WEIERSTRASS' THEOREM WITH WEIGHTS

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ABSTRACT

We characterize the set of functions which can be approximated by continuous functions in the L^∞ norm with respect to almost every weight. This allows to characterize the set of functions which can be approximated by polynomials or by smooth functions for a wide range of weights.

Key Words: Weierstrass' theorem; weight.

1. INTRODUCTION

If I is any compact interval, Weierstrass' Theorem says that $C(I)$ is the largest set of functions which can be approximated by polynomials in the norm $L^\infty(I)$, if we identify, as usual, functions which are equal almost everywhere. There are many generalizations of this theorem (see e.g. the monographs [L], [P], and the references therein).

Our goal is to study the polynomial approximation of functions with the norm $L^\infty(w)$ defined by

$$\|f\|_{L^\infty(w)} := \operatorname{ess\,sup} |f(x)| w(x)\,, \tag{1.1}$$

where w is a weight, i.e. a non-negative measurable function, and we follow the convention $0 \cdot \infty = 0$. Notice that (1.1) is not the usual definition of the L^∞ norm in the context of measure theory, although it is the correct one when working with weights (see e.g. [BO] and [DMS]).

One of the authors studied this problem in [R1], in the case of bounded weights. In the current paper we obtain several improvements of the results in [R1], and besides we manage with general unbounded weights. If w is not bounded, then the polynomials are not in $L^\infty(w)$, in general. Therefore, it is natural to bear in mind the problem of approximation by functions in $C(\mathbf{R})$ or $C^\infty(\mathbf{R})$. An important tool which allows to improve the results in [R1] is a lemma (see Lemma 2.4 in Section 2) which deals with the regularity of functions near the "worst" points of w (in this lemma we study all bad points simultaneously). Another key idea is using covering lemmas similar to the ones in Harmonic Analysis (see Section 3).

Now, let us state the main result. It characterizes the functions which can be approximated by continuous functions, smooth functions or polynomials. Our hypothesis about the weight is not restrictive at all: although we have tried, we have not been able to construct any weight which does not fulfill such condition. We refer to the definitions in the next section.

Theorem 2.1. *Let w be an admissible weight and*

$$\begin{aligned} H_0 := \big\{ f \in L^\infty(w) : \; & f \text{ is continuous to the right at every point of } R^+, \\ & f \text{ is continuous to the left at every point of } R^-, \\ & \text{for each } a \in S^+, \;\; \operatorname*{ess\,lim}_{x\to a^+} |f(x) - f(a)|\, w(x) = 0\,, \\ & \text{for each } a \in S^-, \;\; \operatorname*{ess\,lim}_{x\to a^-} |f(x) - f(a)|\, w(x) = 0 \,\big\}\,. \end{aligned}$$

Then:

(a) *The closure of $C(\mathbf{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is H_0.*

(*b*) *If* $w \in L^\infty_{loc}(\mathbf{R})$*, then the closure of* $C^\infty(\mathbf{R}) \cap L^\infty(w)$ *in* $L^\infty(w)$ *is also* H_0.

(*c*) *If* $\operatorname{supp} w$ *is compact and* $w \in L^\infty(\mathbf{R})$*, then the closure of the space of polynomials is* H_0 *as well.*

(*d*) *If* $f \in H_0 \cap L^1(\operatorname{supp} w)$*,* $S_1^+ \cup S_2^+ \cup S_1^- \cup S_2^-$ *is countable and* $|S| = 0$*, then* f *can be approximated by functions in* $C(\mathbf{R})$ *with the norm* $\| \cdot \|_{L^\infty(w)} + \| \cdot \|_{L^1(\operatorname{supp} w)}$.

If w is not bounded, we can also characterize the completion of smooth functions and polynomials.

Theorem 2.2. *Let us consider a weight* w *with compact support. If* $p_w \equiv 0$*, then the closure of the space of polynomials in* $L^\infty(w)$ *is* $\{0\}$*. If* p_w *is not identically* 0*, the closure of the space of polynomials in* $L^\infty(w)$ *is the set of functions* f *such that* f/p_w *is in the closure of the space of polynomials in* $L^\infty(|p_w|w)$.

The weight $|p_w|w$ is bounded (since $p_w \in L^\infty(w)$) and has compact support; therefore, if $|p_w|w$ is admissible, then by Theorem 2.1 we know which is the closure of the space of polynomials in $L^\infty(|p_w|w)$.

Theorem 2.3. *Let us consider a weight* w *such that there exists a minimal function* f_w *for* w*. Then the closure of* $C^\infty(\mathbf{R})$ *in* $L^\infty(w)$ *is the set of functions* f *such that* f/f_w *is in the closure of* $C^\infty(\mathbf{R})$ *in* $L^\infty(|f_w|w)$.

The weight $|f_w|w$ is locally bounded (since $f_w \in L^\infty_{loc}(w)$); therefore, if $|f_w|w$ is admissible, then by Theorem 2.1 we know which is the closure of $C^\infty(\mathbf{R})$ in $L^\infty(|f_w|w)$.

The simultaneous approximation with the norm $\|\cdot\|_{L^\infty(w)} + \|\cdot\|_{L^1(\operatorname{supp} w)}$ is an important tool to deal with the problem of approximation in weighted Sobolev spaces $W^{k,\infty}(w_0, w_1, \dots, w_k)$. Consequently, Theorem 2.1 is key to characterize the functions which can be approximated by smooth functions or polynomials, in $W^{k,\infty}(w_0, w_1, \dots, w_k)$ (see [PQRT1] and [PQRT2]).

The analogue of Weierstrass' Theorem with the norms $W^{k,p}(\mu_0, \mu_1, \dots, \mu_k)$ (with $1 \le p < \infty$) can be found in [RARP1], [RARP2], [R3]; [APPR] and [RY] deal with the case of curves in the complex plane instead of intervals. The results for $p = 2$ have important consequences in the study of Sobolev orthogonal polynomials (see [LP], [LPP] and [R2]).

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2. APPROXIMATION IN $L^\infty(w)$

Let us start with some definitions.

Definition 2.1. A *weight* w is a measurable function $w : \mathbf{R} \longrightarrow [0, \infty]$. If w is only defined in $A \subset \mathbf{R}$, we set $w := 0$ in $\mathbf{R} \setminus A$.

Definition 2.2. Given a measurable set $A \subset \mathbf{R}$ and a weight w, we define the space $L^\infty(A, w)$ as the space of equivalence classes of measurable functions $f : A \longrightarrow \mathbf{R}$ with respect to the norm

$$\|f\|_{L^\infty(A,w)} := \operatorname*{ess\,sup}_{x \in A} |f(x)| w(x) \,.$$

The main results in this paper can be applied to functions f with complex values, splitting f into its real and imaginary parts. From now on, if we do not specify the set A, we are assuming that $A = \mathbf{R}$; analogously, if we do not make explicit the weight w, we are assuming that $w \equiv 1$.

Let A be a measurable subset of $\mathbf{R}$; we always consider the space $L^1(A)$ with respect to the restriction of the Lebesgue measure on A.

Definition 2.3. Given a measurable set A, we define the *essential closure* of A, as the set

$$\operatorname{ess\,cl} A := \big\{ x \in \mathbf{R} : \ |A \cap (x - \delta, x + \delta)| > 0, \ \ \forall \, \delta > 0 \big\} \,,$$

where $|E|$ denotes the Lebesgue measure of the set E.

Definition 2.4. If A is a measurable set, f is a function defined on A with real values and $a \in \operatorname{ess\,cl} A$, we say that $\operatorname{ess\,lim}_{x \in A,\, x \to a} f(x) = l \in \mathbf{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for almost every $x \in A \cap (a - \delta, a + \delta)$. In a similar way we can define $\operatorname{ess\,lim}_{x \in A,\, x \to a} f(x) = \infty$ and $\operatorname{ess\,lim}_{x \in A,\, x \to a} f(x) = -\infty$. We define the *essential superior limit* and the *essential inferior limit* in A as follows:

$$\operatorname*{ess\,lim\,sup}_{x \in A,\, x \to a} f(x) := \inf_{\delta > 0} \operatorname*{ess\,sup}_{x \in A \cap (a - \delta, a + \delta)} f(x) \,,$$
$$\operatorname*{ess\,lim\,inf}_{x \in A,\, x \to a} f(x) := \sup_{\delta > 0} \operatorname*{ess\,inf}_{x \in A \cap (a - \delta, a + \delta)} f(x) \,.$$

If we do not specify the set A, we are assuming that $A = \mathbf{R}$.

Remarks.

1. The essential superior (or inferior) limit of a function f does not change if we modify f on a set of zero Lebesgue measure.

2. It is well known that

$$\operatorname*{ess\,lim\,sup}_{x \in A,\, x \to a} f(x) \geq \operatorname*{ess\,lim\,inf}_{x \in A,\, x \to a} f(x) \,,$$
$$\operatorname*{ess\,lim}_{x \in A,\, x \to a} f(x) = l \quad \text{ if and only if } \quad \operatorname*{ess\,lim\,sup}_{x \in A,\, x \to a} f(x) = \operatorname*{ess\,lim\,inf}_{x \in A,\, x \to a} f(x) = l \,.$$

3. We impose the condition $a \in \operatorname{ess\,cl} A$ in order to have the unicity of the essential limit. If $a \notin \operatorname{ess\,cl} A$, then every real number is an essential limit for any function f.

Definition 2.5. Given a weight w, the *support* of w, denoted by $\operatorname{supp} w$, is the complement of the greatest open set $G \subset \mathbf{R}$ with $w = 0$ a.e. on G.

It is clear that $\operatorname{supp} w = \operatorname{ess\,cl}\{x \in \mathbf{R} : w(x) > 0\}$. It is also clear that $L^\infty(w) = L^\infty(\operatorname{supp} w, w)$. Since obviously $\operatorname{ess\,cl}(\operatorname{ess\,cl} A) = \operatorname{ess\,cl} A$ and $\operatorname{supp} w = \operatorname{ess\,cl}\{x \in \mathbf{R} : w(x) > 0\}$, it follows that $\operatorname{supp} w = \operatorname{ess\,cl}(\operatorname{supp} w)$. This fact allows to state the following definition.

Definition 2.6. Given a weight w we say that $a \in \operatorname{supp} w$ is a *singularity* of w (or *singular* for w) if

$$\operatorname*{ess\,lim\,inf}_{x\in \operatorname{supp} w,\, x\to a} w(x) = 0\,.$$

We say that a singularity a of w is of *type* 1 if $\operatorname{ess\,lim}_{x\to a} w(x) = 0$.

We say that a singularity a of w is of *type* 2 if $0 < \operatorname{ess\,lim\,sup}_{x\to a} w(x) < \infty$.

We say that a singularity a of w is of *type* 3 if $\operatorname{ess\,lim\,sup}_{x\to a} w(x) = \infty$.

We denote by S and S_i $(i = 1, 2, 3)$, respectively, the set of singularities of w and the set of singularities of w of type i.

We say that $a \in S_i^+$ (respectively $a \in S_i^-$) if a verifies the property in the definition of S_i when we take the limit as $x \to a^+$ (respectively $x \to a^-$). We define $S^+ := S_1^+ \cup S_2^+ \cup S_3^+$ and $S^- := S_1^- \cup S_2^- \cup S_3^-$.

Remark. The sets S and S_3 are closed subsets of $\operatorname{supp} w$.

The current definition of singular point is much more restrictive than the one in [R1]. Consequently, the set of singular points is smaller than in [R1] (recall that $S \subseteq \operatorname{supp} w$; this does not hold with the definition in [R1]): if we consider, for example, a Cantor set $C \subset [0, 1]$ of positive length and take w as the characteristic function of C, we have $S = \emptyset$; however, with the definition of [R1], the set of singular points would be $\mathbf{R}$. This fact is crucial, since singular points make our work more difficult.

Definition 2.7. Given a weight w, we define the *right regular* and *left regular* points of w, respectively, as

$$R^+ := \Big\{a \in \operatorname{supp} w : \operatorname*{ess\,lim\,inf}_{x\in \operatorname{supp} w,\, x\to a^+} w(x) > 0\Big\}\,, \qquad R^- := \Big\{a \in \operatorname{supp} w : \operatorname*{ess\,lim\,inf}_{x\in \operatorname{supp} w,\, x\to a^-} w(x) > 0\Big\}\,.$$

Remark. Notice that $R^+ \cup S_1^+ \cup S_2^+ \cup S_3^+ = \operatorname{supp} w = R^- \cup S_1^- \cup S_2^- \cup S_3^-$.

Definition 2.8. Given a weight w and $\varepsilon > 0$, we define $A_\varepsilon := \{x \in \operatorname{supp} w : w(x) \ge \varepsilon\}$ and $A_\varepsilon^c := \operatorname{supp} w \setminus A_\varepsilon$.

We collect here some useful technical results which were proved in [R1].

Lemma A ([R1, Lemma 2.4]). *If A is a measurable set, we have:*

(1) $\operatorname{ess\,cl} A$ *is a closed set contained in* $\overline{A}$.

(2) $|A \setminus \operatorname{ess\,cl} A| = 0$.

(3) *If f is a measurable function in $A \cup \operatorname{ess\,cl} A$, $a \in \operatorname{ess\,cl} A$ and there exists* $\operatorname{ess\,lim}_{x\in \operatorname{ess\,cl} A,\, x\to a} f(x)$, *then there exists* $\operatorname{ess\,lim}_{x\in A,\, x\to a} f(x)$ *and*

$$\operatorname*{ess\,lim}_{x\in A,\, x\to a} f(x) = \operatorname*{ess\,lim}_{x\in \operatorname{ess\,cl} A,\, x\to a} f(x)\,.$$

(4) *If $|A| > 0$ and f is a continuous function in $\mathbf{R}$ we have*

$$\|f\|_{L^\infty(A)} = \sup_{x\in \operatorname{ess\,cl} A} |f(x)|\,.$$

Lemma B ([R1, Lemma 2.2]). *Let us consider a weight w and $a \in S_1$. Then, every function f in the closure of $C(\mathbf{R}) \cap L^\infty(w)$ with the norm $L^\infty(w)$ verifies*

$$\operatorname*{ess\,lim}_{x\in \operatorname{supp} w,\, x\to a} f(x)\, w(x) = 0\,.$$

Remark. A similar result is true if $a \in S_1^+$ or $a \in S_1^-$.

Lemma C ([R1, Lemma 2.6]). *Let us consider a weight w and $a \in S$. Then, every function f in the closure of $C(\mathbf{R}) \cap L^\infty(w)$ with the norm $L^\infty(w)$ verifies*

$$\inf_{\varepsilon>0} \Big(\operatorname*{ess\,lim\,sup}_{x\in A_\varepsilon^c,\, x\to a} |f(x)|\, w(x) \Big) = 0\,.$$

Lemma D ([R1, Lemma 2.7]). *Let us consider a weight w and $a \in S_1$. If*

$$\inf_{\varepsilon>0} \Big(\operatorname*{ess\,lim\,sup}_{x\in A_\varepsilon^c,\, x\to a} |f(x)|\, w(x) \Big) = 0\,,$$

then we have $\operatorname{ess\,lim}_{x\in \operatorname{supp} w, x\to a} f(x)\, w(x) = 0$.

Remark. A similar result is true if $a \in S_1^+$ or $a \in S_1^-$.

Lemmas B, C and D were proved in [R1] with x in some interval, instead of $x \in \operatorname{supp} w$. However the same proof is still valid.

Next, let us prove some technical lemmas.

Lemma 2.1. *Let us consider a weight w and $a \in \operatorname{supp} w$. If* $\operatorname{ess\,lim\,sup}_{x\in \operatorname{supp} w,\, x\to a} w(x) = l \in (0, \infty]$, *then for every function f in the closure of $C(\mathbf{R}) \cap L^\infty(w)$ with the norm $L^\infty(w)$, we have that*

$$\operatorname*{ess\,lim}_{x\in A_\varepsilon,\, x\to a} f(x) = f(a)\,, \qquad \text{for every} \quad 0 < \varepsilon < l\,.$$

Furthermore $f \in \cap_{\varepsilon>0} C(\operatorname{ess\,cl} A_\varepsilon)$; in particular, f is continuous to the right at each point of R^+ and continuous to the left at each point of R^-.

Remark. Notice that the functions in $L^\infty(w)$ are defined in $\operatorname{supp} w$; therefore, the continuity is referred to this set. Recall that we identify functions which are equal almost everywhere.

Proof. We have for every $\delta > 0$

$$\operatorname*{ess\,sup}_{x \in \operatorname{supp} w \cap (a-\delta, a+\delta)} w(x) \ge l > 0 \,,$$

and then

$$\left| \left\{ x \in \operatorname{supp} w \cap (a-\delta, a+\delta) : \, w(x) \ge \varepsilon \right\} \right| > 0 \,,$$

for every $\delta > 0$ and $0 < \varepsilon < l$. This implies that a belongs to $\operatorname{ess\,cl} A_\varepsilon$, for every $0 < \varepsilon < l$.

If $g \in C(\mathbf{R}) \cap L^\infty(w)$, $0 < \varepsilon < l$ and $\delta > 0$, we have

$$\varepsilon \, \|g\|_{L^\infty(A_\varepsilon \cap [a-\delta, a+\delta])} \le \|g\|_{L^\infty(A_\varepsilon \cap [a-\delta, a+\delta], w)} \,.$$

Since $\operatorname{ess\,cl}(A_\varepsilon \cap [a-\delta, a+\delta])$ is a compact set and $g \in C(\mathbf{R})$, Lemma A (4) gives

$$\varepsilon \cdot \max_{x \in \operatorname{ess\,cl}(A_\varepsilon \cap [a-\delta, a+\delta])} |g(x)| \le \|g\|_{L^\infty(A_\varepsilon \cap [a-\delta, a+\delta], w)} \,.$$

Consequently, if $\{g_n\} \subset C(\mathbf{R}) \cap L^\infty(w)$ converges to f in $L^\infty(w)$, then $\{g_n\}$ converges to f uniformly in $\operatorname{ess\,cl}(A_\varepsilon \cap [a-\delta, a+\delta])$ and $f \in C(\operatorname{ess\,cl}(A_\varepsilon \cap [a-\delta, a+\delta]))$ for every $\delta > 0$. Therefore $f \in C(\operatorname{ess\,cl} A_\varepsilon)$ for every $\varepsilon > 0$. This fact and Lemma A (3) give that, for $0 < \varepsilon < l$, there exists

$$\operatorname*{ess\,lim}_{x \in A_\varepsilon ,\, x \to a} f(x) = \operatorname*{ess\,lim}_{x \in \operatorname{ess\,cl} A_\varepsilon ,\, x \to a} f(x) = \lim_{x \in \operatorname{ess\,cl} A_\varepsilon ,\, x \to a} f(x) = f(a) \,.$$

If $y \in R^+$, then there exists $\varepsilon, \delta > 0$ with $\operatorname{ess\,inf}_{x \in \operatorname{supp} w \cap (y, y+\delta)} w(x) \ge \varepsilon$, and consequently $\operatorname{supp} w \cap [y, y+\delta] \subseteq \operatorname{ess\,cl} A_\varepsilon$. This fact and $f \in C(\operatorname{ess\,cl} A_\varepsilon)$ give that f is continuous to the right at y. If $y \in R^-$, a similar argument allows us to conclude that f is continuous to the left at y.

Definition 2.9. We say that a function g *preserves the continuity of* f if g is continuous to the right at every point in which f is continuous to the right, and g is continuous to the left at every point in which f is continuous to the left.

It is obvious that if g preserves the continuity of f, then g is continuous at every point in which f is continuous.

Lemma 2.2. *Let us consider a weight w. Assume that $a \in S_1^+$ and $a \in \overline{(a, \infty) \setminus S}$. Then, for any fixed $\eta > 0$ and $f \in C(\operatorname{supp} w \setminus S) \cap L^\infty(w)$ with*

$$\inf_{\varepsilon > 0} \Big(\operatorname*{ess\,lim\,sup}_{x \in A_\varepsilon^c ,\, x \to a^+} |f(x)| \, w(x) \Big) = 0 \,,$$

there exist $b \in (a, a+1) \setminus S$ and a function $g \in L^\infty(w) \cap C([a, b])$, preserving the continuity of f, such that $g = f$ in $\operatorname{supp} w \setminus [a, b)$, $\|f - g\|_{L^\infty(w)} < \eta$ (and $\|f - g\|_{L^1(\operatorname{supp} w)} < \eta$ if $f \in L^1(\operatorname{supp} w)$). Furthermore, if f is not continuous to the left at a, g can be chosen with the additional condition $g(a) = 0$ or even $g(a) = \lambda$ for any fixed $\lambda \in \mathbf{R}$.

Remark. A similar result is true if $a \in S_1^-$ and $a \in \overline{(-\infty, a) \setminus S}$.

Proof. Since $a \in \overline{(a,\infty) \setminus S}$ and $(a,\infty) \setminus S$ is an open set, there exist intervals $[y_n^1, y_n] \subset (a, a+1/n) \setminus S$, for each n. We assume first that we can choose $[y_n^1, y_n] \subset \operatorname{supp} w$, for every n. Choosing y_n smaller if it is necessary, we can assume that there exist $\varepsilon_n > 0$ with $[y_n^1, y_n + \varepsilon_n] \subset \operatorname{supp} w \cap ((a, a+1/n) \setminus S)$, for every n; this fact and the last statement of Lemma 2.1 give that $f \in C([y_n^1, y_n + \varepsilon_n])$.

Let us assume that $f(y_n) > 0$. Consider the convex hull C of the set $\{(x,y) \in \mathbf{R}^2 : x \in [y_n^1, y_n] \text{ and } y \geq f(x)\}$. Since $f \in C([y_n^1, y_n])$, we have that $\partial C \setminus (\{x = y_n^1, y > f(y_n^1)\} \cup \{x = y_n, y > f(y_n)\})$ is the graph of a convex function $H_n \in C([y_n^1, y_n])$ with $H_n(y_n^1) = f(y_n^1)$ and $H_n(y_n) = f(y_n)$. Then, we can find a function $h_n \in C([a, y_n])$ with $|h_n| \leq |f|$ and $\operatorname{sgn} h_n = \operatorname{sgn} f$ if $h_n \neq 0$ in $[y_n^1, y_n]$, $h_n(y_n) = f(y_n)$ and $h_n = 0$ in $[a, y_n^1]$: If $H_n(t) = 0$ for some $t \in [y_n^1, y_n)$, we can choose $h_n = 0$ in $[a, t]$ and $h_n = H_n$ in $[t, y_n]$; if $H_n > 0$ in $[y_n^1, y_n]$, we can choose $h_n = 0$ in $[a, s]$ (with $s \in [y_n^1, y_n)$), $h_n = H_n$ in $[t, y_n]$ (with $t \in (s, y_n)$), and h_n a straight line in $[s, t]$.

If $f(y_n) < 0$, we can construct h_n in a similar way. If $f(y_n) = 0$, we can take $h_n = 0$.

If we can not find $[y_n^1, y_n] \subset \operatorname{supp} w$, for every n, then there exist intervals $(y_n, z_n) \subset (a, a+1/n) \setminus \operatorname{supp} w$, for each n, since $(a, a+1/n) \setminus \operatorname{supp} w$ is an open set. Furthermore, we can choose $y_n \in \operatorname{supp} w$ for every n, since $a \in S_1^+$. We define $h_n := 0$ in $[a, y_n]$.

Let us define now the function f_n as

$$f_n(x) := \begin{cases} h_n(x)\,, & \text{if } x \in [a, y_n]\,, \\ f(x)\,, & \text{if } x \in \operatorname{supp} w \setminus [a, y_n]\,. \end{cases}$$

Let us remark that f_n is continuous in $[a, y_n]$ and preserves the continuity of f, except perhaps at $x = a$.

Notice that $|f_n| \leq |f|$ and $\operatorname{sgn} f_n = \operatorname{sgn} f$ if $f_n \neq 0$, in $[a, y_n] \cap \operatorname{supp} w$. Hence

$$\|f - f_n\|_{L^\infty(w)} = \|f - f_n\|_{L^\infty([a,y_n],w)} \leq \|f\|_{L^\infty([a,y_n],w)}\,,$$

and this last expression goes to 0 as $n \to \infty$, since $\operatorname{ess\,lim}_{x \in \operatorname{supp} w,\, x \to a^+} f(x)\, w(x) = 0$, as a consequence of the remark to Lemma D. If $f \in L^1(\operatorname{supp} w)$, we also have

$$\|f - f_n\|_{L^1(\operatorname{supp} w)} = \|f - f_n\|_{L^1([a,y_n] \cap \operatorname{supp} w)} \leq \|f\|_{L^1([a,y_n] \cap \operatorname{supp} w)}\,,$$

and this expression goes to 0 as $n \to \infty$. Notice that $f_n(a) = 0$; it is easy to modify f_n in a small right neighborhood of a in order to have $f_n(a) = \lambda$, for fixed $\lambda \in \mathbf{R}$, since $a \in S_1^+$. We take $\lambda = \operatorname{ess\,lim}_{x \in \operatorname{supp} w, x \to a^-} f(x)$ if this limit exists; then f_n preserves the continuity of f. This finishes the proof of the lemma.

Lemma 2.3. *Let us consider a weight w. Assume that $a \in S_2^+$ and $a \in \overline{(a,\infty) \setminus S}$. Let us fix $\eta > 0$ and $f \in C(\operatorname{supp} w \setminus S) \cap L^\infty(w)$ such that*

(a) $\inf_{\varepsilon>0}\big(\operatorname{ess\,lim\,sup}_{x\in A_\varepsilon^c,\,x\to a^+}|f(x)|\,w(x)\big)=0$,

(b) $\operatorname{ess\,lim}_{x\in A_\varepsilon,\,x\to a^+} f(x)=f(a)$, *for every* $\varepsilon>0$ *small enough.*

Then, there exist $b\in(a,a+1)\setminus S$ *and a function* $g\in L^\infty(w)\cap C([a,b])$, *preserving the continuity of* f, *with* $g=f$ *in* $\operatorname{supp} w\setminus(a,b)$, $\|f-g\|_{L^\infty(w)}<\eta$ *(and* $\|f-g\|_{L^1(\operatorname{supp} w)}<\eta$ *if* $f\in L^1(\operatorname{supp} w)$*).*

Remark. A similar result is true if $a\in S_2^-$ and $a\in\overline{(-\infty,a)\setminus S}$.

Proof. For each natural number n, let us choose $\varepsilon_n>0$ with $\lim_{n\to\infty}\varepsilon_n=0$ and

$$\operatorname*{ess\,lim\,sup}_{x\in A_{\varepsilon_n}^c,\,x\to a^+}|f(x)|\,w(x)<\frac{1}{n}\,.$$

Let us consider now $0<\delta_n<1$ with $\lim_{n\to\infty}\delta_n=0$ and

$$\operatorname*{ess\,sup}_{x\in(a,a+\delta_n)\cap A_{\varepsilon_n}^c}|f(x)|\,w(x)<\frac{1}{n}\,. \tag{2.1}$$

We can take δ_n with the additional property $|f(x)-f(a)|<1/n$ for almost every $x\in(a,a+\delta_n)\cap A_{\varepsilon_n}$.

Since $a\in\overline{(a,\infty)\setminus S}$ and $(a,\infty)\setminus S$ is an open set, there exist intervals $[y_n^1,y_n]\subset(a,a+\delta_n)\setminus S$, for each n. We assume first that we can choose $[y_n^1,y_n]\subset\operatorname{supp} w$, for every n. Choosing y_n smaller if it is necessary, we can assume that there exist $\varepsilon_n>0$ with $[y_n^1,y_n+\varepsilon_n]\subset\operatorname{supp} w\cap((a,a+\delta_n)\setminus S)$, for every n; this fact and the last statement of Lemma 2.1 give that $f\in C([y_n^1,y_n+\varepsilon_n])$.

Let us assume that $f(y_n)>f(a)$. We consider the convex hull C of the set $\{(x,y)\in\mathbf{R}^2/\,x\in[y_n^1,y_n]$ and $y\ge f(x)\}$. Since $f\in C([y_n^1,y_n])$, we have that $\partial C\setminus(\{x=y_n^1,y>f(y_n^1)\}\cup\{x=y_n,y>f(y_n)\})$ is the graph of a convex function $H_n\in C([y_n^1,y_n])$ with $H_n(y_n^1)=f(y_n^1)$ and $H_n(y_n)=f(y_n)$. Then, as in the proof of Lemma 2.2, we can find a function $h_n\in C([a,y_n])$ with $|h_n-f(a)|\le|f-f(a)|$ and $\operatorname{sgn}(h_n-f(a))=\operatorname{sgn}(f-f(a))$ if $h_n\ne f(a)$ in $[y_n^1,y_n]$, $h_n(y_n)=f(y_n)$ and $h_n=f(a)$ in $[a,y_n^1]$.

If $f(y_n)<f(a)$, we can construct h_n in a similar way. If $f(y_n)=f(a)$, we can take $h_n=f(a)$.

If we can not find $[y_n^1,y_n]\subset\operatorname{supp} w$, for every n, then there exist intervals $(y_n,z_n)\subset(a,a+1/n)\setminus\operatorname{supp} w$, for each n, since $(a,a+1/n)\setminus\operatorname{supp} w$ is an open set. Furthermore, we can choose $y_n\in\operatorname{supp} w$ for every n, since $a\in S_1^+$. We define $h_n:=f(a)$ in $[a,y_n]$.

Let us define now the function f_n as

$$f_n(x):=\begin{cases} h_n(x)\,, & \text{if } x\in[a,y_n]\,,\\ f(x)\,, & \text{if } x\in\operatorname{supp} w\setminus[a,y_n]\,.\end{cases}$$

Let us remark that f_n is continuous in $[a,y_n]$ and preserves the continuity of f.

Notice that $|f_n-f(a)|\le|f-f(a)|$ and $\operatorname{sgn}(f_n-f(a))=\operatorname{sgn}(f-f(a))$ if $f_n\ne f(a)$, in $[a,y_n]\cap\operatorname{supp} w$. Recall that $|f(x)-f(a)|<1/n$ for almost every $x\in[a,y_n]\cap A_{\varepsilon_n}$. Hence

$$\|f-f_n\|_{L^\infty([a,y_n]\cap A_{\varepsilon_n},w)}\le 2\|f-f(a)\|_{L^\infty([a,y_n]\cap A_{\varepsilon_n},w)}\le\frac{2}{n}\,\|w\|_{L^\infty([a,y_n])}\,. \tag{2.2}$$

Notice that $\|w\|_{L^\infty([a,y_n])}$ is uniformly bounded for n large enough, since $a \in S_2^+$.

Inequality (2.1) gives

$$\|f - f_n\|_{L^\infty([a,y_n]\cap A^c_{\varepsilon_n},w)} \le 2\|f - f(a)\|_{L^\infty([a,y_n]\cap A^c_{\varepsilon_n},w)} \le 2\|f\|_{L^\infty([a,y_n]\cap A^c_{\varepsilon_n},w)} + 2|f(a)|\varepsilon_n < \frac{2}{n} + 2|f(a)|\varepsilon_n \,.$$

This inequality and (2.2) give

$$\|f - f_n\|_{L^\infty([a,y_n],w)} < \frac{2}{n} + 2|f(a)|\varepsilon_n + \frac{2}{n}\,\|w\|_{L^\infty([a,y_n])} \,.$$

If $f \in L^1(\operatorname{supp} w)$, we also have

$$\|f - f_n\|_{L^1(\operatorname{supp} w)} = \|f - f_n\|_{L^1([a,y_n]\cap \operatorname{supp} w)} \le 2\|f - f(a)\|_{L^1([a,y_n]\cap \operatorname{supp} w)} \,.$$

This finishes the proof.

Lemma 2.4. *Let us consider a weight w, and subsets $T^+ \subseteq S^+ \setminus S_1^+$ and $T^- \subseteq S^- \setminus S_1^-$. Let us take $f \in L^\infty(w)$ such that for every $a \in T^+$,*

(*a*1) $\inf_{\varepsilon>0}\big(\operatorname{ess\,lim\,sup}_{x\in A^c_\varepsilon,\, x\to a^+} |f(x)|\, w(x)\big) = 0$,

(*b*1) $\operatorname{ess\,lim}_{x\in A_\varepsilon,\, x\to a^+} f(x) = f(a) = 0$, *for every $\varepsilon > 0$ small enough,*

and for every $a \in T^-$,

(*a*2) $\inf_{\varepsilon>0}\big(\operatorname{ess\,lim\,sup}_{x\in A^c_\varepsilon,\, x\to a^-} |f(x)|\, w(x)\big) = 0$,

(*b*2) $\operatorname{ess\,lim}_{x\in A_\varepsilon,\, x\to a^-} f(x) = f(a) = 0$, *for every $\varepsilon > 0$ small enough.*

Then, for each $\eta > 0$, there exists a function $g \in L^\infty(w)$ which preserves the continuity of f, is continuous to the right at every point of T^+ and is continuous to the left at every point of T^-, with $\|f - g\|_{L^\infty(w)} \le \eta$ (and $\|f - g\|_{L^1(\operatorname{supp} w)} \le \eta$ if $f \in L^1(\operatorname{supp} w)$ and $|T^+ \cup T^-| = 0$). Furthermore, we have $g = f = 0$ in $T^+ \cup T^-$.

Remark. If $f \in L^\infty(w)$, $\operatorname{ess\,lim}_{x\in A_\varepsilon,\, x\to a^+} f(x) = f(a)$ for every $\varepsilon > 0$ small enough, and $a \in S_3^+$, then $\operatorname{ess\,lim\,sup}_{x\to a^+} w(x) = \infty$ and $\operatorname{ess\,lim}_{x\in A_\varepsilon,\, x\to a^+} f(x) = 0$. A similar result is true for $a \in S_3^-$.

Notice that this result allows to manage simultaneously every point of $S_3^+ \cup S_3^-$, in opposition to lemmas 2.2 and 2.3, which deal only with one point of $S_1^+ \cup S_1^-$ and $S_2^+ \cup S_2^-$.

Proof. The heart of the proof is to modify f in a sequential way; in each step we obtain a smaller function near the points in $S_3^+ \cup S_3^-$.

Fix $\eta > 0$. Conditions (a1) and (b1) give that for any $a \in T^+$ there exist $\varepsilon^+_{a,1}, \delta^+_{a,1} > 0$, such that

$$|f(x)|w(x) < \eta/2\,, \qquad \text{for } a.e.\, x \in [a, a+\delta^+_{a,1}] \cap A^c_{\varepsilon^+_{a,1}}\,,$$
$$|f(x)| < \eta/2\,, \qquad \text{for } a.e.\, x \in [a, a+\delta^+_{a,1}] \cap A_{\varepsilon^+_{a,1}}\,,$$

and $|f(a+\delta^+_{a,1})|<\eta/2$.

In a similar way, for any $a\in T^-$, there exist $\varepsilon^-_{a,1},\delta^-_{a,1}>0$, such that

$$\begin{aligned}|f(x)|w(x)&<\eta/2\,, &\quad \text{for } a.e.\, x\in[a-\delta^-_{a,1},a]\cap A^c_{\varepsilon^-_{a,1}}\,,\\ |f(x)|&<\eta/2\,, &\quad \text{for } a.e.\, x\in[a-\delta^-_{a,1},a]\cap A_{\varepsilon^-_{a,1}}\,,\end{aligned}$$

and $|f(a-\delta^-_{a,1})|<\eta/2$.

If $T_1:=\big\{\big(\cup_{a\in T^+}[a,a+\delta^+_{a,1}]\big)\cup\big(\cup_{a\in T^-}[a-\delta^-_{a,1},a]\big)\big\}\cap\operatorname{supp}w$, and $T_1^c:=\operatorname{supp}w\setminus T_1$, we define

$$g_1(x):=\begin{cases}\max\big\{\min\big\{f(x),\eta/2\big\},-\eta/2\big\}\,, & \text{if } x\in T_1\,,\\ f(x)\,, & \text{if } x\in T_1^c\,.\end{cases}$$

From the definition of $\delta^+_{a,1},\delta^-_{a,1}$, it follows that g_1 preserves the continuity of f: Let us assume that f is continuous to the right at x; if there exists $\varepsilon>0$ with $[x,x+\varepsilon)\cap\operatorname{supp}w\subseteq T_1$ or $[x,x+\varepsilon)\cap\operatorname{supp}w\subseteq T_1^c$, the result is clear; if there exists $\varepsilon>0$ with $(x,x+\varepsilon)\cap\operatorname{supp}w\subseteq T_1^c$ and $x\in T_1$, then $|f(x)|<\eta/2$ and $g_1=f$ in $[x,x+\varepsilon)\cap\operatorname{supp}w$ (if $x=a+\delta^+_{a,1}$, then $|f(x)|<\eta/2$; if $x=a$, then $f(x)=0$); otherwise, there exists a decreasing sequence $\{x_n\}$ converging to x with $|f(x_n)|<\eta/2$, which implies $|f(x)|\le\eta/2$ and, therefore, $g_1(x)=f(x)$; on the one hand, if $g_1(y)=f(y)$, then $|g_1(y)-g_1(x)|=|f(y)-f(x)|$ and on the other hand, there exists $\varepsilon>0$ with $|g_1(y)-g_1(x)|<|f(y)-f(x)|$ for $y\in[x,x+\varepsilon)\cap\operatorname{supp}w$ if $g_1(y)\neq f(y)$. These facts give $|g_1(y)-g_1(x)|\le|f(y)-f(x)|$ for $y\in[x,x+\varepsilon)\cap\operatorname{supp}w$. If f is continuous to the left at x, the argument is similar.

We also have $|g_1|\le|f|$ and $\operatorname{sgn}g_1=\operatorname{sgn}f$. These facts imply that

$$\begin{aligned}\|f-g_1\|_{L^\infty(w)}&=\max\Big\{\sup_{a\in T^+}\|f-g_1\|_{L^\infty([a,a+\delta^+_{a,1}],w)},\sup_{a\in T^-}\|f-g_1\|_{L^\infty([a-\delta^-_{a,1},a],w)}\Big\}\\&=\max\Big\{\sup_{a\in T^+}\|f-g_1\|_{L^\infty([a,a+\delta^+_{a,1}]\cap A^c_{\varepsilon^+_{a,1}},w)},\sup_{a\in T^-}\|f-g_1\|_{L^\infty([a-\delta^-_{a,1},a]\cap A^c_{\varepsilon^-_{a,1}},w)}\Big\}\\&\le\max\Big\{\sup_{a\in T^+}\|f\|_{L^\infty([a,a+\delta^+_{a,1}]\cap A^c_{\varepsilon^+_{a,1}},w)},\sup_{a\in T^-}\|f\|_{L^\infty([a-\delta^-_{a,1},a]\cap A^c_{\varepsilon^-_{a,1}},w)}\Big\}\\&\le\eta/2\,.\end{aligned}$$

We define g_n inductively. Conditions $(a1)$ and $(b1)$ give that for any $a\in T^+$ there exist $0<\varepsilon^+_{a,n}\le\varepsilon^+_{a,n-1}$, $0<\delta^+_{a,n}\le\delta^+_{a,n-1}$, such that

$$\begin{aligned}|f(x)|w(x)&<\eta/2^n\,, &\quad \text{for } a.e.\, x\in[a,a+\delta^+_{a,n}]\cap A^c_{\varepsilon^+_{a,n}}\,,\\ |f(x)|&<\eta/2^n\,, &\quad \text{for } a.e.\, x\in[a,a+\delta^+_{a,n}]\cap A_{\varepsilon^+_{a,n}}\,,\end{aligned}$$

and $|f(a+\delta^+_{a,n})|<\eta/2^n$.

Conditions $(a2)$ and $(b2)$ give that for any $a\in T^-$ there exist $0<\varepsilon^-_{a,n}\le\varepsilon^-_{a,n-1}$, $0<\delta^-_{a,n}\le\delta^-_{a,n-1}$, such that

$$\begin{aligned}|f(x)|w(x)&<\eta/2^n\,, &\quad \text{for } a.e.\, x\in[a-\delta^-_{a,n},a]\cap A^c_{\varepsilon^-_{a,n}}\,,\\ |f(x)|&<\eta/2^n\,, &\quad \text{for } a.e.\, x\in[a-\delta^-_{a,n},a]\cap A_{\varepsilon^-_{a,n}}\,,\end{aligned}$$

and $|f(a-\delta^-_{a,n})| < \eta/2^n$.

If $T_n := \big\{ \big(\cup_{a\in T^+} [a, a+\delta^+_{a,n}] \big) \cup \big(\cup_{a\in T^-} [a-\delta^-_{a,n}, a] \big) \big\} \cap \operatorname{supp} w$, and $T_n^c := \operatorname{supp} w \setminus T_n$, we can define

$$g_n(x) := \begin{cases} \max\big\{ \min\{ g_{n-1}(x), \eta/2^n \}, -\eta/2^n \big\}, & \text{if } x \in T_n\,, \\ g_{n-1}(x)\,, & \text{if } x \in T_n^c\,. \end{cases}$$

From the definition of $\delta^+_{a,n}, \delta^-_{a,n}$, it follows that g_n preserves the continuity of g_{n-1} and, in particular, of f. We also have $|g_n| \le |g_{n-1}| \le |f|$ and $\operatorname{sgn} g_n = \operatorname{sgn} g_{n-1} = \operatorname{sgn} f$. These facts imply that

$$\begin{aligned} \|g_n - g_{n-1}\|_{L^\infty(w)} &= \max\Big\{ \sup_{a\in T^+} \|g_n - g_{n-1}\|_{L^\infty([a,a+\delta^+_{a,n}],w)}, \sup_{a\in T^-} \|g_n - g_{n-1}\|_{L^\infty([a-\delta^-_{a,n},a],w)} \Big\} \\ &= \max\Big\{ \sup_{a\in T^+} \|g_n - g_{n-1}\|_{L^\infty([a,a+\delta^+_{a,n}]\cap A^c_{\varepsilon^+_{a,n}},w)}, \sup_{a\in T^-} \|g_n - g_{n-1}\|_{L^\infty([a-\delta^-_{a,n},a]\cap A^c_{\varepsilon^-_{a,n}},w)} \Big\} \\ &\le \max\Big\{ \sup_{a\in T^+} \|g_{n-1}\|_{L^\infty([a,a+\delta^+_{a,n}]\cap A^c_{\varepsilon^+_{a,n}},w)}, \sup_{a\in T^-} \|g_{n-1}\|_{L^\infty([a-\delta^-_{a,n},a]\cap A^c_{\varepsilon^-_{a,n}},w)} \Big\} \\ &\le \eta/2^n. \end{aligned}$$

Notice that $\|g_n - g_{n-1}\|_{L^\infty(\operatorname{supp} w)} \le \eta/2^n$, since $T_n \subseteq T_{n-1}$. Recall that, for any measurable set $A \subseteq \mathbf{R}$, $L^\infty(A)$ denotes the standard L^∞ space in A with weight equal to 1.

Since $\{|g_n(x)|\}_n$ is decreasing in n, and $\operatorname{sgn} g_n = \operatorname{sgn} f$, we have that $g_n(x)$ converges to some $g(x)$ at every $x \in \operatorname{supp} w$. If $m < n$, we obtain that

$$\|g_n - g_m\|_{L^\infty(w)} \le \eta/2^n + \cdots + \eta/2^{m+1} \le \eta/2^m, \qquad \|g_n - g_m\|_{L^\infty(\operatorname{supp} w)} \le \eta/2^n + \cdots + \eta/2^{m+1} \le \eta/2^m.$$

Therefore $\{g_n\}$ is a Cauchy sequence in $L^\infty(w)$ and $L^\infty(\operatorname{supp} w)$; it follows that $\{g_n\}$ converges to g both in $L^\infty(w)$ and $L^\infty(\operatorname{supp} w)$.

Then $\|f-g\|_{L^\infty(w)} \le \sum_{n=1}^\infty \eta/2^n = \eta$ and g preserves the continuity of f. If $a \in T^+$, given any $\varepsilon > 0$, we can choose n with $\eta/2^n < \varepsilon$; then $|g(x)| \le |g_n(x)| \le \eta/2^n < \varepsilon$ for every $x \in [a, a+\delta^+_{a,n}] \cap \operatorname{supp} w$. In particular, $g(a) = 0$, and hence g is continuous to the right at a. A similar argument gives that $g = 0$ and g is continuous to the left at every point of T^-.

If $f \in L^1(\operatorname{supp} w)$, then there exists $\delta > 0$ such that $\int_E |f| < \eta$ for every measurable set $E \subseteq \operatorname{supp} w$ with $|E| < \delta$. If $|T^+ \cup T^-| = 0$, we can choose $\delta^-_{a,1}, \delta^+_{a,1}$ with the additional property $|T_1| < \delta$. Then $\|f-g\|_{L^1(\operatorname{supp} w)} \le \|f\|_{L^1(T_1)} < \eta$.

Definition 2.10. A weight w is said to be *admissible* if $a \in \overline{(a,\infty)\setminus S}$ for any $a \in S_1^+ \cup S_2^+$, and $a \in \overline{(-\infty,a)\setminus S}$ for any $a \in S_1^- \cup S_2^-$.

In order to characterize the functions which can be approximated in $L^\infty(w)$ by continuous functions, our argument requires that w is admissible. This hypothesis is very weak; in fact, it is difficult to find a non-admissible weight. For a weight to be non-admissible there must exist a whole interval contained in S. In particular, any weight with $|S| = 0$ (for example, of finite total variation) is admissible. Any weight which is equal a.e. to a lower semi-continuous function is admissible; in particular, if there exist pairwise disjoint open intervals $\{I_n\}$ with $w \in C(I_n)$ and $|\operatorname{supp} w \setminus \cup_n I_n| = 0$, then w is admissible. Next, we give an example of Miguel Jiménez of a non-admissible weight; we reproduce it with his kind permission.

Example. Hereby we construct a bounded weight w on $[0,1]$, whose support is the whole interval, with essential inferior limit 0 at every point of the interval of definition and that is not equal 0 almost everywhere. This example is easily extended to the real line as a 1-periodic function.

Express the set of rational numbers lying in $(0,1)$ in form of a sequence $\{r_k\}$, $k = 1, 2, ...$ Define $Y_{k,n} := \left(r_k - 1/2^{n+k+1}, r_k + 1/2^{n+k+1}\right) \cap (0,1)$, $n = 1, 2, ...$ and $Z_n := \bigcup_{k=1}^{\infty} Y_{k,n}$. Then $\{Z_n\}_n$ is a sequence of open sets in $(0,1)$, whose lengths decrease to zero. Define $X_n := [0,1] \setminus Z_n$. Then $\{X_n\}_n$ is a sequence of closed sets in $[0,1]$ whose lengths increase to 1. Set g_n as the characteristic function of the set X_n and $f_n := \sum_{j=1}^{n} g_j/j^2$.

The following properties can be verified without any trouble: $\{f_n\}_n$ is an increasing sequence of positive functions that converges uniformly to a function w on $[0,1]$. The function w is a weight bounded by $\sum_n 1/n^2$. The support of f_n is the set X_n and since the lengths of X_n increase to 1, the support of w is $[0,1]$. For every n and every $x \in [0,1]$, the essential inferior limit of f_n at x is 0. Since $w - f_n \leq 1/n^2$ uniformly, the weihgt w has this same property at x. Finally neither f_n nor w are reduced to 0 almost everywhere.

Notice that this concept of admissible weights is different from the one in [APRR], [RARP1], [RARP2], [R1], [R2], [R3] and [RY].

Proposition 2.1. *If w is an admissible weight, then the closure of $C(\mathbf{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is*

$$\begin{aligned} H := \Big\{ f \in L^\infty(w) : & \; f \text{ is continuous to the right in every point of } R^+, \\ & f \text{ is continuous to the left in every point of } R^-, \\ & \text{for each } a \in S^+, \; \inf_{\varepsilon>0} \Big(\operatorname*{ess\,lim\,sup}_{x\in A_\varepsilon^c,\, x\to a^+} |f(x)|\, w(x) \Big) = 0 \text{ and}, \\ & \text{if } a \notin S_1^+, \; \operatorname*{ess\,lim}_{x\in A_\varepsilon,\, x\to a^+} f(x) = f(a), \text{ for any } \varepsilon > 0 \text{ small enough}, \\ & \text{for each } a \in S^-, \; \inf_{\varepsilon>0} \Big(\operatorname*{ess\,lim\,sup}_{x\in A_\varepsilon^c,\, x\to a^-} |f(x)|\, w(x) \Big) = 0 \text{ and}, \\ & \text{if } a \notin S_1^-, \; \operatorname*{ess\,lim}_{x\in A_\varepsilon,\, x\to a^-} f(x) = f(a), \text{ for any } \varepsilon > 0 \text{ small enough} \Big\}. \end{aligned}$$

If $w \in L^\infty_{loc}(\mathbf{R})$, then the closure of $C^\infty(\mathbf{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is also H. Besides, if $\operatorname{supp} w$ *is compact and $w \in L^\infty(\mathbf{R})$, then the closure of the polynomials is H as well.*

Furthermore, if $f \in H \cap L^1(\operatorname{supp} w)$, $S_1^+ \cup S_2^+ \cup S_1^- \cup S_2^-$ is countable and $|S| = 0$, then f can be approximated by functions in $C(\mathbf{R})$ with the norm $\|\cdot\|_{L^\infty(w)} + \|\cdot\|_{L^1(\operatorname{supp} w)}$.

Remark. Recall that we identify functions which are equal almost everywhere.

Proof. Lemmas 2.1 and C give that H contains $\overline{C(\mathbf{R})\cap L^\infty(w)}$. In order to see that H is contained in $\overline{C(\mathbf{R})\cap L^\infty(w)}$, let us fix $f\in H$ and $\varepsilon>0$.

Lemmas 2.2, 2.3 and 2.4 are the keys in order to obtain a continuous function which approximates f; we only need to paste them in a precise way and in an appropriate order. Another important ingredient in the proof is a covering lemma (Theorem 3.1) which is proved in Section 3, in order to make this proof clearer.

If we apply Lemma 2.4 with $T^+ := S_3^+$ and $T^- := S_3^-$, we obtain a function $g_1\in L^\infty(w)$ which preserves the continuity of f, is continuous to the right at every point of S_3^+ and is continuous to the left at every point of S_3^-, with $\|f-g_1\|_{L^\infty(w)}<\varepsilon/3$ (and $\|f-g_1\|_{L^1(\operatorname{supp} w)}<\varepsilon/3$ if $f\in L^1(\operatorname{supp} w)$, since $|S_3^+\cup S_3^-|=|S|=0$). Recall that $g_1(a)=0$ for every $a\in S_3^+\cup S_3^-$.

Since w is admissible, lemmas 2.2 and 2.3 give that for each $a\in S_3^-\cap(S_1^+\cup S_2^+)$ there exist $b_a\in(a,a+1)\setminus S$ and a function $g_a\in L^\infty(w)\cap C([a,b_a])$, preserving the continuity of g_1, with $g_a=g_1$ in $\operatorname{supp} w\setminus(a,b_a)$, $\|g_1-g_a\|_{L^\infty(w)}<\varepsilon/3$. We define in this case $U_a:=(a,b_a)$. Without loss of generality, we can assume that there are no points of S_3 in U_a, since $\operatorname{ess\,lim\,sup}_{x\to a^+} w(x)<\infty$ implies that w is essentially bounded in a right neighborhood of a.

In a similar way, for each $a\in S_3^+\cap(S_1^-\cup S_2^-)$ there exist $b_a\in(a-1,a)\setminus S$ and a function $g_a\in L^\infty(w)\cap C([b_a,a])$, preserving the continuity of g_1, with $g_a=g_1$ in $\operatorname{supp} w\setminus(b_a,a)$, $\|g_1-g_a\|_{L^\infty(w)}<\varepsilon/3$. We define in this case $U_a:=(b_a,a)$ and we also have $S_3\cap U_a=\emptyset$.

Let us define $A:=(S_3^-\cap(S_1^+\cup S_2^+))\cup(S_3^+\cap(S_1^-\cup S_2^-))$. Since we have $S_3\cap(\cup_{a\in A}U_a)=\emptyset$, we deduce that any U_a intersects at most another neighborhood U_α (in this case, one of them is a right neighborhood and the another one is a left neighborhood). Then, without loss of generality, we can assume that $\{U_a\}_{a\in A}$ are pairwise disjoint (if this was not so, smaller neighborhoods can be taken). This fact implies that A is a countable set, and we can write $A=\cup_n a_n$. Then lemmas 2.2 and 2.3 guarantee that we can choose g_{a_n} with $\|g_1-g_{a_n}\|_{L^1(\operatorname{supp} w)}<2^{-n}\varepsilon/3$ if $f\in L^1(\operatorname{supp} w)$.

We define the function g_2 as

$$g_2(x):=\begin{cases} g_a(x)\,, & \text{if } x\in U_a \text{ for some } a\in A\,,\\ g_1(x)\,, & \text{in other case}\,.\end{cases}$$

We have that $\|f-g_2\|_{L^\infty(w)}<2\varepsilon/3$ (and $\|f-g_2\|_{L^1(\operatorname{supp} w)}<2\varepsilon/3$ if $f\in L^1(\operatorname{supp} w)$).

It is clear that g_2 is continuous in $\operatorname{supp} w$ except perhaps at the points of the set $B:=((S_1^+\cup S_2^+)\setminus S_3^-)\cup((S_1^-\cup S_2^-)\setminus S_3^+)$. Lemmas 2.2 and 2.3 guarantee that for each $a\in B$ there exist $0<r_1(a),r_2(a)<1$ and a function g_a such that, if we define $U_a:=(a-r_1(a),a+r_2(a))$, then $g_a\in L^\infty(w)\cap C(\overline{U_a})$, g_a preserves the continuity of g_2, $g_a=g_2$ in $\operatorname{supp} w\setminus U_a$, and $\|g_2-g_a\|_{L^\infty(w)}<\varepsilon/6$ (if $a\in B\cap R^-$, we take $g_a=g_2$ in $(a-r_1(a),a)$, i.e. g_2 remains unchanged on the left-hand side of the left regular points; if $a\in B\cap R^+$, we take $g_a = g_2$ in $(a, a + r_2(a)))$. Notice that, as in the construction of g_2, we can assume that there are no points of S_3 in $(a - r_1(a), a + r_2(a))$.

Next, let us prove that $r_1(a)$ and $r_2(a)$ can be chosen such that $20/21 \le r_1(a)/r_2(a) \le 21/20$: This is obvious if $r_1(a) = r_2(a)$. Then, without loss of generality, we can assume that $r_1(a) < r_2(a)$; if $a+r_1(a) \notin S$, using lemmas 2.2 and 2.3, we can obtain another approximation h_a of g_2 in the interval $(a-r_1(a), a+r_1(a))$; if $a+r_1(a) \in S$, then $a+r_1(a) \notin S_3^+ \cup S_3^-$, and there is a point $a+r_3(a) \notin S$ as close as we want to $a+r_1(a)$, since w is admissible; then we can obtain another approximation h_a of g_2 in the interval $(a-r_1(a), a+r_3(a))$.

Since $\{U_a\}_{a\in B}$ is an open covering of B, Theorem 3.1 in the next section guarantees that there exists a sequence $\{a_n\} \subset B$ such that $B \subset \cup_n U_{a_n}$, each U_{a_n} intersects at most two U_{a_m}'s, and no U_{a_n} is contained in another U_{a_m}. Consequently, the intersection of two intervals does not meet another interval, i.e. $U_{a_i} \cap U_{a_j} \cap (\cup_{k\neq i,j} U_{a_k}) = \emptyset$.

Let us define $[\alpha_n, \beta_n] := \overline{U}_{a_n}$. Assume that $U_{a_i} \cap U_{a_j} \neq \emptyset$, with $\alpha_i < \alpha_j$; then $\overline{U}_{a_i} \cap \overline{U}_{a_j} = [\alpha_j, \beta_i]$ and $[\alpha_j, \beta_i] \cap U_{a_k} = \emptyset$ for every $k \neq i, j$. We define the functions

$$g_{a_j,a_i}(x) := g_{a_i,a_j}(x) := \frac{\beta_i - x}{\beta_i - \alpha_j}\, g_{a_i}(x) + \frac{x - \alpha_j}{\beta_i - \alpha_j}\, g_{a_j}(x)\,.$$

Notice that $g_{a_i,a_j} \in C([\alpha_j, \beta_i])$ and satisfies $g_{a_i,a_j}(\alpha_j) = g_{a_i}(\alpha_j)$, $g_{a_i,a_j}(\beta_i) = g_{a_j}(\beta_i)$, and

$$\|g_{a_j,a_i} - g_2\|_{L^\infty([\alpha_j,\beta_i],w)} \le \left\| \frac{\beta_i - x}{\beta_i - \alpha_j}\, \big(g_{a_i}(x) - g_2(x)\big) \right\|_{L^\infty([\alpha_j,\beta_i],w)} + \left\| \frac{x - \alpha_j}{\beta_i - \alpha_j}\, \big(g_{a_j}(x) - g_2(x)\big) \right\|_{L^\infty([\alpha_j,\beta_i],w)} < \frac{\varepsilon}{3}.$$

If we define the function g_3 as

$$g(x) := \begin{cases} g_2(x)\,, & \text{if } x \in \operatorname{supp} w \setminus \cup_n U_{a_n}\,, \\ g_{a_i}(x)\,, & \text{if } x \in U_{a_i},\ x \notin \cup_{m\neq i} U_{a_m}\,, \\ g_{a_i,a_j}(x)\,, & \text{if } x \in U_{a_i} \cap U_{a_j}\,, \end{cases}$$

then g_3 is a continuous function in $\operatorname{supp} w$, $\|g_2 - g_3\|_{L^\infty(w)} \le \varepsilon/3$ and $\|f - g_3\|_{L^\infty(w)} < \varepsilon$.

If $f \in L^1(\operatorname{supp} w)$ and B is countable, we can obtain also $\|g_2 - g_3\|_{L^1(\operatorname{supp} w)} < \varepsilon/3$ (in the same way that we obtain the L^1 approximation for g_2), and then $\|f - g_3\|_{L^1(\operatorname{supp} w)} < \varepsilon$.

It is easy to choose a function $g \in L^\infty(w) \cap C(\mathbf{R})$ with $g = g_3$ in $\operatorname{supp} w$. Let us define $g := g_3$ in $\operatorname{supp} w$; then $g \in C(\operatorname{supp} w)$. Since $\operatorname{supp} w$ is a closed set, the complement of $\operatorname{supp} w$ is a countable union of pairwise disjoint open intervals $\mathbf{R} \setminus \operatorname{supp} w = \cup_n (\alpha_n, \beta_n)$. If (α_n, β_n) is bounded, then $\alpha_n, \beta_n \in \operatorname{supp} w$, and we define g in this interval as the function whose graph is the segment joining $(\alpha_n, g_3(\alpha_n))$ with $(\beta_n, g_3(\beta_n))$; if $(\alpha_n, \beta_n) = (-\infty, \beta_n)$ for some n, then $\beta_n \in \operatorname{supp} w$, and we define $g := g_3(\beta_n)$ in this interval; if $(\alpha_n, \beta_n) = (\alpha_n, \infty)$ for some n, then $\alpha_n \in \operatorname{supp} w$, and we define $g := g_3(\alpha_n)$ in this interval. It is clear that this function is continuous in $\mathbf{R}$.

If $\operatorname{supp} w$ is compact and $w \in L^\infty(\mathbf{R})$, the closure of the polynomials is H as well, as a consequence of the classical Weierstrass' Theorem.

If $w \in L^\infty_{loc}(\mathbf{R})$, we split $\mathbf{R}$ into intervals $\mathbf{R} = \cup_{n\in\mathbf{Z}}[2n-1, 2n+2]$. For each $\varepsilon > 0$, there exists $g_n \in C^\infty([2n-1, 2n+2])$ (in fact, we can take g_n as a polynomial) with $\|f - g_n\|_{L^\infty([2n-1,2n+2],w)} < 2^{-|n|-2}\varepsilon$.

Let us consider a partition of unity $\{\phi_n\}$ satisfying: $\sum_{n\in\mathbf{Z}} \phi_n = 1$ in $\mathbf{R}$, $\phi_n|_{[2n,2n+1]} \equiv 1$, $0 \le \phi_n \le 1$ and $\phi_n \in C_c^\infty((2n-1, 2n+2))$. Notice that $g_n\phi_n \in C_c^\infty(\mathbf{R})$; hence the function $g := \sum_n g_n\phi_n$ belongs to $C^\infty(\mathbf{R})$ (since the sum is locally finite) and satisfies

$$\|f - g\|_{L^\infty(w)} = \Big\| f\sum_n \phi_n - \sum_n g_n\phi_n \Big\|_{L^\infty(w)} \le \sum_n \|(f-g_n)\phi_n\|_{L^\infty(w)} < \sum_n 2^{-|n|-2}\varepsilon < \varepsilon \,.$$

We can reformulate Proposition 2.1 as follows:

Theorem 2.1. *Let w be an admissible weight and*

$$\begin{aligned} H_0 := \big\{ f \in L^\infty(w) : & \ f \ \text{ is continuous to the right in every point of } R^+, \\ & f \ \text{ is continuous to the left in every point of } R^-, \\ & \text{for each } a \in S^+, \ \operatorname*{ess\,lim}_{x\to a^+} |f(x) - f(a)|\, w(x) = 0\,, \\ & \text{for each } a \in S^-, \ \operatorname*{ess\,lim}_{x\to a^-} |f(x) - f(a)|\, w(x) = 0 \,\big\}\,. \end{aligned}$$

Then:

(a) The closure of $C(\mathbf{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is H_0.

(b) If $w \in L^\infty_{loc}(\mathbf{R})$, then the closure of $C^\infty(\mathbf{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is also H_0.

(c) If $\operatorname{supp} w$ is compact and $w \in L^\infty(\mathbf{R})$, then the closure of the polynomials is H_0 as well.

(d) If $f \in H_0 \cap L^1(\operatorname{supp} w)$, $S_1^+ \cup S_2^+ \cup S_1^- \cup S_2^-$ is countable and $|S| = 0$, then f can be approximated by functions in $C(\mathbf{R})$ with the norm $\|\cdot\|_{L^\infty(w)} + \|\cdot\|_{L^1(\operatorname{supp} w)}$.

This result improves Theorem 2.1 in [R1], since we remove the hypothesis $w \in L^\infty$. Furthermore, the set of singular points is much smaller than in [R1], since $S \subseteq \operatorname{supp} w$ (see the comment after Definition 2.6). Finally, the hypothesis $|S| = 0$ in [R1] is replaced by the weaker condition of w to be admissible.

Proof. We only need to show the equivalence of the following conditions (a) and (b):

(a) for each $a \in S^+$,

(a.1) $\inf_{\varepsilon>0} \big(\operatorname{ess\,lim\,sup}_{x\in A_\varepsilon^c,\, x\to a^+} |f(x)|\, w(x) \,\big) = 0$,

(a.2) if $a \notin S_1^+$, $\operatorname{ess\,lim}_{x\in A_\varepsilon,\, x\to a^+} f(x) = f(a)$, for $\varepsilon > 0$ small enough,

(b) for each $a \in S^+$, $\operatorname{ess\,lim}_{x\in \operatorname{supp} w,\, x\to a^+} |f(x) - f(a)|\, w(x) = 0$.

(It is direct that (b) is equivalent to $\operatorname{ess\,lim}_{x\to a^+} |f(x) - f(a)|\, w(x) = 0$ for each $a \in S^+$, since $w(x) = 0$ for a.e.$x \notin \operatorname{supp} w$.)

The equivalence of (a) and (b) when $a \in S^-$ is similar.

It is clear that (b) implies (a). Hypothesis (a.1) gives that for each $\eta > 0$, there exist $\varepsilon, \delta > 0$ with $\|f\|_{L^\infty([a,a+\delta]\cap A_\varepsilon^c,w)} < \eta/3$ and $|f(a)|\varepsilon < \eta/3$. By hypothesis (a.2) we can choose δ with the additional condition $\|f - f(a)\|_{L^\infty([a,a+\delta]\cap A_\varepsilon,w)} < \eta/3$. These inequalities imply

$$\|f - f(a)\|_{L^\infty([a,a+\delta],w)} \le \|f\|_{L^\infty([a,a+\delta]\cap A_\varepsilon^c,w)} + |f(a)|\,\varepsilon + \|f - f(a)\|_{L^\infty([a,a+\delta]\cap A_e,w)} < \eta\,.$$

Now we deal with the approximation by polynomials and smooth functions.

Definition 2.11. Given a weight w with compact support, a polynomial $p \in L^\infty(w)$ is said to be *a minimal polynomial for w* if every polynomial in $L^\infty(w)$ is a multiple of p. A minimal polynomial for w is said to be *the minimal polynomial for w* (and we denote it by p_w) if it is 0 or it is monic.

It is clear that there always exists a minimal polynomial for w (although it can be 0): it is sufficient to consider a polynomial in $L^\infty(w)$ of minimal degree. Minimal polynomials for w are unique except for a constant factor; this fact allows to define p_w.

Let us remark that $p_w = 0$ if and only if the unique polynomial in $L^\infty(w)$ is 0.

Theorem 2.2. *Let us consider a weight w with compact support. If $p_w \equiv 0$, then the closure of the space of polynomials in $L^\infty(w)$ is $\{0\}$. If p_w is not identically 0, the closure of the space of polynomials in $L^\infty(w)$ is the set of functions f such that f/p_w is in the closure of the space of polynomials in $L^\infty(|p_w|w)$.*

Remark. The weight $|p_w|w$ is bounded (since $p_w \in L^\infty(w)$) and has compact support. Then we know which is the closure of the space of polynomials in $L^\infty(|p_w|w)$ by Theorem 2.1 (notice that $|p_w|w$ is admissible if w is admissible).

Proof. The first statement is clear, since $p_w = 0$ if and only if the unique polynomial in $L^\infty(w)$ is 0.

We prove now the second statement. First, let us assume that f/p_w is in the closure of the space of polynomials in $L^\infty(|p_w|w)$. Let us choose a sequence of polynomials $\{q_n\}$ with $\|f/p_w - q_n\|_{L^\infty(|p_w|w)} < 1/n$. We have that $\|f - p_w q_n\|_{L^\infty(w)} = \|f/p_w - q_n\|_{L^\infty(|p_w|w)} < 1/n$. Consequently, f belongs to the closure of the space of polynomials in $L^\infty(w)$.

Let us assume now that f/p_w is not in the closure of the space of polynomials in $L^\infty(|p_w|w)$. Then there exists a constant $c > 0$ with $\|f/p_w - p\|_{L^\infty(|p_w|w)} \ge c$ for every polynomial p and, consequently, $\|f - p_w p\|_{L^\infty(w)} = \|f/p_w - p\|_{L^\infty(|p_w|w)} \ge c$ for every polynomial p. Since every polynomial $q \in L^\infty(w)$ can be written as $q = p_w p$ for some polynomial p, we have that f can not be approximated by polynomials in $L^\infty(w)$.

Definition 2.12. Given a weight w, we define the set $T := \{a \in \mathbf{R} :\ \operatorname{ess\,lim\,sup}_{x\to a} w(x) = \infty\} \subset \operatorname{supp} w$.

Let us remark that T is a closed set.

Definition 2.13. Given a weight w, a function $f_w \in C^\infty(\mathbf{R}) \cap L^\infty_{loc}(w)$ is said to be a *minimal function for* w if every function $f \in C^\infty(\mathbf{R}) \cap L^\infty(w)$ can be written as $f = f_w\, g$, with $g \in C^\infty(\mathbf{R})$.

It is clear that minimal functions for w are unique except for a multiplication by a function in $C^\infty(\mathbf{R})$ without zeroes. It is also clear that a minimal function f_w verifies $f_w(x) = 0$ if and only if $x \in T$.

Notice that $\mathbf{R} \setminus T$ is an open nonvoid set, since the case $w \equiv \infty$ is excluded; then there exists some function in $C^\infty(\mathbf{R}) \cap L^\infty(w)$. Consequently, it is not possible that f_w be identically zero.

The same proof of Theorem 2.2, using a minimal function instead of the minimal polynomial, gives the following result.

Theorem 2.3. *Let us consider a weight w such that there exists a minimal function f_w for w. Then the closure of $C^\infty(\mathbf{R})$ in $L^\infty(w)$ is the set of functions f such that f/f_w is in the closure of $C^\infty(\mathbf{R})$ in $L^\infty(|f_w|w)$.*

Remark. The weight $|f_w|w$ is locally bounded (since $f_w \in L^\infty_{loc}(w)$). Then we know by Theorem 2.1, which is the closure of $C^\infty(\mathbf{R})$ in $L^\infty(|f_w|w)$, if $|f_w|w$ is admissible.

In order to use Theorem 2.3 we need a minimal function for w. Let us face the problem of constructing such a minimal function.

Definition 2.14. Given a weight w, a function f_w is said to be a *local minimal function for w at $a \in T$* if $f_w \in C^\infty((a-\varepsilon, a+\varepsilon)) \cap L^\infty((a-\varepsilon, a+\varepsilon), w)$ for some $\varepsilon > 0$, and every function $f \in C^\infty((a-\varepsilon, a+\varepsilon)) \cap L^\infty((a-\varepsilon, a+\varepsilon), w)$ can be written as $f = f_w\, g$, with $g \in C^\infty((a-\varepsilon, a+\varepsilon))$.

It is clear that f_w is a local minimal function for w in a if and only if there exists $\varepsilon > 0$ such that f_w is a minimal function for $w\, \chi_{(a-\varepsilon, a+\varepsilon)}$, where χ_B denotes the characteristic function of the set B.

Proposition 2.2. *Let us consider a weight w. If T is discrete and for every point $a \in T$ there exists a local minimal function $f_{w,a}$ for w in a, then there exists a minimal function f_w for w with $f_w = f_{w,a}$ in a neighborhood of a, for every $a \in T$.*

Proof. Since T is closed and discrete, there is no accumulation point of T; then $T = \{a_n\}_{n\in\Lambda}$, with Λ equal to $\mathbf{Z}$, $\mathbf{Z}^+$, or a finite set, and $\{a_n\}_{n\in\Lambda}$ is a monotonous sequence. Let us consider $\varepsilon_n^0 > 0$, the constant appearing in the definition of local minimal function for f_{w,a_n}. There exists $0 < \varepsilon_n < \varepsilon_n^0$ such that $\{(a_n - \varepsilon_n, a_n + \varepsilon_n)\}_{n\in\Lambda}$ are pairwise disjoint. Let us consider $\phi_n \in C_c^\infty((a_n - \varepsilon_n, a_n + \varepsilon_n))$ with $0 \le \phi_n \le 1$ and $\phi_n = 1$ in $(a_n - \varepsilon_n/2, a_n + \varepsilon_n/2)$; we define also $\phi = 1 - \sum_{n\in\Lambda} \phi_n$.

We show now that $f_w = \phi + \sum_{n\in\Lambda} \phi_n f_{w,a_n}$ is a minimal function for w. Notice first that $f_w = f_{w,a_n}$ in $(a_n - \varepsilon_n/2, a_n + \varepsilon_n/2)$; then, $f_w \in C^\infty(\mathbf{R}) \cap L^\infty_{loc}(w)$, since $w, f_w \in L^\infty_{loc}(\mathbf{R} \setminus \cup_{n\in\Lambda}(a_n - \varepsilon_n/2, a_n + \varepsilon_n/2))$.

Let us consider $f \in C^\infty(\mathbf{R}) \cap L^\infty(w)$. We only need to show that $f/f_w = f/(\phi + \sum_{n\in\Lambda} \phi_n f_{w,a_n}) \in C^\infty(\mathbf{R})$. This function is smooth at every point of $\mathbf{R} \setminus T$, since it is the quotient of two smooth functions with non-vanishing denominator. Notice that $f/f_w = f/f_{w,a_n}$ in $(a_n - \varepsilon_n/2, a_n + \varepsilon_n/2)$; consequently, f/f_w is smooth in a_n, since f_{w,a_n} is a local minimal function for w in a_n.

Definition 2.15. Given a weight w, we say that $a \in T$ has *order* $n \in \mathbf{Z}^+$ if $\operatorname{ess\,lim}_{x\to a,\, x\in \operatorname{supp} w} w(x)|x-a|^{n-1} = \infty$ and $\operatorname{ess\,lim\,sup}_{x\to a} w(x)|x-a|^n < \infty$. We say that $a \in T$ has *finite order* if a has order n for some $n \in \mathbf{Z}^+$.

Proposition 2.3. *Let us consider a weight w and $a \in T$ with order n. Then $(x-a)^n$ is a local minimal function for w in a.*

Proof. First, notice that the condition $\operatorname{ess\,lim\,sup}_{x\to a} w(x)|x-a|^n < \infty$ implies that there exists $\varepsilon > 0$ with $(x-a)^n \in L^\infty((a-\varepsilon, a+\varepsilon), w)$.

We only need to show that for every function $f \in C^\infty((a-\varepsilon, a+\varepsilon)) \cap L^\infty((a-\varepsilon, a+\varepsilon), w)$ we have that $f(x)/(x-a)^n \in C^\infty((a-\varepsilon, a+\varepsilon))$.

Since $\operatorname{ess\,lim\,sup}_{x\to a} |f(x)|w(x) < \infty$ and $\operatorname{ess\,lim}_{x\to a,\, x\in \operatorname{supp} w} w(x)|x-a|^{n-1} = \infty$, then we have that $\operatorname{ess\,lim}_{x\to a,\, x\in \operatorname{supp} w} f(x)/(x-a)^{n-1} = 0$.

As $f \in C^\infty((a-\varepsilon, a+\varepsilon))$, we have that for every $m \geq 0$ there exists

$$\lim_{x\to a} \frac{f(x) - \sum_{k=0}^m f^{(k)}(a)(x-a)^k/k!}{(x-a)^m} = \frac{f^{(m+1)}(a)}{(m+1)!} \ .$$

Then $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$, and we have that $f(x)/(x-a)^n \in C^\infty((a-\varepsilon, a+\varepsilon))$.

Notice that Theorem 2.3 (respectively Theorem 2.2) with propositions 2.2 and 2.3 give the closure of smooth functions (respectively polynomials) in $L^\infty(w)$, if every point of T has finite order (in this case we have that T is discrete).

Our results give that for many unbounded weights the closure of $C^\infty(\mathbf{R})$ in $L^\infty(w)$ is not equal to the closure of $C(\mathbf{R})$ in $L^\infty(w)$.

Proposition 2.4. *Let us consider a weight w such that $w \in L^\infty_{loc}([a-\varepsilon, a) \cup (a, a+\varepsilon])$ and $1/w$ is comparable to the modulus of a local minimal function for w in a. Then the closure of $C^\infty(\mathbf{R})$ in $L^\infty(w)$ is not equal to the closure of $C(\mathbf{R})$ in $L^\infty(w)$.*

Remark. If w is comparable to $|x-a|^{-n}$ in a neighborhood of a, for some $n \in \mathbf{Z}^+$, then $1/w$ is comparable to the modulus of a local minimal function for w in a (we can take $(x-a)^n$ as this minimal function, by Proposition 2.3).

Proof. Without loss of generality, we can assume that $1/w = |f_w|$ in $(a-\varepsilon, a+\varepsilon)$, where f_w is a local minimal function for w in a, and that $f_w \in C^\infty([a-\varepsilon, a+\varepsilon])$. Let us choose a function $\phi \in C_c^\infty((a-\varepsilon, a+\varepsilon))$ with $\phi = 1$ in $(a-\varepsilon/2, a+\varepsilon/2)$.

We see now that the function

$$f(x) := f_w(x)\phi(x)\sin\frac{1}{x-a}$$

is in the closure of $C(\mathbf{R})$ in $L^\infty(w)$ and it is not in the closure of $C^\infty(\mathbf{R})$ in $L^\infty(w)$. Since $\operatorname{supp} f \subset (a-\varepsilon, a+\varepsilon)$, we can assume that $w \equiv 0$ in $\mathbf{R} \setminus [a-\varepsilon, a+\varepsilon]$. Hence the weight w has no singular points, since $1/w = |f_w|$ in $(a-\varepsilon, a+\varepsilon)$ and $f_w \in C^\infty([a-\varepsilon, a+\varepsilon])$.

It is clear that f is in the closure of $C(\mathbf{R})$ in $L^\infty(w)$, since $f \in C(\mathbf{R}) \cap L^\infty(w)$: recall that $T = \{a\}$, since $w \in L^\infty_{loc}([a-\varepsilon, a) \cup (a, a+\varepsilon])$.

The function f/f_w is not in the closure of $C^\infty(\mathbf{R})$ in $L^\infty(1)$, since it is not continuous at a. Then Theorem 2.3 gives that f is not in the closure of $C^\infty(\mathbf{R})$ in $L^\infty(w)$.

3. THE COVERING LEMMAS

The following result is a Besicovitch-Vitali-type lemma; this kind of covering lemma plays an important role in Harmonic Analysis (see e.g. [G]). The proof of Lemma 3.1 follows the classical ideas in the proof of this kind of lemma (see e.g. [G, Chapter 3.2]). However, our situation differs from the standard one: we cover a possibly unbounded set B by intervals which are not centered at points of B; this is the reason why we include the details of the proof. Lemma 3.1 is the main tool in the proof of Theorem 3.1 below.

Lemma 3.1. *Let B be a subset of $\mathbf{R}$ and M a positive number. For each $a \in B$ we are given an open interval $U_a := (a - r_1(a), a + r_2(a))$, with $0 < r_1(a), r_2(a) < M$ and $20/21 \le r_1(a)/r_2(a) \le 21/20$. Then, one can choose a sequence $\{a_n\} \subset B$ such that $B \subset \cup_n U_{a_n}$, and $\{a_n\}$ can be distributed into 42 sequences $\{a_{n_1}\}, \{a_{n_2}\}, \dots, \{a_{n_{42}}\}$ such that for each fixed j we have that $\{U_{a_{n_j}}\}$ are pairwise disjoint.*

Remark. The proof of the lemma allows to obtain a constant greater than $21/20$, but in the proof of Proposition 2.1 we only need a constant greater than 1.

Proof. Let us assume that the lemma is true for bounded sets B, with 14 sequences (instead of 42). If B is not bounded, we can consider the bounded sets $B_k := B \cap [2kM, (2k+2)M]$, for any integer k. Applying the lemma to each B_k, 14 sequences are obtained for each k; since $0 < r_1(a), r_2(a) < M$, an interval corresponding to k can only intersect intervals corresponding to $k-1$, k and $k+1$. Hence, the lemma is true with $3 \cdot 14 = 42$ sequences. Therefore, without loss of generality, we can assume that B is bounded.

For each $a \in B$, let us define $r(a) := \min\{r_1(a), r_2(a)\}$. We choose the sequence $\{a_n\} \subset B$ in the following way: let us consider a_1 with $r(a_1) > \frac{3}{4} \sup \big\{r(a) : a \in B\big\}$; if we have chosen $a_1, \dots, a_n$, let us consider a_{n+1} with $r(a_{n+1}) > \frac{3}{4} \sup \big\{r(a) : a \in B \setminus U_{a_1} \cup \cdots \cup U_{a_n}\big\}$.

In this way we obtain a sequence $\{a_n\} \subset B$. If this sequence is finite, then $B \subset \cup_n U_{a_n}$. If this sequence is infinite, then $\lim_{n\to\infty} r(a_n) = 0$. Seeking a contradiction, suppose that $r(a_n) > \alpha > 0$ for every n. We define $m := 21/20$. Notice that the intervals in the sequence $\big\{(a_n - r_1(a_n)/(3m), a_n + r_2(a_n)/(3m))\big\}_n$ are pairwise disjoint: if $x \in U_{a_n} \cap U_{a_k}$, then $x \in (a_n - r(a_n)/3, a_n + r(a_n)/3) \cap (a_k - r(a_k)/3, a_k + r(a_k)/3)$, since $r_i(a_n)/m \le r(a_n)$. Without loss of generality, we can assume that $a_n < a_k$; therefore, $x - a_n < r(a_n)/3$ and $a_k - x < r(a_k)/3$, and we deduce that $a_k - a_n < r(a_n)/3 + r(a_k)/3$; if we are in the case $k < n$, we also have $r(a_k) > 3r(a_n)/4$ and $r(a_k) < a_k - a_n$, since $a_n \notin U_{a_k}$, and we conclude that $r(a_k) < a_k - a_n < r(a_n)/3 + r(a_k)/3$; hence, $r(a_k) < r(a_n)/2$, which is a contradiction. The case $k > n$ is similar. Therefore, $\lim_{n\to\infty} r(a_n) = 0$. If $a = a_n$ for some n, we have directly $a \in \cup_n U_{a_n}$. If $a \in B \setminus \{a_n\}_n$, then there exists n with $r(a_{n+1}) \le \frac{3}{4} r(a)$, and this implies that $a \in U_{a_1} \cup \cdots \cup U_{a_n}$. Hence, $B \subset \cup_n U_{a_n}$.

In order to prove the second conclusion of the lemma, let us fix U_{a_n} and ask ourselves how many U_{a_k}'s, with $k < n$, intersect U_{a_n}. Such U_{a_k}'s can be classified into two types: those verifying $|a_n - a_k| \le 3mr(a_n)$ (type 1), and those verifying the reverse inequality (type 2). Let us recall that $r(a_k) > 3r(a_n)/4$ for every $k < n$.

We claim that the following is true.

Claim. There is at most one $k < n$ with $U_{a_k} \cap U_{a_n} \ne \emptyset$, $|a_n - a_k| > \frac{5}{2} mr(a_n)$ and $a_k < a_n$. The same is true if we change $a_k < a_n$ by $a_k > a_n$.

Assuming this claim to be true for the moment, we complete the proof. We define now $V_k := \big(a_k - \frac{1}{4} r(a_n), a_k + \frac{1}{4} r(a_n)\big)$ if k is of type 1, and $V_k := \big(a_k^* - \frac{1}{4} r(a_n), a_k^* + \frac{1}{4} r(a_n)\big)$ if k is of type 2, where a_k^* is the point between a_k and a_n at distance $3mr(a_n)$ of a_n.

We have that the sets V_k's are pairwise disjoint: if k_1 and k_2 are both of type 1, this is a consequence of $|a_{k_1} - a_{k_2}| \ge \min\{r(a_{k_1}), r(a_{k_2})\} > \frac{3}{4} r(a_n)$; if k_1 and k_2 are both of type 2, this is a direct consequence of the claim; if k_1 is of type 1 and k_2 is of type 2, the claim gives that $|a_{k_1} - a_{k_2}^*| \ge \frac{1}{2} mr(a_n) > \frac{1}{2} r(a_n)$, and this implies that $V_{a_{k_1}}$ and $V_{a_{k_2}}$ are disjoint.

Now, notice that every V_k is contained in the interval centered in a_n with radius $(3m + \frac{1}{4})r(a_n)$. Since the radius of every V_k is $\frac{1}{4}r(a_n)$, there is at most $12m + 1$ such k's; in fact, there is at most 13 k's with $U_{a_k} \cap U_{a_n} \neq \emptyset$ and $k < n$, since $12m + 1 < 14$.

Hence, $\{a_n\}$ can be distributed into 14 sequences $\{a_{n_1}\}, \{a_{n_2}\}, \dots, \{a_{n_{14}}\}$ such that for each fixed j, $\{U_{a_{n_j}}\}_{n_j}$ are pairwise disjoint.

Proof of the claim. Seeking a contradiction, suppose that there are $k_1, k_2 < n$ with $U_{a_{k_i}} \cap U_{a_n} \neq \emptyset$, $a_n - a_{k_i} > \frac{5}{2}mr(a_n)$ (for $i = 1, 2$) and $a_{k_1} < a_{k_2} < a_n$. Since $a_n - a_{k_2} > \frac{5}{2}mr(a_n)$ by hypothesis, $a_{k_2} \notin U_{a_n}$; if $k_1 < k_2$, we also have that $a_{k_2} \notin U_{a_{k_1}}$ because of the choice of a_{k_2} and, consequently, $U_{a_{k_1}} \cap U_{a_n} = \emptyset$, which is a contradiction. If $k_1 > k_2$, we have that $r(a_{k_2}) > \frac{3}{4}r(a_{k_1}) > \frac{9}{16}r(a_n)$; if we denote by x the distance between a_n and $U_{a_{k_2}}$, we also have $mr(a_{k_2}) + x > a_n - a_{k_2} > \frac{5}{2}mr(a_n)$, i.e.

$$\frac{21}{20}r(a_{k_2}) + x > \frac{21}{8}r(a_n). \tag{3.1}$$

In order to find a contradiction it is sufficient to see that

$$\frac{3}{5}r(a_{k_2}) + x \geq \frac{21}{20}r(a_n), \tag{3.2}$$

since this inequality implies successively (notice that $\frac{3}{5} = 2 - \frac{4}{3}m$)

$$\begin{aligned} 2r(a_{k_2}) + x &\geq \frac{4}{3}mr(a_{k_2}) + mr(a_n), \\ 2r(a_{k_2}) + x &> mr(a_{k_1}) + mr(a_n), \\ a_n - a_{k_1} &> mr(a_{k_1}) + mr(a_n), \\ U_{a_{k_1}} \cap U_{a_n} &= \emptyset. \end{aligned}$$

Notice that $r(a_{k_2}) > \frac{9}{16}r(a_n)$ is equivalent to $\frac{3}{5}r(a_{k_2}) + \frac{57}{80}r(a_n) > \frac{21}{20}r(a_n)$; if $x \geq \frac{57}{80}r(a_n)$, this implies (3.2).

If $x < \frac{57}{80}r(a_n)$, (3.1) guarantees $\frac{21}{20}r(a_{k_2}) + \frac{57}{80}r(a_n) > \frac{21}{8}r(a_n)$.

This inequality implies $r(a_{k_2}) > \frac{51}{28}r(a_n) > \frac{7}{4}r(a_n)$, and this guarantees (3.2).

The following theorem is an improvement of this lemma.

Theorem 3.1. *Let B be a subset of $\mathbf{R}$ and M a positive number. For each $a \in B$ we are given an open interval $U_a := (a - r_1(a), a + r_2(a))$, with $0 < r_1(a), r_2(a) < M$ and $20/21 \leq r_1(a)/r_2(a) \leq 21/20$. Then, one can choose a sequence $\{a_n\} \subset B$ such that $B \subset \cup_n U_{a_n}$, each U_{a_n} intersects at most two U_{a_m}'s, and no U_{a_n} is contained in another U_{a_m}.*

Proof. Let us denote by $\{\alpha_n\}_n$ any sequence of elements of B with the properties in the statement of Lemma 3.1. Since $\{\alpha_n\}_n$ is countable, we can assume that no U_{α_n} is contained in another U_{α_m}; if this is not so, we proceed to remove from the sequence (in a sequential way) those elements whose neighborhood is contained in another U_{α_m}.

We consider the points in $\{\alpha_n\}_n$ such that U_{α_n} intersects U_{α_1}. Notice that there is at most $83 = 1+2(42-1)$ points in $\{\alpha_n\}_n$ (including α_1) with such a property, because no U_{α_n} is contained in another U_{α_m} and Lemma 3.1. Let us denote by $\{\alpha_{n_1},\dots,\alpha_{n_r}\}$ these points ($r \le 83$). Then we can choose at most three $n_{j_1}, n_{j_2}, n_{j_3} \subset \{n_1,\dots,n_r\}$, with $U_{\alpha_{n_1}} \cup \dots \cup U_{\alpha_{n_r}} = U_{\alpha_{n_{j_1}}} \cup U_{\alpha_{n_{j_2}}} \cup U_{\alpha_{n_{j_3}}}$, and such that for any permutation $\{u,v,w\}$ of $\{1,2,3\}$, $U_{\alpha_{n_{j_u}}}$ is not contained in $U_{\alpha_{n_{j_v}}} \cup U_{\alpha_{n_{j_w}}}$. We denote by $\{\alpha_n^1\}$ the subsequence obtained by deleting from $\{\alpha_n\}$ the elements $\{\alpha_{n_1},\dots,\alpha_{n_r}\} \setminus \{\alpha_{n_{j_1}} \cup \alpha_{n_{j_2}} \cup \alpha_{n_{j_3}}\}$. It is clear that $\cup_n U_{\alpha_n} = \cup_n U_{\alpha_n^1}$ and that the points in U_{α_1} are at most in two intervals of $\{U_{\alpha_n^1}\}$ (even though α_1 does not belong to $\{\alpha_n^1\}$ any more).

Let us denote by k the lowest integer greater than 1 with $\alpha_k \in \{\alpha_n^1\}$. The last process can be repeated, with α_k instead of α_1, and $\{\alpha_n^1\}$ instead of $\{\alpha_n\}$, obtaining a subsequence $\{\alpha_n^2\}$ such that $\cup_n U_{\alpha_n} = \cup_n U_{\alpha_n^2}$ and the points in $U_{\alpha_1} \cup U_{\alpha_k}$ are at most in two intervals of $\{U_{\alpha_n^2}\}$.

Iterating this process, we obtain subsequences $\{\alpha_n^1\} \supset \{\alpha_n^2\} \supset \{\alpha_n^3\} \supset \cdots$. Let us denote by $\{a_n\}$ the intersection of such subsequences. We have that $\cup_n U_{\alpha_n} = \cup_n U_{a_n}$ and the points in this set are at most in two intervals of $\{U_{a_n}\}$. Besides, no U_{a_n} is contained in another U_{a_m}. Hence, each U_{a_n} intersects at most two U_{a_m}'s.

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