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APPROXIMATION THEORY FOR WEIGHTED SOBOLEV SPACES ON CURVES

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ABSTRACT. In this paper we present a definition of weighted Sobolev spaces on curves and find general conditions under which the spaces are complete. We also prove the density of the polynomials in these spaces for non-closed compact curves and, finally, we find conditions under which the multiplication operator is bounded on the completion of polynomials. These results have applications to the study of zeroes and asymptotics of Sobolev orthogonal polynomials.

1. Introduction

The topic of weighted Sobolev spaces appears in very different areas of Mathematics going from the partial differential equations to approximation theory (see e.g. [HKM], [K], [Ku], [KO], [KS] and [T]). Some particular cases were studied in [ELW1], [ELW2] and [EL]. Later, we presented a very deep study of general Sobolev spaces in the real line (see [RARP1], [RARP2], [R1], [R2] and [R3]).

Here we are interested in the case of Sobolev spaces with general measures supported on curves in the complex plane.

In the last months of his life, J. J. Guadalupe (Chicho for his friends) showed increasing interest in these problems, planning to collaborate with us.

Sobolev orthogonal polynomials on the unit circle and, more generally, on curves is a topic of recent and increasing interest in approximation theory; see, for example, [CM] and [FMP] (for the unit circle) and [BFM] and [M-F] (for the case of Jordan curves). If γ is a simple and locally absolutely continuous curve, it is clear that the set of holomorphic functions whose norm in the Sobolev space $W^{k,p}(\gamma,\mu)$ is finite is not a Banach space except when the support of μ is finite. In order to obtain a complete space we have to deal with functions which are not holomorphic. Consequently, we need to define $f^{(j)}$ when f is not holomorphic; the precise definition is presented in Section 2.

The zeroes of the Sobolev orthogonal polynomials have been studied in [LP] in the case of a segment on the real line. There it is proved that they are contained in the disk $\{z \in \mathbb{C} : |z| \le 2\|M\|\}$, where $(Mf)(x) = x\,f(x)$ is the multiplication operator.

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Consequently, the set of the zeroes of the Sobolev orthogonal polynomials is bounded if the multiplication operator is bounded. The location of these zeroes allows one to obtain results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [LP]). In [LP] they prove also that if μ is a finite sequentially dominated measure in $[a, b]$, then M is a bounded operator on the completion of polynomials (a measure is sequentially dominated if $\#\operatorname{supp}\mu_0 = \infty$ and $d\mu_j = f_j\, d\mu_{j-1}$ with f_j bounded for $1 \leq j \leq k$). Recently, these results have been improved for measures on compact sets in $\mathbb{C}$ (see [LPP]).

It is not difficult to see that the multiplication operator can also be bounded when the vectorial measure is not sequentially dominated. In Section 8 below other conditions are given in order to have the boundedness of M even on compact sets in $\mathbb{C}$. In [R2] one of the authors obtains a characterization of the boundedness of the operator M for measures in $\mathbb{R}$. Also, in Section 8 (see Theorem 8.1 below) this result is generalized for measures on compact sets in $\mathbb{C}$; therefore this theorem is useful in the study of orthogonal polynomials.

Though we do not have yet the definitions, let us state the main theorems here. The results are numbered according to the section where they appear. The first one gives a sufficient condition under which one obtains a complete Sobolev space. The condition is a bit technical although it is very general, so we prefer to state the theorem in a short version where this condition is denoted by: $(\gamma, \mu) \in \mathcal{C}$. The definition of the class $\mathcal{C}$ is in Section 4, Definition 4.2. The theorem is as follows:

Theorem 5.1. *Let us consider* $1 \leq p \leq \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a locally finite* p*-admissible vectorial measure on* γ *with* $(\gamma, \mu) \in \mathcal{C}$*. Then the Sobolev space* $W^{k,p}(\gamma, \mu)$ *is complete.*

Our main result on the density of polynomials in these spaces is Theorem 6.2. Now, the conditions we need are more restrictive than in Theorem 5.1, but we have found five general types of measures for which it is true. We simply name them by types 1, 2, 3, 4 and 5 and the definitions are in Section 6. These measures include the most usual examples like Jacobi-type weights (that are measures of type 2).

Theorem 6.2. *Let us consider* $1 \leq p < \infty$, $c > 0$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a* p*-admissible vectorial measure on a non-closed compact curve* $\gamma : I \to \mathbb{C}$*. Let us assume that* $\gamma \in W^{k,\infty}(I)$ *and* $|\gamma'| \geq c$*. If* μ *is a measure of type* $1, 2, 3, 4$ *or* 5*, then* P *is dense in the Sobolev space* $W^{k,p}(\gamma, \mu)$*.*

The last result we present here is Theorem 8.1. It gives a necessary and sufficient condition so that the multiplication operator is bounded on the completion of polynomials, $P^{k,p}(E, \mu)$. Here we consider general compact sets $E \subset \mathbb{C}$ instead of curves. The kind of measures that appear here, ESD, is a generalization of sequentially dominated measures. The definition is in Section 6, Definition 6.6.

Theorem 8.1. *Let us consider* $1 \leq p < \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a finite vectorial measure on a compact set* E*. Then, the multiplication operator is bounded on* $P^{k,p}(E, \mu)$ *if and only if there exists a vectorial measure* $\mu' \in ESD$ *such that the*

Sobolev norms in $W^{k,p}(E,\mu)$ *and* $W^{k,p}(E,\mu')$ *are comparable on the space of polynomials* P. *Furthermore, we can choose* $\mu' = (\mu'_0, \dots, \mu'_k)$ *with* $\mu'_j := \mu_j + \mu_{j+1} + \cdots + \mu_k$.

We also obtain results which partially generalize the classical result on density of polynomials in L^p of the unit circle to the context of Sobolev spaces (see Section 7).

Notation. We only consider simple curves in the complex plane which have a locally absolutely continuous parametrization. In the paper, $k \geq 1$ denotes a fixed natural number; z_i are points along a curve $\gamma \subset \mathbb{C}$. All the measures we consider are Borel, positive and locally finite and all the weights are non-negative Borel measurable functions. We can split μ_j as $d\mu_j = d(\mu_j)_s + w_j\, ds$, where $(\mu_j)_s$ is singular with respect to the arc-length measure, w_j is a weight on γ and ds is the differential of arc-length. We always use this terminology for the Radon-Nikodym decomposition of μ_j. We identify a weight w on γ with the measure $w\, ds$. P denotes the set of all polynomials. When every polynomial has finite $W^{k,p}(\gamma,\mu)$-norm, we denote by $P^{k,p}(\gamma,\mu)$ the completion of polynomials with that norm.

If $\gamma : I \longrightarrow \mathbb{C}$ is a non-closed curve and $t_0 \in I$, by a right (respectively, left) neighbourhood of $z_0 = \gamma(t_0)$ in γ we mean the image by γ of $[t_0, t_0+\varepsilon]$ (respectively, $[t_0-\varepsilon, t_0]$) for some $\varepsilon > 0$. If t_0 is the maximum (respectively, minimum) of I we also have left (respectively, right) neighbourhoods of $\gamma(t_0)$.

If $\gamma : I \longrightarrow \mathbb{C}$ is a closed curve and $t_0 \in I$, we can consider its periodic extension $\gamma_0 : \mathbb{R} \longrightarrow \mathbb{C}$, and define left and right neighborhoods of $\gamma(t_0)$ in a similar way.

Finally, the constants (denoted by c or c_i) in the formulae can change from line to line and even in the same line.

The present article is extracted from [APRR], from the same authors. In that paper we can find the proofs that do not appear here; there we can also find another related theorems and all the technical results that we need in our proofs.

We thank the editors of this volume for suggesting us to collaborate with an article related to one of the last topics of interest for J. J. Guadalupe. With this paper we want to honour the memory of our friend Chicho who, tragically, has left us so early.

2. Derivatives along curves

In this section we introduce a definition of *derivative along a curve* extending the usual complex derivative, which will be crucial in the future. As far as we know this concept is new. Recall that every curve in this paper is simple and has a locally absolutely continuous parametrization.

Definition 2.1.

(a) *Let* $I \subseteq \mathbb{R}$ *be any interval and* $\gamma : I \longrightarrow \mathbb{C}$ *be a curve. If* z_1, z_2 *are two distinct points of* $\gamma(I)$, *we denote by* $\int_{z_1}^{z_2} g(\zeta) d\zeta$ *the complex integral of the function* g *along the arc of* γ *joining* z_1 *and* z_2, (*which we denote by* $[z_1, z_2]$). *We also can consider arcs where one or the two extremal points are not included, as* $(z_1, z_2), [z_1, z_2)$ *or* $(z_1, z_2]$. *If* γ *is a closed curve we take the arc of* γ *joining* z_1 *and* z_2 *in the positive sense* (*according to the parametrization*).

(b) *Let z_0 be a fixed point in γ. If γ is compact we say that $f \in AC^k(\gamma)$ if f can be written as*

$$f(z) = q(z) + \int_{z_0}^{z} h(\zeta)\, \frac{(z-\zeta)^{k-1}}{(k-1)!}\, d\zeta\,, \tag{1}$$

for some $h \in L^1(\gamma, ds)$ and some polynomial $q \in P_{k-1}$. If γ is a closed curve we require also the function $h \in L^1(\gamma, ds)$ to verify $\int_\gamma h(\zeta)\, \zeta^i\, d\zeta = 0$, for $0 \le i < k$. When γ is not compact, we say that $f \in AC^k_{loc}(\gamma)$ if it can be split as in (1) *with $h \in L^1_{\mathrm{loc}}(\gamma, ds)$.*

(c) *If $f \in AC^k_{\mathrm{loc}}(\gamma)$ and $z_0 \in \gamma$, we define its derivative f' along γ as*

$$f'(z) = q'(z) + \int_{z_0}^{z} h(\zeta)\, \frac{(z-\zeta)^{k-2}}{(k-2)!}\, d\zeta\,,$$

where $q'(z)$ means the classical derivative of $q(z)$ and $\int_{z_0}^{z} h(\zeta)(z-\zeta)^{-1}/(-1)!\, d\zeta$ means $h(z)$.

Obviously, if γ is a compact real interval, the space $AC^1(\gamma)$ is the set of absolutely continuous functions in γ. If γ is a closed curve and $f \in AC^k(\gamma)$, we have $\int_\gamma h(\zeta)\,(z-\zeta)^{k-1}\, d\zeta = 0$ for every $z \in \gamma$. This property is equivalent to $f^{(j)}$ being continuous in γ for $0 \le j < k$, where $f^{(j)}$ denotes the j-th derivative (according to the previous definition) of f. It is clear that every holomorphic function in a neighbourhood of γ belongs to $AC^k(\gamma)$ for every k.

We also notice that it is natural to define the derivative along γ in this way, since this is the "inverse" of integration:

$$\begin{aligned}\int_{z_0}^{z}\int_{z_0}^{\xi} h(\zeta)\, \frac{(\xi-\zeta)^{k-2}}{(k-2)!}\, d\zeta\, d\xi &= \int_{z_0}^{z}\int_{\zeta}^{z} h(\zeta)\, \frac{(\xi-\zeta)^{k-2}}{(k-2)!}\, d\xi\, d\zeta \\ &= \int_{z_0}^{z} h(\zeta) \left[\frac{(\xi-\zeta)^{k-1}}{(k-1)!}\right]_{\xi=\zeta}^{\xi=z} d\zeta = \int_{z_0}^{z} h(\zeta)\, \frac{(z-\zeta)^{k-1}}{(k-1)!}\, d\zeta\,.\end{aligned}$$

This definition of derivative is independent of the representation of f we are using; moreover, it does not depend on the choice of the point z_0 nor on k. We have the following non-surprising result.

Lemma 2.1. *If $f \in AC^k_{loc}(\gamma)$ and $z_0 \in \gamma$, then*

$$f(z) = q(z) + \int_{z_0}^{z} h(\zeta)\, \frac{(z-\zeta)^{k-1}}{(k-1)!}\, d\zeta\,,$$

where $q(z)$ is the $(k-1)$-th Taylor polynomial of f centered at z_0, i.e.,

$$q(z) = \sum_{j=0}^{k-1} \frac{f^{(j)}(z_0)}{j!}\,(z-z_0)^j, \qquad \text{and} \qquad h(z) = f^{(k)}(z)\,.$$

Definition 2.2. *We say that $f \in C^k(\gamma)$ if $f \in AC^k_{loc}(\gamma)$ and $f^{(k)}$ is continuous in γ.*

This definition of the class $C^k(\gamma)$ coincides with the classical one when $\gamma \in C^k(I)$ and $\gamma' \neq 0$ on I (see Corollary 2.1 in [APRR]).

With our definition of derivatives, we can prove that Leibniz' rule is true. Also, we can prove the chain rule for derivatives. The proofs of these results may be found in [APRR], Lemmas 2.4, 2.5 and 2.6.

3. Sobolev spaces

Obviously one of our main problems is to define the space $W^{k,p}(\gamma,\mu)$. There are two natural definitions:

(1) $W^{k,p}(\gamma,\mu)$ is the biggest space of (classes of) functions f regular enough with $\|f\|_{W^{k,p}(\gamma,\mu)} < \infty$.

(2) $W^{k,p}(\gamma,\mu)$ is the closure of a good set of functions (e.g. $C^\infty(\gamma)$ or P) with the norm $\|\cdot\|_{W^{k,p}(\gamma,\mu)}$.

However both approaches have serious difficulties:

We consider first the approach (1). It is clear that the derivatives $f^{(j)}$ must be derivatives along γ in order to obtain a complete Sobolev space. Therefore we need to restrict the measures μ to a class of p-admissible measures (see Definition 3.6). Roughly speaking, μ is p-admissible if $(\mu_j)_s$, for $1 \le j \le k$, is concentrated in the set of points where $f^{(j)}$ is continuous, for every function f of the space; otherwise $f^{(j)}$ is determined, up to zero-Lebesgue measure sets. Then $(\mu_k)_s$ is identically zero. However, there is no restriction on the support of $(\mu_0)_s$.

This reasonable approach excludes norms appearing in the theory of Sobolev orthogonal polynomials. Even if we work with the simpler case of the weighted Sobolev spaces $W^{k,p}(\gamma,w)$ (measures without singular part) we must impose the condition that w_j belongs to the class B_p (see Definition 3.2 below) in order to have a complete weighted Sobolev space (see [KO], [RARP1]).

The approach (2) is simpler: we know that the completion of every normed space exists (e.g. $(C^\infty(\gamma), \|\cdot\|_{W^{k,p}(\gamma,\mu)})$ or $(P, \|\cdot\|_{W^{k,p}(\gamma,\mu)})$), but we have two difficulties. The first one is evident: we do not get an explicit description of the Sobolev functions as in (1) (in Section 6 there are several theorems which prove that both definitions of Sobolev space are the same for p-admissible measures). The second problem is worse: The completion of a normed space is by definition a set of equivalence classes of Cauchy sequences. In many cases this completion is not a function space (see Theorem 3.1 in [R2] and its Remark).

However, since we need to work with the multiplication operator in $P^{k,p}(\gamma,\mu)$, we have to choose this second approach if μ is not p-admissible. First of all, we explain the definition of generalized Sobolev space on a curve. Let us start with some preliminary technical definitions.

Definition 3.1. *We say that two functions u, v are comparable on the set $A \subseteq \gamma$ if there are positive constants c_1, c_2 such that $c_1 v(x) \le u(x) \le c_2 v(x)$ for almost every $x \in A$. Since measures and norms are functions on measurable sets and vectors, respectively, we can talk about comparable measures and comparable norms. We say* *that two vectorial weights or vectorial measures are comparable if each component is comparable.*

In what follows, the symbol $a \asymp b$ means that a and b are comparable for a and b functions, measures or norms.

Obviously, the spaces $L^p(A, \mu)$ and $L^p(A, \nu)$ are the same and have comparable norms if μ and ν are comparable on A. Therefore, in order to obtain our results we can replace a measure μ by any comparable measure ν.

To define a Sobolev space along a curve γ we consider first a class of weights which plays an important role in our results.

Definition 3.2. *If $1 \le p \le \infty$, we say that a weight w belongs to $B_p([z_1, z_2])$ if and only if*

$$w^{-1} \in L^{1/(p-1)}([z_1, z_2])\,, \qquad \textit{if } p < \infty\,,$$
$$w^{-1} \in L^{1}([z_1, z_2])\,, \qquad \textit{if } p = \infty\,.$$

Also, if J is any arc of γ we say that $w \in B_p(J)$ if $w \in B_p(J_0)$ for every compact arc $J_0 \subseteq J$. We say that a weight belongs to $B_p(J)$, where J is a union of disjoint arcs $\cup_{i\in A} J_i$, if it belongs to $B_p(J_i)$, for $i \in A$.

If the curve γ is $\mathbb{R}$, then $B_p(\mathbb{R})$ contains the classical $A_p(\mathbb{R})$ weights appearing in Harmonic Analysis (see [Mu1] or [GR]). The classes $B_p(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$, and $A_p(\mathbb{R}^n)$ $(1 < p < \infty)$ have been used in other definitions of weighted Sobolev spaces on $\mathbb{R}^n$ in [KO] and [K] respectively.

Definition 3.3. *Let us consider $1 \le p \le \infty$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ defined on the curve γ. For $0 \le j \le k$ we define the open set*

$$\Omega_j := \big\{\, z \in \gamma : \exists \textit{ an open neighbourhood } V \textit{ of } z \textit{ on the curve } \gamma \textit{ with } w_j \in B_p(V)\, \big\}\,.$$

Remark. Observe that we always have $w_j \in B_p(\Omega_j)$ for any $1 \le p \le \infty$ and $0 \le j \le k$. In fact, Ω_j is the greatest open set U with $w_j \in B_p(U)$. Obviously, Ω_j depends on μ and p, although p and μ do not appear explicitly in the symbol Ω_j. Applying Hölder inequality it is easy to check that if $f^{(j)} \in L^p(\Omega_j, w_j)$ with $1 \le j \le k$, then $f^{(j)} \in L^1_{\mathrm{loc}}(\Omega_j)$ and $f^{(j-1)} \in AC^1_{\mathrm{loc}}(\Omega_j)$.

The definitions below also depend on μ and p, although μ and p may not appear explicitly.

Let us consider $1 \le p \le \infty$, a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ and $z_0 \in \gamma$. We can modify the measure μ in a neighbourhood of z_0, using the following version of Muckenhoupt inequality on curves. This modified measure is equivalent in some sense to the original one (see Theorem 4.1).

Theorem 3.1 (Muckenhoupt inequality on curves)**.** *Let us consider $1 \le p \le \infty$, $[z_0, z_1\,] \subseteq \gamma$ and μ_0, μ_1 measures in $(z_0, z_1\,]$. Assume also $(\mu_0)_s \equiv 0$ if $p = \infty$. Then there exists a positive constant c such that*

$$\Big\| \int_z^{z_1} g(\zeta)\, d\zeta \,\Big\|_{L^p((z_0,z_1],\, \mu_0)} \le c\, \|g\|_{L^p((z_0,z_1],\, \mu_1)}\,, \tag{2}$$

for any measurable function g in $(z_0, z_1\,]$, if and only if

$$(3)\qquad \begin{aligned} &\sup_{\zeta\in(z_0,z_1)} \mu_0\big((z_0,\zeta]\big)\big\|w_1^{-1}\big\|_{L^{1/(p-1)}([\zeta,z_1])} < \infty\,, && \text{if } 1\le p<\infty\,,\\ &\operatorname*{ess\,sup}_{\zeta\in(z_0,z_1)} w_0(\zeta)\int_\zeta^{z_1} w_1(\xi)^{-1}\,|d\xi| < \infty\,, && \text{if } p=\infty\,, \end{aligned}$$

where ess sup *refers to the arc-length.*

Remark. This inequality is already known for γ contained in the real line (see [Mu2], [M, p. 44] for $1 \le p < \infty$, and [RARP1, Lemma 3.2] for the case $p = \infty$).

Definition 3.4. *A vectorial measure $\overline{\mu} = (\overline{\mu}_0, \dots, \overline{\mu}_k)$ is a right completion of a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ with respect to $z_0 \in \gamma$ in a right neighbourhood $[z_0, z_1]$, if $\overline{\mu}_k = \mu_k$ in γ, $\overline{\mu}_j = \mu_j$ in the complement of $(z_0, z_1]$ and*

$$\overline{\mu}_j = \mu_j + \tilde{\mu}_j\,, \qquad \text{in } (z_0,z_1] \text{ for } 0\le j<k\,,$$

where $\tilde{\mu}_j$ is any measure satisfying:
(i) $\tilde{\mu}_j((z_0, z_1]) < \infty$ *if* $1 \le p < \infty$,
(ii) $(\tilde{\mu}_j)_s \equiv 0$ *and* $\tilde{w}_j \in L^\infty([z_0, z_1])$ *if* $p = \infty$,
(iii) $\Lambda_p(\tilde{\mu}_j, \overline{\mu}_{j+1}) < \infty$, *where*

$$\begin{aligned} \Lambda_p(\nu,\sigma) &:= \sup_{\zeta\in(z_0,z_1)} \nu\big((z_0,\zeta]\big)\Big\|\Big(\frac{d\sigma}{ds}\Big)^{-1}\Big\|_{L^{1/(p-1)}([\zeta,z_1])}\,, && \text{if } 1\le p<\infty\,,\\ \Lambda_\infty(\nu,\sigma) &:= \operatorname*{ess\,sup}_{\zeta\in(z_0,z_1)} \frac{d\nu}{ds}(\zeta)\int_\zeta^{z_1}\Big(\frac{d\sigma}{ds}\Big)^{-1}(\xi)\,|d\xi|\,. \end{aligned}$$

The Muckenhoupt inequality guarantees that if $f^{(j)} \in L^p(\mu_j)$ and $f^{(j+1)} \in L^p(\overline{\mu}_{j+1})$, then $f^{(j)} \in L^p(\overline{\mu}_j)$. Therefore, $f \in W^{k,p}(\gamma, \overline{\mu})$ if and only if $f \in W^{k,p}(\gamma, \mu)$ (see Theorem 4.1 for further results). If we work with absolutely continuous measures, we also say that a vectorial weight $\overline{w}$ is a completion of μ (or of w). Some examples of completions may be found in [RARP1].

We can define a left completion of μ with respect to z_0 in a similar way.

Definition 3.5. *For $1 \le p \le \infty$ and a vectorial measure μ, we say that a point $z_0 \in \gamma$ is right j-regular (respectively, left j-regular), if there exist a right completion $\overline{\mu}$ (respectively, left completion) of μ in $[z_0, z_1]$ and $j < i \le k$ such that $\overline{w}_i \in B_p([z_0, z_1])$ (respectively, $B_p([z_1, z_0])$). Also, we say that a point $z_0 \in \gamma$ is j-regular, if it is right and left j-regular.*

Remarks.

1. A point $z_0 \in \gamma$ is right j-regular (respectively, left j-regular), if at least one of the following properties is verified:

(a) There exist a right (respectively, left) neighbourhood $[z_0, z_1]$ (respectively, $[z_1, z_0]$) and $j < i \le k$ such that $w_i \in B_p([z_0, z_1])$ (respectively, $B_p([z_1, z_0])$). Here we have chosen $\tilde{w}_j = 0$.

(b) There exist a right (respectively, left) neighbourhood $[z_0, z_1]$ (respectively, $[z_1, z_0]$) and $j < i \le k$, $\alpha > 0$, $\delta < \delta_p$ with $\delta_p := (i - j)p - 1$ if $1 \le p < \infty$ and

$\delta_\infty := i-j-1$, such that $w_i(z) \geq \alpha\,|z-z_0|^\delta$, for almost every $z \in [z_0, z_1]$ (respectively, $[z_1, z_0]$) and we have $|z - z_0| \asymp |\gamma^{-1}(z) - \gamma^{-1}(z_0)|$ in $[z_0, z_1]$ (respectively, $[z_1, z_0]$), when γ is the arc-length parametrization. See Lemma 3.4 in [RARP1].

2. If z_0 is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0 \leq i \leq j$.

3. It is easy to prove that we can take $i = j + 1$ in this definition.

Let us introduce some more notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $z \in \Omega^{(j)}$ if and only if z is j-regular, we say that $z^+ \in \Omega^{(j)}$ if and only if z is right j-regular, and we say that $z^- \in \Omega^{(j)}$ if and only if z is left j-regular. Obviously, $\Omega^{(k)} = \emptyset$ and $\Omega_{j+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(j)}$. Observe that $\Omega^{(j)}$ depends on p (see Definition 3.5).

Remark. If $0 \leq j < k$ and J is an arc in γ, $J \subseteq \Omega^{(j)}$, then the set $J \setminus (\Omega_{j+1} \cup \cdots \cup \Omega_k)$ is discrete (see the Remark before Definition 7 in [RARP1]).

Definition 3.6. *We say that the vectorial measure $\mu = (\mu_0, \ldots, \mu_k)$ is p-admissible if*

$$(\mu_j - (w_j)|_{\Omega_j})(\gamma \setminus \Omega^{(j)}) = 0, \qquad \text{for } 1 \leq j \leq k.$$

We say that μ is strongly p-admissible if $\operatorname{supp}(\mu_j - (w_j)|_{\Omega_j}) \subseteq \Omega^{(j)}$, *for $1 \leq j \leq k$.*

We use the letter p in p-admissible in order to emphasize the dependence on p (recall that $\Omega^{(j)}$ depends on p).

Remarks.

1. There is no condition on μ_0.

2. We have $w_k = 0$ in almost every $z \in \gamma \setminus \Omega_k$ and $(\mu_k)_s \equiv 0$, since $\Omega^{(k)} = \emptyset$.

3. Every absolutely continuous measure w with $w_j(z) = 0$ in almost every $z \in \gamma \setminus \Omega_j$ for $1 \leq j \leq k$ is p-admissible.

4. Recall that we are identifying w_j with the measure $w_j\, ds$.

5. This definition is more general than Definition 8 in [RARP1]; there we always assume $w_j(z) = 0$ in $\gamma \setminus \Omega_j$. There exist weights which do not satisfy this reasonable condition: Consider a Cantor set C in $[0, 1]$ with positive length and define $w_1 := 1$ in C and $w_1(x) := \operatorname{dist}(x, C)$ if $x \in \mathbb{R} \setminus C$; it is clear that $\Omega_1 = \mathbb{R} \setminus C$ and $w_1 = 1$ in C.

Definition 3.7 (Sobolev space)**.** *Let us consider $1 \leq p \leq \infty$ and $\mu = (\mu_0, \ldots, \mu_k)$ a p-admissible vectorial measure. We define the Sobolev space $W^{k,p}(\gamma, \mu)$ as the space of equivalence classes of*

$$V^{k,p}(\gamma, \mu) := \Big\{ f : \gamma \to \mathbb{C} \;/\; f^{(j)} \in AC^1_{\text{loc}}(\Omega^{(j)}) \text{ for } 0 \leq j < k \text{ and } \|f^{(j)}\|_{L^p(\gamma, \mu_j)} < \infty \text{ for } 0 \leq j \leq k \Big\},$$

with respect to the seminorm

$$\|f\|_{W^{k,p}(\gamma,\mu)} := \Big(\sum_{j=0}^{k} \|f^{(j)}\|^p_{L^p(\gamma,\mu_j)} \Big)^{1/p}, \qquad \text{for } 1 \leq p < \infty\,,$$

and

$$\|f\|_{W^{k,\infty}(\gamma,\mu)} := \max_{0\le j\le k} \|f^{(j)}\|_{L^\infty(\gamma,\mu_j)}\,,$$

where

$$\|g\|_{L^\infty(\gamma,\mu_j)} := \max\Big\{ \operatorname*{ess\,sup}_{z\in\gamma} |g(z)|\, w_j(z)\,,\ \sup_{z\in \operatorname{supp}(\mu_j)_s} |g(z)| \Big\}$$

and we assume the usual convention $\sup\emptyset = -\infty$.

Remark. It is natural to ask for $f^{(j)} \in AC^1_{\text{loc}}(\Omega^{(j)})$, since when the $(\mu_j)_s$-measure of the set where $|f^{(j)}|$ is not continuous is positive, the integral $\int |f^{(j)}|^p\, d(\mu_j)_s$ does not make sense.

4. Some technical results

In these results rely the hardest part of the proofs of our theorems. There are some concepts of particular importance that appear in these results. They are in the following definitions.

Definition 4.1. *Let us consider* $1\le p\le\infty$ *and* μ *a* p*-admissible vectorial measure on* γ. *Let us define the space* $\mathcal{K}(\gamma,\mu)$ *as*

$$\mathcal{K}(\gamma,\mu) := \Big\{ g:\Omega^{(0)} \longrightarrow \mathbb{C} /\ g\in V^{k,p}\big(\gamma,\mu|_{\Omega^{(0)}}\big),\ \|g\|_{W^{k,p}(\gamma,\,\mu|_{\Omega^{(0)}})} = 0 \Big\}\,.$$

$\mathcal{K}(\gamma,\mu)$ is the equivalence class of 0 in $W^{k,p}(\gamma,\,\mu|_{\Omega^{(0)}})$. It plays an important role in the study of the multiplication operator in Sobolev spaces (see [RARP2] and Theorem 8.3 below) and in the following definition, which will be crucial in the study of Sobolev spaces (see [RARP1], [RARP2] and theorems 4.1, 4.2 and 5.1 below).

Definition 4.2. *Let us consider* $1\le p\le\infty$ *and* μ *a* p*-admissible vectorial measure on* γ. *We say that* (γ,μ) *belongs to the class* $\mathcal{C}_0$ *if there exist compact sets* M_n, *which are a finite union of compact arcs in* γ, *such that*

(i) M_n *intersects at most a finite number of connected components of* $\Omega_1\cup\cdots\cup\Omega_k$,
(ii) $\mathcal{K}(M_n,\mu)=\{0\}$,
(iii) $M_n\subseteq M_{n+1}$,
(iv) $\cup_n M_n = \Omega^{(0)}$.

We say that (γ,μ) *belongs to the class* $\mathcal{C}$ *if there exists a measure* $\mu_0' = \mu_0 + \sum_{m\in D} c_m\delta_{z_m}$ *with* $c_m>0$, $\{z_m\}\subset\Omega^{(0)}$, $D\subseteq\mathbb{N}$ *and* $(\gamma,\mu')\in\mathcal{C}_0$, *where* $\mu' = (\mu_0',\mu_1,\dots,\mu_k)$ *is minimal in the following sense: there exists* $\{M_n\}$ *corresponding to* $(\gamma,\mu')\in\mathcal{C}_0$ *such that if* $\mu_0'' = \mu_0' - c_{m_0}\delta_{z_{m_0}}$ *with* $m_0\in D$ *and* $\mu'' = (\mu_0'',\mu_1,\dots,\mu_k)$, *then* $\mathcal{K}(M_n,\mu'')\ne\{0\}$ *if* $z_{m_0}\in M_n$.

Remarks.

1. The condition $(\gamma,\mu)\in\mathcal{C}$ is not very restrictive. In fact, the proof of Theorem 4.1 (see [APRR]) shows that if $\Omega^{(0)}\setminus(\Omega_1\cup\cdots\cup\Omega_k)$ has only a finite number of points in each connected component of $\Omega^{(0)}$, then $(\gamma,\mu)\in\mathcal{C}$. Furthermore, if $\mathcal{K}(\gamma,\mu)=\{0\}$, we have $(\gamma,\mu)\in\mathcal{C}_0$.

2. If $(\gamma,\mu)\in\mathcal{C}_0$, then $(\gamma,\mu)\in\mathcal{C}$, with $\mu'=\mu$.

3. The proof of Theorem 4.1 shows that if for every connected component Λ of $\Omega_1\cup\cdots\cup\Omega_k$ we have $\mathcal{K}(\overline{\Lambda},\mu)=\{0\}$, then $(\gamma,\mu)\in\mathcal{C}_0$. Condition $\#\operatorname{supp}\mu_0|_{\overline{\Lambda}\cap\Omega^{(0)}}\geq k$ implies $\mathcal{K}(\overline{\Lambda},\mu)=\{0\}$.

The next results play a central role in the theory of Sobolev spaces on curves.

Theorem 4.1. *Let us consider $1\leq p\leq\infty$ and $\mu=(\mu_0,\dots,\mu_k)$ a locally finite p-admissible vectorial measure on γ. Let K_j be a finite union of compact arcs contained in $\Omega^{(j)}$, for $0\leq j<k$ and $\overline{\mu}$ a right (or left) completion of μ. Then:*

(a) *If $(\gamma,\mu)\in\mathcal{C}_0$ there exist positive constants $c_1=c_1(K_0,\dots,K_{k-1})$ and $c_2=c_2(\overline{\mu},K_0,\dots,K_{k-1})$ such that, $\forall\, g\in V^{k,p}(\gamma,\mu)$,*

$$c_1\sum_{j=0}^{k-1}\|g^{(j)}\|_{L^\infty(K_j)}\leq\|g\|_{W^{k,p}(\gamma,\,\mu)},\qquad c_2\,\|g\|_{W^{k,p}(\gamma,\,\overline{\mu})}\leq\|g\|_{W^{k,p}(\gamma,\,\mu)}.$$

(b) *If $(\gamma,\mu)\in\mathcal{C}$ there exist positive constants $c_3=c_3(K_0,\dots,K_{k-1})$ and $c_4=c_4(\overline{\mu},K_0,\dots,K_{k-1})$ such that for every $g\in V^{k,p}(\gamma,\mu)$, there exists $g_0\in V^{k,p}(\gamma,\mu)$, independent of $K_0,\dots,K_{k-1}$, c_3, c_4 and $\overline{\mu}$, with*

$$\|g_0-g\|_{W^{k,p}(\gamma,\,\mu)}=0\,,$$
$$c_3\sum_{j=0}^{k-1}\|g_0^{(j)}\|_{L^\infty(K_j)}\leq\|g_0\|_{W^{k,p}(\gamma,\,\mu)}=\|g\|_{W^{k,p}(\gamma,\,\mu)},$$
$$c_4\,\|g_0\|_{W^{k,p}(\gamma,\,\overline{\mu})}\leq\|g\|_{W^{k,p}(\gamma,\,\mu)}.$$

Furthermore, if g_0,f_0 are, respectively, these representatives of g,f, we have, with the same constants c_3,c_4,

$$c_3\sum_{j=0}^{k-1}\|g_0^{(j)}-f_0^{(j)}\|_{L^\infty(K_j)}\leq\|g-f\|_{W^{k,p}(\gamma,\,\mu)},$$
$$c_4\,\|g_0-f_0\|_{W^{k,p}(\gamma,\,\overline{\mu})}\leq\|g-f\|_{W^{k,p}(\gamma,\,\mu)}.$$

The proof follows the argument in the proof of Theorem 4.3 in [RARP1] and needs some additional technical results on curves.

Theorem 4.2. *Let us consider $1\leq p\leq\infty$ and μ a locally finite p-admissible vectorial measure on γ. Let K_j be a finite union of compact arcs contained in $\Omega^{(j)}$, for $0\leq j<k$. Then:*

(a) *If $(\gamma,\mu)\in\mathcal{C}_0$ there exists a positive constant $c_1=c_1(K_0,\dots,K_{k-1})$ such that*

$$c_1\sum_{j=0}^{k-1}\|g^{(j+1)}\|_{L^1(K_j)}\leq\|g\|_{W^{k,p}(\gamma,\,\mu)},\qquad\forall\, g\in V^{k,p}(\gamma,\mu).$$

(b) *If $(\gamma,\mu)\in\mathcal{C}$ there exists a positive constant $c_2=c_2(K_0,\dots,K_{k-1})$ such that for every $g\in V^{k,p}(\gamma,\mu)$, there exists $g_0\in V^{k,p}(\gamma,\mu)$ (the same function as in*

Theorem 4.1), with

$$\|g_0 - g\|_{W^{k,p}(\gamma,\,\mu)} = 0\,,$$

$$c_2 \sum_{j=0}^{k-1} \|g_0^{(j+1)}\|_{L^1(K_j)} \le \|g_0\|_{W^{k,p}(\gamma,\,\mu)} = \|g\|_{W^{k,p}(\gamma,\,\mu)}.$$

Furthermore, if g_0, f_0 *are, respectively, these representatives of* g, f*, we have with the same constant* c_2

$$c_2 \sum_{j=0}^{k-1} \|g_0^{(j+1)} - f_0^{(j+1)}\|_{L^1(K_j)} \le \|g - f\|_{W^{k,p}(\gamma,\,\mu)}.$$

Proof. We only prove part (b) since (a) is simpler. Given a function $g \in V^{k,p}(\gamma,\mu)$, let us choose g_0 as in Theorem 4.1(b). Fix $0 \le j < k$. Since $K_j \subseteq \Omega^{(j)}$, given any point $z \in K_j$, there exist an arc J_z and a completion $\overline{w}^z$ of w with $z \in J_z$ and $\overline{w}^z_{j+1} \in B_p(J_z)$. The compactness of K_j gives that there exists a finite set of points $z_1, \ldots, z_l$ with $K_j \subseteq J_{z_1} \cup \cdots \cup J_{z_l}$.

If we define $w^*_{j+1} := \sum_{i=1}^{l} \overline{w}^{z_i}_{j+1} \chi_{J_{z_i}}$, the second inequality in Theorem 4.1(b) gives, since $w^*_{j+1} \in B_p(K_j)$,

$$c\,\|g_0^{(j+1)}\|_{L^1(K_j)} \le c\,\|g_0^{(j+1)}\|_{L^p(K_j, w^*_{j+1})} \le \|g_0\|_{W^{k,p}(\gamma,\,\mu)}\,,$$

and this finishes the proof of the first inequality. The proof of the second one is similar. □

5. Completeness

Theorem 5.1. *Let us consider* $1 \le p \le \infty$ *and* $\mu = (\mu_0, \ldots, \mu_k)$ *a locally finite* p*-admissible vectorial measure on* γ *with* $(\gamma,\mu) \in \mathcal{C}$*. Then the Sobolev space* $W^{k,p}(\gamma,\mu)$ *is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $W^{k,p}(\gamma,\mu)$. For each n, let us choose a representative of the class of $f_n \in W^{k,p}(\gamma,\mu)$ (which we also denote by f_n) as in theorems 4.1 and 4.2. Therefore, for each $0 \le j \le k$, $\{f_n^{(j)}\}$ is a Cauchy sequence in $L^p(\gamma,\mu_j)$ and it converges to a function $g_j \in L^p(\gamma,\mu_j)$. We only need to prove that, for each $0 \le j \le k-1$, g_j is (perhaps modified in a set of zero μ_j-measure) a function belonging to $AC^1_{\text{loc}}(\Omega^{(j)})$ such that $g_j' = g_{j+1}$ in $\Omega^{(j)}$.

Let us consider any compact arc $K \subseteq \Omega^{(j)}$ (K can be the whole curve γ if $\Omega^{(j)} = \gamma$ and it is a compact curve). By theorems 4.1(b) and 4.2(b) we know that there exists a positive constant c such that for every $n, m \in \mathbb{N}$

$$\|f_n^{(j)} - f_m^{(j)}\|_{L^\infty(K)} + \|f_n^{(j+1)} - f_m^{(j+1)}\|_{L^1(K)} \le c \sum_{i=0}^{k} \|f_n^{(i)} - f_m^{(i)}\|_{L^p(\gamma,\,\mu_i)}\,.$$

As $\{f_n^{(j)}\} \subset C(K)$, there exists a function $h_j \in C(K)$ such that

$$c\,\|f_n^{(j)} - h_j\|_{L^\infty(K)} \le \sum_{i=0}^{k} \|f_n^{(i)} - g_i\|_{L^p(\gamma,\,\mu_i)}\,.$$

Since we can take as K any compact arc contained in $\Omega^{(j)}$, we obtain that the function h_j can be extended to $\Omega^{(j)}$ and we have in fact $h_j \in C(\Omega^{(j)})$. It is obvious that $g_j = h_j$ in $\Omega^{(j)}$ (except for at most a set of zero μ_j-measure), since $f_n^{(j)}$ converges to g_j in the norm of $L^p(\gamma, \mu_j)$ and to h_j uniformly on each compact arc $K \subseteq \Omega^{(j)}$. Therefore we can assume that $g_j \in C(\Omega^{(j)})$.

Let us see now that $g_j' = g_{j+1}$ in K. We have for $z, z_0 \in K$ that

$$f_n^{(j)}(z) = f_n^{(j)}(z_0) + \int_{z_0}^{z} f_n^{(j+1)}(\zeta)\, d\zeta\,.$$

The uniform convergence of $f_n^{(j)}$ in K and the L^1-convergence of $f_n^{(j+1)}$ in K give that

$$g_j(z) = g_j(z_0) + \int_{z_0}^{z} g_{j+1}(\zeta)\, d\zeta\,.$$

□

6. Density

We do not have a density theorem as general as Theorem 5.1, but Theorem 6.1 covers many important cases. We need some previous definitions.

Definition 6.1. *Consider $1 \le p < \infty$, a compact curve γ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ on γ. We say that μ is of type 1 if it is finite and p-admissible on γ and $w_k \in B_p(\gamma)$.*

Definition 6.2. *Consider $1 \le p < \infty$, a non-closed compact curve $\gamma = [z_1, z_2]$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ on γ. We say that μ is of type 2 if it is finite and strongly p-admissible on γ and there exist points along the curve $z_1 \le \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4 \le z_2$ and integers $k_1, k_2 \ge 0$ such that*

(1) $w_k \in B_p([\zeta_1, \zeta_4])$,

(2) if $z_1 < \zeta_1$, then w_j is comparable to a non-decreasing weight in $[z_1, \zeta_2]$, for $k_1 \le j \le k$,

(3) if $\zeta_4 < z_2$, then w_j is comparable to a non-increasing weight in $[\zeta_3, z_2]$, for $k_2 \le j \le k$,

(4) z_1 is right $(k_1 - 1)$-regular if $k_1 > 0$ and z_2 is left $(k_2 - 1)$-regular if $k_2 > 0$.

Definition 6.3. *Consider $1 \le p < \infty$, a compact curve γ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ on γ. We say that μ is of type 3 if it is finite and p-admissible on γ and there exist $z_0 \in \gamma$, an open neighbourhood V of z_0 in γ, an integer $0 \le r < k$ and a positive constant c such that*

(1) $d\mu_{j+1}(z) \le c\,|z - z_0|^p d\mu_j(z)$ on V, for $r \le j < k$,
(2) $w_k \in B_p(\gamma \setminus \{z_0\})$,
(3) if $r > 0$, z_0 is $(r-1)$-regular.

Remark. Condition (1) means that μ_{j+1} is absolutely continuous with respect to μ_j on V and its Radon-Nikodym derivative is less than or equal to $c\,|z - z_0|^p$.

Definition 6.4. *Consider $1 \le p < \infty$, a compact curve γ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ on γ. We say that μ is of type* 4 *if it is finite and p-admissible on γ and there exist $z_0 \in \gamma$, an open neighbourhood V of z_0 on γ and a positive constant c such that*

(1) *if $p > 1$, $w_k(z) \le c\,|z - z_0|^{p-1}$ for almost every $z \in V$; if $p = 1$, w_k can be modified in a set of zero length in such a way that $\lim_{z \to z_0} w_k(z) = 0$,*
(2) *$w_k \in B_p(\gamma \setminus \{z_0\})$,*
(3) *if $k > 1$, z_0 is $(k-2)$-regular.*

Definition 6.5. *Consider $1 \le p < \infty$, a non-closed compact curve $\gamma = [z_1, z_2]$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ on γ. We say that μ is of type* 5 *if it is finite and p-admissible on γ and verifies*

(1) *$w_k \in B_p((z_1, z_2))$,*
(2) *if $k > 1$, z_1 is right $(k-2)$-regular and z_2 is left $(k-2)$-regular.*

We want to remark that the types of measures in [RARP2] and here do not coincide.

Our theorems on density use in their proofs a new concept of measures, which we define now.

Definition 6.6. *A vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ on the complex plane belongs to ESD (extended sequentially dominated) if there exists a positive constant c such that $\mu_{j+1} \le c\mu_j$ for $0 \le j < k$.*

Remark. If $\mu \in ESD$ is a p-admissible vectorial measure on a curve γ, then $(\gamma, \mu) \in \mathcal{C}_0$ (see Remark 3 to Definition 4.2). A vectorial measure μ is sequentially dominated if and only if $\mu \in ESD$ and $\#\operatorname{supp}\mu_0 = \infty$. If $\mu \in ESD$, 0 is the unique polynomial q with $\|q\|_{W^{k,p}(\mathbb{C},\mu)} = 0$ if and only if $\#\operatorname{supp}\mu_0 = \infty$.

Now, let us state our results.

Theorem 6.1. *Let us consider $1 \le p < \infty$, $c > 0$ and $\mu = (\mu_0, \dots, \mu_k)$ a p-admissible vectorial measure on a compact curve $\gamma : I \to \mathbb{C}$. Let us assume that $\gamma \in W^{k,\infty}(I)$ and $|\gamma'| \ge c$. If μ is a measure of type* 1, 2, 3, 4 *or* 5*, then $AC^k(I)$ is dense in the Sobolev space $W^{k,p}(\gamma, \mu)$. Furthermore, if $\gamma \in C^\infty(I)$, then $C^\infty(\gamma)$ is dense in $W^{k,p}(\gamma, \mu)$.*

To prove this theorem we assume that the measure $\mu \in ESD$ (this can be done by lemmas 6.1 and 6.2 in [APRR]). Then, we can find another measure, μ^*, such that the spaces $W^{k,p}(I, \mu^*)$ and $W^{k,p}(\gamma, \mu)$ are isomorphic as normed spaces (μ^* is the pullback of μ by γ). Therefore, we can apply the results on density in intervals (see Theorem 4.1 in [RARP2] and theorems 3.3 and 3.4 in [R3]). This is the basis of the proof.

Theorem 6.2. *Let us consider $1 \le p < \infty$, $c > 0$ and $\mu = (\mu_0, \dots, \mu_k)$ a p-admissible vectorial measure on a non-closed compact curve $\gamma : I \to \mathbb{C}$. Let us assume that $\gamma \in W^{k,\infty}(I)$ and $|\gamma'| \ge c$. If μ is a measure of type* 1, 2, 3, 4 *or* 5*, then P is dense in the Sobolev space $W^{k,p}(\gamma, \mu)$.*

Proof. Let $f_0 \in V^{k,p}(\gamma,\mu)$. By Theorem 6.1 we can approximate f_0 by a function $f \in AC^k(\gamma)$. Let g be a continuous function approximating $f^{(k)}$ in the $L^p(\gamma,\mu_k)$ and the $L^1(\gamma)$ norms (see [R3, Lemma 3.1]). Since γ is non-closed, we can choose a polynomial q approximating g in $L^\infty(\gamma)$ (and therefore in the $L^p(\gamma,\mu_k)$ and the $L^1(\gamma)$ norms). If $z_0 \in \gamma$, the following function approximates f:

$$Q(z) := \sum_{j=0}^{k-1} f^{(j)}(z_0)\frac{(z-z_0)^j}{j!} + \int_{z_0}^{z} q(\zeta)\frac{(z-\zeta)^{k-1}}{(k-1)!}\,d\zeta\,.$$

□

7. Density in analytic closed curves

First of all, let us translate Szegö condition to our context of curves.

Definition 7.1. *A scalar measure μ on an analytic closed curve γ with absolutely continuous part w verifies the Szegö condition if*

$$\int_\gamma \log w(z)\,|dz| > -\infty\,.$$

The following theorem of Kolmogorov-Krein-Szegö is well known (see e.g. [G, pp. 135–137]).

Theorem A. *Let us consider $1 \le p < \infty$ and a finite scalar measure μ on $\partial\mathbb{D}$. Then the polynomials are dense in $L^p(\partial\mathbb{D},\mu)$ if and only if μ does not verify the Szegö condition.*

We can prove the following consequence of Theorem A.

Corollary 7.1. *Let us consider $1 \le p < \infty$ and a finite scalar measure μ on an analytic closed curve γ. Then the polynomials are dense in $L^p(\gamma,\mu)$ if and only if μ does not verify the Szegö condition.*

In the case of Sobolev spaces, the natural translation of this result would be that when one of the components, μ_j, of the vectorial measure μ verifies the Szegö condition, then the polynomials are not dense in $W^{k,p}(\gamma,\mu)$. In change, we obtain something better.

Theorem 7.1. *Let us consider $1 \le p < \infty$, $\mu = (\mu_0,\dots,\mu_k)$ a finite p-admissible vectorial measure on an analytic closed curve γ, with $(\gamma,\mu) \in \mathcal{C}_0$ and $\overline{\mu}$ a finite sum of completions of μ. Let us assume that $\mu \in ESD$ if $k \ge 2$. If for some $0 \le j \le k$ the measure $\overline{\mu}_j$ verifies the Szegö condition, then the polynomials are not dense in $W^{k,p}(\gamma,\mu)$.*

This is also true, in particular, for $\overline{\mu} = \mu$. In the same line we have the following result.

Theorem 7.2. *Let us consider $1 \le p < \infty$, a fixed integer $0 \le j \le k$, $\mu = (\mu_0,\dots,\mu_k)$ a finite p-admissible vectorial measure on an analytic closed curve γ,* *with $(\gamma,\mu)\in\mathcal{C}_0$ and K a finite union of compact arcs with $K\subseteq\Omega^{(j)}$. Let us assume that $\mu\in ESD$ if $k\geq 2$. If the measure μ_j verifies*

$$\int_{\gamma\setminus K}\log w_j(z)\,|dz|>-\infty\,,$$

then the polynomials are not dense in $W^{k,p}(\gamma,\mu)$.

When $\gamma=\partial\mathbb{D}$, these two last theorems are true even without the hypothesis $\mu\in ESD$ for $k\geq 2$ (see Theorem 7.1 and Corollary 7.2 in [APRR]).

8. Multiplication operator

First of all, let us see some remarks about the definition of the multiplication operator. In this section we only consider measures such that every polynomial has finite Sobolev norm. Recall that when every polynomial has finite $W^{k,p}(E,\mu)$-norm, we denote by $P^{k,p}(E,\mu)$ the completion of P with that norm. We start with a definition which has sense for measures defined on arbitrary measurable sets E (not necessarily curves).

Definition 8.1. *If μ is a vectorial measure on the Borel set $E\subseteq\mathbb{C}$, we say that the multiplication operator is well defined in $P^{k,p}(E,\mu)$ if given any sequence $\{s_n\}$ of polynomials converging to 0 in $W^{k,p}(E,\mu)$, then $\{zs_n\}$ also converges to 0 in $W^{k,p}(E,\mu)$. In this case, if $\{q_n\}\in P^{k,p}(E,\mu)$, we define $M(\{q_n\}):=\{zq_n\}$. If we choose another Cauchy sequence $\{r_n\}$ representing the same element in $P^{k,p}(E,\mu)$ (i.e. $\{q_n-r_n\}$ converges to 0 in $W^{k,p}(E,\mu)$), then $\{zq_n\}$ and $\{zr_n\}$ represent the same element in $P^{k,p}(E,\mu)$ (since $\{z(q_n-r_n)\}$ converges to 0 in $W^{k,p}(E,\mu)$).*

We can also think of another definition which is as natural in the case of curves:

Definition 8.2. *If μ is a p-admissible vectorial measure on γ (and hence $W^{k,p}(\gamma,\mu)$ is a space of classes of functions), we say that the multiplication operator is well defined in $W^{k,p}(\gamma,\mu)$ if given any function $h\in V^{k,p}(\gamma,\mu)$ with $\|h\|_{W^{k,p}(\gamma,\mu)}=0$, we have $\|zh\|_{W^{k,p}(\gamma,\mu)}=0$. In this case, if $[f]$ is an equivalence class in $W^{k,p}(\gamma,\mu)$, we define $M([f]):=[zf]$. If we choose another representative g of $[f]$ (i.e. $\|f-g\|_{W^{k,p}(\gamma,\mu)}=0$) we have $[zf]=[zg]$, since $\|z(f-g)\|_{W^{k,p}(\gamma,\mu)}=0$.*

Although both definitions are natural, it is possible for a p-admissible measure μ with $W^{k,p}(\gamma,\mu)=P^{k,p}(\gamma,\mu)$ that M is well defined in $W^{k,p}(\gamma,\mu)$ and not well defined in $P^{k,p}(\gamma,\mu)$ (see Lemma 8.1 and Theorem 8.3). The following lemma characterizes the spaces $P^{k,p}(E,\mu)$ with M well defined.

Lemma 8.1. *Let us consider $1\leq p<\infty$ and $\mu=(\mu_0,\dots,\mu_k)$ a vectorial measure on a measurable set $E\subseteq\mathbb{C}$. The following facts are equivalent:*

1. *The multiplication operator is well defined in $P^{k,p}(E,\mu)$.*
2. *The multiplication operator is bounded in $P^{k,p}(E,\mu)$.*
3. *There exists a positive constant c such that*

$$\|zq\|_{W^{k,p}(E,\mu)}\leq c\,\|q\|_{W^{k,p}(E,\mu)}\,,\qquad \textit{for every } q\in P\,.$$

The following result characterizes the boundedness of the multiplication operator.

Theorem 8.1. *Let us consider* $1 \le p < \infty$ *and* $\mu = (\mu_0, \dots, \mu_k)$ *a finite vectorial measure on a compact set* E. *Then, the multiplication operator is bounded in* $P^{k,p}(E,\mu)$ *if and only if there exists a vectorial measure* $\mu' \in ESD$ *such that the Sobolev norms in* $W^{k,p}(E,\mu)$ *and* $W^{k,p}(E,\mu')$ *are comparable on* P. *Furthermore, we can choose* $\mu' = (\mu'_0, \dots, \mu'_k)$ *with* $\mu'_j := \mu_j + \mu_{j+1} + \cdots + \mu_k$.

Remark. In order to apply Theorem 8.1, if $E = \gamma$ is a curve, the best way to deduce that $\| \cdot \|_{W^{k,p}(\gamma,\mu)}$ and $\| \cdot \|_{W^{k,p}(\gamma,\mu')}$ are comparable is to prove that μ' can be obtained by a finite number of completions of μ (in that case we can use Theorem 4.1).

If we consider the case of a curve $E = \gamma$, we have the following results.

Theorem 8.2. *Let us consider* $1 \le p < \infty$ *and a* p*-admissible vectorial measure* μ *on a compact curve* γ. *If* μ *is of type* 1, 2 *or* 3, *and the multiplication operator is well defined in* $W^{k,p}(\gamma,\mu)$, *then it is bounded on* $P^{k,p}(\gamma,\mu)$.

In this situation Theorem 6.2 gives $P^{k,p}(\gamma,\mu) = W^{k,p}(\gamma,\mu)$ if $\gamma : I \to \mathbb{C}$ is a non-closed curve with $|\gamma'| \ge c$ and $\gamma \in W^{k,\infty}(I)$. In this case the multiplication operator is bounded in $W^{k,p}(\gamma,\mu)$.

Theorem 8.3. *Let us consider* $1 \le p < \infty$ *and a* p*-admissible vectorial measure* μ *on* γ. *Assume that* $zf(z) \in V^{k,p}(\gamma,\mu)$ *for every* $f \in V^{k,p}(\gamma,\mu)$. *Then the multiplication operator is well defined in* $W^{k,p}(\gamma,\mu)$ *if and only if* $\mathcal{K}(\gamma,\mu) = \{0\}$.

Theorem 8.4. *Let us consider* $1 \le p < \infty$ *and a finite* p*-admissible vectorial measure* μ *on a compact curve* γ. *Assume that* $(\gamma,\mu) \in \mathcal{C}_0$ *and that for each* $1 \le j \le k$ *we have* $\mu_j(\gamma \setminus (J_{j-1} \cup K_{j-1})) = 0$, *where* K_{j-1} *is a finite union of compact arcs contained in* $\Omega^{(j-1)}$, *and* J_{j-1} *is a measurable set with* $\mu_j \le c\,\mu_{j-1}$ *in* J_{j-1}. *Then the multiplication operator is bounded in* $P^{k,p}(\gamma,\mu)$.

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