Uniform asymptotic estimates of hypergeometric functions appearing in Potential Theory

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1. Introduction.

In [F], G. B. Folland obtained an expansion in spherical harmonics of the Poisson-Szegö kernel for the unit ball $\mathcal{B}$ in $\mathbf{C}^n$,

$$\mathcal{P}_n(z,w) = \frac{1}{\omega_{2n}} \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}} \,, \qquad z \in \mathcal{B}\,,\ w \in \partial\mathcal{B}\,,$$

where $\langle z,w\rangle$ denotes the standard scalar product in $\mathbf{C}^n$

$$\langle z,w\rangle = z_1\,\overline{w}_1 + \cdots + z_n\,\overline{w}_n \,,$$

and ω_{2n} is the $(2n-1)$-dimensional Lebesgue measure of the unit sphere of $\mathbf{C}^n$.

Let $\Delta_{\mathcal{B}}$ denote the Laplace-Beltrami operator associated to the Bergman metric on $\mathcal{B}$,

$$\Delta_{\mathcal{B}} = \frac{4}{n+1}\,(1-|z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - z_i\,\bar{z}_j)\,\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \,.$$

$\Delta_{\mathcal{B}}$ is the basic invariant differential operator on the symmetric space $SU(n,1)/U(n) \approx \mathcal{B}$. The solution of the Dirichlet problem

$$\begin{cases} \Delta_{\mathcal{B}} u = 0\,, & \text{in } \mathcal{B}\,, \\ u = f\,, & \text{in } \partial\mathcal{B}\,, \end{cases} \tag{1.1}$$

with continuous boundary data f is given by the following representation formula

$$u(z) = \int_{\partial\mathcal{B}} \mathcal{P}_n(z,w)\, f(w)\, dw \,.$$

If $\mathcal{H}_n^{p,q}$ denotes the linear space of restrictions to $\partial\mathcal{B}$ of harmonic polynomials $g(z,\bar{z})$ on $\mathbf{C}^n$ which are homogeneous of degree p in z and degree q in $\bar{z}$, the solution of the Dirichlet problem (1.1), with $f \in \mathcal{H}_n^{p,q}$, is given by

$$u(r\eta) = S_n^{p,q}(r)\, f(\eta)\,, \qquad 0 \le r \le 1\,,\ \eta \in \partial\mathcal{B}\,, \tag{1.2}$$

where

$$S_n^{p,q}(r) = r^{p+q}\,\frac{F(p,q;p+q+n;r^2)}{F(p,q;p+q+n;1)} \,.$$

By $F(a,b;c;t)$ we denote the usual Gauss hypergeometric function

$$F(a,b;c;t) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\,\frac{t^k}{k!} \,,$$

where $(u)_k$ is the Pochhammer symbol,

$$(u)_k = u\,(u+1)\cdots(u+k-1) = \frac{\Gamma(u+k)}{\Gamma(u)} \,.$$

The formula (1.2) points to the crucial role of $S_n^{p,q}$ in the expansion of the Poisson-Szegö kernel in spherical harmonics. In fact,

$$\mathcal{P}_n(r\eta,w) = \sum_{p,q=0}^{\infty} S_n^{p,q}(r)\, H_n^{p,q}(\langle \eta,w\rangle)\,, \tag{1.3}$$

where $H_n^{p,q}(\langle\,\cdot\,,w\rangle) \in \mathcal{H}_n^{p,q}$ is the zonal harmonic with pole w, *cf.* [F].

If one wants to use the expansion in spherical harmonics, then one is required to know *uniform* estimates in the variable t of $F(p,q;p+q+n;t)$ when the parameters p,q grow, in order to obtain bounds of integrals involving $S_n^{p,q}$ (see *e.g.* Theorem 2 below). For $q=p+a$, with a bounded, Watson [W] [L, p. 237] gave the asymptotic behaviour of such an F. However, we will need more general estimates.

In this paper, we study the asymptotic behaviour of

$$F(q,mq;q+mq+n;t)$$

and we obtain the following uniform estimate, where $B(\,\cdot\,,\,\cdot\,)$ denotes the Beta function:

Theorem 1. *There exists a universal constant C, not depending on n,p,u,m,z, such that, for all real numbers, $u,p\geq 0$, $m,n\geq 1$, $0\leq z<1$, if we denote*

$$G=F(p+u,mp+1;(m+1)p+u+n+1;z)\,B(mp+1,p+u+n),$$

then

$$G\geq C\,L\,,$$

where

$$L=t_0^{mp+1}(1-m(1-t_0))^{p+u}(1-t_0)^{n-1}\left(\frac{1-z}{a^2-b^2z}\right)^{1/4}\frac{1}{m\sqrt{p+1}}$$

and

$$t_0=\frac{a+bz-\sqrt{(1-z)\,(a^2-b^2z)}}{2z}=\frac{2}{a+bz+\sqrt{(1-z)\,(a^2-b^2z)}}\,,$$

$$a=1+\frac{1}{m}\,,\qquad b=1-\frac{1}{m}\,.$$

Besides, this result is sharp in the sense that

$$\lim_{p\to\infty}\frac{G}{L}=\sqrt{2\pi}\,.$$

By making the choices $u=1/m$, $p+u=q$, we have the following

Corollary. *There exists a universal constant C, not depending on n,q,m,z, such that, for all real numbers, $m,n\geq 1$, $q\geq 1/m$, $0\leq z<1$, if we denote*

$$G=F(q,mq;q+mq+n;z)\,B(mq,q+n)\,,$$

then

$$G\geq C\,L$$

where

$$L=t_0^{mq}(1-m(1-t_0))^{q}(1-t_0)^{n-1}\left(\frac{1-z}{a^2-b^2z}\right)^{1/4}\frac{1}{m\sqrt{q+1}}\,.$$

Observe that without loss of generality we can suppose $m\geq 1$, because of the symmetry of the hypergeometric function in the two first parameters.

It is not possible to obtain a similar uniform upper bound of F because L is zero for $z=1$. However usually the hard inequalities involve lower bounds.

One could think that the hypothesis $p=mq$ is too restrictive, but this is enough in order to prove some results in which p and q grow independently (see Theorem 2 below). On the other hand, Theorem 2 is sharp.

This uniform estimation of $S_n^{p,q}$ allow us to obtain an integral expression for the α-energy of a complex measure supported in $\partial\mathcal{B}$. We recall that the α-energy is defined as follows:

$$J_\alpha(\mu) = \iint_{\partial\mathcal{B}\times\partial\mathcal{B}} \Phi_\alpha(d(x,y))\, d\overline{\mu}(x)\, d\mu(y)\,,$$

where

$$\Phi_\alpha(t) = \begin{cases} \log\dfrac{1}{t}\,, & \text{if } \alpha = 0\,,\\[2mm] \dfrac{1}{t^\alpha}\,, & \text{if } 0 < \alpha < 2n\,, \end{cases}$$

and $d(x,y)$ is a distance in $\partial\mathcal{B}$.

More concretely, we have obtained in [FPR2] the following result.

Theorem A ([FPR2]). *If μ is a complex measure supported on $\partial\mathcal{B}$ and $d(z,w) = |1-\langle z,w\rangle|^{1/2}$, we have for $0<\alpha<2n$, that*

$$J_\alpha(\mu) \asymp \int_0^1 \left\{ \int_{\partial\mathcal{B}} |\mathcal{P}_\mu(r\xi)|^2\, d\xi \right\} r^{\alpha/2-1}(1-r^2)^{n-\alpha/2-1}\, dr\,, \tag{1.3}$$

where $\asymp$ means that the quotient of the two terms is between two constants which can depend on n and α, and $\mathcal{P}_\mu$ denotes the invariant Poisson extension of μ, which we recall is defined as follows

$$\mathcal{P}_\mu(z) = \int_{\partial\mathcal{B}} \mathcal{P}_n(z,w)\, d\mu(w)\,, \qquad z\in\mathcal{B}\,.$$

Theorem A is one of the keys to obtain a capacity distortion result [FPR2] under inner functions. Recall that if E is a closed subset of $\partial\mathcal{B}$, then

$$(cap_\alpha(E))^{-1} = \inf\{J_\alpha(\mu):\ \mu \text{ a probability measure supported on } E\}\,.$$

Recall also that an *inner function* is a bounded holomorphic function from the unit ball $\mathcal{B}$ of $\mathbf{C}^n$ into the unit disk Δ of the complex plane such that the radial boundary values have modulus 1 almost everywhere. If E is a non empty Borel subset of $\partial\Delta$, we denote by $f^{-1}(E)$ the following subset of $\partial\mathcal{B}$

$$f^{-1}(E) = \{\xi\in\partial\mathcal{B}:\ \lim_{r\to 1} f(r\xi) \text{ exists and belongs to } E\}\,.$$

Theorem B. [FPR2] *If f is inner in the unit ball of $\mathbf{C}^n$, $f(0)=0$, and E is a Borel subset of $\partial\Delta$, we have:*

i) *If $0<\alpha<2$ (and also $\alpha=0$ if $n=1$), then*

$$cap_{2n-2+\alpha}(f^{-1}(E)) \ge C(n,\alpha)\, cap_\alpha(E)\,.$$

ii) *If $\alpha=0$ and $n\ge 2$, then*

$$\frac{1}{cap_{2n-2}(f^{-1}(E))} \le C(n)\left(1+\log\frac{1}{cap_0(E)}\right).$$

Corollary. *With the same hypotheses of Theorem B, we have*

$$Dim(f^{-1}(E)) \ge Dim(E) + 2n-2\,,$$

where Dim denotes Hausdorff dimension with respect to the distance $d(z,w) = |1-\langle z,w\rangle|^{1/2}$.

These two theorems translate to the distance $d(z,w) = |1 - \langle z,w\rangle|^{1/2}$ in $\partial\mathcal{B}$ the corresponding results [FPR1] for the euclidean distance. It is interesting to remark that in the euclidean case the analogue of (1.3) is an equality. On the other hand, these results have a lot of applications [FP1], [FP2], [FPR1].

The heart of the proof of Theorem A is to reduce it to the following:

Theorem 2. *For all non negative integers p, q, n ($n \geq 1$) and for all β, $0 < \beta < n/2$, we have, with constants which only depend on n, β, that*

$$I = \int_0^1 \left(\frac{F(z)}{F(1)}\right)^2 z^{p+q+\beta-1}(1-z)^{n-2\beta-1}dz \asymp \frac{\Gamma(p+\beta)\Gamma(q+\beta)}{\Gamma(p+n-\beta)\Gamma(q+n-\beta)} \,,$$

where $F(z)$ is the hypergeometric function $F(p,q;p+q+n;z)$.

The outline of the paper is as follows. In Section 2 we give the proof of Theorem 1. We will prove Theorem 2 in Sections 3 and 4. In Section 5 we will give an open question.

Notations. By C we will denote a constant, which sometimes can depend on n and β, that can change its value from line to line and even in the same line. The expression $A \asymp B$ will mean that there exists a constant C, depending at most on n and β, such that $C^{-1} \leq A/B \leq C$. Finally, $A \sim B$ when $x \to a$, means that $\lim_{x\to a} A/B = 1$.

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2. Proof of Theorem 1.

Theorem 1. *There exists a universal constant C, not depending on n, p, u, m, z, such that, for all real numbers, $u, p \geq 0$, $m, n \geq 1$, $0 \leq z < 1$, if we denote*

$$G = F(p+u, mp+1; (m+1)p+u+n+1; z)\, B(mp+1, p+u+n),$$

then

$$G \geq C\, L\,,$$

where

$$L = t_0^{mp+1}(1-m(1-t_0))^{p+u}(1-t_0)^{n-1}\left(\frac{1-z}{a^2-b^2z}\right)^{1/4}\frac{1}{m\sqrt{p+1}}$$

and

$$t_0 = \frac{a+bz-\sqrt{(1-z)\,(a^2-b^2z)}}{2z} = \frac{2}{a+bz+\sqrt{(1-z)\,(a^2-b^2z)}}\,, \tag{2.1}$$

$$a = 1+\frac{1}{m}\,, \qquad b = 1-\frac{1}{m}\,.$$

Besides,

$$\lim_{p\to\infty}\frac{G}{L} = \sqrt{2\pi}\,.$$

In order to prove the Theorem 1 we will need the following well-known integral expression [S, p. 20] [L, p. 99]

$$G = \int_0^1 t^{mp}(1-t)^{p+u+n-1}(1-zt)^{-p-u}\,dt\,.$$

Accordingly, we can write

$$G = \int_0^1 e^{pf(t)}g(t)\,dt\,,$$

where

$$f(t) = \log\frac{t^m(1-t)}{1-zt} \qquad \text{and} \qquad g(t) = \frac{(1-t)^{u+n-1}}{(1-zt)^u}\,.$$

Observe that the function f has a unique maximum t_0 in $[0,1]$, given by (2.1).

The classical Laplace's method (see e.g. [O], [Wo]) for asymptotic expansions gives that the principal contribution of the integrand of G is located in a neighborhood of t_0. Consequently, it will be useful to have at our disposal some expressions involving t_0.

Lemma 2.1. *If t_0 is defined by (2.1) we have the following formulae*

$$z\,t_0^2 = (a+bz)\,t_0 - 1\,, \tag{2.2}$$

$$2\,z\,t_0 = a+bz-\sqrt{(1-z)\,(a^2-b^2z)}\,, \tag{2.3}$$

$$2\,(1-zt_0) = b\,(1-z)+\sqrt{(1-z)\,(a^2-b^2z)} = t_0\Big(a\,(1-z)+\sqrt{(1-z)\,(a^2-b^2z)}\Big)\,, \tag{2.4}$$

$$1-t_0 = \frac{\sqrt{(1-z)\,(a^2-b^2z)}-a\,(1-z)}{2z} = \frac{t_0}{2}\left(\sqrt{(1-z)\,(a^2-b^2z)}-b\,(1-z)\right), \tag{2.5}$$

$$(1-t_0)\,(1-zt_0) = \frac{t_0}{m}\,(1-z)\,, \tag{2.6}$$

$$\frac{1-t_0}{1-zt_0} = 1-m\,(1-t_0)\,, \tag{2.7}$$

$$f''(t_0) = -\frac{m^2}{t_0^2}\sqrt{\frac{a^2-b^2z}{1-z}}\,, \tag{2.8}$$

Proof. In order to find t_0 we need, of course, to solve the equation $f'(t)=0$. This equation is equivalent to (2.2). The identities (2.3)-(2-7) can be obtained by an elementary argument if we recall (2.1) and the definition of a and b. More concretely, (2.6) and (2.7) use (2.2). To obtain (2.8) we use (2.6) and (2.3) in the following way

$$\begin{aligned} f''(t_0) &= -\frac{m}{t_0^2} - \frac{1}{(1-t_0)^2} + \frac{z^2}{(1-zt_0)^2} \\ &\underset{(2.6)}{=} -\frac{m^2}{t_0^2} \frac{\frac{1}{m}(1-z)^2 + (1-zt_0)^2 - z^2(1-t_0)^2}{(1-z)^2} \\ &= -\frac{m^2}{t_0^2} \frac{a+bz-2zt_0}{1-z} \\ &\underset{(2.3)}{=} -\frac{m^2}{t_0^2} \frac{\sqrt{(1-z)\,(a^2-b^2z)}}{1-z} . \qquad \text{Q.E.D.} \end{aligned}$$

Proof of Theorem 1. Following Laplace's method (see e.g. [O], [Wo]), we define a new variable τ by the equation

$$f(t_0) - f(t) = \tau^2 \tag{2.9}$$

and the condition that τ must be an increasing function of t.

Using the Taylor's polynomial of degree 2 of f in t_0, we obtain that if we define h by $t=h(\tau)$, we have

$$h'(0) = \sqrt{\frac{-2}{f''(t_0)}} . \tag{2.10}$$

Then

$$G = e^{pf(t_0)} \int_{-\infty}^{\infty} e^{-p\tau^2} g(h(\tau))\, h'(\tau)\, d\tau . \tag{2.11}$$

If we use (2.9) and (2.10), we have that as $p\to\infty$

$$G \sim e^{pf(t_0)}\, g(h(0))\, h'(0) \int_{-\infty}^{\infty} e^{-p\tau^2}\, d\tau = e^{pf(t_0)}\, g(t_0) \sqrt{\frac{-2\pi}{f''(t_0)\, p}} .$$

Then, using (2.8), we obtain

$$G \sim \Big(\frac{t_0^m(1-t_0)}{1-zt_0}\Big)^p \Big(\frac{1-t_0}{1-zt_0}\Big)^u (1-t_0)^{n-1} \sqrt{\frac{2\pi}{p}\,\frac{t_0^2}{m^2}\sqrt{\frac{1-z}{a^2-b^2z}}} .$$

The identity (2.7) gives

$$G \sim t_0^{mp+1}(1-m\,(1-t_0))^{p+u}(1-t_0)^{n-1}\frac{1}{m}\sqrt{\frac{2\pi}{p}\sqrt{\frac{1-z}{a^2-b^2z}}} \sim \sqrt{2\pi}\, L .$$

This proves the last part of Theorem 1. To prove the main part of Theorem 1 we need to estimate $g(h(\tau))$ and $h'(\tau)$ near 0. These estimates must be *uniform* in n, p, u, m and z.

For each $0 < \varepsilon < 1$ we define

$$t = (1-\varepsilon)\, t_0 \,, \tag{2.12}$$

$$x = b + \sqrt{\frac{a^2 - b^2 z}{1-z}} \geq 2\,, \qquad \text{if } \ 0 \leq z < 1\,, \tag{2.13}$$

$$w = 1 + \frac{m\varepsilon}{2}\, x \geq 1 + m\,\varepsilon\,, \qquad \text{if } \ 0 \leq z < 1\,. \tag{2.14}$$

We need to estimate

$$\tau^2 = f(t_0) - f(t) = \log\Big(\frac{1}{(1-\varepsilon)^m}\,\frac{1-t_0}{1-(1-\varepsilon)\,t_0}\,\frac{1-(1-\varepsilon)\,zt_0}{1-zt_0}\Big)\,. \tag{2.15}$$

A computation gives, using (2.4), that

$$\frac{1-(1-\varepsilon)\,zt_0}{1-zt_0} = 1 - m\,\varepsilon + \frac{m\,\varepsilon}{2}\,x = w - m\,\varepsilon\,, \tag{2.16}$$

and also, using (2.5), that

$$\frac{1-t_0}{1-(1-\varepsilon)\,t_0} = \frac{1}{1+\dfrac{m\,\varepsilon}{2}\,x} = \frac{1}{w}\,, \tag{2.17}$$

where x, w are defined by (2.13) and (2.14). If we substitute (2.16) and (2.17) in (2.15) we obtain

$$f(t_0) - f(t) = \log\Big(\frac{1}{(1-\varepsilon)^m}\Big(1 - \frac{m\,\varepsilon}{w}\Big)\Big) \geq \log\frac{1}{(1-\varepsilon)^m(1+m\,\varepsilon)}\,. \tag{2.18}$$

We wish to show that

$$h'(\tau) \geq K\,h'(0)\,, \qquad \text{for all } \ \tau \in [\tau_1, 0]\,, \tag{2.19}$$

for some constants $K > 0$ and $\tau_1 < 0$ which are independent of n, p, u, m and z. In order to obtain this inequality consider the function $H = h^{-1}$ (*i.e.* $H(t)^2 = f(t_0) - f(t)$). Then, (2.19) is equivalent to the inequality

$$\frac{1}{H'(t)} \geq \frac{K}{H'(t_0)} = K\sqrt{\frac{-2}{f''(t_0)}}\,, \tag{2.20}$$

for all $t \in [t_1, t_0]$, with $t_1 = h(\tau_1)$.

Since we are working with $t < t_0$, we have that

$$H(t) = -\sqrt{f(t_0) - f(t)}\,.$$

And recalling (2.8), (2.12), (2.13) and (2.14), we see that to prove (2.20) is equivalent to prove that

$$\frac{4\,(f(t_0) - f(t))}{f'(t)^2} \geq 2\,K^2\,\frac{t_0^2}{m^2}\,\frac{1}{x-b} = \frac{2\,K^2\,t_0^2}{m^2}\,\frac{1}{\dfrac{2}{m\,\varepsilon}\,(w-1) - b}\,. \tag{2.21}$$

On the other hand if t is given by (2.12), computations give, with the help of (2.6), (2.16) and (2.17), that

$$\begin{aligned} f'(t) &= \frac{m}{(1-\varepsilon)t_0} - \frac{1}{1-(1-\varepsilon)t_0} + \frac{z}{1-(1-\varepsilon)zt_0} \\ &\underset{\substack{(2.16)\\(2.17)}}{=} \frac{m}{(1-\varepsilon)t_0} - \frac{1}{w(1-t_0)} + \frac{z}{(w-m\,\varepsilon)(1-zt_0)} \\ &\underset{(2.6)}{=} \frac{m}{t_0}\Big(\frac{1}{1-\varepsilon} - \frac{w\,(1-z) - m\,\varepsilon\,(1-zt_0)}{w\,(w-m\,\varepsilon)(1-z)}\Big)\,, \end{aligned}$$

and so if we use (2.4) and (2.14) to obtain

$$1 - z t_0 = \frac{x}{2}\,(1-z)\,,$$

we find that

$$f'(t) = \frac{m}{t_0}\Big(\frac{1}{1-\varepsilon} - \frac{1}{w\,(w-m\,\varepsilon)}\Big)\,. \tag{2.22}$$

Substituting (2.18) and (2.22) into the inequality (2.21), we obtain that (2.19) is equivalent to

$$M(w) = \log\Big(\frac{1}{(1-\varepsilon)^m}\,\Big(1-\frac{m\,\varepsilon}{w}\Big)\Big) - \frac{K^2}{\dfrac{4}{m\,\varepsilon}\,(w-1)-2b}\,\Big(\frac{1}{1-\varepsilon} - \frac{1}{w\,(w-m\,\varepsilon)}\Big)^2 \geq 0\,, \tag{2.23}$$

for all $w \geq 1 + m\,\varepsilon$ and $\varepsilon \leq \varepsilon_1$.

In order to show (2.23) the next lemma plays an important role.

Lemma 2.2. *For all* $0 < \varepsilon < 1$, $m > 0$, $K \leq \sqrt{(1-\varepsilon)/3}$, $w \geq 1 + m\,\varepsilon$, *we have* $M'(w) > 0$.

In the proof of Lemma 2.2 we will need the next inequality:

Lemma 2.3. *For all* $\varepsilon, m > 0$, $w \geq 1 + m\,\varepsilon$, *we have*

$$\frac{w\,(w-m\,\varepsilon)-(1-\varepsilon)}{w-1-m\,\varepsilon\,b/2} \leq w+2\,. \tag{2.24}$$

Proof of Lemma 2.3. The restrictions $1 + m\,\varepsilon \leq w$ and $b < 1$ give

$$1 + m\,\varepsilon \leq w + w\,m\,\varepsilon\,(1-b/2)\,.$$

This inequality can be transformed, using the fact that $m = mb + 1$, into

$$1 + \varepsilon + m\,\varepsilon\,b \leq w + w\,m\,\varepsilon - w\,m\,\varepsilon\,b/2\,,$$

which is equivalent to

$$w\,(w-m\,\varepsilon)-(1-\varepsilon) \leq (w+2)\,(w-1-m\,\varepsilon\,b/2)\,.$$

Therefore, we obtain (2.24) by observing that $w - 1 - m\,\varepsilon\,b/2 \geq m\,\varepsilon - m\,\varepsilon\,b/2 > 0$. Q.E.D.

Proof of Lemma 2.2. We have that

$$M'(w) = \frac{1}{w-m\,\varepsilon} - \frac{1}{w} + \frac{K^2\,m\,\varepsilon}{4}\Big[\frac{1}{(w-1-m\,\varepsilon\,b/2)^2}\Big(\frac{1}{1-\varepsilon}-\frac{1}{w\,(w-m\,\varepsilon)}\Big)^2 - \frac{2}{w-1-m\,\varepsilon\,b/2}\Big(\frac{1}{1-\varepsilon}-\frac{1}{w\,(w-m\,\varepsilon)}\Big)\,\frac{2\,w-m\,\varepsilon}{w^2\,(w-m\,\varepsilon)^2}\Big]\,.$$

Then

$$M'(w) \geq \frac{m\,\varepsilon}{w\,(w-m\,\varepsilon)} - \frac{K^2\,m\,\varepsilon}{2\,(w-1-m\,\varepsilon\,b/2)}\Big(\frac{1}{1-\varepsilon}-\frac{1}{w\,(w-m\,\varepsilon)}\Big)\,\frac{2\,w-m\,\varepsilon}{w^2\,(w-m\,\varepsilon)^2}\,. \tag{2.25}$$

We can bound, with the help of (2.24), the term

$$\begin{aligned}\frac{1}{w-1-m\,\varepsilon\,b/2}\Big(\frac{1}{1-\varepsilon}-\frac{1}{w\,(w-m\,\varepsilon)}\Big) &= \frac{1}{(1-\varepsilon)\,w\,(w-m\,\varepsilon)}\,\frac{w\,(w-m\,\varepsilon)-(1-\varepsilon)}{w-1-m\,\varepsilon\,b/2}\\ &\leq \frac{w+2}{(1-\varepsilon)\,w\,(w-m\,\varepsilon)}\,.\end{aligned} \tag{2.26}$$

We can also obtain an upper bound of the the term

$$\frac{2\,w - m\,\varepsilon}{w^2\,(w-m\,\varepsilon)^2} < \frac{2\,w}{w^2} = \frac{2}{w}\,. \tag{2.27}$$

Substituting (2.26) and (2.27) into (2.25), we obtain

$$\begin{aligned} M'(w) &> \frac{m\,\varepsilon}{w\,(w-m\,\varepsilon)} - \frac{K^2\,m\,\varepsilon}{w}\,\frac{w+2}{(1-\varepsilon)\,w\,(w-m\,\varepsilon)} \\ &= \frac{m\,\varepsilon}{w\,(w-m\,\varepsilon)}\Big(1-\frac{K^2}{1-\varepsilon}\Big(1+\frac{2}{w}\Big)\Big)\,. \end{aligned}$$

The hypothesis on K in Lemma 2.2 gives that $K^2 \le (1-\varepsilon)/3$, and then

$$\frac{K^2}{1-\varepsilon}\Big(1+\frac{2}{w}\Big) \le 1\,.$$

This implies $M'(w) > 0$. Q.E.D.

Consequently, if $K \le \sqrt{(1-\varepsilon)/3}$, we have that

$$M(w) \ge M(1+m\,\varepsilon)\,,$$

and so, we only need to prove that $N(\varepsilon) = M(1+m\,\varepsilon) \ge 0$.

Lemma 2.4. *For all* $0<\varepsilon<1$, $m>0$, $K \le 1-\varepsilon$, *we have that* $N(\varepsilon) \ge N(0) = 0\,.$

Proof of Lemma 2.4. It is enough to show that $N'(\varepsilon) > 0$. Recall that

$$N(\varepsilon) = M(1+m\,\varepsilon) = \log\Big(\frac{1}{(1-\varepsilon)^m(1+m\,\varepsilon)}\Big) - \frac{K^2}{2a}\Big(\frac{1}{1-\varepsilon}-\frac{1}{1+m\,\varepsilon}\Big)^2\,.$$

Therefore,

$$N'(\varepsilon) = \frac{m}{1-\varepsilon} - \frac{m}{1+m\,\varepsilon} - \frac{K^2}{a}\Big(\frac{1}{1-\varepsilon}-\frac{1}{1+m\,\varepsilon}\Big)\Big(\frac{1}{(1-\varepsilon)^2}+\frac{m}{(1+m\,\varepsilon)^2}\Big)\,.$$

Using the fact that

$$\frac{1}{1-\varepsilon}-\frac{1}{1+m\,\varepsilon} = \frac{m\,\varepsilon\,a}{(1-\varepsilon)\,(1+m\,\varepsilon)}\,,$$

we have

$$N'(\varepsilon) = \frac{m\,\varepsilon}{(1-\varepsilon)\,(1+m\,\varepsilon)}\Big(m\,a - K^2\Big(\frac{1}{(1-\varepsilon)^2}+\frac{m}{(1+m\,\varepsilon)^2}\Big)\Big)\,.$$

The hypothesis $K^2 \le (1-\varepsilon)^2$ gives

$$\frac{K^2}{(1-\varepsilon)^2}\Big(1+m\,\frac{(1-\varepsilon)^2}{(1+m\,\varepsilon)^2}\Big) < 1+m = m\,a\,,$$

and this implies $N'(\varepsilon) > 0$. Q.E.D.

It is convenient to make a back-up of our results. *We have showed that if* $0<\varepsilon<1$, $m>0$, $K \le \min\{\sqrt{(1-\varepsilon)/3}, 1-\varepsilon\}$, *for* $t = (1-\varepsilon)\,t_0\,,$

$$\frac{1}{H'(t)} \ge \frac{K}{H'(t_0)}\,.$$

Take $0 < \varepsilon \le \varepsilon_0 < 1$ and $K = \min\{\sqrt{(1-\varepsilon_0)/3}, 1-\varepsilon_0\} \le \min\{\sqrt{(1-\varepsilon)/3}, 1-\varepsilon\}$. Then we have

$$h'(H(t)) \ge K\, h'(0)\,, \qquad \text{for all } t \in [(1-\varepsilon_0)\, t_0, t_0]\,.$$

Then (2.18) gives, if $m \ge 1$,

$$H((1-\varepsilon_0)\, t_0)^2 = f(t_0) - f((1-\varepsilon_0)\, t_0) \ge \log \frac{1}{(1-\varepsilon_0)^m (1+m\, \varepsilon_0)} \ge \log \frac{1}{1-\varepsilon_0^2} \equiv \tau_1^2\ ,$$

where the last inequality is true since $m \ge 1$. Of course, $H((1-\varepsilon_0)\, t_0)$ and τ_1 are negative numbers and we have

$$H((1-\varepsilon_0)\, t_0) \le \tau_1\ ,$$

and then

$$h'(\tau) \ge K\, h'(0)\,, \qquad \text{for all } \ \tau \in [\tau_1, 0]\,.$$

Therefore (2.11) and the positivity of the integrand give that

$$\begin{aligned} G &\ge e^{pf(t_0)} \int_{\tau_1}^{0} e^{-p\tau^2} g(h(\tau))\, h'(\tau)\, d\tau \\ &\ge K\, h'(0)\, e^{pf(t_0)} \int_{\tau_1}^{0} e^{-p\tau^2} g(h(\tau))\, d\tau\,. \end{aligned}$$

Observe that $h(\tau)$ is an increasing function on τ and $g(t)$ is a decreasing function on t (because $n \ge 1$). Then

$$G \ge K\, e^{pf(t_0)} g(t_0)\, h'(0) \int_{\tau_1}^{0} e^{-p\tau^2}\, d\tau \ge \frac{C}{\sqrt{p+1}}\, e^{pf(t_0)}\, g(t_0)\, h'(0)\,.$$

This finishes the proof of Theorem 1 by observing that

$$\lim_{p\to\infty} \frac{\displaystyle\int_{\tau_1}^{0} e^{-p\tau^2}\, d\tau}{\displaystyle\int_{-\infty}^{\infty} e^{-p\tau^2}\, d\tau} = \frac{1}{2}\,.$$

Q.E.D.

3. Proof of Theorem 2. First part.

In this Section we will prove one half of Theorem 2. More concretely:

Theorem 2.1. *There exists a positive constant C, depending only on n and β, such that for all nonnegative integers p, q, n ($n \geq 1$) and for all β, $0 < \beta < n/2$, we have*

$$I = \int_0^1 \left(\frac{F(z)}{F(1)}\right)^2 z^{p+q+\beta-1}(1-z)^{n-2\beta-1}dz \;\geq\; C\;\frac{\Gamma(p+\beta)\Gamma(q+\beta)}{\Gamma(p+n-\beta)\Gamma(q+n-\beta)},$$

where $F(z)$ is the hypergeometric function $F(p, q; p+q+n; z)$.

If p or q are 0, F is the constant 1, and I is the Beta function $B(p+q+\beta, n-2\beta)$. So we can assume that p and q are not zero.

By the symmetry of the hypergeometric function in the two first variables, it is enough to prove the inequality for $p \geq q$. Let $p = mq$, with $m \geq 1$.

The corollary following Theorem 1 gives that

$$F(z)\, B(mq, q+n) \geq C\, L\,.$$

Gauss summation formula [S, p. 28], [L, p. 99], gives

$$F(1) = \frac{\Gamma(mq+q+n)\,\Gamma(n)}{\Gamma(mq+n)\,\Gamma(q+n)}\,,$$

and therefore

$$F(z)\, B(mq, q+n) = \frac{F(z)}{F(1)}\,\frac{\Gamma(mq)\,\Gamma(n)}{\Gamma(mq+n)} \leq \frac{F(z)}{F(1)}\,\frac{C}{(mq)^n}\,,$$

where we have used the following well-known fact,

Proposition 3.1. *For all u, v fixed real numbers, we have that*

$$\frac{\Gamma(x+u)}{\Gamma(x+v)} \sim \frac{1}{x^{v-u}}\,, \qquad \text{when} \;\; x \to +\infty\,.$$

Hence,

$$\frac{F(z)}{F(1)} \geq C(n)\, m^{n-1} q^{n-1/2}\, t_0^{mq}(1-m(1-t_0))^q(1-t_0)^{n-1}\Big(\frac{1-z}{a^2-b^2 z}\Big)^{1/4},$$

where $a = 1 + 1/m$ and $b = 1 - 1/m$.

Then we have

$$I \geq C\, m^{2n-2} q^{2n-1}\, J,$$

where

$$J = \int_0^1 e^{qf(z)} g(z)\, dz,$$

and

$$f(z) = \log\left(t_0^{2m}(1-m(1-t_0))^2 z^{m+1}\right),$$

$$g(z) = \frac{(1-t_0)^{2n-2}}{\sqrt{a^2-b^2 z}}\; z^{\beta-1}(1-z)^{n-2\beta-1/2}\,.$$

The functions t_0 and f are increasing; observe that

$$\frac{d}{dz}\Big(\frac{2}{t_0}\Big) = b - \frac{b^2(1-z)+2/m}{\sqrt{(1-z)\,(a^2-b^2z)}}\,,$$

and that this function is negative because

$$b\sqrt{(1-z)\,(a^2-b^2z)} \le 2\,\frac{1}{\sqrt{m}}\,b\sqrt{1-z} \le \frac{1}{m} + b^2(1-z)\,.$$

Following Laplace's method ([O], [Wo]) we introduce the new variable $\tau = -f(z)$; then, if $z = h(\tau)$, we have

$$J = \int_0^\infty e^{-q\tau} g(h(\tau))\,|h'(\tau)|\,d\tau\,.$$

In order to bound J we need some estimates for the function

$$r(z) = \frac{2/m}{\sqrt{4/m + b^2(1-z)} + a\sqrt{1-z}}\,.$$

The function r is increasing for $0 \le z \le 1$; then we have that $1/(m+1) \le r(z) \le 1/\sqrt{m}$. For each k, such that $\sqrt{m}/(m+1) \le k \le 1$, there is a unique $0 \le z_m \le 1$ such that $r(z_m) = k/\sqrt{m}$. A computation shows that

(3.1) $$\sqrt{1-z_m} = \frac{1}{2k}(a\sqrt{m} - \sqrt{b^2m+4k^2})\,.$$

and

$$z_m = \frac{1}{4k^2}\Big(2a\sqrt{b^2m^2+4k^2m} - m(a^2+b^2)\Big)\,.$$

We need the following lemma in order to prove that there is an interval $[0, A]$ for the variable τ, for some universal constant A, in which the estimates are valid.

In what follows we choose $k = (\sqrt{65}-1)/8$ and z_m such that $r(z_m) = k/\sqrt{m}$ for this particular k.

Lemma 3.1. *If τ_m is defined as $\tau_m = -f(z_m)$, there is a universal positive constant A such that $\tau_m \ge A$ for all $m \ge 1$.*

In order to prove this result we need some inequalities.

Lemma 3.2. *We have, for all $z \in [z_m, 1]$, and for all $m \ge 1$, that*

(3.2.A) $$k\sqrt{\frac{1-z}{m}} \le 1 - t_0 \le \sqrt{\frac{1-z}{m}}\,,$$

(3.2.B) $$1 - \sqrt{m(1-z)} \le 1 - m\,(1-t_0) \le 1 - k\sqrt{m(1-z)}\,,$$

(3.2.C) $$1 - t_0 \le 1 - t_0(z_m) < 2\,\frac{1-k^2}{m} < \frac{2}{m}\,,$$

(3.2.D) $$\sqrt{m(1-z)} \le \sqrt{m(1-z_m)} < 2\,\frac{1-k^2}{k} = \frac{1}{2}\,,$$

(3.2.E) $$z_m \in \Big[\frac{3}{4}\,, 1\Big]\,,$$

(3.2.F) $$k\sqrt{\frac{1-z}{m}} \le -\log t_0 \le 2\sqrt{\frac{1-z}{m}}\,,$$

(3.2.G) $$k\sqrt{m(1-z)} \le -\log(1-m(1-t_0)) \le 2\sqrt{m(1-z)}\,,$$

(3.2.H) $$0 \le -\log z \le \sqrt{\frac{1-z}{m}}\,,$$

(3.2.I) $$a^2 - b^2 z \le \frac{5}{m}\,,$$

(3.2.J) $$4k\sqrt{m(1-z)} \le \tau \le 10\sqrt{m(1-z)}\,,$$

Proof of Lemma 3.2. A straigthforward computation shows, using (2.5), that

$$1 - t_0 = r(z)\sqrt{1-z}\,. \tag{3.3}$$

This proves (3.2.A) and (3.2.B), since r is an increasing function for $0 \le z \le 1$.

Since $t_0 = t_0(z)$ is an increasing function of z, we have, using the fact that $r(z_m) = k/\sqrt{m}$ and also (3.1), that

$$1 - t_0 \le 1 - t_0(z_m) = r(z_m)\sqrt{1-z_m} = k\sqrt{\frac{1-z_m}{m}} = \frac{1}{2}\left(a - \sqrt{b^2 + \frac{4k^2}{m}}\right)$$

and so

$$1 - t_0 \le 1 - t_0(z_m) = \frac{1}{2}\,\frac{\dfrac{4}{m} - \dfrac{4k^2}{m}}{a + \sqrt{b^2 + \dfrac{4k^2}{m}}} < 2\,\frac{1-k^2}{m}\,.$$

which proves (3.2.C).

In order to prove (3.2.D) it is enough to observe that (3.3) and (3.2.C) give

$$\sqrt{m(1-z_m)} = \frac{m}{k}\,(1 - t_0(z_m)) < 2\,\frac{1-k^2}{k}$$

and this last number is equal to $1/2$ because of our choice of the constant k.

(3.2.E) follows directly from (3.2.D).

(3.2.A) gives

$$1 - \sqrt{\frac{1-z}{m}} \le t_0 \le 1 - k\sqrt{\frac{1-z}{m}}\,.$$

If we use the inequalities

$$x \le -\log(1-x) \le \frac{x}{1-x}\,, \qquad \text{for all } \; x \in (0,1)\,,$$

and we observe (see (3.2.D)) that $\sqrt{(1-z)/m} \le 1/2$, we obtain (3.2.F).

(3.2.G) can be deduced like (3.2.F) using (3.2.B) instead of (3.2.A).

The inequality (3.2.H) follows from

$$\begin{aligned} -\log z &\le \frac{1-z}{z} \le \frac{4}{3}\,(1-z) \\ &= \frac{4}{3}\sqrt{m(1-z)}\sqrt{\frac{1-z}{m}} \le \sqrt{\frac{1-z}{m}}\,, \end{aligned}$$

where we have used (3.2.E) and (3.2.D).

(3.2.I) can be proved using (3.2.D) in the following way

$$a^2 - b^2 z = a^2 - b^2 + b^2(1-z) \le \frac{4}{m} + 1 - z_m \le \frac{5}{m}\,.$$

Finally, (3.2.J) follows from (3.2.F), (3.2.G) and (3.2.H). Q.E.D.

Proof of Lemma 3.1. The inequality (3.2.J) with $z = z_m$ gives

$$\tau_m \ge 4k\sqrt{m(1-z_m)}\,.$$

On the other hand, (3.1) allows to compute

$$\lim_{m\to\infty} \sqrt{m(1-z_m)} = \lim_{m\to\infty} \frac{m}{2\,k}\left(a - \sqrt{b^2 + \frac{4k^2}{m}}\right)$$
$$= \lim_{m\to\infty} \frac{m}{2\,k} \frac{\frac{4}{m} - \frac{4k^2}{m}}{a + \sqrt{b^2 + \frac{4k^2}{m}}} = \frac{1-k^2}{k} = \frac{1}{4}\,,$$

where the last equality is true because of our choice of k.

Since $\tau_m > 0$ for all $m \geq 1$ and $\liminf_{m\to\infty} \tau_m \geq k$, we have that

$$A = \inf_m \tau_m > 0\,. \qquad \text{Q.E.D.}$$

Lemma 3.3. *If $z \in [z_m, 1]$, the derivative with respect to z of the function t_0 satisfies*

$$t_0'(z) \leq \frac{2}{\sqrt{m(1-z)}}\,. \tag{3.4}$$

Proof. Recall (see (2.1)) that

$$t_0(z) = \frac{a}{2z} + \frac{b}{2} - \frac{\sqrt{a^2 - b^2 z}}{2z}\sqrt{1-z}\,.$$

Therefore,

$$t_0'(z) = \frac{-a}{2z^2} + \frac{2a^2 - b^2 z}{4z^2\sqrt{a^2 - b^2 z}}\sqrt{1-z} + \frac{\sqrt{a^2 - b^2 z}}{2z}\frac{1}{2\sqrt{1-z}}\,.$$

Hence,

$$t_0'(z) \leq \frac{a^2}{2z^2\sqrt{a^2 - b^2 z}}\sqrt{1-z} + \frac{1}{\sqrt{m}}\frac{1}{\sqrt{1-z}}\,, \tag{3.5}$$

where we have used (3.2.E) and (3.2.I). On the other hand, using that $a^2 - b^2 = 4/m$ and (3.2.E), we have that

$$\frac{a^2}{2z^2\sqrt{a^2 - b^2 z}}\sqrt{1-z} \leq \frac{a^2}{2z^2\sqrt{a^2 - b^2}}\sqrt{1-z} \leq 2\sqrt{m(1-z)}\,.$$

Besides, using (3.2.D), we deduce

$$2\sqrt{m(1-z)} \leq \frac{1}{\sqrt{m(1-z)}}\,.$$

Finally, substituting these two last inequalities in (3.5), we obtain (3.4). Q.E.D.

Lemma 3.4. *For all $z \in [z_m, 1]$ we have that*

$$f'(z) \leq C\sqrt{\frac{m}{1-z}}\,. \tag{3.6}$$

Proof. Recall that

$$f(z) = \log\left(t_0^{2m}(1 - m(1-t_0))^2 z^{m+1}\right).$$

Hence,

$$f'(z) = 2m\frac{t_0'}{t_0} + 2m\frac{t_0'}{1 - m(1-t_0)} + \frac{m+1}{z}\,.$$

Using (3.2.A) and (3.2.D), we have that $t_0 \geq 1/2$. Similarly, using (3.2.B) and (3.2.D) again, one deduces that $1 - m(1 - t_0) \geq 1/2$. Therefore, if we recall (3.2.E) and (3.4), we obtain that

$$f'(z) \leq 4m\,(t_0' + t_0' + 1) \leq C\sqrt{\frac{m}{1-z}}\,. \qquad \text{Q.E.D.}$$

Lemma 3.5. *Let $A = \inf_m \tau_m$. Then, for all $\tau \in [0, A]$, we have that*

$$|h'(\tau)| \geq \frac{C\,\tau}{m}\,, \tag{3.7}$$

$$g(h(\tau)) \geq \frac{C\,\tau^{4n-4\beta-3}}{m^{3n-2\beta-3}}\,. \tag{3.8}$$

PROOF. First, recalling that $h = (-f)^{-1}$ and using (3.6) and (3.2.J), we have that

$$|h'(\tau)| = \frac{1}{f'(z)} \geq C\sqrt{\frac{1-z}{m}} \geq \frac{C\,\tau}{m}\,.$$

This proves (3.7). Secondly, recall also that

$$g(h(\tau)) = g(z) = \frac{(1-t_0)^{2n-2}}{\sqrt{a^2 - b^2 z}}\, z^{\beta-1}(1-z)^{n-2\beta-1/2}\,.$$

Therefore, using (3.2.A), (3.2.I) and (3.2.E), we have that

$$g(h(\tau)) \geq C\left(\frac{1-z}{m}\right)^{n-1}\sqrt{m}\,(1-z)^{n-2\beta-1/2} = C\,\frac{(1-z)^{2n-2\beta-3/2}}{m^{n-3/2}}\,,$$

and so, (3.2.J) gives the result. Q.E.D.

PROOF OF THEOREM 2.1. Recall that we need a lower bound of the integral

$$J = \int_0^\infty e^{-q\tau}\, g(h(\tau))\,|h'(\tau)|\,d\tau\,.$$

Using Lemma 3.5 and the positivity of the integrand we have that

$$\begin{aligned} J &\geq \int_0^A e^{-q\tau}\, g(h(\tau))\,|h'(\tau)|\,d\tau \\ &\geq \frac{C}{m^{3n-2\beta-2}}\int_0^A e^{-q\tau}\,\tau^{4n-4\beta-2}\,d\tau \\ &\geq \frac{C}{m^{3n-2\beta-2}}\,\frac{\Gamma(4n-4\beta-1)}{q^{4n-4\beta-1}}\,, \end{aligned} \tag{3.9}$$

where we have used the elementary fact that

$$\lim_{q\to\infty} \frac{\displaystyle\int_0^A e^{-q\tau}\,\tau^{4n-4\beta-2}\,d\tau}{\displaystyle\int_0^\infty e^{-q\tau}\,\tau^{4n-4\beta-2}\,d\tau} = 1\,.$$

Finally, recalling that $I \geq C\,m^{2n-2}\,q^{2n-1}\,J$, and using (3.9), we obtain that

$$I \geq C\,\frac{1}{m^{n-2\beta}}\,\frac{1}{q^{2n-4\beta}} = \frac{C}{(p\,q)^{n-2\beta}}\,. \tag{3.10}$$

Finally, (3.10) and Proposition 3.1 give Theorem 2.1. Q.E.D.

4. Proof of Theorem 2. Second part.

To finish the proof of Theorem 2, we need only to prove the reverse inequality.

Theorem 2.2. *There exists a positive constant C, depending only on n and β, such that for all nonnegative integers p, q, n ($n \geq 1$) and for all β, $0 < \beta < n/2$, we have*

$$I = \int_0^1 \left(\frac{F(z)}{F(1)} \right)^2 z^{p+q+\beta-1}(1-z)^{n-2\beta-1} dz \leq C \frac{\Gamma(p+\beta)\Gamma(q+\beta)}{\Gamma(p+n-\beta)\Gamma(q+n-\beta)}.$$

In order to prove Theorem 2.2 we will need some lemmas.

Lemma 4.1. *For p, q, n, z as in Theorem 2, we have*

$$\big(B(q,p+n)\, F(p,q;p+q+n;z) \big)^2 \leq B(q,n)\, B(q,2p+n)\, F(2p,q;2p+q+n;z)\,.$$

Proof. We have ([S, p. 20], [L, p. 99]) that

$$\begin{aligned} B(q,p+n)\, F(p,q;p+q+n;z) &= \int_0^1 t^{q-1}(1-t)^{p+n-1}(1-zt)^{-p}\, dt \\ &= \int_0^1 t^{(q-1)/2}(1-t)^{(n-1)/2} t^{(q-1)/2}(1-t)^{p+(n-1)/2}(1-zt)^{-p}\, dt\,, \end{aligned}$$

and so, using the Cauchy-Schwarz inequality,

$$\begin{aligned} \big(B(q,p+n)\, F(p,q;p+q+n;z) \big)^2 &\leq \Big(\int_0^1 t^{q-1}(1-t)^{n-1}\, dt \Big) \Big(\int_0^1 t^{q-1}(1-t)^{2p+n-1}(1-zt)^{-2p}\, dt \Big) \\ &= B(q,n)\, B(q,2p+n)\, F(2p,q;2p+q+n;z)\,. \qquad \text{Q.E.D.} \end{aligned}$$

We will denote by ${}_3F_2(a,b,c;d,e;z)$ the following generalized hypergeometric function

$${}_3F_2(a,b,c;d,e;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \, \frac{z^k}{k!}\,.$$

We have that

Lemma 4.2. *There exist constants C_1, C_2, depending only on n and β such that*

$$C_1 \leq \frac{{}_3F_2(2p,q,p+q+\beta;2p+q+n,p+q+n-\beta;z)}{F(2p,q;2p+q+2n-2\beta;z)} \leq C_2\,.$$

Proof. By comparing the k-th terms of each series we have that

$$\begin{aligned} Q_k &\equiv \frac{\dfrac{(2p)_k (q)_k (p+q+\beta)_k}{(2p+q+n)_k (p+q+n-\beta)_k} \, \dfrac{z^k}{k!}}{\dfrac{(2p)_k (q)_k}{(2p+q+2n-2\beta)_k} \, \dfrac{z^k}{k!}} \\ &= \frac{(2p+q+2n-2\beta)_k}{(2p+q+n)_k} \, \frac{(p+q+\beta)_k}{(p+q+n-\beta)_k} \\ &= \frac{\Gamma(2p+q+2n-2\beta+k)\, \Gamma(p+q+\beta+k)}{\Gamma(2p+q+n+k)\, \Gamma(p+q+n-\beta+k)} \, \frac{\Gamma(2p+q+n)\, \Gamma(p+q+n-\beta)}{\Gamma(2p+q+2n-2\beta)\, \Gamma(p+q+\beta)}\,. \end{aligned}$$

If we denote

$$A(p,q) \equiv \frac{\Gamma(2p+q+n)\,\Gamma(p+q+n-\beta)}{\Gamma(2p+q+2n-2\beta)\,\Gamma(p+q+\beta)}\ ,$$

using again Proposition 3.1, we have that

$$A(p,q) \sim \frac{(p+q)^{n-2\beta}}{(2p+q)^{n-2\beta}}\ , \qquad \text{if}\ \ p+q\to\infty\,,$$

and so there exists a constant $C=C(n,\beta)$ such that

$$C^{-1} \le A(p,q) \le C\,, \qquad \text{for all}\ \ p,q\ge 0\,.$$

Also

$$C^{-1} \le A(p,q+k) \le C\,, \qquad \text{for all}\ \ p,q,k\ge 0\,.$$

Therefore,

$$C^{-2} \le Q_k = \frac{A(p,q)}{A(p,q+k)} \le C^2\,, \qquad \text{for all}\ \ k\ge 0\,,$$

and this implies the lemma.

Proof of Theorem 2.2. Gauss summation formula ([S, p. 28], [L, p. 99]) gives

$$F(1) = \frac{\Gamma(p+q+n)\,\Gamma(n)}{\Gamma(p+n)\,\Gamma(q+n)} = \frac{B(q,n)}{B(q,p+n)}\ ,$$

and therefore

$$\begin{aligned}
I &= \frac{1}{B(q,n)^2}\int_0^1 \big(F(z)\,B(q,p+n)\big)^2 z^{p+q+\beta-1}(1-z)^{n-2\beta-1}\,dz\\
&\underset{\text{Lemma 4.1}}{\le} \frac{B(q,2p+n)}{B(q,n)}\int_0^1 F(2p,q;2p+q+n;z)\,z^{p+q+\beta-1}(1-z)^{n-2\beta-1}\,dz\\
&= \frac{B(q,2p+n)}{B(q,n)}\int_0^1 \sum_{k=0}^\infty \frac{(2p)_k(q)_k}{(2p+q+n)_k\,k!}\ z^{k+p+q+\beta-1}(1-z)^{n-2\beta-1}\,dz\\
&= \frac{B(q,2p+n)}{B(q,n)}\sum_{k=0}^\infty \frac{(2p)_k(q)_k}{(2p+q+n)_k\,k!}\ \frac{\Gamma(k+p+q+\beta)\,\Gamma(n-2\beta)}{\Gamma(k+p+q+n-\beta)}\\
&= \frac{B(q,2p+n)}{B(q,n)}\ B(p+q+\beta,n-2\beta)\,{}_3F_2(2p,q,p+q+\beta;2p+q+n,p+q+n-\beta;1)\\
&\underset{\text{Lemma 4.2}}{\le} C\ \frac{\Gamma(2p+n)\,\Gamma(q+n)}{\Gamma(2p+q+n)}\ \frac{\Gamma(p+q+\beta)}{\Gamma(p+q+n-\beta)}\ F(2p,q;2p+q+2n-2\beta;1)\\
&= C\,\frac{\Gamma(2p+n)\,\Gamma(q+n)\,\Gamma(p+q+\beta)}{\Gamma(2p+q+n)\,\Gamma(p+q+n-\beta)}\ \frac{\Gamma(2p+q+2n-2\beta)}{\Gamma(2p+2n-2\beta)\,\Gamma(q+2n-2\beta)}\\
&\le C\ \frac{(2p+q+1)^{n-2\beta}}{(2p+1)^{n-2\beta}(q+1)^{n-2\beta}(p+q+1)^{n-2\beta}}\\
&\le \frac{C}{(p+1)^{n-2\beta}(q+1)^{n-2\beta}}\\
&\le C\ \frac{\Gamma(p+\beta)\,\Gamma(q+\beta)}{\Gamma(p+n-\beta)\,\Gamma(q+n-\beta)}\ ,
\end{aligned}$$

where we have used again Gauss summation formula and twice Proposition 3.1. Q.E.D.

5. An open question.

In this section we formulate an open question which refers to estimates of the square of an hypergeometric function:

Is true that

$$\big(F(p,q;p+q+n;z)\big)^2 \asymp F(2p,2q;2p+2q+2n-1/2;z)\,,$$

for p, q, n positive integers, $0 \le z \le 1$?

We know three cases in which this is true: if $n = 1/2$ (though $1/2$ is not an integer!) as a consequence of Clausen formula, see *e.g.* [S, p. 75]; if $z = 1$ (using Gauss summation formula) or $z = 0$; if p or q is zero. On the other hand, we have a formal argument based on the asymptotic behaviour of the hypergeometric function stated in Theorem 1, which would give a positive answer to the question above.

If this question would be true, this would simplify considerably the proof of Theorem 2 by using the ideas contained in the proof of Theorem 2.2.

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