STRUCTURE THEOREM FOR RIEMANNIAN SURFACES WITH ARBITRARY CURVATURE

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Abstract. In this paper we prove that any Riemannian surface, with no restriction of curvature at all, can be decomposed into blocks belonging just to some of these types: generalized Y-pieces, generalized funnels and halfplanes.
Key words and phrases: Riemannian surface; decomposition of surfaces; arbitrary curvature.

1. Introduction.

The Classification Theorem of compact surfaces says that every orientable compact topological surface is homeomorphic either to a sphere or to a "torus" of genus $g \geq 1$ (see e.g. [9]).

We say that the closure of a three-holed sphere (which is a bordered compact topological surface whose border is the union of three pairwise disjoint simple closed curves) is a *topological Y-piece*. A Y-piece can be visualized as a tubing with the shape of the letter Y. A *cylinder* is a bordered topological surface homeomorphic to $\mathbb{S}^1 \times [0, \infty)$, where $\mathbb{S}^1$ is the unit circle.

We refer to the next section for precise definitions and background.

The Classification Theorem of compact surfaces says, in other words, that every orientable compact topological surface except for the sphere and the torus (of genus 1) can be obtained by gluing topological Y-pieces along their boundaries.

In [1], the Classification Theorem is generalized to noncompact surfaces in the following way:

Theorem 1.1. ([1, Theorem 1.1]) *Every complete orientable topological surface which is homeomorphic neither to the sphere nor to the plane nor to the torus is the union (with pairwise disjoint interiors) of topological Y-pieces and cylinders.*

The following result is the most important in [1] and is a geometric version of the theorem above for complete surfaces with constant negative curvature. In this case we have more information about the basic blocks of the surface: the surface can be decomposed in such a way that the boundary of the blocks is the union of at most three simple closed geodesics. Since the Riemannian structure is more restrictive than the topological one, an additional piece is necessary in order to achieve the decomposition: the halfplane.

We state now this result for Riemannian surfaces.

Theorem 1.2. ([1, Theorem 1.2]) *Every complete orientable Riemannian surface with constant curvature $K = -1$, which is not the punctured disk, is the union (with pairwise disjoint interiors) of generalized Y-pieces, funnels and halfplanes.*

In the applications of Theorem 1.2, it is a crucial fact that the boundaries of the generalized Y-pieces are simple closed geodesics. There is a clear reason for this: it is very easy to cut and paste surfaces along such kind of curves.

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Furthermore, closed geodesics in a Riemannian surface S are geometrical objects interesting by themselves. Since they are the periodic orbits of the dynamical system associated to S on its unit tangent bundle, they provide tools to study the geodesic flow, just like the fixed points of an automorphism helps to study it. Lastly, the closed geodesics are becoming more and more important in the study of heat and wave equations, and of the spectrum of S. The lengths of all closed geodesics determine largely the spectrum. Conversely, the spectrum determines completely the lengths of the closed geodesics (see [4], [8], [5]).

In this paper we prove the conclusion of Theorem 1.2 with no restriction of curvature at all (see Theorem 4.3 forward). In our context, we require that the boundaries of the generalized Y-pieces are minimizing simple closed geodesics (in its free homotopy class). Although if $K = -1$ the property of minimization ever holds (see e.g. Theorem 3.7 and Lemma 3.9), this is obviously false for arbitrary curvature.

Theorem 4.8 is a sharp version of Theorem 4.3 for Riemannian surfaces with curvature $K \le -c^2 < 0$.

Finally, Theorems 4.10 and 4.11 are versions, respectively, of Theorems 4.3 and 4.8 in the context of bordered Riemannian surfaces.

J. L. Fernández and M. V. Melián (see [7]) proved the following result which helps to understand the behavior of geodesics in surfaces with curvature $K = -1$.

Theorem 1.3. ([7, Theorem 1]) *Let S be a complete orientable Riemannian surface with curvature $K = -1$. There are three possibilities:*

(i) S has finite area. Then for every $p \in S$ there is exactly a countable collection of directions in $\mathcal{E}(p)$.

(ii) S is transient. Then for every $p \in S$, $\mathcal{E}(p)$ has full measure.

(iii) S is recurrent and of infinite area. Then for every $p \in S$, $\mathcal{E}(p)$ has zero length but its Hausdorff dimension is 1.

A surface is said to be *transient* (respectively, *recurrent*) if Brownian motion of S is transient (respectively, recurrent). Also, we define $\mathcal{E}(p)$ as the set of unitary directions v in the tangent plane of S at p such that the unit speed geodesic emanating from p in the direction of v, *escapes* to infinity.

Just like Theorem 1.2 played an important role in the proof of Theorem 1.3, we are sure that Theorems 4.3 and 4.8 will be crucial in order to generalize this latest result to surfaces with curvature $K \le -c^2 < 0$.

The argument in the proof of Theorem 4.3 is quite alike to the one in the proof of Theorem 1.2. Unfortunately, *every* standard fact used in the proof of Theorem 1.2 is false when there is no restriction of curvature. Hence, it was unavoidable both to state definitions for the new objects appearing in our current context and to prove alternate results valid for arbitrary curvature. This work has provided some results with intrinsic interest, as Theorems 3.7 and 3.11.

One can think that in the decomposition of Theorem 4.3 we might not need halfplanes. There is an example in [1] which shows that, even with curvature $K = -1$, we do need them. The necessity of halfplanes is, in fact, one of the most difficult parts in the proof of this theorem.

The outline of the paper is as follows. Section 2 presents some definitions and technical results which we will need. We prove some additional technical results in Section 3. Section 4 is dedicated to the main results.

Notations. We denote by $L_M(\gamma)$ the length of a curve γ in a Riemannian manifold M. If there is no possible confusion, we usually write $L(\gamma)$.

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2. Background in Riemannian manifolds.

Definition 2.1. *Any divergent curve $\sigma : [0, \infty) \longrightarrow Y$, where Y is a noncompact Hausdorff space, determines an* end *E of Y. Given a compact set F of Y, one defines $E(F)$ to be the arc component of $Y \setminus F$ that contains a terminal segment $\sigma([a, \infty))$ of σ. A set $U \subset Y$ is a* neighborhood of an end *E if U contains $E(F)$ for some compact set F of Y. An end E in a surface S is* collared *if E has a neighborhood homeomorphic to $(0, \infty) \times \mathbb{S}^1$. A neighborhood U of E will be called* Riemannian collared *if there exists a C^1 diffeomorphism $X : (0, \infty) \times \mathbb{S}^1 \longrightarrow U$ such that the metric in U relative to the coordinate system X is written $ds^2 = dr^2 + G(r, \theta)^2 d\theta^2$, where G is a positive continuous function. A sequence of curves $\{C_n\}$* converges to *E if* *for any neighborhood U of E we have $C_n \subset U$ for sufficiently large n. We say that a closed curve γ* bounds *a collared end E in S if some arc component of $S \setminus \gamma$ is a neighborhood U of E.*

It follows directly from the metric expression $ds^2 = dr^2 + G(r,\theta)^2 d\theta^2$ of a Riemannian collared parametrization that the r-parameter curves have unit speed and minimize the distance between any two of their points. Consequently the r-parameter curves are geodesics of S. If the curvature K satisfies $K \le 0$, then G is a C^∞ function of r for each fixed θ and satisfies the Jacobi equation

$$\frac{\partial^2 G}{\partial r^2}(r,\theta) + K(r,\theta)G(r,\theta) = 0\,,$$

where $K(r,\theta)$ is the curvature of S at $X(r,\theta)$ ([6, p. 17]).

Every manifold is connected, C^∞ and satisfy the second axiom of countability (has a countable basis for its topology). In a Riemannian surface we always assume that the Riemannian metric is C^∞ unless perhaps in some simple closed geodesics, each of them bounding a collared end, where we allow the metric to be C^1 and piecewise C^∞, with the "singularities" along these geodesics. Then the curvature is a (possibly discontinuous) function along these geodesics.

Geodesic always means local geodesic (unless we say explicitly something else).

Definition 2.2. *Given a Riemannian surface S, a geodesic γ in S, and a continuous unit vector field ξ along γ, orthogonal to γ, we define the* Fermi coordinates *based on γ as the map $Y(r,\theta) := \exp_{\gamma(\theta)} r\xi(\theta)$.*

It is well known that the Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dr^2 + G(r,\theta)^2\, d\theta^2$, where $G(r,\theta)$ is the solution of the scalar equation

$$\frac{\partial^2 G}{\partial r^2}(r,\theta) + K(r,\theta)G(r,\theta) = 0\,, \qquad G(0,\theta) = 1\,, \qquad \frac{\partial G}{\partial r}(0,\theta) = 0\,,$$

(see e.g. [3, p. 247]).

We will need the following three results.

Theorem 2.3. ([6, Theorem 4.2]) *Let S be a complete Riemannian surface with $K \le 0$ and E an end of S. Then the following are equivalent:*

(1) *E is a collared end.*
(2) *E is a Riemannian collared end.*
(3) *There exists a sequence $\{C_n\}$ of continuous piecewise smooth closed curves converging to E such that $\{C_n\}$ belongs to a single nontrivial free homotopy class.*

It is clear that (2) can be deduced from (1) since S verifies $K \le 0$. However, since (1) and (3) are topological conditions, we have the following result without conditions on K.

Theorem 2.4. *Let S be a complete Riemannian surface and E an end of S. Then E is a collared end if and only if there exists a sequence $\{C_n\}$ of continuous piecewise smooth closed curves converging to E such that $\{C_n\}$ belongs to a single nontrivial free homotopy class.*

Theorem 2.5. ([2, Theorem (5.16)]) *Giving a sequence of rectifiable curves $\{\alpha_k\}$ contained in a compact set of a Riemannian manifold M with $\{L(\alpha_k)\}$ a bounded sequence, there exists a subsequence of curves (which we also call $\{\alpha_k\}$ for simplicity), a rectifiable curve α, and parametrizations $x_k : [0,1] \longrightarrow X$ of α_k and $x : [0,1] \longrightarrow X$ of α, such that $\{x_k\}$ converges uniformly to x in $[0,1]$ and*

$$L(\alpha) \le \liminf_{k\to\infty} L(\alpha_k)\,.$$

In fact, Theorem (5.16) in [2] is stronger than Theorem 2.5, but this statement is general enough for our purposes.

Definition 2.6. *Given a n-dimensional Riemannian manifold M and a closed curve α in M, we define the* length of the freely homotopy class $[\alpha]$ *as*

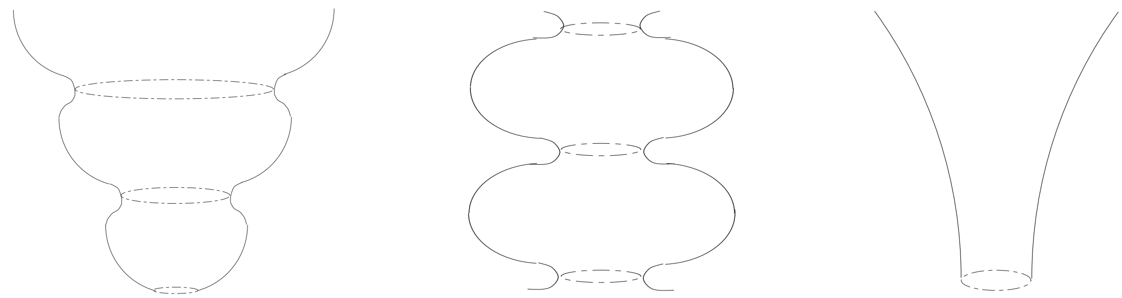
$$L([\alpha]) := \inf\big\{L(\sigma) : \sigma \in [\alpha]\big\}\,.$$

The curve α is minimizing *if $L(\alpha) = L([\alpha])$.*

A minimizing sequence for α *is a sequence of closed curves $\{\alpha_k\} \subset [\alpha]$ such that $\lim_{k\to\infty} L(\alpha_k) = L([\alpha])$.*

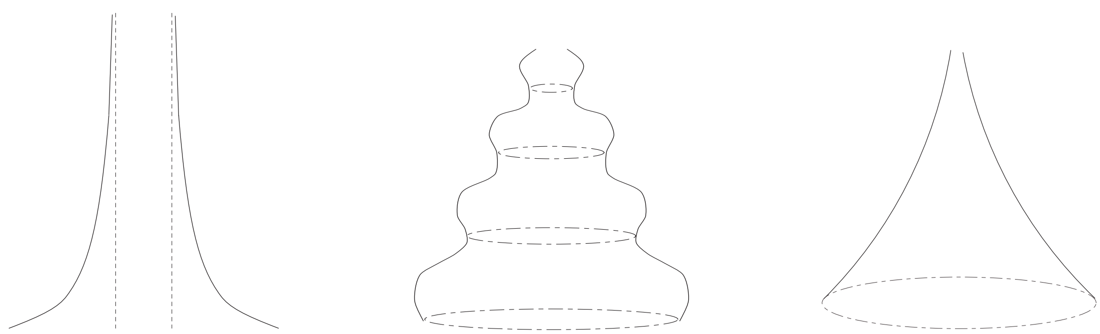
Definition 2.7. *A* halfplane *is a bordered Riemannian surface which is simply connected and whose border is a unique nonclosed simple geodesic.*

A generalized funnel *is a bordered Riemannian surface which is a neighborhood of a collared end and whose border is a minimizing simple closed geodesic. A* funnel *is a generalized funnel such that there does not exist another simple closed geodesic freely homotopic to the border of the funnel.*



(a) Generalized funnel (b) Generalized funnel (c) Funnel

A generalized puncture *is a collared end whose fundamental group is generated by a simple closed curve σ and there is no minimizing closed geodesic $\gamma \in [\sigma]$. A* puncture *is a generalized puncture such that $L([\sigma]) = 0$ and there is no closed geodesic in $[\sigma]$.*



(d) Generalized puncture (e) Generalized puncture (f) Puncture

A bordered or nonbordered surface is doubly connected *if its fundamental group is isomorphic to $\mathbb{Z}$. Every generalized funnel and every generalized puncture are doubly connected surfaces.*

A geodesic domain *G is a bordered Riemannian surface (which is neither simply nor doubly connected) with finitely generated fundamental group and such that ∂G consists of finitely many minimizing simple closed geodesics, and it may contain generalized punctures but not generalized funnels.*

A Y-piece *is a compact bordered Riemannian surface which is topologically a sphere without three open disks and whose boundary curves are minimizing simple closed geodesics. They are a standard tool to construct Riemannian surfaces. A clear description of these Y-pieces and their use is given in* [3, chapter X.3].

A generalized Y-piece *is a bordered or nonbordered Riemannian surface which is topologically a sphere without three open disks, such that there exist integers $n, m \geq 0$ with $n + m = 3$, so that the border are n minimizing simple closed geodesics and there are m generalized punctures.*

Notice that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \leq m \leq 3$. It is clear that every generalized Y-piece is a geodesic domain (unless $m = 3$, in which case it has no border). Furthermore, every geodesic domain is a finite union (with pairwise disjoint interiors) of generalized Y-pieces (see Proposition 4.1 below).

We say that the set A is exhausted *by* $\{A_n\}$ *if* $A_n \subseteq A_{n+1}$ *for every* n *and* $A = \cup_n A_n$.

We say that a bordered Riemannian surface S is simple *if the border of S is a (finite or infinite) union of pairwise disjoint simple closed geodesics.*

3. Technical results.

Lemma 3.1. *Let us consider a n-dimensional complete Riemannian manifold M and an homotopically nontrivial closed curve α in M. If there exists a minimizing sequence $\{\alpha_k\}$ for α contained in a compact set, then there exists a minimizing closed geodesic $\gamma \in [\alpha]$.*

Proof. Since $\{L(\alpha_k)\}$ is convergent, it is a bounded sequence. By Theorem 2.5, there exists a subsequence of curves (which we also call $\{\alpha_k\}$ for simplicity), a rectifiable curve γ, and parametrizations $x_k : [0,1] \longrightarrow M$ of α_k and $x : [0,1] \longrightarrow M$ of γ, such that $\{x_k\}$ converges uniformly to x in $[0,1]$ and

$$L(\gamma) \le \liminf_{k\to\infty} L(\alpha_k) = L([\alpha])\,.$$

The curve γ is closed since each α_k is a closed curve and $x(0) = \lim_{k\to\infty} x_k(0) = \lim_{k\to\infty} x_k(1) = x(1)$. Then, in order to finish the proof of the lemma, it is enough to show that $\gamma \in [\alpha]$, since then γ attains the minimum length in its homotopy class, and it must be a geodesic.

We can assume that x_k and x are 1-periodic functions in $\mathbb{R}$. For each $t \in [0,1]$ let us consider $r_t > 0$ small enough to guarantee that the ball $B(x(t), r_t)$ in M is simply connected. For each $t \in [0,1]$, let us denote by J_t the connected component of $\gamma \cap B(x(t), r_t)$ which contains $x(t)$. Since γ is a compact topological space and $\{J_t\}_{t\in[0,1]}$ is an open covering of γ, there exist $0 \le t_1 < t_2 < \cdots < t_{m-1} < t_m \le 1$ such that $\gamma \subset \cup_{j=1}^m J_{t_j}$. Choosing a subset of $\{t_1, t_2, \dots, t_m\}$ if it is necessary, without loss of generality we can assume that the subcovering $\{J_{t_j}\}_{j=1}^m$ is minimal in the following sense: each $y \in \gamma$ belongs at most to two sets of $\{J_{t_j}\}_{j=1}^m$.

Let us consider $0 \le s_1 < \cdots < s_{m-1} < 1$ and $s_m \in (s_{m-1}, s_1 + 1)$ such that

$$x(s_1) \in J_{t_1} \cap J_{t_2}, \dots, x(s_{m-1}) \in J_{t_{m-1}} \cap J_{t_m}, x(s_m) \in J_{t_m} \cap J_{t_1}\,.$$

Hence,

$$x(s_m), x(s_1) \in B(x(t_1), r_{t_1}),\ x(s_1), x(s_2) \in B(x(t_2), r_{t_2}),\ \dots,\ x(s_{m-1}), x(s_m) \in B(x(t_m), r_{t_m}).$$

Since $\{x_k\}$ converges uniformly to x in $\mathbb{R}$, there exists k_0 such that

$$x_k([s_m, s_1]) \subset B(x(t_1), r_{t_1}),\ x_k([s_1, s_2]) \subset B(x(t_2), r_{t_2}),\ \dots,\ x_k([s_{m-1}, s_m]) \subset B(x(t_m), r_{t_m}),$$

for every $k \ge k_0$.

This proves that $\gamma \in [\alpha_k]$ for every $k \ge k_0$ (since each ball $B(x(t), r_t)$ is simply connected), and then $\gamma \in [\alpha]$. This finishes the proof of the lemma. □

Proposition 3.2. *Let us consider a complete Riemannian surface S and an homotopically nontrivial closed curve α in S. Then, one and only one of the two following possibilities holds:*

(1) *There exists a minimizing closed geodesic $\gamma \in [\alpha]$.*

(2) *The curve α bounds a generalized puncture E in S. Furthermore, if S is not doubly connected, then any minimizing sequence for α converges to E.*

Proof. If there exists a minimizing sequence $\{\alpha_k\}$ for α contained in a compact set, then Lemma 3.1 gives (1).

Otherwise, every minimizing sequence $\{\alpha_k\}$ for α escapes from any compact set. Hence, there not exists any minimizing closed geodesic in $[\alpha]$.

If S is doubly connected, then α bounds a collared end E (in fact, α bounds exactly two collared ends). This collared end E is a generalized puncture since there not exists any minimizing closed geodesic in $[\alpha]$.

If S is not doubly connected, then any minimizing sequence $\{\alpha_k\}$ for α converges to an end E. Since the curves $\{\alpha_k\}$ belong to a single nontrivial free homotopy class, Theorem 2.4 gives that E is a collared end in S. Hence, α bounds a collared end in S, which must be a generalized puncture since there not exists any minimizing closed geodesic in $[\alpha]$. □

In order to deal with bordered surfaces, we need the following results.

Lemma 3.3. *Any simple complete bordered Riemannian surface S is a subset of a complete Riemannian surface R, which can be obtained by attaching a neighborhood of a collared end to each simple closed geodesic $\gamma \subseteq \partial S$, with the following properties:*

(1) *If σ is a closed curve in R which is not contained in S, then there exists $\sigma_0 \subseteq (S \cap \sigma) \cup \partial S \subset S$ with $\sigma_0 \in [\sigma]$ and $L(\sigma_0) < L(\sigma)$.*

(2) *A closed geodesic is minimizing in R if and only if it is minimizing in S (in particular, it is contained in S).*

(3) *If σ is a closed curve in S, then $L_S([\sigma]) = L_R([\sigma])$.*

(4) *If σ is a closed curve in S, and $\{\sigma_k\}$ is a minimizing sequence for σ verifying $\{\sigma_k\} \subseteq (R \setminus S) \cup K$, with K a compact subset of S, then there exists a minimizing closed geodesic in $[\sigma]$.*

(5) *The curvature satisfies $K = -1$ in $R \setminus S$.*

(6) *The fundamental group of R is isomorphic to the fundamental group of S.*

(7) *If S is not doubly connected, then there exists a minimizing simple closed geodesic in $[\gamma_0]$ for each simple closed geodesic $\gamma_0 \subseteq \partial S$.*

Proof. The border of S is a (finite or infinite) union of pairwise disjoint simple closed geodesics. Let us fix a closed geodesic $\gamma_0 \subseteq \partial S$ with length l. We can consider the Fermi coordinates based on γ_0. The Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dr^2 + G(r,\theta)^2\, d\theta^2$, with $G(r,\theta)$ a l-periodic function in θ defined in $[-r_0, 0] \times \mathbb{R}$, for some $r_0 > 0$. We have $G(0,\theta) = 1$ and $\partial G/\partial r(0,\theta) = 0$ for every $\theta \in \mathbb{R}$. If we define $G(r,\theta) := \cosh r$ in $(0,\infty) \times \mathbb{R}$, then it is C^1 (and even piecewise C^∞) in $[-r_0,\infty) \times \mathbb{R}$, and l-periodic in θ. These coordinates $(r,\theta) \in [-r_0,\infty) \times \mathbb{R}$, with the Riemannian metric $ds^2 = dr^2 + G(r,\theta)^2\, d\theta^2$, attach a neighborhood of a collared end F to γ_0; by this way we get a C^∞ surface. We have that $K(r,\theta) = -1$ in $(0,\infty) \times \mathbb{R}$. We also have the following properties:

(a) Any homotopically nontrivial closed curve σ in F verifies $L(\gamma_0) < L(\sigma)$:

Without loss of generality we can assume that σ can be parametrized in Fermi coordinates based on γ_0 as $\sigma(\theta) = (r(\theta), \theta)$, with $\theta \in [0,l]$. Then,

$$L(\sigma) = \int_0^l \sqrt{r'(\theta)^2 + \cosh^2 r(\theta)}\; d\theta \geq \int_0^l \cosh r(\theta)\, d\theta > \int_0^l d\theta = l = L(\gamma_0)\,.$$

(b) Given any closed curve σ intersecting S and the interior of F, there exists $\sigma_0 \in [\sigma]$ contained in S verifying $L(\sigma_0) < L(\sigma)$:

We can construct this curve in the following way: given any subarc a of σ contained in F and joining two points $p, q \in \gamma_0$, we replace it by the subarc of γ_0 joining p, q, which is homotopic to a. The argument above gives $L(\sigma_0) < L(\sigma)$.

We define R as the surface obtained by attaching this neighborhood of a collared end to each closed geodesic in ∂S.

Properties (a) and (b) give that if σ is a closed curve in R which is not contained in S, then there exists $\sigma_0 \subseteq (S \cap \sigma) \cup \partial S \subset S$ with $\sigma_0 \in [\sigma]$ and $L(\sigma_0) < L(\sigma)$. This finishes the proof of (1), (5) and (6).

Now, the statements (2) and (3) are direct consequences of (1).

We prove now (4). If σ is a closed curve in S, and $\{\sigma_k\}$ is a minimizing sequence for σ verifying $\{\sigma_k\} \subseteq (R \setminus S) \cup K$, with K a compact subset of S, by (1) there exists $\{\sigma_k^0\} \subseteq K$ with $\sigma_k^0 \in [\sigma]$ and $L(\sigma_k^0) \leq L(\sigma_k)$. Then $\{\sigma_k^0\}$ is a minimizing sequence for σ contained in a compact set and Lemma 3.1 gives that there exists a minimizing closed geodesic in $[\sigma]$.

In order to prove (7), fix a simple closed geodesic $\gamma_0 \subseteq \partial S$. Let us call F the neighborhood of a collared end in R with $\partial F = \gamma_0$. Seeking for a contradiction, assume that there not exists a minimizing closed geodesic in $[\gamma_0]$. Since R is not doubly connected, by Proposition 3.2 γ_0 bounds a generalized puncture E in R and any minimizing sequence for α converges to E. Since R is not doubly connected, F is a neighborhood of E, and for any minimizing sequence $\{\alpha_k\}$ for $[\gamma_0]$ there exists N with $\alpha_k \subset F$ for every $k \geq N$. By (a) we have $L(\gamma_0) < L(\alpha_k)$ for every $k \geq N$, which is the required contradiction. □

Using Lemma 3.3, Proposition 3.2 can be generalized to simple bordered Riemannian surfaces.

Proposition 3.4. *Let us consider a simple complete bordered Riemannian surface S and an homotopically nontrivial closed curve α in S. Then, one and only one of the two following possibilities holds:*

(1) *There exists a minimizing closed geodesic $\gamma \in [\alpha]$.*

(2) *The curve α bounds a generalized puncture E in S and any minimizing sequence for α converges to E.*

Proof. By Lemma 3.3, S is a subset of a complete Riemannian surface R.

Assume first that S is not doubly connected (then R is not doubly connected).

By Lemma 3.3 (2), if there exists a minimizing closed geodesic $\gamma \in [\alpha]$ in R, then $\gamma \in S$.

If there not exist such minimizing geodesic, by Proposition 3.2 the curve α bounds a generalized puncture E in R and any minimizing sequence for α converges to E. By Lemma 3.3 (7), α can not be freely homotopic to any closed geodesic in ∂S, and therefore $E \subset S$.

Assume now that S is doubly connected (then R is also doubly connected). The curve α bounds exactly two collared ends in R.

Since S is doubly connected, ∂S can be either a simple closed geodesic or two simple closed geodesics.

Assume first that ∂S is a simple closed geodesic γ_0. Then α bounds the unique collared end E in S. Consider a minimizing sequence $\{\alpha_n\}$ for $[\alpha]$ in R. If (2) does not hold, then either α does not bound a generalized puncture (and then (1) holds) or there exists a neighborhood U of E and a subsequence $\{\alpha_{n_k}\}$ with $\alpha_{n_k} \nsubseteq U$ for every k. Since $\{L(\alpha_n)\}$ is a bounded sequence, without loss of generality we can assume that $\alpha_{n_k} \cap U = \varnothing$ for every k. Since S is doubly connected, $R = S \cup F$ with $\partial F = \partial S = \gamma_0$, and the complement of U in R is $F \cup K$, where K is a compact set in S. Therefore, $\alpha_{n_k} \subset F \cup K$ for every k and by Lemma 3.3 (4) there exists a minimizing closed geodesic in $[\alpha]$. Then (1) also holds.

Assume now that ∂S is the union of two simple closed geodesics γ_1, γ_2. Then, $R = S \cup F_1 \cup F_2$ with $\partial F_j = \gamma_j$ $(j = 1, 2)$ and S is compact. Consider a minimizing sequence $\{\alpha_n\}$ for $[\alpha]$ in R. By Lemma 3.3 (1), there exists a minimizing sequence $\{\alpha_n^0\} \subset S$ for $[\alpha]$. Since S is compact, Lemma 3.1 gives that there exists a minimizing closed geodesic $\gamma \in [\alpha]$. □

Lemma 3.5. *Let us consider a positive constant c and two functions y_0, y, satisfying respectively $y_0'' = c^2 y_0$, $y_0(t_0) > 0$, $y_0'(t_0) > 0$, and*

$$\begin{cases} y''(t) \ge c^2 y(t) > 0\,, & \textit{if } t \ge t_0\,, \\ y(t_0) = y_0(t_0)\,, \\ y'(t_0) \ge y_0'(t_0)\,. \end{cases}$$

Then, $y(t) \ge y_0(t)$ for every $t \ge t_0$.

Proof. Since $y''(t) > 0$ if $t \ge t_0$, then y' is an increasing function in $[t_0, \infty)$. This fact and $y'(t_0) \ge y_0'(t_0) > 0$ give $y'(t) \ge y'(t_0) > 0$ for every $t \ge t_0$. Then, for every $t \ge t_0$, we can deduce

$$\begin{aligned} y''(t) &\ge c^2 y(t)\,, \\ y''(t)y'(t) &\ge c^2 y(t) y'(t)\,, \\ y'(t)^2 - y'(t_0)^2 &\ge c^2 \big(y(t)^2 - y(t_0)^2\big)\,, \\ y'(t) &\ge \sqrt{c^2\big(y(t)^2 - y(t_0)^2\big) + y'(t_0)^2}\,. \end{aligned}$$

For each fixed $\varepsilon \in (0, y_0'(t_0))$, we define the function y_ε as the unique solution of

$$\begin{cases} y_\varepsilon''(t) = c^2 y_\varepsilon(t)\,, & \text{if } t \ge t_0\,, \\ y_\varepsilon(t_0) = y_0(t_0)\,, \\ y_\varepsilon'(t_0) = y_0'(t_0) - \varepsilon > 0\,. \end{cases}$$

Then $y'(t_0) > y_\varepsilon'(t_0) > 0$. Using the same argument above in the case of y_ε (with equality instead of inequality) we obtain that $y_\varepsilon'(t) \ge y_\varepsilon'(t_0) > 0$ for every $t \ge t_0$ and

$$y_\varepsilon'(t) = \sqrt{c^2\big(y_\varepsilon(t)^2 - y_\varepsilon(t_0)^2\big) + y_\varepsilon'(t_0)^2}\,.$$

We prove now that $y(t) \ge y_\varepsilon(t)$ for every $t \ge t_0$. Seeking for a contradiction, suppose that $y(t) < y_\varepsilon(t)$ for some $t > t_0$. Then, we can define $t_1 := \min\{t > t_0 : y(t) = y_\varepsilon(t)\}$; this minimum is attained since

$y(t_0) = y_\varepsilon(t_0)$ and $y'(t_0) > y'_\varepsilon(t_0)$; consequently, $y(t) > y_\varepsilon(t) > 0$ for every $t \in (t_0, t_1)$, and

$$\begin{aligned} y_\varepsilon(t_1) - y_\varepsilon(t_0) &= \int_{t_0}^{t_1} y'_\varepsilon(t)\, dt = \int_{t_0}^{t_1} \sqrt{c^2\big(y_\varepsilon(t)^2 - y_\varepsilon(t_0)^2\big) + y'_\varepsilon(t_0)^2}\, dt \\ &< \int_{t_0}^{t_1} \sqrt{c^2\big(y(t)^2 - y(t_0)^2\big) + y'(t_0)^2}\, dt \le \int_{t_0}^{t_1} y'(t)\, dt = y(t_1) - y(t_0) = y_\varepsilon(t_1) - y_\varepsilon(t_0)\,. \end{aligned}$$

This is a contradiction and we have proved that $y(t) \ge y_\varepsilon(t)$ for every $t \ge t_0$. It is easy to check that

$$y_\varepsilon(t) = y_0(t_0) \cosh c(t - t_0) + \frac{y'_0(t_0) - \varepsilon}{c} \sinh c(t - t_0)\,,$$

for every $t, \varepsilon \in \mathbb{R}$. Hence

$$y(t) \ge y_0(t_0) \cosh c(t - t_0) + \frac{y'_0(t_0) - \varepsilon}{c} \sinh c(t - t_0)\,,$$

for every $t \ge t_0$ and $\varepsilon \in (0, y'_0(t_0))$. If $\varepsilon \to 0$, we obtain

$$y(t) \ge y_0(t_0) \cosh c(t - t_0) + \frac{y'_0(t_0)}{c} \sinh c(t - t_0) = y_0(t)\,,$$

for every $t \ge t_0$. This finishes the proof of the lemma. ☐

Lemma 3.5 has the following direct consequence.

Corollary 3.6. *Let us consider a positive constant c and a function y satisfying $y''(t) \ge c^2 y(t) > 0$ and $y'(t_0) > 0$. Then*

$$y(t) \ge y(t_0) \cosh c(t - t_0)\,,$$

for every $t \ge t_0$.

Proof. Let us consider the function y_0 with

$$\begin{cases} y''_0(t) = c^2 y_0(t)\,, & \text{if } t \ge t_0\,, \\ y_0(t_0) = y(t_0)\,, \\ y'_0(t_0) = y'(t_0)\,. \end{cases}$$

The first inequality in the following expression is obtained by applying Lemma 3.5 and the first equality by solving the above differential equation:

$$\begin{aligned} y(t) \ge y_0(t) &= y_0(t_0) \cosh c(t - t_0) + \frac{y'_0(t_0)}{c} \sinh c(t - t_0) \\ &\ge y_0(t_0) \cosh c(t - t_0) = y(t_0) \cosh c(t - t_0)\,, \end{aligned}$$

for every $t \ge t_0$. ☐

The following result assures that if $K \le -c^2 < 0$, there always exists a closed geodesic in every free homotopy class, except for punctures, in which is impossible to have one.

Theorem 3.7. *Let us consider a Riemannian surface S, which can be either simple bordered or without border. Besides, S must be complete and with curvature $K \le -c^2 < 0$. Fix an homotopically nontrivial closed curve α in S. Then there exists a minimizing closed geodesic $\gamma \in [\alpha]$ if and only if $L([\alpha]) > 0$.*

Remark 3.8. *The conclusion of this Theorem does not hold if we replace the hypothesis $K \le -c^2 < 0$ by the weaker one $K < 0$, as shows the revolution surface of the graph of $f(x) = 1 + e^x$ around the horizontal axis (with the standard metric induced by the Euclidean metric in $\mathbb{R}^3$).*

Proof. We deal first with nonbordered surfaces S.

If $L([\alpha]) = 0$, it is clear that there does not exist a closed geodesic $\gamma \in [\alpha]$ with $L(\gamma) = L([\alpha]) = 0$, since α is an homotopically nontrivial closed curve in S.

Let us assume now that $L([\alpha]) > 0$.

Assume first that S is not doubly connected. Seeking for a contradiction, suppose that there not exist a closed geodesic $\gamma \in [\alpha]$. Then, by Proposition 3.2, the curve α bounds a generalized puncture E in S. Since the curvature satisfies $K \le -c^2 < 0$, this end E is a Riemannian collared end, by Theorem 2.3.

For each r_0 we define g_{r_0} as the closed curve $\{r = r_0\}$. It is easy to check that $l(r) := L(g_r) = \int_0^{2\pi} G(r,\theta)\, d\theta$ satisfies $l''(r) \ge c^2 l(r)$:

$$\frac{\partial^2 G}{\partial r^2}\,(r,\theta) = -K(r,\theta)G(r,\theta) \ge c^2 G(r,\theta) > 0$$

implies

$$l''(r) = \int_0^{2\pi} \frac{\partial^2 G}{\partial r^2}\,(r,\theta)\, d\theta \ge \int_0^{2\pi} c^2 G(r,\theta)\, d\theta = c^2 l(r) > 0\,.$$

Since $L([\alpha]) > 0$, there exist positive constants c_0, r_1 with $l(r) \ge c_0$ for every $r \ge r_1$. Hence, for every $r \ge r_1$,

$$l'(r) = l'(r_1) + \int_{r_1}^{r} l''(t)\, dt \ge l'(r_1) + \int_{r_1}^{r} c^2 c_0\, dt = l'(r_1) + c^2 c_0 (r - r_1)\,,$$

and consequently $\lim_{r\to\infty} l'(r) = \infty$. Since

$$\lim_{r\to\infty} \int_0^{2\pi} \frac{\partial G}{\partial r}\,(r,\theta)\, d\theta = \lim_{r\to\infty} l'(r) = \infty\,,$$

there exist $r_2 \ge r_1$ and a set $A \subset [0, 2\pi]$ with positive Lebesgue measure such that $\partial G/\partial r(r_2,\theta) > 0$ for every $\theta \in A$. Since $\partial^2 G/\partial r^2(r,\theta) > 0$, the function $\partial G/\partial r(r,\theta)$ increases in $r \ge r_2$ for each fixed $\theta \in A$, and consequently $\partial G/\partial r(r,\theta) \ge \partial G/\partial r(r_2,\theta) > 0$ for every $\theta \in A$ and $r \ge r_2$. Hence, $G(r,\theta)$ increases in $r \ge r_2$ for each fixed $\theta \in A$. By Corollary 3.6, $G(r,\theta) \ge G(r_2,\theta) \cosh c(r - r_2)$ for every $\theta \in A$ and $r \ge r_2$.

Let us consider a curve σ parametrized in the Riemannian collared end as $\sigma(\theta) = (r(\theta),\theta)$, with $\theta \in [0, 2\pi]$ and $r(\theta) \ge R \ge r_2$. Then

$$\begin{aligned} L(\sigma) &= \int_0^{2\pi} \sqrt{r'(\theta)^2 + G(r(\theta),\theta)^2}\, d\theta \ge \int_0^{2\pi} G(r(\theta),\theta)\, d\theta \ge \int_A G(r(\theta),\theta)\, d\theta \\ &\ge \int_A G(R,\theta)\, d\theta \ge \cosh c(R - r_2) \int_A G(r_2,\theta)\, d\theta\,. \end{aligned}$$

Since $\int_A G(r_2,\theta)\, d\theta$ is a positive constant independent of R, there exists $r_3 > r_2$ with

$$\cosh c(r_3 - r_2) \int_A G(r_2,\theta)\, d\theta > L(\alpha)\,.$$

Hence, given any curve σ parametrized in the Riemannian collared end as $\sigma(\theta) = (r(\theta),\theta)$, with $\theta \in [0, 2\pi]$ and $r(\theta) \ge r_3$, we have $L(\sigma) > L(\alpha)$. Consequently, any curve $\sigma \in [\alpha]$ contained in the region $\{r \ge r_3\}$ verifies $L(\sigma) > L(\alpha)$. Then a minimizing sequence for α can not converge to E. This fact contradicts Proposition 3.2.

If S is doubly connected, the argument is similar except for the fact that α bounds two collared ends. Therefore, a minimizing sequence might not converge to an end in S; but we can always extract a subsequence converging to some end in S.

Assume now that S is simple bordered. By Lemma 3.3, S is a subset of a complete Riemannian surface R. The previous argument gives the desired result in R. Then, Lemma 3.3 implies the result in S. □

The following lemma is a well known result, but we include a direct proof by the sake of completeness.

Lemma 3.9. *Let us consider a Riemannian surface S, which can be either simple bordered or without border. Besides, S must be complete and with curvature $K < 0$. Then in each free homotopy class there exists at most a closed geodesic, and if there exists, then it is minimizing. Consequently, every generalized funnel is a funnel.*

Proof. By Lemma 3.3, without loss of generality we can assume that S is nonbordered.

Seeking for a contradiction, suppose that there exist two freely homotopic closed geodesics γ_1, γ_2.

If γ_1 and γ_2 intersect at some point, they can not be tangent at this point, since they are geodesics. Then, γ_1 and γ_2 intersect at least at another point, since they are freely homotopic. Therefore, some segment of

γ_1 and some segment of γ_2 determine a geodesic "bigon" B (a polygon with two sides) with interior angles $\alpha, \beta > 0$. Gauss-Bonnet Formula gives

$$\iint_B K\, dA = \alpha + \beta > 0\,,$$

which is a contradiction with $K < 0$.

Then γ_1 and γ_2 do not intersect. We consider the geodesic segment σ joining $x_1 \in \gamma_1$ with $x_2 \in \gamma_2$, which gives the minimum distance between γ_1 and γ_2; then σ meets orthogonally to γ_1 and to γ_2. Let us consider a universal covering map $\pi : \tilde{S} \longrightarrow S$. Fix a lift $\tilde{\gamma}_1$ of γ_1 starting in $\tilde{x}_1$, a lift $\tilde{\sigma}$ of σ starting in $\tilde{x}_1$, and finishing in $\tilde{x}_2$, and a lift $\tilde{\gamma}_2$ of γ_2 starting in $\tilde{x}_2$. Then $\tilde{\sigma}$ meets orthogonally to $\tilde{\gamma}_1$ and to $\tilde{\gamma}_2$. If we denote by y_1 and y_2, respectively, the endpoints of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, there exists a covering isometry $T : \tilde{S} \longrightarrow \tilde{S}$ with $T(\tilde{x}_1) = y_1$ and $T(\tilde{x}_2) = y_2$. We also have that with $T(\tilde{\sigma})$ joins y_1 and y_2, and meets orthogonally to $\tilde{\gamma}_1$ and to $\tilde{\gamma}_2$. Consequently, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\tilde{\sigma}$ and $T(\tilde{\sigma})$ bound a geodesical quadrilateral Q in $\tilde{S}$ with four right angles. Gauss-Bonnet Formula gives

$$-\iint_Q K\, dA = 2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = 0\,,$$

which is a contradiction with $K < 0$.

Then in each free homotopy class there exists at most a closed geodesic.

Consider now a simple closed geodesic γ. We need to prove that $L(\gamma) = L([\gamma])$. Let us define $l := L(\gamma)$.

The Riemannian metric can be expressed in Fermi coordinates based on γ as $ds^2 = dr^2 + G(r,\theta)^2\, d\theta^2$, where $G(r,\theta)$ satisfies

$$\frac{\partial^2 G}{\partial r^2}\,(r,\theta) + K(r,\theta) G(r,\theta) = 0\,, \qquad G(0,\theta) = 1\,, \qquad \frac{\partial G}{\partial r}\,(0,\theta) = 0\,.$$

Since $\partial^2 G/\partial r^2(r,\theta) = -K(r,\theta)G(r,\theta) > 0$, it follows that $G(r,\theta)$ is a convex function on r for each fixed $\theta \in [0,l]$; since $\partial G/\partial r(0,\theta) = 0$, we deduce that for each fixed $\theta \in [0,l]$, $G(r,\theta)$ attains its minimum value 1 at $r = 0$.

We prove now that any curve $\sigma \in [\gamma]$ verifies $L(\sigma) \geq L(\gamma)$. Let us consider a fixed curve $\sigma \in [\gamma]$. Without loss of generality we can assume that σ can be parametrized in Fermi coordinates as $\sigma(\theta) = (r(\theta), \theta)$, with $\theta \in [0,l]$. Then,

$$L(\sigma) = \int_0^l \sqrt{r'(\theta)^2 + G(r(\theta),\theta)^2}\; d\theta \geq \int_0^l G(r(\theta),\theta)\, d\theta \geq \int_0^l G(0,\theta)\, d\theta = \int_0^l d\theta = l = L(\gamma)\,.$$

This shows that $L(\gamma) = L([\gamma])$ and hence γ is minimizing.

In order to prove the last part of the lemma, consider now a generalized funnel F in S. We have proved that there does not exist another closed geodesic freely homotopic to the boundary of the generalized funnel. Hence, the generalized funnel is a funnel. □

Lemma 3.10. *Let us consider a Riemannian surface S, which can be either simple bordered or without border. Besides, S must be complete and with curvature $K \leq -c^2 < 0$. Then, every generalized puncture is a puncture.*

Proof. By Lemma 3.3, without loss of generality we can assume that S is nonbordered.

Seeking for a contradiction, consider a generalized puncture which is not a puncture. Then, its fundamental group is generated by a simple closed curve σ, there is no minimizing simple closed geodesic $\gamma \in [\sigma]$, and we have either:

(*i*) $L([\sigma]) > 0$; then by Theorem 3.7 there exists a minimizing simple closed geodesic $\gamma \in [\sigma]$, which is a contradiction,

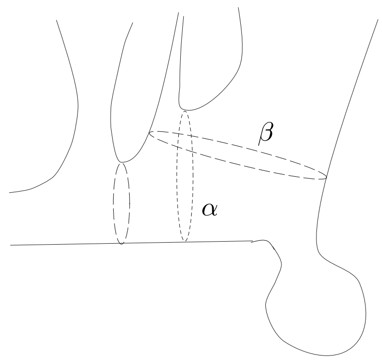
or

(*ii*) $L([\sigma]) = 0$ and there exists a simple closed geodesic $\gamma \in [\sigma]$; then by Lemma 3.9 γ is minimizing, which is a contradiction. □

Theorem 3.11. *Let us consider a complete Riemannian surface S and two disjoint nontrivial piecewise smooth simple closed curves a and b in S, which are not freely homotopic. If $\alpha \in [a]$ and $\beta \in [b]$ are any choice of minimizing closed geodesics, then they are disjoint as well.*

Furthermore, if $\alpha \neq \beta$ are freely homotopic minimizing simple closed geodesics in S, then they are disjoint.

Remark 3.12. *We have some examples that show that the conclusion of the previous Theorem does not hold if either α or β are not minimizing geodesics. See for example the figure below:*



Proof. First, let us assume that the curves a and b are not freely homotopic. Without loss of generality we can assume that a and b are disjoint nontrivial smooth simple closed curves in S, since in other case we can modify them slightly in order to obtain all these facts. Since $\alpha \in [a]$ and S is a surface, by Baer's Theorem (see e.g. [11] or [10]) there exists an isotopy, that is to say, a continuous family of diffeomorphisms $f_t : S \longrightarrow S$, such that f_0 is the identity, and $f_1(a) = \alpha$. Let us define $b_1 := f_1(b)$, which is a simple closed curve freely homotopic to b. As f_1 is bijective and a and b are disjoint curves, then b_1 and α are disjoint too.

Seeking for a contradiction, let us assume that $\alpha \cap \beta \neq \varnothing$. If they do intersect each other tangentially then they should coincide, and this is not possible since they are not freely homotopic. Therefore, they must intersect transversally. Since $b_1 \in [\beta]$, there exists an smooth homotopy $F : A \longrightarrow S$, where A is the annulus $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$, such that $F(e^{i\theta}) = b_1(\theta)$ and $F(2e^{i\theta}) = \beta(\theta)$, with $\theta \in [0, 2\pi]$.

No connected component γ of $F^{-1}(\alpha)$ can be a closed curve in A, since it should be either trivial or freely homotopic to $F^{-1}(\beta)$. In this case, $F(\gamma) = \alpha$ would also be either trivial or homotopic to $F(F^{-1}(\beta)) = \beta$, and this is contradiction with our hypothesis. Therefore, $F^{-1}(\alpha)$ must contain an arc σ joining two points z_1 and z_2 in $\{z \in \mathbb{C} : |z| = 2\} = F^{-1}(\beta)$. As A is a planar domain, one of the two arcs joining z_1 and z_2 in $F^{-1}(\beta)$ (let us denote such arc by η), is homotopic to σ. This fact implies that the geodesics α and β intersect in $F(z_1)$ and $F(z_2)$ and that $F(\sigma)$ and $F(\eta)$ are homotopic. Since α and β are minimizing, $L(F(\sigma)) = L(F(\eta))$ and so, from α we can construct a new curve $\tilde{\alpha} \in [\alpha]$ with the same length by replacing the arc $F(\sigma)$ by the arc $F(\eta)$. By means of a smooth modification in small neighborhoods of $F(z_1)$ and $F(z_2)$ we can obtain a shorter curve freely homotopic to α, which is contradiction with the fact that α is minimizing.

Now, we will deal with the second part of the Theorem. Once again we are going to seek for a contradiction: let us assume that $\alpha \cap \beta \neq \varnothing$. If they do intersect in a single point, as they are in the same homotopy class, they must intersect each other tangentially and therefore $\alpha = \beta$. If they do intersect in several points, then they must intersect transversally. The argument in the previous case allows to obtain a shorter curve freely homotopic to α, which is contradiction with the fact that α is minimizing. □

4. The main results.

Proposition 4.1. *Every geodesic domain in any complete orientable Riemannian surface is a finite union (with pairwise disjoint interiors) of generalized Y-pieces.*

Remark 4.2. *The argument in the proof of Proposition 4.1 also proves the following:*

Every complete orientable Riemannian surface without generalized funnels and with finitely generated fundamental group, which is neither simply nor doubly connected nor homeomorphic to a torus, is a finite union (with pairwise disjoint interiors) of generalized Y-pieces.

Proof. Let us fix a geodesic domain G in a complete orientable Riemannian surface S. We denote by $\gamma_1, \gamma_2, \dots, \gamma_k$ the minimizing simple closed geodesics in ∂G. Since G is a simple complete bordered Riemannian surface, by Lemma 3.3 it is a subset of a complete Riemannian surface R.

In particular, R is a complete orientable topological surface. Since R contains a geodesic domain, R is neither simply nor doubly connected nor homeomorphic to a torus. Then, by Theorem 1.1, R is the union of topological Y-pieces $\{Y_n\}$ and cylinders $\{C_n\}$. The fundamental group of R is isomorphic to the fundamental group of G, and therefore it is finitely generated; then there are only a finite number of topological Y-pieces and cylinders. We denote by $\{\eta_m\} \subset R$ the set of pairwise disjoint simple closed curves in $\cup_n \partial Y_n$. Without loss of generality we can assume that the curves are numbered such that $\eta_j \in [\gamma_j]$ for each $1 \le j \le k$.

We want to change the curves η_j by minimizing simple closed geodesics. For each $1 \le m \le k$, we replace η_m by γ_m (Lemma 3.3 gives that γ_m are also minimizing simple closed geodesic in R). For each $m > k$, let us choose a minimizing simple closed geodesic $\gamma_m \in [\eta_m]$, if it exists. In other case, by Proposition 3.2, the curve η_m bounds a generalized puncture in S and we define $\gamma_m := \varnothing$. By Theorem 3.11, the minimizing simple closed geodesics $\{\gamma_m\} \subset G$ are pairwise disjoint; then, they split G in the required finite union of generalized Y-pieces (if for some m we have $\gamma_m = \varnothing$, the corresponding Y-piece in R is a generalized Y-piece in G). □

The following theorem is the main result of this paper. It generalizes an already known result for constant negatively curved surfaces to arbitrary surfaces with no restricition of curvature at all.

Theorem 4.3. *Every complete orientable Riemannian surface which is neither simply nor doubly connected nor homeomorphic to a torus is the union (with pairwise disjoint interiors) of generalized Y-pieces, generalized funnels and halfplanes.*

Remark 4.4. *If there are several freely homotopic minimizing simple closed geodesics which bound a generalized funnel, we will see in the proof that any of them can be chosen as border of this generalized funnel.*

Proof. We assume first that the fundamental group of S is finitely generated. If S has not generalized funnels, then Remark 4.2 gives the result. If S has generalized funnels $\{F_j\}$, then the closure of $S \setminus \cup_j F_j$ is a geodesic domain; Proposition 4.1 gives the result in this case.

Let us consider a surface S with infinitely generated fundamental group, and fix a point $p \in S$. Next, we will take an increasing sequence of positive numbers $\{r_n\}$ so that $\lim_{n\to\infty} r_n = \infty$. For each r_n we intend to associate a geodesic domain G_n to the closed ball $B(r_n)$ centered in p with radius r_n.

The boundary of $B(r)$ is a union of pairwise disjoint simple closed curves except for $r \in A$ with A a countable set. Since S is of infinite type, we can always find a positive number $r_1 \notin A$ such that the fundamental group of the ball $B(r_1)$ has, at least, two generators. We choose $r_n \notin A$ with $r_n > \max\{r_{n-1}, n\}$. As $r_n > r_1$, the fundamental group of $B(r_n)$ has, at least, two generators as well. Since $r_n \notin A$, the boundary of $B(r_n)$ is a union of pairwise disjoint simple closed curves $\{\eta_i^n\}_{i\in I_n}$. In order to construct its geodesic domain G_n, our goal is to relate a minimizing geodesic γ_i^n to each curve $\eta_i^n \subseteq \partial B(r_n)$, and we do it inductively as it follows. There are two possibilities:

(1) There does not exist any minimizing simple closed geodesic in $[\eta_i^n]$ or there exists $j \in I_n$ $(j \ne i)$ such that $\eta_j^n \in [\eta_i^n]$. In this case $\gamma_i^n := \varnothing$.
(2) There exists at least one minimizing simple closed geodesic in $[\eta_i^n]$ and there does not exist $j \in I_n$ $(j \ne i)$ such that $\eta_j^n \in [\eta_i^n]$. If $n > 1$ and there exists $j \in I_{n-1}$ such that $\gamma_j^{n-1} \in [\eta_i^n]$, then $\gamma_i^n := \gamma_j^{n-1}$. Otherwise (notice that this situation includes the case $n = 1$), choose as γ_i^n any of the minimizing simple closed geodesics in $[\eta_i^n]$.

G_n is the geodesic domain limited by all these geodesics $\{\gamma_i^n\}_{i\in I_n}$. By construction, $G_n \subseteq G_{n+1}$.

Before going on with the proof, we need the following lemma:

Lemma 4.5. *If there exists some positive number N such that γ is a minimizing simple closed geodesic contained in ∂G_n for every $n > N$, then γ is the border of a generalized funnel in S.*

Proof. For $n > N$, let us consider the simple closed curve $\eta_n \subseteq \partial B(r_n)$ which is freely homotopic to γ. Since $\lim_{n\to\infty} \operatorname{dist}(p, \eta_n) = \lim_{n\to\infty} r_n = \infty$, and η_n belongs to a single nontrivial freely homotopy class for every $n > N$, Theorem 2.4 gives that $\{\eta_n\}$ converges to a collared end F; since its border γ is a minimizing simple closed geodesic, F must be a generalized funnel. □

Now, let us continue with the proof of Theorem 4.3. We can take a subsequence of radii (by simplicity of the notation we will denote this subsequence just like the whole sequence) such that $G_n \subset G_{n+1}$ and besides, $\partial G_n \cap \partial G_{n+1}$ is either the empty set or a union of minimizing simple closed geodesics, each of them is the border of a generalized funnel.

Let us define H_n as the closed set obtained as the union of G_n and the generalized funnels whose borders belong to ∂G_n, and $H := \cup_n H_n$. Notice that due to the properties of G_n, each H_n is contained in the interior of H_{n+1}. If $S = H$, then there is nothing else to prove. Otherwise, $S \setminus H$ is a closed non-empty set.

By Proposition 4.1, each connected component of the closure of $G_{n+1} \setminus G_n$ is a finite union (with pairwise disjoint interiors) of generalized Y-pieces. Therefore, in order to finish the proof, we just have to see that every connected component J of $S \setminus H$ is a halfplane, that is to say: J is a simply connected set and $\partial J \subseteq \partial H$ is a unique nonclosed simple geodesic. From now on, by simplicity in the notation and as there is no possible confusion, we will denote $\gamma_i^n \subset \partial H_n$ by γ_n.

Next, we state a lemma that we will need along the proof:

Lemma 4.6. *Let σ be a nontrivial simple closed curve in S. If there is a minimizing simple closed geodesic $\gamma \in [\sigma]$ contained in $B(r_n)$, then either γ is contained in G_n or it is freely homotopic to some geodesic in ∂G_n.*

Proof. Let us assume that γ is not contained in G_n. Then, there are two possibilities: either $\gamma \cap G_n = \varnothing$ or $\gamma \cap \partial G_n \neq \varnothing$. In the first case, γ is contained in a doubly connected set whose borders are $\gamma_n \subset \partial G_n$ and $\eta_n \subset \partial B(r_n)$ and therefore γ is freely homotopic to both of them.

We finish the proof by showing that γ cannot intersect ∂G_n. Seeking for a contradiction, assume that $\gamma \cap \partial G_n \neq \varnothing$; then, $\gamma \cap \gamma_n \neq \varnothing$ for some minimizing closed geodesic $\gamma_n \subset \partial G_n$. Since γ and γ_n are minimizing simple closed geodesics, Theorem 3.11 implies that $\gamma \cap \eta_n \neq \varnothing$, where η_n is the closed curve in $\partial B(r_n)$ with $\gamma_n \in [\eta_n]$. This is the required contradiction, since $\gamma \subset B(r_n)$. □

Going on with proof of Theorem 4.3 we will see that J is simply connected. In order to do so, let us prove that every simple closed curve contained in J must be trivial, since every topological obstacle must be in H: Let us consider a nontrivial simple closed curve σ in J. By Proposition 3.2, there are two possibilities:

1. There exists a minimizing simple closed geodesic $\gamma \in [\sigma]$. Consider r_n with $\gamma \subset B(r_n)$; we prove now that $\gamma \subset H_{n+1}$. If γ is not contained in G_n, then by Lemma 4.6 it is freely homotopic to some geodesic in ∂G_n. If γ bounds a generalized funnel, then $\gamma \subset H_n$. If a curve in ∂G_n is freely homotopic to some curve in ∂G_{n+1} then it must bound a generalized funnel. Since γ does not bound a generalized funnel, it is not freely homotopic to any geodesic in ∂G_{n+1}; hence, Lemma 4.6 gives $\gamma \subset G_{n+1}$, since $\gamma \subset B(r_n) \subset B(r_{n+1})$. Hence, in any case, $\gamma \subset H_{n+1}$, and it is not freely homotopic to any curve in ∂H_{n+1}. Therefore, σ must intersect H_{n+1}, which is a contradiction with $\sigma \subset J$.
2. The curve σ bounds a generalized puncture. Then, there exists some neighborhood of this collared end contained in some H_n. Since σ is not freely homotopic to any geodesic in ∂H_n, σ must intersect H_n, which is again a contradiction with $\sigma \subset J$.

Next, we will prove that ∂J is a geodesic. Let us fix a point $q \in \partial J$; we want to prove that q belongs to a geodesic arc γ contained in ∂J: There exist points $q_n \in \gamma_n \subseteq \partial H_n$ converging to q. Let us consider now the sequence $\{v_n\}$ of tangent vectors to γ_n in q_n. Notice that this latest sequence must converge to a certain vector v, since otherwise the geodesics γ_n would intersect. As geodesics are solutions of a system of ordinary differential equations and the initial data $\{q_n, v_n\}$ converge to $\{q, v\}$, then the geodesics γ_n converge uniformly in some neighborhood U of q to the geodesic γ whose tangent vector in q is v.

Now, let us prove that $\gamma \cap U$ is contained in ∂J. Choose a point $q' \in \gamma \cap U$. On the one hand, $q' \notin \operatorname{ext} H$, since there exists a sequence of points in $\gamma_n \subset H_n$ converging to q'. On the other hand, q' does not belong to H either, since if it did, there would exist some H_m containing q' for some positive integer m, and this is a contradiction with H_n contained in the interior of H_{n+1} for every n.

From the previous argument we also deduce that this geodesic can be prolonged to infinity at both sides: if it had an endpoint p, it is obvious that $p \in \partial J$, but as we have just seen, for every point in the boundary of J there exists a neighborhood U such that $U \cap \partial J$ is a geodesic arc containing p.

In order to see that every connected component $\gamma \subseteq \partial J$ is simple, we will prove that it is not closed and it does not intersect itself transversally: If γ were a simple closed geodesic, it would be compact and as γ_n locally uniformly converge to γ then there would exist a positive integer N and a collar C for γ such that $\gamma_n \subset C$ for every $n \geq N$, and therefore $\gamma_n \in [\gamma]$, which is a contradiction. If γ intersected itself transversally, there would exist some positive integer N such that each γ_n would intersect itself as well for every $n \geq N$, and this is not possible since they are all simple. This same argument also proves that two different geodesics contained in ∂H must be simple and pairwise disjoint.

To finish, there is only one fact to prove: ∂J consists of just one geodesic. Let us assume that there exist two simple geodesics $\sigma_1, \sigma_2 \subset \partial J$. Let us consider two points $q_1 \in \sigma_1, q_2 \in \sigma_2$, two simple connected neighborhoods V_1, V_2 of q_1 and q_2 respectively, two simple closed geodesics $\gamma_{n_1} \subset \partial H_{n_1}, \gamma_{n_2} \subset \partial H_{n_2}$, with $\gamma_{n_1} \cap V_1 \neq \varnothing, \gamma_{n_2} \cap V_2 \neq \varnothing$ and $n_1 \neq n_2$, and curves $\eta_1 \subset V_1, \eta_2 \subset V_2$ joining, respectively, γ_{n_1} with q_1 and q_2 with γ_{n_2}. As J is path-connected, it is possible to construct the three following curves: $\eta_3 \subset J$ joining q_1 and q_2, $\eta := \eta_1 + \eta_3 + \eta_2$ and the closed curve $\beta := \eta + \gamma_{n_2} - \eta + \gamma_{n_1}$. Since β cannot bound a generalized puncture, every closed geodesic $\gamma \in [\beta]$ verifies $\gamma \cap \sigma_1 \neq \varnothing$ and $\gamma \cap \sigma_2 \neq \varnothing$; in particular, this means that every minimizing simple closed geodesic do intersect ∂J. But, by Lemma 4.6, there must exist a minimizing closed geodesic in $[\beta]$ entirely contained in H, which is a contradiction. □

In fact, the proof of Theorem 4.3 gives the following result.

Theorem 4.7. *Every complete orientable Riemannian surface which is neither simply nor doubly connected nor homeomorphic to a torus is the union (with pairwise disjoint interiors) of generalized funnels, halfplanes and a set G which can be exhausted by geodesic domains.*

The curvature of a Riemannian surface homeomorphic to a torus can not verify $K < 0$; then, Theorem 4.3, Lemma 3.9 and Lemma 3.10 give directly the following result.

Theorem 4.8. *Every complete orientable Riemannian surface with curvature $K \leq -c^2 < 0$, which is neither simply nor doubly connected is the union (with pairwise disjoint interiors) of generalized Y-pieces, funnels and halfplanes. Furthermore, every generalized puncture is a puncture.*

In order to deal with bordered surfaces, we need a last definition.

Definition 4.9. *A* finite cylinder *is a bordered Riemannian surface which is homeomorphic to $\mathbb{S}^1 \times [0, 1]$, whose border is the union of two simple closed geodesics, and at least one of them is minimizing.*

Theorem 4.10. *Every simple complete orientable bordered Riemannian surface which is neither simply nor doubly connected is the union (with pairwise disjoint interiors) of generalized Y-pieces, finite cylinders, generalized funnels and halfplanes.*

Furthermore, there is a bijection between finite cylinders in the decomposition and nonminimizing simple closed geodesics in the border.

Proof. Let S be a simple complete orientable bordered Riemannian surface which is not simply nor doubly connected, whose border is the union of simple closed geodesics $\{\gamma_i\}_{i\in I}$. By applying Lemma 3.3 we can construct another complete Riemannian surface R by gluing a neighborhood F_i of a collared end to each γ_i, such that $R = \cup_{i\in I} F_i \cup S$. It is obvious that R is not homeomorphic to a torus, since it is not compact.

By Theorem 4.3 we know that R is the union (with pairwise disjoint interiors) of generalized Y-pieces, generalized funnels and halfplanes. Furthermore, by Remark 4.4, if γ_i is a minimizing simple closed geodesic in S, for some $i \in I$, we can choose the decomposition in such a way that γ_i belongs to the border of a generalized funnel.

If γ_i is a nonminimizing simple closed geodesic in S for some $i \in I$, Lemma 3.3 ((7) and (2)) guarantees both that there exists the minimizing simple closed geodesic $\gamma_i^0 \in [\gamma_i]$ and that it is contained in S. Then, the funnel F_i in this decomposition intersects S in a finite cylinder whose border is $\gamma_i^0 \cup \gamma_i$.

Consequently, we obtain the desired decomposition in S if we restrict to S this decomposition in R. □

Theorem 4.11. *Every simple complete orientable bordered Riemannian surface with curvature* $K \le -c^2 < 0$*, is the union (with pairwise disjoint interiors) of generalized Y-pieces, funnels and halfplanes. Furthermore, every generalized puncture is a puncture.*

Proof. The proof follows the argument of the proof of Theorem 4.10, using Theorem 4.8, instead of Theorem 4.3.

Lemma 3.9 gives that in each free homotopy class there exists at most a closed geodesic. Consequently, there are not finite cylinders in the decomposition.

We only need to study the simple complete orientable bordered Riemannian surfaces with curvature $K \le -c^2 < 0$ which are simply or doubly connected.

Let S be such a surface. S can not be simply connected: Seeking for a contradiction, suppose that S is simply connected; then ∂S can be considered a geodesic triangle with three angles equal to π, and Gauss-Bonnet Formula gives

$$-\iint_S K\, dA = \pi - \pi - \pi - \pi = -2\pi\,,$$

which is a contradiction with $K < 0$.

If S is doubly connected, then Lemma 3.9 gives that ∂S is a single simple closed geodesic and besides it is minimizing. Then, S is a funnel. □

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