 days for the first period (see Table 2) and in up to three different cases. The maximum value of m is equal to .1398 (this corresponds to an l value of .0752). For the second period and the same index, arbitrage is feasible every day for up to three different available premium quotes. This time the maximum value of m is .375 ($l = .2308$). This reflects a riskless benefit that amounts to a 37.5% of the total monetary value of the sold assets (or a 23.08% of the total monetary value of all traded assets). With respect to other indices, California and Eastern include a unique position that allows for arbitrage hedging in the first period (the maximum value of m is .2105 and .0698, respectively) and the same may be said about the Eastern index in the second period (maximum $m = .1023$). No misspricings were found for the Southeastern index. In any event, the number of available positions were notably low for these last three indices (see Table 1). Thus, note that for a significant percentage of days, agents

[23]This analysis has not been implemented in cases where $c_{j_0} = \infty$.

could analyze the bid-ask spread and offer more efficient prices in some cases without assuming any kind of risk. This fact should lead to smaller spreads.

For illustration purposes, we show in Table 3 the optimal (maximum m value) second type arbitrage portfolio detected on date 07/24/98 for the Eastern September-98 Index (Strategy 1). Exhibit 1 plots the portfolio payoffs pattern for different levels of the final index value.

Table 3
Optimal Second Type Arbitrage on July 24, 1998 for the Eastern September-98 Index

This table shows the optimal second type arbitrage opportunity corresponding to date 7/24/1998 and the Eastern September-98 Index. CA 20 40 and PU 50 stand for a call spread and a put with relevant exercise prices as indicated, respectively. All derivatives available for trading together with their bid and ask prices are reported. The price for a zero-coupon bond (risk-less asset) with a maturity value of one point is also given. All prices are expressed in points with a value of $200. Bold face is used to indicate those assets involved in the detected arbitrage portfolio, and the bought or sold units are given in parenthesis beside the affected price (a negative sign indicates a sale). The last two rows give the portfolio price and the m value. This arbitrage was detected when the bid quote was set equal to the ask for the PU 50 derivative. The same arbitrage strategy was detected for 9 days.

Asset	Bid	Ask
Bond	0.93867188	**0.93867188 (100)**
CA 20 40	**3 (-4)**	n.a.
CA 40 60	**2.5 (-1)**	3.5
CA 150 200	2	n.a.
PU 50	30	**45 (-2)**
Portfolio Price	-10.6328	
Value of m	0.1017	

As better quotes could have been offered in some cases, this fact led us to try to compute the adjustments that could have been implemented in the bid (an increase) and ask (a decrease) premiums. To this aim we followed the next algorithm. Focusing on one of the quotes which gave rise to the above arbitrage opportunities, we appropriately moved up or down the implied quote only for a tick and then searched again for arbitrage opportunities. This process was iterated until reaching a total removal of the riskless arbitrage hedging. We carried over this algorithm for each price independently. The corresponding price and spread final adjustments are given in Table 4. For the National Annual derivatives, our results show price changes that range from 2.5% to 100% along with spread reductions ranging from 5% to 56.52%. Significant adjustments were also possible for the other indices.

It should be mentioned that some refinements of this procedure point out that a more adjusted set of prices may still be reached. If the above algorithm is not carried out *ceteris paribus*, that is, if we keep the final adjusted premium before moving to the next one, we find that new arbitrage hedging strategies could appear, thereby leading to possible further reductions in the spread.[24]

In summary, although the market quotes studied here do not permit to gain arbitrage profits to anyone obliged to incur in the cost of the bid-ask spread, better relative pricing

[24] As the ordering might be relevant in this case, we do not report our results.

by (strong) arbitrage methods is possible in the PCS options market for the studied periods. This is important for two reasons. First, this information is useful for traders since the whole arbitrage portfolio may be shown. Second, frictionless pricing theory suggests that competition among traders should lead to a situation with correct relative prices, that is, a set of quotes that should exhaust any exploitable possibility of making money without any sort of risk.

Table 4
Bid-Ask Spread Reduction

This table shows the bid-ask spread reduction that can be implemented for each derivative in order to remove the second type arbitrage strategies previously detected. Those assets involved are reported on the left side; additionally, in parenthesis we indicate whether changes correspond to the ask quote (a), bid quote (b) or both bid and ask quotes (b/a). CA 40 60 stands for a call spread with exercise prices 40 and 60. PS 40 60 stands for a put spread with relevant exercise prices as indicated. The price change and the spread reduction are both given in ticks (*i.e.* $20 or one tenth of a point) and in percentage terms. For each asset some descriptive statistics have been computed over those days and quotes for which changes were possible.

	Price Change						Spread Reduction					
	Tick			Per cent			Ticks			Per cent		
Index & Derivative	Mean	Min.	Max.	Mean	Min.	Max.	Mean	Min.	Max.	Mean	Min.	Max.
Panel A: First Period												
NAT ANN 98				10.09	5.17	22.41				16.35	5.41	56.52
CA 80 100 (a)	5	3	13	8.62	5.17	22.41	5	3	13	19.80	10.34	56.52
CA 100 120 (b/a)	3.5	2	5	12.69	10.00	15.38	3.5	2	5	9.66	5.41	13.89
CA 120 140 (a)	5.71	5	10	10.39	9.09	18.18	5	5	5	12.50	12.50	12.50
EST SEP 98												
PS 40 60 (a)	14	14	14	8.24	8.24	8.24		n.a.			n.a.	
CAL ANN 98												
CA 100 200 (b)	5.54	4	8	18.46	13.33	26.67	4	4	4	13.33	13.33	13.33
Panel B: Second Period												
NAT ANN 98				26.33	2.50	100.00				13.45	5.00	33.33
CA 40 60 (b)	3	3	3	4.00	4.00	4.00		n.a.			n.a.	
CA 60 80 (b)	18	11	25	72.00	44.00	100.00	18	11	25	25.13	16.92	33.33
CA 100 120 (b)	6	6	6	60.00	60.00	60.00	6	6	6	15.00	15.00	15.00
CA 100 200 (a)	5	5	5	2.50	2.50	2.50	5	5	5	5.00	5.00	5.00
CA 120 140 (b/a)	5.41	4	6	37.35	6.67	60.00	7.11	6	10	15.19	13.33	20.00
EST SEP 98												
PS 0 50 (a)	53.56	52	54	11.90	11.56	12.00	53.56	52	54	35.70	34.67	36.00

Some factors related to the implementation of the detected arbitrage strategies and not considered so far might explain those relative misspricings. One of them is the existence of transaction costs. On this point, it should be kept in mind that measures m and l represent relative arbitrage profits, and the levels achieved by these measures are high enough to reflect earns after discounting transaction costs.

Another factor is due to the number of units of each asset needed to implement some arbitrage strategies. This number might not be available for trading. We do not have any piece of information about the volume associated to the quotes gathered by the CBOT that constitute our sample. Nevertheless, a comparison between the volume corresponding to the transactions made during our sample periods and the number of derivatives needed to implement the detected arbitrage strategies lead us to think that in most cases this does not seem to be a real problem.

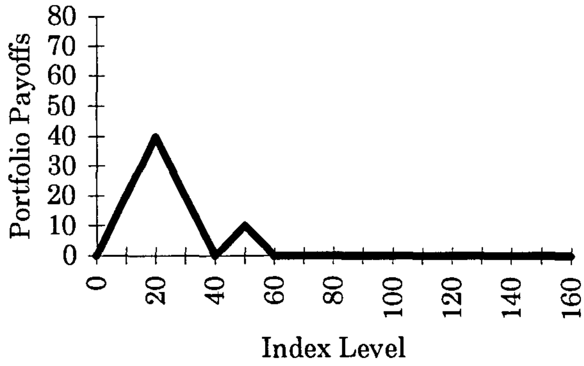

Exhibit 1: Final Payoffs of Strategy 1

4.2. Weak arbitrage and hedging portfolios

Suppose that for a fixed $j_0 \in \{2, 3, ..., n\}$ it is still obtained $m = 0$ after assuming that v_{j_0} is the ask price (respectively, c_{j_0} is the bid price). Then, problem (2.4) (respectively, after the modification $x_{j_0} = -1$) has been solved in order to analyze how the real bid price v_{j_0} (ask price c_{j_0}) can be improved. This will be case when the achieved solution guarantees a return R or very close to R. If so, investors can offer a new bid (ask) $v_{j_0} \leq v'_{j_0} \leq c_{j_0}$ and the solution of (2.4) provides a portfolio that will almost guarantee the riskless return R if a new agent accepts the new price. Furthermore, this strategy could lead to great returns in some states of the world, and thus, it could be interesting for many investors.

Following this procedure hedging strategies were obtained and they were grouped into first type arbitrage opportunities (with a guaranteed return equal to R and payoffs greater than one in at least one state of the world) and other optimal hedging strategies. Both, the guaranteed net return and the maximum possible net return, were computed for each detected position available for hedging and mean values are given in Table 5. We also report the corresponding mean values after substractioning the return guaranteed by the risk-free asset. For some states of the world, extraordinarily large returns might be obtained (e.g., there were first type arbitrage opportunities that involved selling one call spread 80/100 and gave rise to a possible net return of 2, 215.63). The minimum net return equals that of the risk-free asset for almost all cases.

Thus, an important part of the available positions might have been hedged by means of weak (and strong) arbitrage or other optimal strategies leading in many cases to possible returns exceeding largely that of the risk-free asset. Note that this has been feasible even in a situation in which the underlying index is not tradable, and put derivatives are seldom available. Considerations akin to the ones pointed out at the end of the previous subsection, regarding the proper interpretation of these results, also apply here.

Table 5
Optimal Hedging Portfolios

CA 40 60 stands for a call spread with exercise prices 40 and 60, and similarly for the other possible exercise prices. CB denotes a butterfly call spread with relevant exercise prices as indicated. For each derivative, the number of days for which a hedging strategy was available is given and in parenthes it is indicated whether the hedged derivative is bought (b) or sold (s) at the optimal hedging portfolios. Guaranteed and maximum returns in average terms along with the corresponding excesses over the risk free rate ,R, are also given.

	Optimal First Type Arbitrage Opport.					Other Optimal Hedging Portfolios				
	Days	Mean Guaranteed Return (%)		Mean Maximum Return (%)		Days	Mean Guaranteed Return (%)		Mean Maximum Return (%)	
Index & Derivative		Return	Excess	Return	Excess		Return	Excess	Return	Excess
Panel A: First Period										
NAT ANN 98										
CA 40 60						16 (b)	10.35	0.00	10.35	0.00
						1 (s)	10.62	0.00	10.62	0.00
CA 120 140	29 (s)	10.27	0.00	2,215.63	2,205.36					
CAL ANN 98										
CA 80 100	1 (s)	9.74	0.00	2,204.53	2,194.79					
Panel B: Second Period										
NAT ANN 98										
CA 60 80	5 (b)	8.23	0.00	2,172.77	2,164.54					
CB 40 60 80 100	1 (s)	8.11	0.00	2,170.27	2,162.16	6 (s)	6.57	-1.46	2,137.97	2,129.93

Again, for illustration purposes, Table 6 shows the optimal hedging portfolio (weak arbitrage) on date 07/01/98 for the National Annual-98 Index (Strategy 2). Exhibit 2 plots the portfolio payoffs pattern for different levels of the final index value.

Table 6
Optimal Hedging Portfolio
(Weak Arbitrage) on July 1, 1998
for the National Annual-98 Index

This table shows the optimal hedging portfolio detected on date 7/1/1998 for the National Annual-98 Index. CA 30 50 stands for a call spread with exercise prices 30 and 50, and similarly for the other possible exercise prices. CB denotes a butterfly call spread with relevant exercise prices as indicated. The reported portfolio is a weak arbitrage portfolio which permits to hedge the purchase of the CA 60 80 derivative. All derivatives available for trading together with their bid and ask prices are reported. The price for a zero-coupon bond (risk-less asset) with a maturity value of one point is also given. All prices are expressed in points with a value of $200. Bold face is used to indicate those assets involved in the detected hedging portfolio, and the bought and sold units are given in parenthesis (a negative sign indicates a sale). The last row gives the portfolio price. The same portfolio was available for 5 consecutive days.

Derivative	Bid	Ask
Bond	0.92351562	**0.92351562 (1)**
CA 30 50	10	n.a.
CA 40 60	**7.5 (-1)**	n.a.
CA 60 80	**2.5 (1)**	9
CA 80 100	n.a.	7
CA 100 120	1	5
CA 100 150	n.a.	12
CA 100 200	n.a.	20
CA 120 140	n.a.	5.5
CA 150 200	4	7.5
CA 180 200	0.4	1.8
CA 200 250	n.a.	4
CA 250 300	0.5	2.5
CB 40 60 80 100	n.a.	**5 (1)**
Portfolio Price	0.92351562	

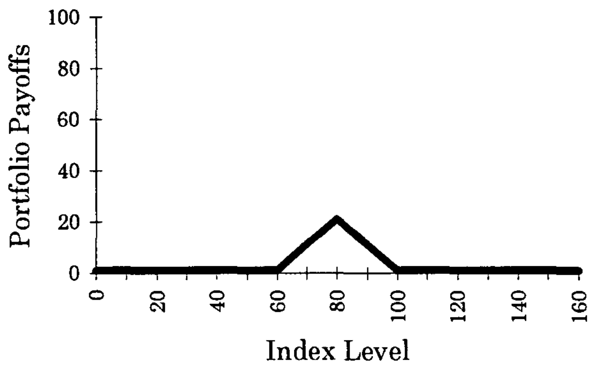

Exhibit 2: Final Payoffs of Strategy 2

4.3. Evaluating the index real distribution and the risk-neutral probability measure

To asses catastrophe options from an actuarial point of view, an analysis of the distribution of possible future values of the underlying indices is required. There are essentially two approaches to forming such probability assessments. One is to use computer simulation of scenarios based on a vast amount of meteorological, seismological and economic information. The other relies on statistical modelling based on historical data.

This subsection is partially devoted to the analysis of the distributional properties of the National Annual Index to be used in the rest of the paper and for this matter we concentrate on the statistical analysis of historical catastrophe data. We develope a nonparametric simulation procedure in order to obtain the expected final payoffs. This method does not require any distributional assumption, instead 'it lets the data talk'. Furthermore, it relies on the empirical distribution of waiting times and their associated losses thereby avoiding the traditional shortage of data that is faced when using exclusively the empirical distribution of the final historical values of the index.

In addition, from a financial perspective, once we have exhausted the arbitrage and hedging pricing approaches, risk considerations come into place and therefore the use of probability assessments is also necessary. As stated by Theorems 3 and 4, the underlying index real distribution is required in order to solve minimum variance problems. In this case, variables and parameters (dates t_0 or t_1, the riskless return, securities, prices etc.) are introduced by the procedures already mentioned, but the set of states of the world must be enlarged. Now this set must incorporate all the index feasible final (at the expiration date t_1) values and not only the derivatives striking prices.

Fix a day t_0 and, consequently, let us assume that all the parameters are fixed. To determine the final distribution of the underlying index W, we proceed as follows. First of all, we consider the empirical distribution of random variables T, "*time between two consecutive catastrophes*", and L, "*losses caused by a specific catastrophe*". It is assumed that T and L can achieve several values with probabilities according to the empirical frequencies obtained from the real data described in the third section. Later, we simulate several values $T_1, T_2, ..., T_s$ of T till $\sum_{i=1}^{s-1} T_i \leq t_1 - t_0$ and $\sum_{i=1}^{s} T_i > t_1 - t_0$, and $s-1$ values $L_1, L_2, ..., L_{s-1}$ of L. Each specific result L_i is incorporated if and only if $L_i \geq 25$ million of dollars, and we take $L_i = 0$ otherwise. If W_0 is the index value at t_0, the simulation process provides the total value $W = W_0 + \sum_{i=1}^{s-1} L_i$ which is translated into index points. The whole simulation process is repeated a high number of times in order to attain a numerical distribution of W.

The risk neutral probabilities, defined in (2.2), have been determined too. We have followed the general method proposed by Hansen and Jagannathan (1997).[25] Hence, fix a day t_0 and all the parameters of the problem. Suppose that the simulation process has been already implemented and, therefore, the (real) probability measure $\mu = (\mu_1, \mu_2, ..., \mu_k)$ is known. Then, the (risk-neutral) measure λ is obtained by minimizing $\sum_{i=1}^{k} (\lambda_i - \mu_i)^2$ among the row-matrixes λ such that $\sum_{i=1}^{k} \lambda_i = 1$, $\lambda_i \geq 0$, $i = 1, 2, ..., k$, and (2.3) holds.

Once the measures μ and λ have been determined, we can give two theoretical prices per security. The first one, $\frac{1}{R} E^{\mu}(S_j)$, $j = 1, 2, ..., n$, is the *Pure Premium*, usual in Actuarial Sciences. The second (see (2.3)),$\frac{1}{R} E^{\lambda}(S_j)$, is deduced from a financial point of view by considering real quotes and applying the most important topics on static asset pricing models.

Above procedures have been implemented for all the possible values of the current date t_0 (every day of our sample periods).

We restrict our observations of the value of insured losses to the sample period $1990 - 1997$ for several reasons. Exhibit 3 shows the annual number of catastrophes and quantity of their associated losses since 1973. As the figure suggests, while the annual number of catastrophes remains reasonably stable along the period, there seems to be a general positive trend, and a structural change in the behavior of the value of insured losses associated with each catastrophe since hurricane Hugo hit US in 1989. However, these patterns may be only illusory, and due to the fact that insured losses are affected by multiple variables ignored so far like population growth, development, changes in insurance coverage, number of premiums, and inflation. When a sufficient long period of time is considered, the adjustments of the loss series for these variables tend to homogenize it, approximating past losses to more recent ones.[26] Thus, a reasonable approximation to the adjusted series can be obtained by concentrating on recent (unadjusted) losses. Moreover, any adjustment of the series, which become necessary when a long period of losses is considered, turns out to be quite cumbersome and methodologically questionable.

With regard to the number of catastrophes, the whole sample since 1973 has been used. This is expected to surmount the difficulty of getting good estimates based on small samples for the probability of low frequency events such as catastrophes.

[25]The Hansen and Jagannathan (1997) method extends the procedure provided by Rubinstein (1994) and by Jackwerth and Rubinstein (1996) to study the effect of the volatility smile on the risk neutral probability measure.

[26]An illustration of this effect may be found in Litzenberger *et al.* (1996). These authors adjust historical loss ratios for both, the increase in population and the market penetration of catastrophe coverage.

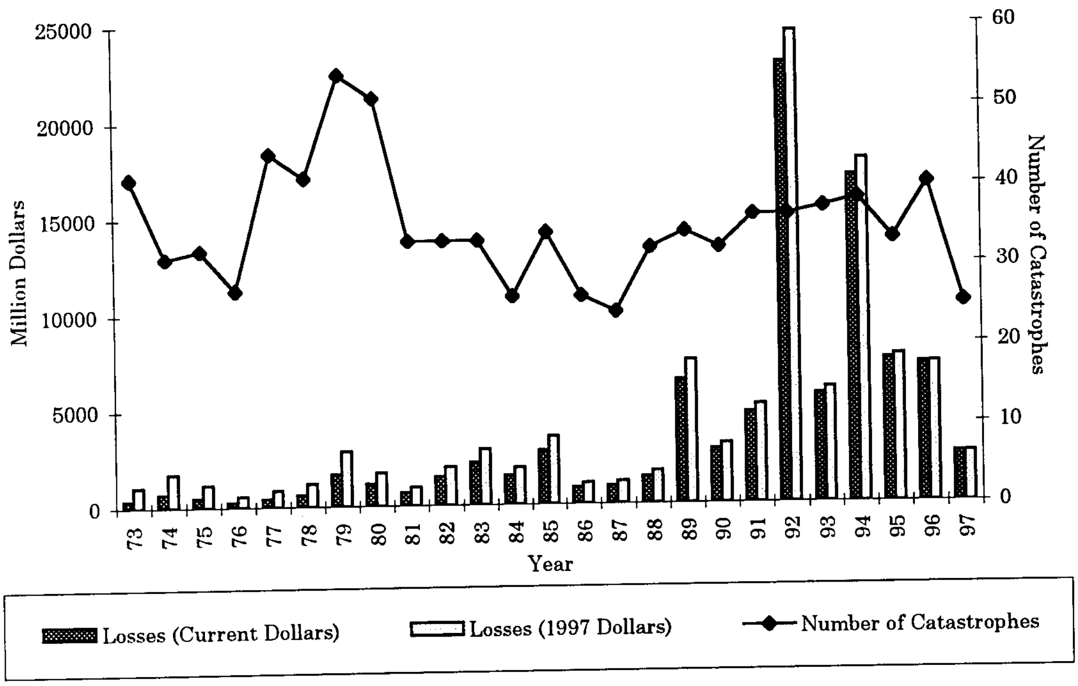

Exhibit 3: Number of catastrophes and amount of losses (1973-1997)

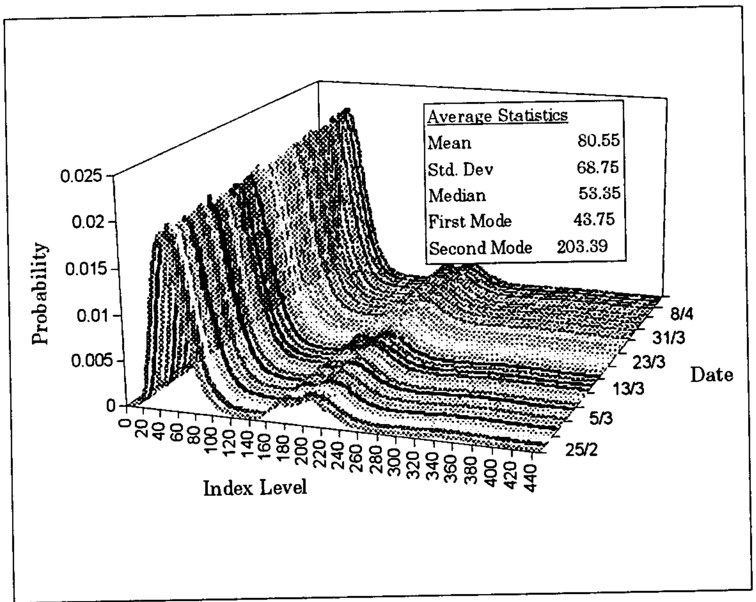

Exhibit 4: Probability distributions of the index final value (first period)

The simulation process described above has been implemented in order to estimate the probability distributions of the final (end of 1999) value of the National Annual-98 index for every day in both periods. A number of 100,000 replications has been used for each day. Results are given in Exhibits 4 and 5. As the figures show, the probability mass is mainly accumulated around the index levels lying approximately in the intervals 20-100 and 160-240. The distributions are bimodal or even trimodal. Note also that as days go by and the final date approaches, the probability mass tends to concentrate due to the accompanying uncertainty reduction.

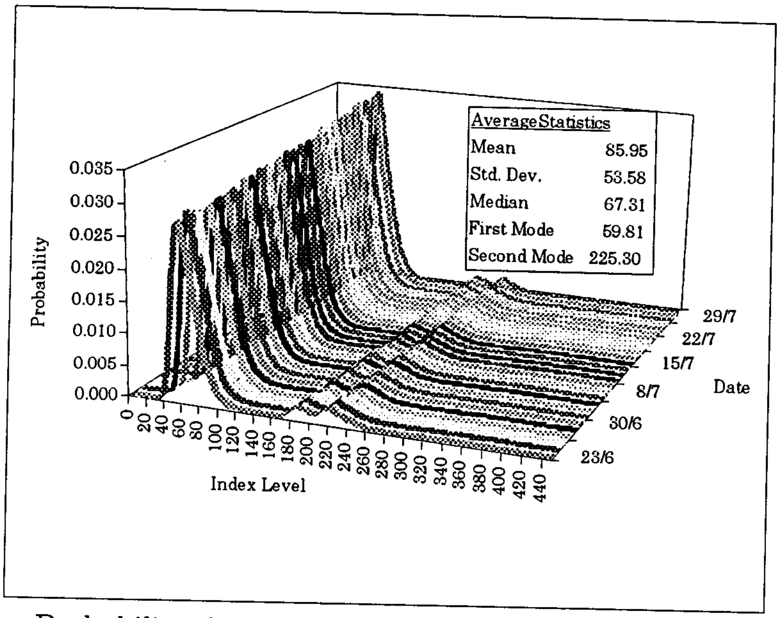

Exhibit 5: Probability distributions of the index final value (second period)

Once the required distributions have been estimated, the next step is to compute the discounted expected options payoffs (pure premiums). These are given in Table 7. Notice that the call spreads 100/120 and 120/140 have almost identical discounted expected payoffs. This is due to the general lack of probability mass in the interval 100-140. In general, pure premiums lie around options real quotes.

We now use this result to infer some conclusions on the individual prices of these options, based on our estimated future probabilities. First, mean midpoints of the bid and ask quotes being around mean pure premiums seem to indicate that, in average terms, transactions in this market could have been made at reasonable prices, close to the actuarial 'fair' ones during the sample periods. These are good news for those participating in the market that hedge their catastrophe insurance risks: this market seems to be an attractive alternative to traditional reinsurance, for instance, as it offers reinsurance at 'fair' prices. However, those willing to participate in the (re)insurance market by selling options seeking for attractive returns over the risk-free rate might find difficult to get them, at least when trading with individual options.

Table 7
Theoretical Prices for the National Annual-98 Index Derivatives

This table shows the average theoretical prices corresponding to the National Annual-98 derivatives available for trading on any day during the sample periods. CA 40 60 stands for a call spread with exercise prices 40 and 60, and similarly for the other possible exercise prices. CB denotes a butterfly call spread with relevant exercise prices as indicated. The first two columns give the mean pure (actuarial) premiums and the mean risk-neutral (financial) prices. The third column displays the absolute value of the difference between the previous two columns and the last two columns show the mean real bid and ask quotes for comparison purposes. All figures are given in points, each point with a value of two hundred dollars. We also report the mean euclidian distance between the measures μ and λ (standard deviation in parenthesis) in the last row of each panel.

	Mean Theoretical Prices			Mean Real Quotes	
Derivatives	Pure Premium	Risk-Neutral Price	Discrepancy	Bid (Days)	Ask (Days)
	Panel A: First Period				
CA 40 60	10.42	10.46	0.038	8.61 (36)	9.47 (17)
CA 60 80	5.48	6.10	0.617	6.09 (36)	7.31 (34)
CA 80 100	3.70	4.35	0.644	4.12 (36)	5.96 (36)
CB 40 60 80 100	6.72	6.11	0.606	3.88 (36)	6.00 (1)
CA 100 120	3.32	3.36	0.440	1.60 (36)	5.40 (36)
CA 120 140	3.26	3.48	0.221	1.50 (1)	5.50 (36)
CA 150 200	6.79	6.08	0.712	n.a. (0)	8.00 (28)
CA 180 200	2.18	1.80	0.381	0.73 (36)	1.80 (36)
CA 250 300	0.82	0.74	0.071	0.74 (36)	2.50 (36)
Mean Square Distance (Std. Dev.):	0.000317 (0.000515)				
	Panel B: Second Period				
CA 30 50	18.17	18.19	0.024	10.00 (11)	n.a. (0)
CA 40 60	17.03	17.11	0.082	9.48 (27)	n.a. (0)
CA 60 80	8.66	8.97	0.318	4.89 (27)	9.63 (27)
CA 80 100	3.44	3.97	0.535	3.87 (3)	7.42 (26)
CB 40 60 80 100	13.92	13.40	0.525	5.00 (15)	11.25 (16)
CA 100 120	2.26	2.94	0.685	2.45 (22)	5.67 (27)
CA 100 150	5.32	7.20	1.878	n.a. (0)	12.00 (22)
CA 100 200	10.18	13.80	3.629	10.00 (7)	20.00 (21)
CA 120 140	2.11	2.77	0.658	1.00 (18)	5.59 (27)
CA 150 200	4.89	6.46	1.572	4.00 (21)	7.50 (22)
CA 180 200	1.87	1.80	0.065	0.40 (9)	1.80 (9)
CA 200 250	2.24	3.12	0.881	n.a. (0)	4.30 (15)
CA 250 300	0.37	1.46	1.088	0.50 (27)	2.95 (20)
Mean Square Distance (Std. Dev.):	0.000052 (0.000050)				

Second, for our sample periods, conclusions inferred from spread midpoints may substantially change when the real bid-ask spread is taken into account . On the one hand, in general, those hedgers able to buy at bid prices will find this market more attractive than the usual reinsurance (for most options mean bid prices are lower than the mean actuarial ones) but things might turn the other way round for hedgers that buy at the ask. On the other hand, keeping aside, at the moment, risk considerations, investors seeking for high returns should try to buy close to the bid or sell close to the ask.

Third, investors searching for new investment/financing opportunities also attend to risk considerations in their decisions. Analyzing risk and return on an option by option basis hardly makes sense as it is in a portfolio choice context where risk/return opportunities are fully understood (it is in this context where risk-pooling benefits, for instance, come into place). These are the topics considered in the next subsection.

To finish with the present subsection, we solve the minimization program that gives a possible vector of risk-neutral probabilities and we use them to obtain the corresponding theoretical risk-neutral prices. The latter are also given in Table 7.[27] We also report the

[27]Risk-neutral prices certainly verify the bid and ask quote restrictions day by day, even though the

average value of the objective function for both periods, and the resulting discrepancies between pure (actuarial) premiums and prices calculated with the risk-neutral probabilities. As these figures show, it might well be concluded that real prices, as summarized by the linear pricing rule extracted from them, are reasonably close to their 'fair' value.

4.4. Looking for well diversified portfolios

Consider an arbitrary date t_0, and the linear pricing rule π such that $\pi_j = \frac{1}{R}E^{\lambda}(S_j)$, $j = 1, 2, ..., n$. Then, the (unique) discount factor d of Theorem 3 may be easily found by means of the following conditions

$$\begin{cases} d^t \in Span(A) \\ \sum_{i=1}^{k} a_{ij} d_i \mu_i = \pi_j, j = 1, 2, ..., n \end{cases}$$

According to Theorem 4, the minimum-variance strategies are obtained by selling the portfolio $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)$ such that $A\tilde{x}^t = d^t$. The portfolio $\tilde{x}^* = (0, \tilde{x}_2, ..., \tilde{x}_n)$[28] will also provide a very useful information. Depending on the sign of its theoretical price $\sum_{j=2}^{n} \pi_j \tilde{x}_j$, we know when variance-averse individuals must sell or buy derivatives in their reinsurance-linked portfolios. As it will be shown, it was obtained that almost all the days investors in the PCS options market must sell derivatives (the sign is positive).[29]

For the empirical analysis we first excluded those redundant derivatives for each day. See Table 8 for a summary of the main results. Regarding to the $\tilde{x}$ portfolio, this was mainly composed by bonds (87.98% and 68.36% in average terms for the first and the second periods, respectively) and its mean (gross) return equals .9460 and .9096.[30] The portfolio $\tilde{x}^*$ has a positive value for 77.78% of the days in the first period and 81.48% in the second. Thus, risk-averse investors should view the PCS options market as a very profitable source of capital that allows them to finance their investments in other markets.

In order to further illustrate the relationship between probabilities μ and λ, and the portfolio $\tilde{x}^*$ payoffs (appropriately standardized), these variables are plotted at Exhibits 6 and 7 for a representative day of both periods. It is clear that the portfolio final payoff becomes very negative only for states of the world (index final values) with slight probability.

reported mean risk-neutral prices do not have to lie inside the mean bid-ask spread, as either the bid or the ask quote may not be present every day these prices are computed (we use the term 'risk-neutral prices' as has become usual in finance).

[28] *i.e.* the portfolio $\tilde{x}$ once the bond has been excluded

[29] Remind that we are considering investors willing to participate in the catastrophe insurance market through PCS options in order to benefit from their new risk/return opportunities when included in diversified stock and bond portfolios. Of course, insurers, who must hedge their proper liabilities when the final index value becomes large, follow different portfolio criteria.

[30] Recall that $\tilde{x}$ is the portfolio that variance-averse investors should sell and combine with the risk-free asset in order to get the desired attainable return with the minimum variance.

Table 8
Well Diversified Portfolios

The first two rows show the risk/return characteristics of the entire portfolio by providing the expected return and its standard deviation, respectively (returns defined as payoffs divided by the risk neutral price). The third row gives the weight of $\tilde{x}^*$(the derivatives portfolio) over the entire diversified portfolio, while the fourth row shows the price of $\tilde{x}^*$ in index points. Finally, the number of days in which $\tilde{x}$ has positive price is indicated in the last row.

Portfolio Characteristics	Descriptive Statistics				
	Mean	Std. Dev.	Min.	Median	Max.
Panel A: First Period					
Expected Return	0.9460	0.0809	0.8262	0.9330	1.0589
Std.Dev. of the Return	5.1902	10.8703	1.4038	1.7278	64.1093
Cat derivatives weight	0.1202	0.1875	-0.2386	0.1319	0.3374
Price of $\tilde{x}^*$	0.1249	0.1930	-0.2345	0.1185	0.3664
Days with positive price	28 (77.78%)				
Panel B: Second Period					
Expected Return	0.9096	0.1753	0.6622	0.9911	1.0826
Std.Dev. of the Return	1.4279	1.4056	0.3566	0.9453	5.5590
Cat derivatives weight	0.3164	0.2907	-0.0980	0.5053	0.8079
Price of $\tilde{x}^*$	0.3808	0.3519	-0.0845	0.5727	0.8648
Days with positive price	22 (81.48%)				

Table 9
An Example of
Two Well Diversified Portfolios

This table shows the asset weights of the derivatives portfolio $\tilde{x}^*$ for two selected days. Weights were calculated as value invested in the asset divided by the portfolio value (a negative sign indicates a sale). The corresponding risk-neutral prices are also reported. The weights for these derivative portfolios over the entire (bond included) portfolios are 0.29 and 0.63 for panels A and B, respectively.

Panel A: Date 2/25/1998 (1st. Period)			Panel B: Date 7/30/1998 (2nd. Period)		
Derivative	Weight	Risk-Neutral Price	Derivative	Weight	Risk-Neutral Price
CA 40 60	0.06	11.66	CA 40 60	0.36	17.88
CA 60 80	0.05	7.00	CA 60 80	-0.02	9.84
CA 80 100	0.32	5.00	CA 80 100	-0.08	4.43
CA 100 120	3.92	4.25	CA 100 120	1.74	3.50
CA 120 140	-1.52	3.83	CA 100 150	-4.16	8.43
CA 150 200	-2.58	6.23	CA 120 140	4.24	3.32
CA 180 200	0.72	1.80	CA 150 200	-1.59	7.5
CA 250 300	0.00	0.80	CA 250 300	0.52	2.70
Portfolio	1	0.30	Portfolio	1	0.86

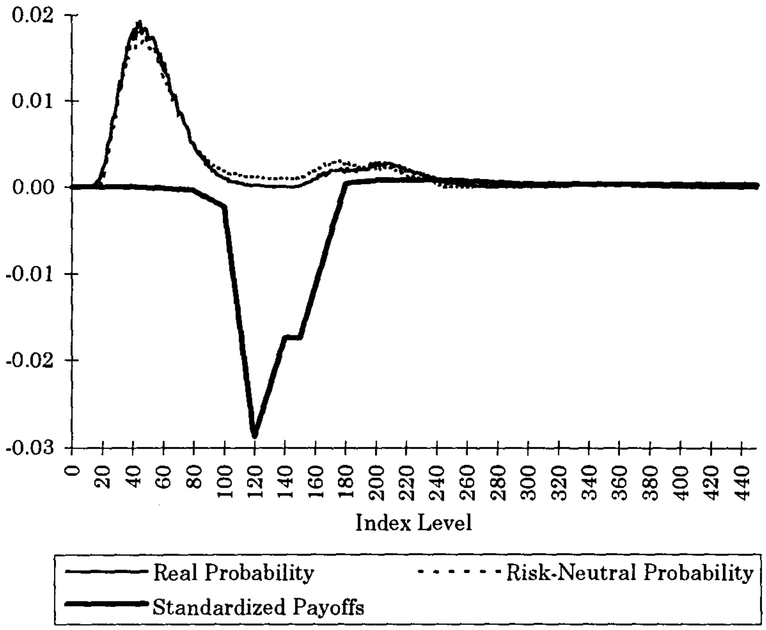

Exhibit 6: Standardized payoffs of portfolio $\tilde{x}^*$ on date 2/25/98 along with the corresponding risk-neutral and real probabilities

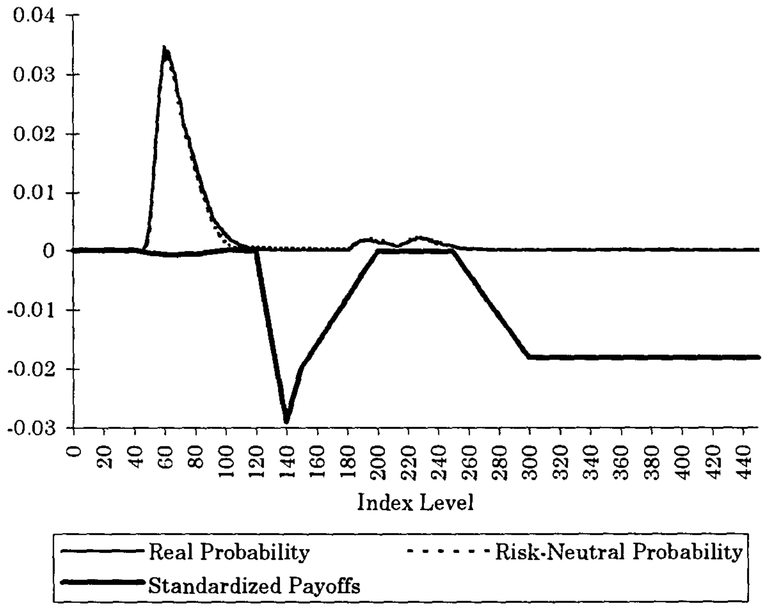

Exhibit 7: Standardized payoffs of portfolio $\tilde{x}^*$ on date 7/30/98 along with the corresponding risk-neutral and real probabilities

5. Concluding Remarks

All along the paper it has been shown that arbitrage arguments, in a static setting, very often allows to price catastrophe-linked derivatives. Furthermore, the procedure to detect arbitrage portfolios has been provided and some illustrative examples have been presented.

Hedging arguments have also been applied and, again, they have shown many possibilities to price these derivatives. Concrete procedures to detect hedged portfolios and concrete examples have been given.

The Theory of Portfolio Selection also yields suitable strategies to invest. Moreover, if the bid-ask spread is reduced by arbitrage, the real quotes available in the market show very significant particularities. Linear pricing rules compatible with the quotes usually imply theoretical prices quite close to the actuarial pure premiums. However, even though a well diversified portfolio (in a variance aversion context) is composed by different catastrophe-linked derivatives in short and long positions, its price is usually negative (*i.e.*, the total price of the sold derivatives is greater than the total price of the purchased ones) and this capital must be invested in shares and bonds.

The above comments lead to significant implications. Insurers can consider the market to buy reinsurances and hedge their liabilities. The paid price may be adequate and not high with respect to the pure premiums. On the contrary, variance averse investors whose risk does not depend on the indices can use catastrophe-linked derivatives to compose portfolios with negative price that must be invested in other type of assets. This makes the market very interesting because it allows to appropriately diversify among many investors the risk due to catastrophic events. Anyway, we must notice that latter properties hold for linear pricing rules and, therefore, it is important to reduce the real bid-ask spreads observed in the market. As said above, it is possible by arbitrage and hedging arguments.

The applied methodology seems to reveal two interesting advantages. It is useful for traders because practical criteria and strategies to invest are provided. Moreover, it seems to be general enough to apply in many other contexts. Only two properties are needed. The underlying uncertainty must be easily identified (arbitrage and hedging), and the probability space that explains its behavior has to be determined with precision (variance aversion).

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The usual caveat applies.

References

[1] Balbás A. and M.J. Muñoz (1998). "Measuring the Degree of Fulfillment of the Law of One Price. Applications to Financial Market Integration". *Investigaciones Económicas, 22, 2, 153-177.*

[2] Balbás A., P.J. Guerra and M.J. Muñoz (1998). "Measuring the Arbitrage Opportunities in an Intertemporal Dynamic Asset Pricing Model". *Applied Decision Analysis, Kluwer, Boston, 159-172.*

[3] Canter M.S., J.B. Cole and R.L. Sandor (1996). "Insurance Derivatives: A New Asset Class for the Capital Markets and a New Hedging Tool for the Insurance Industry". *Journal of Derivatives, 4, 2, 89-105.*

[4] Cummins J.D. and H. Geman (1995). "Pricing Catastrophe Insurance Futures and Call Spreads: An Arbitrage Approach". *The Journal of Fixed Income, 4, 4, 46-57.*

[5] Chamberlain G. and M. Rothschild (1983). "Arbitrage, Factor Structure and Mean-Variance Analysis of Large Assets". *Econometrica, 51, 5, 1281-1304.*

[6] D'Arcy S.P. and V.G. France (1992). "Catastrophe Futures: A Better Hedge for Insurers". *The Journal of Risk and Insurance, 59, 4, 575-601.*

[7] Duffie D. (1988). "Security Markets: Stochastic Models". *Academic Press.*

[8] Garman M.B. (1976). "An Algebra for Evaluating Hedge Portfolios". *Journal of Financial Economics, 3, 4, 403-427.*

[9] Geman H. and M. Yor (1997). "Stochastic Time Changes in Catastrophe Option Pricing". *Insurance Mathematics and Economics, 21, 3, 185-194.*

[10] Hansen L.P. and R. Jagannathan (1997). "Assessing Specification Errors in Stochastic Discount Factor Models". *The Journal of Finance, 52, 2, 567-590.*

[11] Hansen L.P. and S.F. Richard (1987). "The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models". *Econometrica, 55, 3, 587-613.*

[12] Ingersoll J.E. (1987). "Theory of Financial Decision Making". *Rowman & Littlefield Publishers Inc.*

[13] Jackwerth J.C. and M. Rubinstein (1996). "Recovering Probability Distributions from Option Prices". *The Journal of Finance, 50, 6, 1611-1631.*

[14] Jaffee D.M. and T. Russell (1997). "Catastrophe Insurance, Capital Markets and Insurable Risk". *The Journal of Risk and Insurance, 64, 2, 205-230.*

[15] Jouini E. and H. Kallal (1995). "Martingales and Arbitrage in Securities Markets with Transaction Costs". *Journal of Economic Theory, 66, 1, 178-197.*

[16] Litzenberger R.H., R.D. Beaglehole and C.E. Reynolds (1996). "Assessing Catastrophe Reinsurance-Linked Securities as a New Asset Class". *Journal of Portfolio Management, Special Issue, 76-86.*

[17] Prisman E.J. (1986). "Valuation of Risky Assets in Arbitrage Free Economies with Frictions". *The Journal of Finance, 41, 3, 545-56.*

[18] O'Brien T. (1997). "Hedging Strategies Using Catastrophe Insurance Options". *Insurance Mathematics and Economics, 21, 2, 153-162.*

[19] Rubinstein M. (1994). "Implied Binomial Trees". *The Journal of Finance, 69, 3, 771-818.*

[20] Tomas M.T. (1998). "A Note on Pricing PCS Single-Event Options". *Derivatives Quarterly, 4, 3, 23-28.*