WEIERSTRASS' THEOREM IN WEIGHTED SOBOLEV SPACES WITH k DERIVATIVES

ANA PORTILLA[(1)], YAMILET QUINTANA, JOSE M. RODRIGUEZ[(1)(2)] AND EVA TOURIS[(1)(3)]

ABSTRACT. We characterize the set of functions which can be approximated by smooth functions and by polynomials with the norm

$$\|f\|_{W^{k,\infty}(w)} := \sum_{j=0}^{k} \|f^{(j)}\|_{L^\infty(w_j)} ,$$

for a wide range of (even non-bounded) weights w_j's. We allow a great deal of independence among the weights w_j's.

Key words and phrases: Weierstrass' theorem; weight; Sobolev spaces; weighted Sobolev spaces.

1. INTRODUCTION.

If I is any compact interval, Weierstrass' Theorem says that $C(I)$ is the largest set of functions which can be approximated by polynomials in the norm $L^\infty(I)$, if we identify, as usual, functions which are equal almost everywhere. There are many generalizations of this theorem (see e.g. the monographs [18], [23], and the references therein).

In [28] and [24] we study the same problem with the norm $L^\infty(w)$ defined by

$$\|f\|_{L^\infty(w)} := \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|w(x) , \tag{1}$$

where w is a weight, i.e. a non-negative measurable function and we use the convention $0\cdot\infty = 0$. In [24] we improve the theorems in [28], obtaining sharp results for a large class of weights (see Theorem 2.1 below). Notice that (1) is not the usual definition of the L^∞ norm in the context of measure theory, although it is the correct one when working with weights (see e.g. [3] and [6]).

Considering weighted norms $L^\infty(w)$ has been proved to be interesting mainly because of two reasons: on the one hand, it allows to wider the set of approximable functions (since the functions in $L^\infty(w)$ can have singularities where the weight tends to zero); and, on the other one, it is possible to find functions which approximate f whose qualitative behaviour is similar to the one of f at those points where the weight tends to infinity.

If $w = (w_0, \dots , w_k)$ is a vectorial weight, we study this approximation problem with the Sobolev norm $W^{k,\infty}(w)$ defined by

$$\|f\|_{W^{k,\infty}(w)} := \sum_{j=0}^{k} \left\|f^{(j)}\right\|_{L^\infty(w_j)} . \tag{2}$$

Weighted Sobolev spaces are an interesting topic in many fields of Mathematics, as Approximation Theory, Partial Differential Equations (with or without Numerical Methods), and Quasiconformal and Quasiregular maps (see e.g. [11], [12], [13], [14], [15], [16] and [17]). In particular, in [12] and [13], the authors showed

---

Date: November 19, 2004.
2000 AMS Subject Classification: 41A10, 46E35, 46G10.
(1) Research partially supported by a grant from DGI (BFM 2003-04870), Spain.
(2) Research partially supported by a grant from DGI (BFM 2003-06335-C03-02), Spain.
(3) Research partially supported by a grant from DGI (BFM 2000-0022), Spain.

that the expansions with Sobolev orthogonal polynomials can avoid the Gibbs phenomenon which appears with classical orthogonal series in L^2. In [8], [7] and [9] the authors study some interesting examples of Sobolev spaces for $p = 2$ with respect to general measures instead of weights, in relation with Ordinary Differential Equations and Sobolev Orthogonal Polynomials. The papers [26], [27], [28], [29] and [30] are the beginning of a theory of Sobolev spaces with respect to general measures for $1 \le p \le \infty$. This theory plays an important role in the location of the zeroes of the Sobolev orthogonal polynomials (see [19], [20], [27] and [29]). The location of these zeroes allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [19]). The papers [1], [2], [4], [10], [20] and [31] deal with Sobolev spaces on curves and more general subsets of the complex plane.

One of the authors studied the problem of approximation with the Sobolev norm (2) in [28], for bounded weights. We also study this problem in [25] for $k = 1$. In the current paper we obtain several results for any k; in most cases, the theorems are new, even for $k = 1$; besides, we manage with general unbounded weights, and we allow a great deal of independence among the weights.

If w is not bounded, then the polynomials are not in $W^{k,\infty}(w)$, in general. Therefore, it is natural to bear in mind the problem of approximation by functions in $C^k(\mathbb{R})$ or $C^\infty(\mathbb{R})$.

The fundamental results of this paper guarantee that a function f belongs to the closure of the space of polynomials (respectively, smooth functions) in the norm $W^{k,\infty}(w)$ if and only if $f^{(j)}$ belongs to the closure of smooth functions in the norm $L^\infty(w_j)$, for every $0 \le j \le k$. See Section 3 (respectively, sections 4 and 5) for the precise statement of the theorems.

The results of this paper are more valuable thanks to Theorem 5.3 (see Section 5) which allows to deal with weights which can be obtained by "gluing" simpler ones.

The analogue of Weierstrass' Theorem with the norms $W^{k,p}(\mu)$ (with $1 \le p < \infty$ and μ a vectorial measure) can be founded in [27] and [30] on the real line, and in [1] and [31] on curves in the complex plane.

The outline of the paper is as follows. Section 2 is dedicated to the definitions and theorems for the case $k = 0$, which are proved in [24]; we also include in this section the definition of weighted Sobolev space and a version of Muckenhoupt inequality which will be useful. We prove the theorems on approximation by polynomials in Section 3. Section 4 presents most interesting results on approximation by smooth functions. Some complementary results, which require more background can be founded in Section 5.

Now we present the notation we use.

Notation. If A is a Borel set, $|A|$, χ_A and $\overline{A}$ denote, respectively, the Lebesgue measure, the characteristic function and the closure of A. By $f^{(j)}$ we mean the j-th distributional derivative of f. $\mathbb{P}$ denotes the set of polynomials. We say that an n-dimensional vector satisfies a one-dimensional property if each coordinate satisfies this property. Finally, the constants in the formulae can vary from line to line and even in the same line.

2. Previous Results.

It is clear that our approximation results in $W^{k,\infty}(w_0, \dots, w_k)$ must be based on approximation results in $L^\infty(w_j)$: if f can be approximated by polynomials in $W^{k,\infty}(w_0, \dots, w_k)$, then $f^{(j)}$ can be approximated by polynomials in $L^\infty(w_j)$ for each $0 \le j \le k$. We describe here the very general approximation results in $L^\infty(w)$, which appear in [24] and [25].

Let us start with some definitions.

Definition 2.1. *A* weight *w is a measurable function $w : \mathbb{R} \longrightarrow [0, \infty]$. If w is only defined on $A \subset \mathbb{R}$, we set $w := 0$ in $\mathbb{R} \setminus A$.*

Definition 2.2. *Given a measurable set $A \subset \mathbb{R}$ and a weight w, we define the space $L^\infty(A, w)$ as the space of equivalence classes of measurable functions $f : A \longrightarrow \mathbb{R}$ with respect to the norm*

$$\|f\|_{L^\infty(A,w)} := \operatorname{ess\,sup}_{x\in A} |f(x)| w(x)\,.$$

The theorems in this paper can be applied to functions f with complex values, splitting f into its real and imaginary parts. From now on, if we do not specify the set A, we are assuming that $A=\mathbb{R}$; analogously, if we do not make explicit the weight w, we are assuming that $w\equiv 1$.

Let A be a measurable subset of $\mathbb{R}$; we always consider the space $L^1(A)$, with respect to the restriction of the Lebesgue measure on A.

Definition 2.3. *Given a measurable set A, we define the* essential closure *of A, as the set*

$$\operatorname{ess\,cl}A := \{x\in\mathbb{R} :\, |A\cap(x-\delta,x+\delta)|>0,\ \ \forall\,\delta>0\}\,,$$

where $|E|$ denotes the Lebesgue measure of E.

Definition 2.4. *If A is a measurable set, f is a function defined on A with real values and $a\in\operatorname{ess\,cl}A$, we say that $\operatorname{ess\,lim}_{x\in A,\,x\to a} f(x)=l\in\mathbb{R}$ if for every $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-l|<\varepsilon$ for almost every $x\in A\cap(a-\delta,a+\delta)$. In a similar way we can define $\operatorname{ess\,lim}_{x\in A,\,x\to a} f(x)=\infty$ and $\operatorname{ess\,lim}_{x\in A,\,x\to a} f(x)=-\infty$. We define the* essential superior limit *and the* essential inferior limit *on A as follows:*

$$\begin{aligned}\operatorname{ess\,lim\,sup}_{x\in A,\,x\to a} f(x) &:= \inf_{\delta>0}\ \operatorname{ess\,sup}_{x\in A\cap(a-\delta,a+\delta)} f(x)\,,\\ \operatorname{ess\,lim\,inf}_{x\in A,\,x\to a} f(x) &:= \sup_{\delta>0}\ \operatorname{ess\,inf}_{x\in A\cap(a-\delta,a+\delta)} f(x)\,.\end{aligned}$$

Remark 2.1.

1. *The essential superior (or inferior) limit of a function f does not change if we modify f on a set of zero Lebesgue measure.*
2. *When we say that there exists a essential limit (or essential superior limit or essential inferior limit), we are assuming that it is finite.*
3. *It is well known that*

$$\operatorname{ess\,lim\,sup}_{x\in A,\,x\to a} f(x)\quad\geq\quad \operatorname{ess\,lim\,inf}_{x\in A,\,x\to a} f(x)\,,$$

$$\operatorname{ess\,lim}_{x\in A,\,x\to a} f(x)=l \quad \textit{if and only if}\quad \operatorname{ess\,lim\,sup}_{x\in A,\,x\to a} f(x)=\operatorname{ess\,lim\,inf}_{x\in A,\,x\to a} f(x)=l\,.$$

4. *We impose the condition $a\in\operatorname{ess\,cl}A$ in order to have the unicity of the essential limit. If $a\notin\operatorname{ess\,cl}A$, then every real number is an essential limit for any function f.*

Definition 2.5. *Given a weight w, the* support *of w, denoted by $\operatorname{supp} w$, is the complement of the largest open set $G\subset\mathbb{R}$ with $w=0$ a.e. on G.*

Definition 2.6. *Given a weight w we say that $a\in\operatorname{supp} w$ is a* singularity *of w if*

$$\operatorname{ess\,lim\,inf}_{x\in\operatorname{supp} w,\,x\to a} w(x)=0\,.$$

We denote by $S(w)$ the set of singularities of w.

We say that $a\in S^+(w)$ (respectively, $a\in S^-(w)$) if $\operatorname{ess\,lim\,inf}_{x\in\operatorname{supp} w,\,x\to a^+} w(x)=0$ (respectively, $\operatorname{ess\,lim\,inf}_{x\in\operatorname{supp} w,\,x\to a^-} w(x)=0$).

Definition 2.7. *Given a weight w, we define the* right regular *and* left regular *points of w, respectively, as*

$$\begin{aligned}R^+(w) &:= \left\{a\in\operatorname{supp} w :\ \operatorname{ess\,lim\,inf}_{x\in\operatorname{supp} w,\,x\to a^+} w(x)>0\right\}\,,\\ R^-(w) &:= \left\{a\in\operatorname{supp} w :\ \operatorname{ess\,lim\,inf}_{x\in\operatorname{supp} w,\,x\to a^-} w(x)>0\right\}\,.\end{aligned}$$

We say that a is a regular point of w if $a\in R(w):=R^+(w)\cap R^-(w)$.

It is easy to check that $R(w)$ is an open set.

We collect here some useful technical results which were proved in [24] and [25].

Theorem 2.1. *([25], Theorem 2.1) Let w be any weight and*

$$H_0 := \left\{ \begin{array}{r} f \in L^\infty(w) : f \;\text{ is continuous to the right at every point of } R^+(w), \\ f \;\text{ is continuous to the left at every point of } R^-(w), \\ \text{for each } a \in S^+(w), \quad \operatorname{ess\,lim}_{x\to a^+} |f(x)-f(a)|\, w(x) = 0, \\ \text{for each } a \in S^-(w), \quad \operatorname{ess\,lim}_{x\to a^-} |f(x)-f(a)|\, w(x) = 0 \end{array} \right\}.$$

Then:

(a) *The closure of $C(\mathbb{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is H_0.*
(b) *If $w \in L^\infty_{loc}(\mathbb{R})$, then the closure of $C^\infty(\mathbb{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is also H_0.*
(c) *If $\operatorname{supp} w$ is compact and $w \in L^\infty(\mathbb{R})$, then the closure of the space of polynomials is H_0 as well.*

Remark 2.2.

1. *Recall that we identify functions which are equal almost everywhere.*
2. *Let us fix $x_1, \dots, x_m \in R(w)$. The proof of this theorem allows to get approximating functions to f coinciding with f in some neighborhood of $\{x_1, \dots, x_m\}$.*

Theorem 2.1 has the following direct consequence.

Corollary 2.1. *Let us consider $\alpha_1 < \cdots < \alpha_n$ and any weight w in $[\alpha_1, \alpha_n]$. Then, f belongs to the closure of $C([\alpha_1, \alpha_n]) \cap L^\infty([\alpha_1, \alpha_n], w)$ in $L^\infty([\alpha_1, \alpha_n], w)$ if and only if f belongs to the closure of $C([\alpha_m, \alpha_{m+1}]) \cap L^\infty([\alpha_m, \alpha_{m+1}], w)$ in $L^\infty([\alpha_m, \alpha_{m+1}], w)$ for every $1 \le m < n$.*

Definition 2.8. *Given a weight w with compact support, a polynomial $p \in L^\infty(w)$ is said to be a* minimal polynomial *for w if every polynomial in $L^\infty(w)$ is a multiple of p. A minimal polynomial for w is said to be* the minimal polynomial *for w (and we denote it by p_w) if it is* 0 *or it is monic.*

It is clear that there always exists a minimal polynomial for w (although it can be 0): it is sufficient to consider a polynomial in $L^\infty(w)$ of minimal degree. Minimal polynomials for w are unique unless multiplication by constants; this fact allows to define p_w.

Let us remark that $p_w = 0$ if and only if the unique polynomial in $L^\infty(w)$ is 0.

Theorem 2.1 and the following result characterizes the closure of the space of polynomials in $L^\infty(w)$, if w has compact support, since then $|p_w| w \in L^\infty(\mathbb{R})$.

Theorem 2.2. *([24], Theorem 2.2) Let us consider a weight w with compact support. If $p_w \equiv 0$, then the closure of the space of polynomials in $L^\infty(w)$ is $\{0\}$. If p_w is not identically* 0*, then the closure of the space of polynomials in $L^\infty(w)$ is the set of functions f such that f/p_w is in the closure of the space of polynomials in $L^\infty(|p_w| w)$.*

We deal now with the definition of the Sobolev space $W^{k,\infty}(w)$, for a vectorial weight $w = (w_0, \dots, w_k)$.

We follow the approach in [16]. First of all, notice that the distributional derivative of a function f in Ω is a function belonging to $L^1_{loc}(\Omega)$. If $f' \in L^\infty(\Omega, w_1)$, in order to get the inclusion

$$L^\infty(\Omega, w_1) \subseteq L^1_{loc}(\Omega),$$

a sufficient condition, is that the weight w_1 satisfies $1/w_1 \in L^1_{loc}(\Omega)$ (see e.g. the proof of Theorem 4.1 below). Consequently, $f \in AC_{loc}(\Omega)$, i.e. f is an absolutely continuous function on every compact interval contained in Ω.

Given a vectorial weight $w = (w_0, \dots, w_k)$, let us denote by Ω_j, for $0 < j \le k$, the largest set (which is a union of intervals) such that $1/w_j \in L^1_{loc}(\Omega_j)$. We always require that $\operatorname{supp} w_j = \overline{\Omega}_j$, for $0 < j \le k$. We define *the Sobolev space* $W^{k,\infty}(w)$, as the set of all (equivalence classes of) functions f defined in $\operatorname{supp} w_0 \cup \Omega_1 \cup \cdots \cup \Omega_k$, such that the weak derivative $f^{(j-1)}$ belongs to $AC_{loc}(\Omega_j)$, for $0 < j \le k$, and $f^{(j)}$ belongs to $L^\infty(w_j)$, for $0 \le j \le k$.

With this definition, the weighted Sobolev space $W^{k,\infty}(w)$ is a Banach space (see [16], Section 3). In general, this is not true without our hypotheses (see some examples in [16]).

3. Approximation by polynomials.

Lemma 3.1. *Let us fix an interval $[\alpha,\beta]$, a positive integer s, a function p_0 belonging to $L^\infty([\alpha,\beta])$ with $p_0 \neq 0$ a.e. in $[\alpha,\beta]$, and $\{g_i\}_{i=1}^s$ a subset of functions of $L^2([\alpha,\beta]) \setminus \{0\}$, such that for every $1 < i \le s$, the function g_i is linearly independent of $\{g_1,\dots,g_{i-1}\}$.*

Let $c^1,\dots,c^s$, be real numbers satisfying the following system of linear equations on $\{c^m\}_{m=1}^s$

$$\sum_{m=1}^{s} c^m \int_\alpha^\beta p_0 g_i h_m = 0, \qquad \forall\ 1 \le i \le s. \tag{3}$$

Then there exist polynomials $h_1,\dots,h_s$, such that the determinant Δ_s of the coefficient matrix of the linear system (3) on $c^1,\dots,c^s$ is not zero.

Remark 3.1.

1. *Since $\Delta_s \neq 0$, none of the polynomials $h_1,\dots,h_s$ can be identically zero.*
2. *When talking about linear independence, we consider the functions as equivalence classes in L^2, that is to say, a function is linearly dependent of some others when it is equal to a linear combination of them almost everywhere.*

Proof. Let us prove the lemma by induction on m. We will show that for every $1 \le m < s$, there exists a polynomial h_{m+1} such that, together with the polynomials $h_1,\dots,h_m$ chosen in the previous steps, the minor Δ_{m+1} consisting of the $m+1$ first rows and columns of the coefficient matrix of (3), is not zero.

If $m=1$, since $g_1 \in L^2([\alpha,\beta]) \setminus \{0\}$, and $p_0 \neq 0$ a.e. in $[\alpha,\beta]$, the functional $\Lambda_1(F) := \int_\alpha^\beta F\, p_0 g_1$ is not identically zero in $L^2([\alpha,\beta])$ (Λ_1 is well defined on $L^2([\alpha,\beta])$ since $p_0 \in L^\infty([\alpha,\beta])$ and $g_1 \in L^2([\alpha,\beta])$); hence, as the polynomials are dense in $L^2([\alpha,\beta])$, there exists a polynomial h_1 with $\Lambda_1(h_1) = \int_\alpha^\beta p_0 g_1 h_1 \neq 0$.

If $m=2$, we must show that there exists a polynomial h_2 such that

$$\Delta_2 := \begin{vmatrix} \int_\alpha^\beta p_0 g_1 h_1 & \int_\alpha^\beta p_0 g_1 h_2 \\ \int_\alpha^\beta p_0 g_2 h_1 & \int_\alpha^\beta p_0 g_2 h_2 \end{vmatrix} \neq 0,$$

that is to say,

$$\Delta_2 = A_{12} \int_\alpha^\beta p_0 g_1 h_2 + A_{22} \int_\alpha^\beta p_0 g_2 h_2 \neq 0,$$

where $A_{12} = -\int_\alpha^\beta p_0 g_2 h_1$ and $A_{22} = \int_\alpha^\beta p_0 g_1 h_1 \neq 0$.

Let us define the function

$$u_2(x) := A_{12} p_0(x) g_1(x) + A_{22} p_0(x) g_2(x), \qquad \forall\ x \in [\alpha,\beta],$$

which is not zero at a positive measured subset of $[\alpha,\beta]$, since $A_{22} \neq 0$, g_2 is linearly independent of g_1, and $p_0 \neq 0$ a.e. in $[\alpha,\beta]$. We can define as well

$$\Lambda_2(F) := \int_\alpha^\beta F u_2, \qquad \forall\ F \in L^2([\alpha,\beta]),$$

since $p_0 \in L^\infty([\alpha,\beta])$ and $g_i \in L^2([\alpha,\beta])$ imply $u_2 \in L^2([\alpha,\beta])$. As Λ_2 is not identically zero in $L^2([\alpha,\beta])$ and the polynomials are dense in $L^2([\alpha,\beta])$, there exists a polynomial h_2 with $\Delta_2 = \Lambda_2(h_2) \neq 0$.

Let us assume the result to be true for m and let us prove it for $m+1$. Then,

$$\Delta_{m+1} = \sum_{i=1}^{m+1} A_{i,m+1} \int_\alpha^\beta p_0 g_i h_{m+1},$$

where $A_{i,m+1}$ ($1 \le i \le m+1$) are the minors corresponding to the expansion of Δ_{m+1} along the last column (with the proper sign in each case). Notice that $A_{m+1,m+1} \neq 0$, by induction hypothesis.

Now, let us define the function u_{m+1} on the interval $[\alpha, \beta]$ and the linear functional Λ_{m+1} on $L^2([\alpha, \beta])$ similarly to the previous case:

$$u_{m+1}(x) := \sum_{i=1}^{m+1} A_{i,m+1} p_0(x) g_i(x)$$

and

$$\Lambda_{m+1}(F) := \int_\alpha^\beta F u_{m+1}, \qquad \forall \; F \in L^2([\alpha, \beta]).$$

The function u_{m+1} is not 0 at a positive measured subset of $[\alpha, \beta]$, since $A_{m+1,m+1} \neq 0$, g_{m+1} is linearly independent of $\{g_1, \dots, g_m\}$, and $p_0 \neq 0$ a.e. in $[\alpha, \beta]$; therefore Λ_{m+1} is not identically zero on $L^2([\alpha, \beta])$ and it follows that there exists a polynomial h_{m+1} such that $\Delta_{m+1} = \Lambda_{m+1}(h_{m+1}) \neq 0$.

□

Lemma 3.2. *Let us consider $a, b, u_1, \dots, u_r \in [\alpha, \beta]$ and $f \in L^1([\alpha, \beta])$. Then,*

$$\int_a^b \int_{u_1}^{x_1} \cdots \int_{u_r}^{x_r} f(x_{r+1}) \quad dx_{r+1} \cdots dx_2 \, dx_1 = \int_a^b f(x) \, \frac{(b-x)^r}{r!} \, dx$$
$$+ \; \sum_h k_h^{r-1}(r) \int_{J_h^{r-1}(r)} f(x) \, x^{r-1} \, dx + \cdots + \sum_h k_h^0(r) \int_{J_h^0(r)} f(x) \, dx \, ,$$

where every sum is finite, $k_h^j(r)$ are real numbers, and $J_h^j(r)$ are subintervals of $[\alpha, \beta]$, whose endpoints belong to the set $\{a, u_1, \dots, u_r\}$.

Proof. We will prove the result by induction on r. Let us check that it is true for $r = 1$. Applying Fubini's theorem,

$$\begin{aligned} \int_a^b \int_{u_1}^{x_1} f(x_2) \, dx_2 \, dx_1 &= \int_a^b \int_a^{x_1} f(x_2) \, dx_2 \, dx_1 + \int_a^b \int_{u_1}^a f(x_2) \, dx_2 \, dx_1 \\ &= \int_a^b f(x_2) \int_{x_2}^b dx_1 \, dx_2 + (b-a) \int_{u_1}^a f(x) \, dx \\ &= \int_a^b f(x)(b-x) \, dx + (b-a) \int_{u_1}^a f(x) \, dx \, . \end{aligned}$$

Let us assume now that the lemma is true for r and let us check it for $r+1$. Applying the induction hypothesis to the function $\int_{u_{r+1}}^{x_{r+1}} f(x_{r+2}) \, dx_{r+2}$, we get that:

$$\begin{aligned} &\int_a^b \int_{u_1}^{x_1} \cdots \int_{u_r}^{x_r} \int_{u_{r+1}}^{x_{r+1}} f(x_{r+1}) \quad dx_{r+2} \, dx_{r+1} \cdots dx_2 \, dx_1 \\ &\qquad = \int_a^b \int_{u_{r+1}}^{x_{r+1}} f(x_{r+2}) \, dx_{r+2} \, \frac{(b - x_{r+1})^r}{r!} \, dx_{r+1} \\ &\qquad + \; \sum_h k_h^{r-1}(r) \int_{J_h^{r-1}(r)} \int_{u_{r+1}}^{x_{r+1}} f(x_{r+2}) \, dx_{r+2} \, x_{r+1}^{r-1} \, dx_{r+1} \\ &\qquad + \; \cdots + \sum_h k_h^0(r) \int_{J_h^0(r)} \int_{u_{r+1}}^{x_{r+1}} f(x_{r+2}) \, dx_{r+2} \, dx_{r+1} \, . \end{aligned}$$

Let us deal with every term separately. On the one hand,

$$\begin{aligned}
&\int_a^b \int_{u_{r+1}}^{x_{r+1}} f(x_{r+2})\, dx_{r+2}\, \frac{(b-x_{r+1})^r}{r!}\, dx_{r+1} \\
&\quad = \int_a^b \int_a^{x_{r+1}} f(x_{r+2})\, dx_{r+2}\, \frac{(b-x_{r+1})^r}{r!}\, dx_{r+1} + \int_a^b \int_{u_{r+1}}^{a} f(x_{r+2})\, dx_{r+2}\, \frac{(b-x_{r+1})^r}{r!}\, dx_{r+1} \\
&\quad = \int_a^b f(x_{r+2}) \int_{x_{r+2}}^b \frac{(b-x_{r+1})^r}{r!}\, dx_{r+1}\, dx_{r+2} + \frac{(b-a)^{r+1}}{(r+1)!} \int_{u_{r+1}}^a f(x)\, dx \\
&\quad = \int_a^b f(x) \frac{(b-x)^{r+1}}{(r+1)!}\, dx + \frac{(b-a)^{r+1}}{(r+1)!} \int_{u_{r+1}}^a f(x)\, dx\,.
\end{aligned}$$

On the other, if $J_h^j(r) = [A,B]$, with $0 \le j \le r-1$, then

$$\begin{aligned}
&\int_{J_h^j(r)} \int_{u_{r+1}}^{x_{r+1}} f(x_{r+2}) \quad dx_{r+2}\, x_{r+1}^j\, dx_{r+1} \\
&\quad = \int_A^B \int_A^{x_{r+1}} f(x_{r+2})\, dx_{r+2}\, x_{r+1}^j\, dx_{r+1} + \int_A^B \int_{u_{r+1}}^{A} f(x_{r+2})\, dx_{r+2}\, x_{r+1}^j\, dx_{r+1} \\
&\quad = \int_A^B f(x_{r+2}) \int_{x_{r+2}}^B x_{r+1}^j\, dx_{r+1}\, dx_{r+2} + \frac{B^{j+1}-A^{j+1}}{j+1} \int_{u_{r+1}}^A f(x)\, dx \\
&\quad = \frac{B^{j+1}}{j+1} \int_A^B f(x)\, dx - \int_A^B f(x) \frac{x^{j+1}}{j+1}\, dx + \frac{B^{j+1}-A^{j+1}}{j+1} \int_{u_{r+1}}^A f(x)\, dx\,.
\end{aligned}$$

This finishes the proof of the lemma, since $j+1 \le r$.

$\square$

In order to control a function by means of its derivative, we are going to need the following version (a proof can be found in [26], Lemma 3.2) of Muckenhoupt's inequality (see [22] or [21], p. 44).

Lemma 3.3. *Let w_0, w_1 be weights on $[\alpha,\beta]$ and $a \in [\alpha,\beta]$. Then, there exists a positive constant c such that*

$$\left\| \int_a^x g(t)\, dt \right\|_{L^\infty([\alpha,\beta],w_0)} \le c\, \|g\|_{L^\infty([\alpha,\beta],w_1)},$$

for every function g on $[\alpha,\beta]$, if and only if

$$\operatorname{ess\,sup}_{\alpha<x<\beta} w_0(x) \left| \int_a^x 1/w_1 \right| < \infty\,.$$

Theorem 3.1. *Let be a vectorial weight $w = (w_0, \dots, w_k)$ on $[\alpha,\beta]$ satisfying:*

(i) $\int_\alpha^\beta 1/w_k < \infty$.

(ii) $w_j \in L^\infty_{loc}([\alpha,\beta] \setminus \{a_1^j, \dots, a_{m_j}^j\})$, *for every* $0 \le j < k$.

(iii) $w_j(x) \left| \int_{a_i^j}^x 1/(1+w_{j+1}) \right| \le c$, *a.e. in some neighborhood of* a_i^j, *for every* $1 \le i \le m_j$, $0 \le j \le k-2$, *and* $w_{k-1}(x) \left| \int_{a_i^{k-1}}^x 1/w_k \right| \le c$, *a.e. in some neighborhood of* a_i^{k-1}, *for every* $1 \le i \le m_{k-1}$.

Then the closure of the space of polynomials in $W^{k,\infty}(w)$ is

$$H_1 := \left\{ f \in W^{k,\infty}(w) :\, f^{(k)} \in \overline{\mathbb{P} \cap L^\infty(w_k)}^{L^\infty(w_k)} \right\}.$$

Remark 3.2.

1. *Hypothesis* (ii) *is not restrictive at all, since if* $\operatorname{ess\,lim\,sup}_{x\to a} w_j(x) = \infty$ *for an infinite number of points* $a \in \mathbb{R}$*, for some* $0 \le j < k$*, then* 0 *is the only polynomial in* $L^\infty(w_j)$*, and it is trivial to find the closure of the space of polynomials in* $W^{k,\infty}(w)$*.*
2. *Hypothesis* (iii) *appears frequently in the applications: It is usual to consider weights* $w_j(x) = |x-a|^{\alpha_j}$ *in a neighborhood of* a *(this is the case of the Jacobi weights or the weights in Part One of* [15]*). In this case, hypothesis* (iii) *at* a *is equivalent to* $\alpha_j \ge -1$ *if* $\alpha_{j+1} \ge 0$*,* $\alpha_j \ge \alpha_{j+1} - 1$ *if* $\alpha_{j+1} < 0$*, for* $0 \le j \le k-2$*, and* $\alpha_{k-1} \ge \alpha_k - 1$*. In fact, it is usual to have* $\alpha_j = \alpha_{j+1} - 1$ *if* $0 \le j < k$*.*
3. *Notice that hypothesis* (iii) *is much weaker than* $w_j(x)\left|\int_{a_i^j}^x 1/w_{j+1}\right| \le c$*, appearing in Lemma 3.3, since some* w_{j+1} *are allowed to be* 0*.*
4. *The possibility of some* w_j *to be bounded is, naturally, allowed. That is to say,* $\{a_1^j, \dots, a_{m_j}^j\}$ *might be the empty set.*

Proof. Whether 0 is the only polynomial in $L^\infty(w_k)$, the result is obvious (if $f^{(k)} = 0$, then f is a polynomial). Therefore, without loss of generality, we can assume that some non trivial polynomial is in $L^\infty(w_k)$.

It is obvious that the closure of the space of polynomials in $W^{k,\infty}(w)$ is contained in H_1.

Then, it suffices to prove that every function in H_1 can be approximated by polynomials in the norm $W^{k,\infty}(w)$. Let us consider then, $f \in H_1$ and $\{p_n\}_n$ a sequence of polynomials converging to $f^{(k)}$ in the norm $L^\infty(w_k)$. From the sequence $\{p_n\}_n$ we will construct another one of polynomials converging to f in the norm $W^{k,\infty}(w)$.

The key idea in order to carry out such a process, is to find, from p_n, a polynomial $q_{n,k}$ in M, where M is the space of polynomials which have a primitive of order k in $W^{k,\infty}(w)$. If $\mathbb{P}$ were a Hilbert space and M a closed subspace, it would suffice to take as $q_{n,k}$ the orthogonal projection of p_n on M. However, since our norms do not come from an inner product, the problem is much more complicated; fortunately, thanks to the three previous lemmas, we will find a finite set of polynomials B in $L^\infty(w_k)$, such that $q_{n,k}$ can be expressed as a linear combination of p_n and elements of B.

Without loss of generality, we can assume that $\operatorname{ess\,lim\,sup}_{x\to a_i^j} w_j(x) = \infty$, for every $1 \le i \le m_j$, $0 \le j < k$, since if $\operatorname{ess\,lim\,sup}_{x\to a_i^j} w_j(x) < \infty$, for some a_i^j, it is enough to remove it from the list $\{a_i^j : 1 \le i \le m_j,\ 0 \le j < k\}$. Analogously, such points can be assumed to be ordered, that is to say, that $a_1^j < \cdots < a_{m_j}^j$, for every $0 \le j < k$ with $m_j \ge 2$.

Since $1/w_k \in L^1([\alpha,\beta])$, for every function $g \in W^{k,\infty}(w)$ it follows that

$$\int_\alpha^\beta |g^{(k)}| = \int_\alpha^\beta |g^{(k)}| \frac{w_k}{w_k} \le \|g^{(k)}\|_{L^\infty(w_k)} \int_\alpha^\beta \frac{1}{w_k} < \infty,$$

and therefore $g^{(k-1)} \in AC([\alpha,\beta])$, and $g \in C^{k-1}([\alpha,\beta])$.

On the other hand, $\operatorname{ess\,lim\,sup}_{x\to a_i^j} w_j(x) = \infty$, for every $1 \le i \le m_j$, $0 \le j < k$ and $g^{(j)} \in L^\infty(w_j)$, imply that $g^{(j)}(a_i^j) = 0$, for every $1 \le i \le m_j$, $0 \le j < k$ (it makes sense to talk about the value of $g^{(j)}$ at a_i^j since $g^{(j)}$ is a continuous function). As a consequence of the remarks above, we have that $\int_{a_i^j}^{a_{i+1}^j} g^{(j+1)} = g^{(j)}(a_{i+1}^j) - g^{(j)}(a_i^j) = 0$, for every $1 \le i < m_j$, $0 \le j < k$, with $m_j \ge 2$ and every $g \in W^{k,\infty}(w)$.

If $w_j \in L^\infty([\alpha,\beta])$, for some $0 \le j < k$, we define $a_1^j := \alpha$. Firstly, we will construct (from $\{p_n\}_n$) a sequence of polynomials $\{q_{n,k}\}_n$ which converges to $f^{(k)}$ in the norm $L^\infty(w_k)$, with the additional property

$$\int_{a_i^j}^{a_{i+1}^j} q_{n,j+1} = 0, \qquad \forall\ 1 \le i < m_j, 0 \le j < k, \tag{4}$$

where

$$q_{n,j}(x) := f^{(j)}(a_1^j) + \int_{a_1^j}^x q_{n,j+1}, \qquad \forall\ 0 \le j < k.$$

Later we will prove that the sequence of polynomials $\{q_{n,j}\}_n$ converges to $f^{(j)}$ in the norm $L^\infty(w_j)$; the property (4) will exactly guarantee that $q_{n,j}$ is in $L^\infty(w_j)$. This will be the major advantage of $q_{n,k}$ over p_n.

Obviously, in (4) we will only bear in mind the equations related to those j with $m_j \geq 2$. These equations could be rewrited as

$$\int_{a_i^j}^{a_{i+1}^j} \int_{a_1^{j+1}}^{x_{j+1}} \cdots \int_{a_1^{k-1}}^{x_{k-1}} q_{n,k}(x_k)\, dx_k \cdots dx_{j+2}\, dx_{j+1} + H_j(f) = 0, \tag{5}$$

where H_j is a linear operator like $H_j(f) = \sum_{i=j}^{k-1} \alpha_i^j f^{(i)}(a_1^i)$, with α_i^j real numbers just depending on $\{a_i^j, a_{i+1}^j, a_1^{j+1}, \dots, a_1^{k-1}\}$.

Now we will use the lemmas 3.1 and 3.2 so as to prove that it is possible to construct the sequence $\{q_{n,k}\}_k$ verifying (4). Let us consider $p_0 := p_{w_k}$, the minimal polynomial of $L^\infty(w_k)$ (p_{w_k} is not identically zero, since $L^\infty(w_k)$ contains non trivial polynomials), the intervals $I_i^j := [a_i^j, a_{i+1}^j]$ when $m_j \geq 2$, and $s := \sum_{j=0}^{k-1} m_j - k$ (if $w_j \in L^\infty([\alpha,\beta])$, we define $m_j := 1$, so that s is the total number of intervals I_i^j considered). As $a_1^j < \cdots < a_{m_j}^j$, for every $0 \leq j < k$ with $m_j \geq 2$, it follows that the intervals $I_1^j, \dots, I_{m_j-1}^j$, have disjoint interior, for every $0 \leq j < k$ with $m_j \geq 2$.

Let us define now functions g_i^j if $m_j \geq 2$. The Lemma 3.2 allows us to assure that

$$\begin{aligned}\int_{a_i^j}^{a_{i+1}^j} \int_{a_1^{j+1}}^{x_1} \cdots \int_{a_1^{k-1}}^{x_{k-j-1}} F(x_{k-j}) \quad dx_{k-j} \cdots dx_2\, dx_1 &= \int_{a_i^j}^{a_{i+1}^j} F(t)\, \frac{(a_{i+1}^j - t)^{k-j-1}}{(k-j-1)!}\, dt \\ &+ \sum_h k_h^{k-j-2}(i,j) \int_{J_h^{k-j-2}(i,j)} F(t)\, t^{k-j-2}\, dt + \cdots \\ &+ \sum_h k_h^0(i,j) \int_{J_h^0(i,j)} F(t)\, dt,\end{aligned}$$

for every $F \in L^1([\alpha,\beta])$, where every sum is finite. For every $1 \leq i < m_j$, $0 \leq j < k$, with $m_j \geq 2$, we define

$$\begin{aligned}g_i^j(t) \;:=\; & \frac{(a_{i+1}^j - t)^{k-j-1}}{(k-j-1)!}\, \chi_{I_i^j}(t) \\ & + \sum_h k_h^{k-j-2}(i,j)\, t^{k-j-1}\, \chi_{J_h^{k-j-2}(i,j)}(t) + \cdots + \sum_h k_h^0(i,j) \chi_{J_h^0(i,j)}(t).\end{aligned}$$

Then, for every $F \in L^1([\alpha,\beta])$,

$$\int_{a_i^j}^{a_{i+1}^j} \int_{a_1^{j+1}}^{x_1} \cdots \int_{a_1^{k-1}}^{x_{k-j-1}} F(x_{k-j})\, dx_{k-j} \cdots dx_2\, dx_1 = \int_\alpha^\beta F g_i^j. \tag{6}$$

Changing F by $q_{n,k}$ in this equality, we get that (5) (and therefore (4)) can be equivalently rewritten as

$$\int_\alpha^\beta q_{n,k}\, g_i^j + H_j(f) = 0. \tag{7}$$

Let us define the functions $\{g_1, \dots, g_s\}$ as the functions in the list

$$\{g_1^{k-1}, g_2^{k-1}, \dots, g_{m_{k-1}-1}^{k-1}, \dots, g_1^1, g_2^1, \dots, g_{m_1-1}^1, g_1^0, g_2^0, \dots, g_{m_0-1}^0\},$$

in that precise order.

It is obvious that these functions satisfy the hypothesis of Lemma 3.1: $g_i^j \in L^2([\alpha,\beta]) \setminus \{0\}$; besides, for every pair i_0, j_0, the function $g_{i_0}^{j_0}$ is linearly independent of

$$\{g_1^{k-1}, g_2^{k-1}, \dots, g_{m_{k-1}-1}^{k-1}, \dots, g_1^{j_0+1}, g_2^{j_0+1}, \dots, g_{m_{j_0+1}-1}^{j_0+1}, g_1^{j_0}, g_2^{j_0}, \dots, g_{i_0-1}^{j_0}\},$$

since $g_{i_0}^{j_0}$ is equal to $\chi_{I_{i_0}^{j_0}}$ multiplied by a polynomial of degree $k-j_0-1$ plus a finite number of characteristic functions multiplied by polynomials whose degree is lesser than $k-j_0-1$, g_i^j (with $j_0<j<k$) is a finite linear combination of characteristic functions multiplied by polynomials whose degree is lesser or equal than $k-j-1<k-j_0-1$, and every interval $I_i^{j_0}$ with $i\neq i_0$ intersects $I_{i_0}^{j_0}$ at an only point at most.

Therefore, Lemma 3.1 implies that there exist polynomials $h_1,\dots,h_s$, such that the determinant Δ_s of the coefficient matrix of the following linear system on $c^1,\dots,c^s$ is not zero:

$$\sum_{m=1}^{s} c^m \int_\alpha^\beta p_{w_k} h_m g_i^j = 0, \qquad \forall\ 1\le i<m_j,\ 0\le j<k. \tag{8}$$

Let us define now

$$q_{n,k} := p_n - c_n^1\, p_{w_k} h_1 - c_n^2\, p_{w_k} h_2 - \cdots - c_n^s\, p_{w_k} h_s,$$

where $c_n^1, c_n^2, \dots, c_n^s$, must verify (7): These coefficients can be chosen as the only solution of the linear system

$$\sum_{m=1}^{s} c_n^m \int_\alpha^\beta p_{w_k} h_m g_i^j = \int_\alpha^\beta p_n g_i^j + H_j(f), \qquad \forall\ 1\le i<m_j,\ 0\le j<k,$$

since the coefficient matrix is the same as the one of the the system (8). Hence, those $q_{n,k}$ so defined verify (4).

Notice that our argument allows us to construct $q_{n,k}$ as a linear combination of $p_n, p_{w_k}h_1,\ \dots, p_{w_k}h_s$, so that the dependence on n of $q_{n,k}$ is just shown through p_n and the coefficients of $p_{w_k}h_1,\dots,p_{w_k}h_s$. Therefore, the functions $p_{w_k}h_1,\dots,p_{w_k}h_s$, play the same role in our normed space than the one that a base of the orthogonal space to M would play in a Hilbert space. That is the thorough reason why the effort to guarantee their existence is worth it.

At sight of (iii), it turns out to be natural to define the weights $v_j := 1 + w_j$ for $0\le j<k$ and $v_k := w_k$. These weights have an advantage over w_j since they verify:

(i') $\int_\alpha^\beta 1/v_j < \infty$, for every $0\le j\le k$.

(iii') $v_j(x)\left|\int_{a_i^j}^x 1/v_{j+1}\right| \le c'$, a.e. in some neighborhood of a_i^j, for every $1\le i\le m_j$, $0\le j<k$.

Let us show that the polynomials $\{q_{n,0}\}_n$ converge to f in the norm $W^{k,\infty}(v_0,\dots,v_k)$, and, therefore, they converge to f in the norm $W^{k,\infty}(w)$.

Let us define $E_{n,j} := f^{(j)} - q_{n,j}$ for every $0\le j\le k$. Thus

$$E_{n,j}(x) = f^{(j)}(x) - q_{n,j}(x) = \int_{a_1^j}^x \big(f^{(j+1)} - q_{n,j+1}\big) = \int_{a_1^j}^x E_{n,j+1}, \qquad \forall\ 0\le j<k. \tag{9}$$

Since $\int_{a_i^j}^{a_{i+1}^j} f^{(j+1)} = f^{(j)}(a_{i+1}^j) - f^{(j)}(a_i^j) = 0$, and $\int_{a_i^j}^{a_{i+1}^j} q_{n,j+1} = 0$ from the definition of $q_{n,k}$ it follows that

$$\int_{a_i^j}^{a_{i+1}^j} E_{n,j+1} = 0. \tag{10}$$

Particularly $E_{n,j}(a_i^j) = 0$, for every $1\le i<m_j$, $0\le j<k$, since $E_{n,j}(a_1^j)=0$.

The equalities (6), (9) and (10) allow to deduce $\int_\alpha^\beta E_{n,k} g_i^j = 0$, for every $1\le i<m_j$, $0\le j<k$, and thus the coefficients $\{c_n^1,\dots,c_n^s\}$ are themselves the only solution of the linear system

$$\sum_{m=1}^{s} c_n^m \int_\alpha^\beta p_{w_k} h_m g_i^j = \int_\alpha^\beta (p_n - f^{(k)}) g_i^j, \qquad \forall\ 1\le i<m_j,\ 0\le j<k.$$

As the right terms of this system verify

$$\left|\int_\alpha^\beta (p_n - f^{(k)}) g_i^j\right| \le \left\|g_i^j\right\|_{L^\infty([\alpha,\beta])} \left\|p_n - f^{(k)}\right\|_{L^1([\alpha,\beta])} \le \left\|g_i^j\right\|_{L^\infty([\alpha,\beta])} \left\|p_n - f^{(k)}\right\|_{L^\infty(w_k)} \int_\alpha^\beta \frac{1}{w_k} \longrightarrow 0,$$

as n tends to infinity, and the coefficient matrix is independent of n, then, applying Kramer's rule $\lim_{n\to\infty} c_n^m = 0$, for every $1 \le m \le s$. Therefore,

$$\|E_{n,k}\|_{L^\infty(w_k)} = \left\| f^{(k)} - q_{n,k} \right\|_{L^\infty(w_k)} \le \left\| f^{(k)} - p_n \right\|_{L^\infty(w_k)} + \sum_{m=1}^{s} |c_n^m| \, \|p_{w_k} h_m\|_{L^\infty(w_k)} \longrightarrow 0, \tag{11}$$

as n tends to infinity. Hence, $\{q_{n,k}\}_n$ converges to $f^{(k)}$ in $L^\infty(v_k)$. Let us see now that $\{q_{n,0}\}_n$ converges to f in $W^{k,\infty}(v_0,\dots,v_k)$.

Next, let us see that

$$\|E_{n,j}\|_{L^\infty(v_j)} \le c_j \, \|E_{n,j+1}\|_{L^\infty(v_{j+1})}, \qquad \forall\ 0 \le j < k.$$

This inequality and (11) give that $\{q_{n,0}\}_n$ converges to f in $W^{k,\infty}(v_0,\dots,v_k)$, which finishes the proof of the theorem.

Firstly let us assume that $w_j \notin L^\infty([\alpha,\beta])$. Let us choose a partition of $[\alpha,\beta]$ by means of m_j compact intervals $H_1^j,\dots,H_{m_j}^j$, such that a_i^j belongs just to H_i^j, for $1 \le i \le m_j$. The hypothesis (i'), (ii) and (iii') guarantee that $v_j(x)\big|\int_{a_i^j}^x 1/v_{j+1}\big| \le c_j^1$ for almost every $x \in H_i^j$, for every $1 \le i \le m_j$.

If $w_j \in L^\infty([\alpha,\beta])$, then we define $H_1^j := [\alpha,\beta]$ (remember that $a_1^j := \alpha$). The hypothesis (i') and $w_j \in L^\infty([\alpha,\beta])$ guarantee as well that $v_j(x)\big|\int_{a_1^j}^x 1/v_{j+1}\big| \le c_j^1$ for almost every $x \in H_1^j$.

Therefore, whether or not w_j is bounded, Lemma 3.3 implies that

$$\|E_{n,j}\|_{L^\infty(H_i^j,v_j)} \le c_j \|E_{n,j+1}\|_{L^\infty(H_i^j,v_{j+1})},$$

since $E_{n,j}(a_i^j) = 0$ for every $1 \le i \le m_j$. Then

$$\|E_{n,j}\|_{L^\infty(v_j)} \le c_j \, \|E_{n,j+1}\|_{L^\infty(v_{j+1})}, \qquad \forall\ 0 \le j < k.$$

This finishes the proof.

□

4. Approximation by smooth functions.

Definition 4.1. *We say that a vectorial weight $w = (w_0,\dots,w_k)$ in $[a,b]$ is of* type 1 *if $1/w_k \in L^1([a,b])$ and $w_0,\dots,w_{k-1} \in L^\infty([a,b])$.*

We say that u, v, are *comparable* functions in the set A if there exists a positive constant c such that $c^{-1}u \le v \le cu$ a.e. in A. It is clear that $L^\infty(u)$ and $L^\infty(v)$ are the same space and have equivalent norms if u and v are comparable weights.

Definition 4.2. *We say that a vectorial weight $w = (w_0,\dots,w_k)$ in $[a,b]$ is of* type 2 *if there exist real numbers $a \le a_1 < a_2 < a_3 < a_4 \le b$ such that*

1. *$1/w_k \in L^1([a_1,a_4])$, and $w_0,\dots,w_{k-1} \in L^\infty([a,b])$,*
2. *if $a < a_1$, then w_j is comparable to a finite non-decreasing weight in $[a,a_2]$, for $0 \le j \le k$,*
3. *if $a_4 < b$, then w_j is comparable to a finite non-increasing weight in $[a_3,b]$, for $0 \le j \le k$.*

Observe that the weights of type 1 are also of type 2.

In the following theorems we describe the closure of smooth functions in Sobolev spaces with weights of types 1 and 2 in compact intervals.

Theorem 4.1. *Let us consider a vectorial weight $w = (w_0,\dots,w_k)$ of type 1 in a compact interval $I = [a,b]$. Then the closure of $\mathbb{P} \cap W^{k,\infty}(I,w)$, $C^\infty(\mathbb{R}) \cap W^{k,\infty}(I,w)$ and $C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$ are, respectively,*

$$\begin{aligned} H_1 &:= \left\{ f \in W^{k,\infty}(I,w) : \, f^{(k)} \in \overline{\mathbb{P} \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\}, \\ H_2 &:= \left\{ f \in W^{k,\infty}(I,w) : \, f^{(k)} \in \overline{C^\infty(I) \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\}, \\ H_3 &:= \left\{ f \in W^{k,\infty}(I,w) : \, f^{(k)} \in \overline{C(I) \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\}. \end{aligned}$$

Remark 4.1.

1. *Let us observe that Theorem 4.1 characterizes the closure of $C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$, $C^\infty(\mathbb{R}) \cap W^{k,\infty}(I,w)$ and $\mathbb{P} \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$, in terms of the similar problem in $L^\infty(I,w_k)$. This question is completely solved by theorems 2.1 and 2.2 for the closure of $C(\mathbb{R}) \cap L^\infty(I,w_k)$ and $\mathbb{P} \cap L^\infty(I,w_k)$. Theorem 2.3 in* [24] *also characterizes the closure of $C^\infty(\mathbb{R}) \cap L^\infty(I,w_k)$, for many weights w_k.*
2. *If $w_k \in L^\infty(I)$, then the closure of $C^k(\mathbb{R})$, $\mathbb{P}$ and $C^\infty(\mathbb{R})$ are the same. This is a consequence of Bernstein's proof of Weierstrass' Theorem (see e.g.* [5]*, p. 113), which gives a sequence of polynomials converging uniformly up to the k-th derivative for any function in $C^k(I)$.*

Proof. First of all, let us prove that $H_3 = \overline{C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)}^{W^{k,\infty}(I,w)}$. The inclusion

$$\overline{C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)}^{W^{k,\infty}(I,w)} \subseteq H_3$$

is obvious. Let us consider now a function $f \in H_3$, and let us show that it can be approximated by functions in $C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$ with the norm of $W^{k,\infty}(I,w)$.

Let $g \in C(\mathbb{R})$ be a function which approximates $f^{(k)}$ in $L^\infty(I,w_k)$ norm. We consider the function

$$h(x) := \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \int_a^x g(t) \frac{(x-t)^{k-1}}{(k-1)!}\, dt \,.$$

Obviously we have that

$$f^{(j)}(x) - h^{(j)}(x) = \int_a^x \left(f^{(k)}(t) - g(t) \right) \frac{(x-t)^{k-j-1}}{(k-j-1)!}\, dt \,, \quad \text{for} \quad j = 0, \dots, k-1.$$

This gives the inequalities

$$\begin{aligned} \left| f^{(j)}(x) - h^{(j)}(x) \right| &\le \int_a^x \left| f^{(k)}(t) - g(t) \right| \frac{|x-t|^{k-j-1}}{(k-j-1)!}\, dt \\ &\le c_1 \int_a^b \left| f^{(k)}(t) - g(t) \right| \frac{w_k(t)}{w_k(t)}\, dt \le c_1 \big\| 1/w_k \big\|_{L^1(I)} \left\| f^{(k)} - g \right\|_{L^\infty(I,w_k)}, \end{aligned}$$

for $j = 0, \dots, k-1$, since $1/w_k \in L^1(I)$.

Consequently,

$$\| f - h \|_{W^{k,\infty}(I,w)} \le c_2 \left\| f^{(k)} - g \right\|_{L^\infty(I,w_k)}, \qquad \text{with } h \in C^k(\mathbb{R})\,.$$

In the other cases the proof is similar. Notice that the nature of the function h depends on the choice of the function g, that is to say, if $g \in C^\infty(\mathbb{R})$ (respectively, $g \in \mathbb{P}$) approximates f in $L^\infty(I,w_k)$, then $h \in C^\infty(\mathbb{R})$ (respectively, $h \in \mathbb{P}$).

□

Cut and paste functions is a useful method to decompose complicated functions in several simpler ones. In order to do this the partitions of unity are natural tools. The following result guarantees that this technical device preserves the Sobolev spaces. To state this result in an abstract and independent way will allows to simplify the proofs of theorems 4.2, 4.3 and 5.1.

PROPOSITION 4.1. *Let us consider a vectorial weight $w = (w_0, \dots, w_k)$. Assume that K is a finite union of compact intervals $J_1, \dots, J_n$ and that for every J_m there is an integer $0 \le k_m \le k$ verifying $1/w_{k_m} \in L^1(J_m)$, if $k_m > 0$, and $w_j = 0$ a.e. in J_m for $k_m < j \le k$, if $k_m < k$.*

(a) *If $w_1, \dots, w_k \in L^\infty(K)$, then $fg \in W^{k,\infty}(w)$ for every $f \in W^{k,\infty}(w)$ and $g \in C^k(\mathbb{R})$ with $\operatorname{supp} g' \subseteq K$.*
(b) *If furthermore $f^{(k_m)}$ belongs to the closure of $C(J_m) \cap L^\infty(J_m, w_{k_m})$ in $L^\infty(J_m, w_{k_m})$ for some $1 \le m \le n$, then $(fg)^{(j)}$ belongs to the closure of $C(J_m) \cap L^\infty(J_m, w_j)$ in $L^\infty(J_m, w_j)$ for every $0 \le j \le k_m$.*

Proof. Let us fix $f \in W^{k,\infty}(w)$ and $g \in C^k(\mathbb{R})$ with $\operatorname{supp} g' \subseteq K$.

First, let us show that fg belongs to $W^{k,\infty}(w)$. It is clear that fg belongs to $L^\infty(w_0)$, since $g \in L^\infty(\mathbb{R})$: it is constant in each connected component of $\mathbb{R} \setminus K$ and it is bounded in the compact set K. The same argument allows to deduce that fg belongs to $W^{k,\infty}(I, w)$ for each connected component I of $\mathbb{R} \setminus K$. Then we only need to prove that fg belongs to $W^{k,\infty}(J_m, w)$ for each m. If $k_m = 0$, we have the result, since $W^{k,\infty}(J_m, w) = L^\infty(J_m, w_0)$.

Let us fix now m with $k_m > 0$. Then $1/w_{k_m} \in L^1(J_m)$, and $w_j = 0$ a.e. in J_m for $k_m < j \le k$, if $k_m < k$. Since $f \in W^{k,\infty}(J_m, w) = W^{k_m,\infty}(J_m, w_0, \dots, w_{k_m})$, the definition of weighted Sobolev space allows to conclude that f and fg belongs to $C^{k_m-1}(J_m)$. Consequently, for each $0 < j \le k_m$, we have that $(fg)^{(j)}$ is the sum of a continuous function and $f^{(j)}g$ in J_m. Then, we conclude that $(fg)^{(j)}$ belongs to $L^\infty(J_m, w_j)$, since $w_j, g \in L^\infty(J_m)$. This finishes the proof of (a).

Let us assume now that $f^{(k_m)}$ belongs to the closure of $C(J_m) \cap L^\infty(J_m, w_{k_m})$ in $L^\infty(J_m, w_{k_m})$ for some $1 \le m \le n$. We prove now that $(fg)^{(j)}$ belongs to the closure of $C(J_m) \cap L^\infty(J_m, w_j)$ in $L^\infty(J_m, w_j)$ for every $0 \le j \le k_m$.

The result is direct if $k_m = 0$, using Theorem 2.1. Let us fix now m with $k_m > 0$.

As we have seen, $(fg)^{(j)}$ is continuous in J_m if $0 \le j < k_m$. We also have that $(fg)^{(k_m)}$ is the sum of a continuous function and $f^{(k_m)}g$ in J_m. Using Theorem 2.1, it is easy to check that $(fg)^{(k_m)}$ verifies the properties that guarantee that it belongs to the closure of $C(J_m) \cap L^\infty(J_m, w_{k_m})$ in $L^\infty(J_m, w_{k_m})$: the continuity properties hold directly, and the limits are 0 since $w_{k_m}, g \in L^\infty(J_m)$. This finishes the proof. □

THEOREM 4.2. *Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ of type 2 in a compact interval $I = [a, b]$. Then the closure of $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)$ in $W^{k,\infty}(I, w)$ is*

$$H_4 := \left\{ f \in W^{k,\infty}(I, w) : \, f^{(j)} \in \overline{C(I) \cap L^\infty(I, w_j)}^{L^\infty(I, w_j)} \text{ for } 0 \le j \le k \right\}.$$

Proof. It is clear that the closure of $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)$ in $W^{k,\infty}(I, w)$ is contained in H_4. Let us consider now a function $f \in H_4$; we want to see that it can be approximated by functions in $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)$ with the norm of $W^{k,\infty}(I, w)$.

Let us consider a partition of unity $\{\psi_1, \psi_2, \psi_3\} \subseteq C_c^\infty(\mathbb{R})$ in I satisfying: $\psi_1 + \psi_2 + \psi_3 = 1$ in I, $\psi_1|_{[a,a_1]} \equiv 1$, $\psi_2|_{[a_4,b]} \equiv 1$, $\psi_3|_{[a_2,a_3]} \equiv 1$, $\operatorname{supp} \psi_1 \subseteq [a, a_2 - \delta]$, $\operatorname{supp} \psi_2 \subseteq [a_3 + \delta, b]$, $\operatorname{supp} \psi_3 \subseteq [a_1 + \delta, a_4 - \delta]$, for some $\delta > 0$. We consider also the functions $f_i = f\psi_i$ for $i = 1, 2, 3$. If $a = a_1$ and $a_4 < b$ (or $a_4 = b$ and $a < a_1$), we consider a partition of unity with only two functions. If $a = a_1$ and $a_4 = b$, then w is a weight of type 1 in I, and we can apply Theorem 4.1. Then we only consider the case $a < a_1$ and $a_4 < b$, since the other cases are easier.

Without loss of generality, we can assume that w_j is a finite non-decreasing weight in $[a, a_2]$, and a finite non-increasing weight in $[a_3, b]$, for $0 \le j \le k$.

Observe that each f_i belongs to $W^{k,\infty}(I, w)$ by Proposition 4.1, since $1/w_k \in L^1([a_1, a_4])$, $\operatorname{supp} \psi_i' \subseteq [a_1, a_2] \cup [a_3, a_4]$, and $w_1, \dots, w_k \in L^\infty([a_1, a_2] \cup [a_3, a_4])$, because the weights w_j are monotonous.

Since $f^{(k)}$ belongs to the closure of $C([a_1, a_2] \cup [a_3, a_4]) \cap L^\infty([a_1, a_2] \cup [a_3, a_4], w_k)$ in $L^\infty([a_1, a_2] \cup [a_3, a_4], w_k)$, then Proposition 4.1 also implies that $f_i^{(j)}$ belongs to the closure of $C([a_1, a_2] \cup [a_3, a_4]) \cap L^\infty([a_1, a_2] \cup [a_3, a_4], w_j)$ in $L^\infty([a_1, a_2] \cup [a_3, a_4], w_j)$ for every $0 \le j \le k$ and $1 \le i \le 3$.

Let us observe that $f_i^{(j)}$ is equal either to $f^{(j)}$ or to 0 in each interval $[a, a_1], [a_2, a_3], [a_4, b]$, for any $0 \le j \le k$. Then Corollary 2.1 allows to deduce that $f_i^{(j)}$ belongs to the closure of $C(I) \cap L^\infty(I, w_j)$ in $L^\infty(I, w_j)$ for every $0 \le j \le k$.

It is enough to show that each f_i can be approximated in $W^{k,\infty}(I, w)$ by functions belonging to $C^k(I)$, since $f = f_1 + f_2 + f_3$ in I.

(1) Approximation of f_1.

For fixed $0 \le j \le k$, let us consider the functions $g_\lambda(x) := f_1^{(j)}(x + \lambda)$ with $0 < \lambda < \delta$. It is clear that g_λ also belongs to $L^\infty([a, b], w_j)$, since $w_j|_{[a,a_2]}$ is non-decreasing for $0 \le j \le k$ and supp $f_1^{(j)} \subseteq [a, a_2 - \delta]$.

Next, we show that g_λ tends to $f_1^{(j)}$ in $L^\infty(I, w_j)$ as $\lambda \to 0^+$. We need to estimate

$$J(\lambda) := \left\| f_1^{(j)} - g_\lambda \right\|_{L^\infty(I,w_j)} = \operatorname{ess\,sup}_{x \in [a,a_2]} \left| f_1^{(j)}(x) - g_\lambda(x) \right| w_j(x),$$

since $f_1^{(j)}(x) = g_\lambda(x) = 0$ for $x \ge a_2$ and $0 < \lambda < \delta$.

We define $\alpha_j := \max\{x \in [a, b] : \ w_j(t) = 0 \ \text{ for } \ a.e.t \in [a, x]\}$.

If $\alpha_j \ge a_2$, we obtain $J(\lambda) = 0$.We deal now with the case $\alpha_j < a_2$.

Theorem 2.1 guarantees that $f^{(j)} \in C((\alpha_j, a_2])$ and then $f_1^{(j)} \in C((\alpha_j, b])$.

Let us assume that $\lim_{x \to \alpha_j^+} w_j(x) > 0$. Hence, Theorem 2.1 implies that $f_1^{(j)} \in C([\alpha_j, b])$ and consequently $\lim_{\lambda \to 0^+} J(\lambda) = 0$, since $f_1^{(j)}$ is uniformly continuous in $C([\alpha_j, b])$ and $w_j \le w_j(a_2)\chi_{[\alpha_j,a_2]}$ in $[a, a_2]$. If we do not have $\lim_{x \to \alpha_j^+} w_j(x) > 0$, then $\lim_{x \to \alpha_j^+} w_j(x) = 0$, since w_j is a non-decreasing weight in $[a, a_2]$.

Since $f_1^{(j)}$ belongs to the closure of $C(I) \cap L^\infty(I, w_j)$ in $L^\infty(I, w_j)$ and $\lim_{x \to \alpha_j^+} w_j(x) = 0$, Theorem 2.1 implies that $\operatorname{ess\,lim}_{x \to \alpha_j^+} f_1^{(j)}(x) w_j(x) = 0$. In fact, we can deduce $\lim_{x \to \alpha_j^+} f_1^{(j)}(x) w_j(x) = 0$, since w_j is a finite non-decreasing weight in $[a, a_2]$ and $f_1^{(j)} \in C((\alpha_j, b])$. Consequently there exists $0 < \delta_1 \le \delta$ such that $\left| f_1^{(j)}(x) \right| w_j(x) < \varepsilon/3$, whenever $x \in (\alpha_j, \alpha_j + 2\delta_1]$. Then

$$\left| f_1^{(j)}(x) - g_\lambda(x) \right| w_j(x) \le \left| f_1^{(j)}(x) w_j(x) - g_\lambda(x) w_j(x + \lambda) \right| + |g_\lambda(x) w_j(x + \lambda) - g_\lambda(x) w_j(x)| < \varepsilon,$$

for any $x \in (\alpha_j, \alpha_j + \delta_1]$ and $0 < \lambda < \delta_1$, since

$$\left| f_1^{(j)}(x) w_j(x) - g_\lambda(x) w_j(x + \lambda) \right| \le \left| f_1^{(j)}(x) \right| w_j(x) + |g_\lambda(x)|\, w_j(x + \lambda) < \frac{2\varepsilon}{3},$$

and

$$|g_\lambda(x) w_j(x + \lambda) - g_\lambda(x) w_j(x)| \le |g_\lambda(x)|\, w_j(x + \lambda) < \frac{\varepsilon}{3},$$

because the weight w_j is non-decreasing.

Using the uniform continuity of $f_1^{(j)}$ in $[\alpha_j + \delta_1, a_2]$, we have that there exists $0 < \delta_2 \le \delta_1$ such that

$$\left| f_1^{(j)}(x) - g_\lambda(x) \right| w_j(x) \le w_j(a_2) \left| f_1^{(j)}(x) - g_\lambda(x) \right| < \varepsilon,$$

for every $x \in [\alpha_j + \delta_1, a_2]$ if $0 < \lambda < \delta_2$; that is to say, $J(\lambda) = \left\| f_1^{(j)} - g_\lambda \right\|_{L^\infty([\alpha_j,a_2],w_j)} \le \varepsilon$.

Then, it is enough to approximate $(f_1)_\lambda(x) := f_1(x + \lambda)$ in $W^{k,\infty}(I, w)$ for $\lambda > 0$ small enough.

Without loss of generality, we can assume that $a = \min_j \alpha_j$, since in other case we can consider the interval $[\min_j \alpha_j, b]$ instead of $[a, b]$. Then, f is continuous in $(a, a_2]$ and, consequently, f_1 is continuous in $(a, b]$.

Let $\{\phi_t\}_{t>0}$ be an usual approximation of the identity: $\phi_t(x) = t^{-1}\phi(t^{-1}x)$ for all $x \in \mathbb{R}, t > 0$, with $\phi \in C_c^\infty((-1, 1))$ verifying $\phi \ge 0$ and $\int \phi = 1$. Set u_t the convolution $u_t := (f_1)_\lambda * \phi_t$, with $0 < t < \lambda/2 < \delta/2$. Then $u_t \in C^\infty(I)$, since $(f_1)_\lambda \in C([a - \lambda/2, b]) \subset L^1([a - \lambda/2, b])$. We have to use $(f_1)_\lambda$ instead of f_1 because of this good property. We define $v_t := u_t^{(j)} = g_\lambda * \phi_t$ for some fixed $0 \le j \le k$. We only need to check that v_t approximates g_λ in $L^\infty(I, w_j)$ as $t \to 0^+$. But

$$\begin{aligned}
\|v_t - g_\lambda\|_{L^\infty(I,w_j)} &= \operatorname{ess\,sup}_{x\in I} \left| \int_{-t}^{t} g_\lambda(x-y)\phi_t(y)\,dy - \int_{-t}^{t} g_\lambda(x)\phi_t(y)\,dy \right| w_j(x) \\
&\le \int_{-t}^{t} \operatorname{ess\,sup}_{x\in I} |g_\lambda(x-y) - g_\lambda(x)|\, w_j(x)\phi_t(y)\,dy \\
&\le \sup_{|y|\le t} \left\{ \operatorname{ess\,sup}_{x\in I} \left| f_1^{(j)}(x) - g_\lambda(x-y) \right| w_j(x) + \operatorname{ess\,sup}_{x\in I} \left| f_1^{(j)}(x) - g_\lambda(x) \right| w_j(x) \right\} \int_{-t}^{t} \phi_t(y)\,dy \\
&= \sup_{|y|\le t} \{ J(\lambda - y) + J(\lambda) \} \le 2 \sup_{0<s<2\lambda} J(s),
\end{aligned}$$

and this last term tends to zero since $J(\lambda) \to 0$ as $\lambda \to 0^+$. Therefore, given $\varepsilon > 0$, there is a function $f_{1,\varepsilon} \in C^\infty(I)$ such that $\|f_1 - f_{1,\varepsilon}\|_{W^{k,\infty}(I,w)} < \varepsilon$.

(2) Approximation of f_2.

We obtain the result applying a symmetric argument to (1).

(3) Approximation of f_3.

It is a consequence of Theorem 4.1:

We define $w_k^* := w_k + \chi_{[a,a_1+\delta]\cup[a_4-\delta,b]}$ and $w^* := (w_0, \dots, w_{k-1}, w_k^*)$; since $1/w_k^* \in L^1(I)$, we have that w^* is a weight of type 1 in I. Let us observe that $f_3 \in W^{k,\infty}(I, w^*)$, since $\operatorname{supp} f_3 \subseteq [a_1 + \delta, a_4 - \delta]$. Then $f_3^{(k)}$ belongs to the closure of $C(I) \cap L^\infty(w_k^*)$ in $L^\infty(w_k^*)$ by Corollary 2.1: we have seen that $f_3^{(k)}$ belongs to the closure of $C([a_1+\delta, a_4-\delta]) \cap L^\infty([a_1+\delta, a_4-\delta], w_k^*)$ in $L^\infty([a_1+\delta, a_4-\delta], w_k^*) = L^\infty([a_1+\delta, a_4-\delta], w_k)$, and $f_3^{(k)} = 0$ in $[a, a_1 + \delta] \cup [a_4 - \delta, b]$.

Hence, Theorem 4.1 implies that f_3 can be approximated by functions in $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w^*)$ with the norm of $W^{k,\infty}(I, w^*)$. Therefore, f_3 can be approximated by functions in $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)$ with the norm of $W^{k,\infty}(I, w)$, since $w_j \le w_j^*$ for every $0 \le j \le k$.

□

The following result allows to deal with weights which can be obtained by "gluing" simpler ones.

THEOREM 4.3. *Let us consider strictly increasing sequences of real numbers $\{a_n\}$, $\{b_n\}$ (n belonging to a finite set, to $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$) with $b_{n-1} < a_{n+1} < b_n$ for every n. Let $w = (w_0, \dots, w_k)$ be a vectorial weight in the interval $I := \cup_n [a_n, b_n]$. Let us assume also that for each n we have either w is of type 1 in $[a_n, b_n]$, or $1/w_k \in L^\infty([a_n, b_n])$. Then the closure of $C^k(I) \cap W^{k,\infty}(I, w)$ in $W^{k,\infty}(I, w)$ is*

$$H_3 := \left\{ f \in W^{k,\infty}(I, w) : \, f^{(k)} \in \overline{C(I) \cap L^\infty(I, w_k)}^{L^\infty(I,w_k)} \right\}.$$

REMARK 4.2.

1. *The hypothesis $1/w_k \in L^\infty([a_n, b_n])$ is stronger than $1/w_k \in L^1([a_n, b_n])$; however, here we do not have the hypothesis $w_0, \dots, w_{k-1} \in L^\infty([a_n, b_n])$ which is required for weights of type 1.*
2. *The hypothesis $1/w_k \in L^\infty([a_n, b_n])$ is very restrictive, but we only need it in a subset of the interval I. Notice that we are considering also weights of type 1 in other subintervals, so in this way Theorem 4.3 gives a general enough criterion.*
3. *Let us observe that we do not require any technical hypothesis which are usual in this kind of theorems (see, for example, Theorem 5.3).*

Proof. We prove the non-trivial implication. Given any fixed $f \in H_3$, we will find functions in $C^k([a_n, b_n]) \cap W^{k,\infty}([a_n, b_n], w)$ approximating f; next, we will paste them in an appropriate way.

Without loss of generality we can assume that $w_0 \ge c_n > 0$ in $[a_n, b_n]$, since in other case we can change w_0 by $w_0^* := w_0 + \sum_n c_n \chi_{[a_n,b_n]}$, where $\{c_n\}_n$ are chosen such that $(c_{n-1} + c_n + c_{n+1})\|f\|_{L^\infty([a_n,b_n])} \le 1$ (recall that $f^{(k)} \in L^1([a_n, b_n])$, since $1/w_k \in L^1([a_n, b_n])$, and hence $f \in C([a_n, b_n])$). Then $f \in W^{k,\infty}(I, w^*)$

if $w_j^* := w_j$ for $1 \le j \le k$, since $\|f\|_{W^{k,\infty}(I,w^*)} \le \|f\|_{W^{k,\infty}(I,w)} + 1$. It is clear that it is more difficult to approximate f in $W^{k,\infty}(I, w^*)$ than in $W^{k,\infty}(I, w)$.

If for some n we have $1/w_k \in L^\infty([a_n, b_n])$, then there is no singularity of w_k in $[a_n, b_n]$; consequently, $f^{(k)} \in C([a_n, b_n])$ by Theorem 2.1, and therefore $f \in C^k([a_n, b_n])$. Hence, we can choose f as its own approximating function in this interval.

We consider now an interval $[a_n, b_n]$ with w of type 1 in $[a_n, b_n]$. Next, we prove that if $\operatorname{ess\,lim\,sup}_{t\to x} w_k(t) = \infty$ for every $x \in [a_n, b_{n-1}] \cup [a_{n+1}, b_n]$, then we can choose approximating functions to f in $W^{k,\infty}([a_n, b_n], w)$ which are equal to f in $[a_n, b_{n-1}] \cup [a_{n+1}, b_n]$:

If $\operatorname{ess\,lim\,sup}_{t\to x} w_k(t) = \infty$ for every $x \in [a_n, b_n]$, then any continuous function in $L^\infty([a_n, b_n], w_k)$ is zero in this interval. Consequently, $f^{(k)} = 0$ in $[a_n, b_n]$, since $f \in H_3$. Hence, f is a polynomial in this interval and we can choose f as its own approximating function in $[a_n, b_n]$.

If $\operatorname{ess\,lim\,sup}_{t\to x_0} w_k(t) < \infty$ for some $x_0 \in [b_{n-1}, a_{n+1}]$, we can choose some interval J_n with $x_0 \in J_n \subset [b_{n-1}, a_{n+1}]$ and $w_k \in L^\infty(J_n)$. Let us consider approximating functions $\{f_l\}_l$ to $f^{(k)}$ in $L^\infty([a_n, b_n], w_k)$.

Let us choose a function $p_0 \in C_c(J_n)$ such that $p_0 > 0$ in the interior of J_n. Since $w_k \in L^\infty(J_n)$, we deduce that $p_0 \in L^\infty(w_k)$. We define

$$v_l := f_l - c_l^1 p_0 h_1 - \cdots - c_l^k p_0 h_k \,,$$

where the functions $h_1, \dots, h_k$, and the constants $c_l^1, \dots, c_l^k$, are chosen as follows: If $g_i(t) := (b_n - t)^{i-1}$ for $1 \le i \le k$, Lemma 3.1 guarantees that there exist polynomials $h_1, \dots, h_k$, such that the determinant of the coefficient matrix of the following linear system on $\{c_l^m\}_{1\le m\le k}$ is not zero (since $\operatorname{supp} p_0 = J_n$, the interval $[a_n, b_n]$ in the left hand side of (12) can be substituted by J_n in order to apply Lemma 3.1):

$$\sum_{m=1}^{k} c_l^m \int_{a_n}^{b_n} p_0 g_i h_m = \int_{a_n}^{b_n} (f_l - f^{(k)}) g_i \,, \qquad \forall\ 1 \le i \le k \,. \tag{12}$$

Hence, we can compute $\{c_l^m\}_{1\le m\le k}$ verifying this linear system, using the Kramer's rule. We consider the functions $\{v_l\}_l$ with this choice of $h_1, \dots, h_k$, and $c_l^1, \dots, c_l^k$. It is clear that $\{v_l\}_l \subset C([a_n, b_n]) \cap L^\infty([a_n, b_n], w_k)$, since $p_0 \in C([a_n, b_n]) \cap L^\infty([a_n, b_n], w_k)$.

Therefore, $\int_{a_n}^{b_n} v_l g_i = \int_{a_n}^{b_n} f^{(k)} g_i$ for all $1 \le i \le k$. Let us define

$$V_l(x) := \sum_{i=0}^{k-1} \frac{f^{(i)}(a_n)}{i!} (x - a_n)^i + \int_{a_n}^{x} v_l(t) \frac{(x-t)^{k-1}}{(k-1)!}\, dt \,.$$

It is clear that $V_l^{(j)}(a_n) = f^{(j)}(a_n)$, for all $0 \le j < k$. Since $\operatorname{ess\,lim\,sup}_{t\to x} w_k(t) = \infty$ for every $x \in [a_n, b_{n-1}]$, we have $v_l = f^{(k)} = 0$ in $[a_n, b_{n-1}]$, and consequently $V_l = f$ in $[a_n, b_{n-1}]$.

We have, for $0 \le j < k$,

$$\begin{aligned} V_l^{(j)}(b_n) &= \sum_{i=j}^{k-1} \frac{f^{(i)}(a_n)}{(i-j)!} (b_n - a_n)^{i-j} + \int_{a_n}^{b_n} v_l(t) \frac{(b_n - t)^{k-j-1}}{(k-j-1)!}\, dt \\ &= \sum_{i=j}^{k-1} \frac{f^{(i)}(a_n)}{(i-j)!} (b_n - a_n)^{i-j} + \int_{a_n}^{b_n} f^{(k)}(t) \frac{(b_n - t)^{k-j-1}}{(k-j-1)!}\, dt = f^{(j)}(b_n) \,. \end{aligned}$$

Since $\operatorname{ess\,lim\,sup}_{t\to x} w_k(t) = \infty$ for every $x \in [a_{n+1}, b_n]$, we have $v_l = f^{(k)} = 0$ in this interval, and consequently $V_l = f$ in $[a_{n+1}, b_n]$.

In order to see that V_l converges to f in $W^{k,\infty}([a_n, b_n], w)$, we prove first that v_l converges to $f^{(k)}$ in $L^\infty([a_n, b_n], w_k)$ and in $L^1([a_n, b_n])$. We have that

$$\left\| f^{(k)} - f_l \right\|_{L^1([a_n,b_n])} = \int_{a_n}^{b_n} \left| f^{(k)} - f_l \right| \frac{w_k}{w_k} \le \left\| f^{(k)} - f_l \right\|_{L^\infty([a_n,b_n],w_k)} \int_{a_n}^{b_n} \frac{1}{w_k} \longrightarrow 0 \,,$$

as l tends to infinity. Since f_l converges to $f^{(k)}$ in $L^1([a_n, b_n])$, we deduce that the right terms of the linear system (12) tend to zero when l tends to infinity. Since the coefficient matrix of (12) does not depend on l, this fact implies that $\lim_{l\to\infty} c_l^m = 0$, for all $1 \le m \le k$. Consequently, v_l converges to $f^{(k)}$ in $L^\infty([a_n, b_n], w_k)$ and in $L^1([a_n, b_n])$, since f_l converges to $f^{(k)}$ in $L^\infty([a_n, b_n], w_k)$ and in $L^1([a_n, b_n])$.

Then, for any $0 \le j < k$ and $x \in [a_n, b_n]$, we deduce

$$\begin{aligned}
\left| f^{(j)}(x) - V_l^{(j)}(x) \right| &= \left| \int_{a_n}^{x} \left(f^{(k)}(t) - v_l(t) \right) \frac{(x-t)^{k-j-1}}{(k-j-1)!}\, dt \right| \\
&\le \int_{a_n}^{b_n} \left| f^{(k)}(t) - v_l(t) \right| \frac{|x-t|^{k-j-1}}{(k-j-1)!}\, dt \\
&\le c_1 \left\| f^{(k)} - v_l \right\|_{L^1([a_n,b_n])} \le c_2 \left\| f^{(k)} - v_l \right\|_{L^\infty([a_n,b_n],w_k)} .
\end{aligned}$$

Since $w_0, \dots, w_{k-1} \in L^\infty([a_n, b_n])$ (recall that w is of type 1 in $[a_n, b_n]$), V_l converges to f in $W^{k,\infty}([a_n, b_n], w)$, and this fact finishes this part of the proof.

In a similar way, a simpler argument shows the following: If w is of type 1 in $[a_n, b_n]$ and $\operatorname{ess\,lim\,sup}_{t\to x} w_k(t) = \infty$ for every $x \in [a_n, b_{n-1}]$ (respectively $[a_{n+1}, b_n]$), then we can choose approximating functions to f in $W^{k,\infty}([a_n, b_n], w)$ which are equal to f in $[a_n, b_{n-1}]$ (respectively $[a_{n+1}, b_n]$).

We have described how to choose the approximating functions to f in $W^{k,\infty}([a_n, b_n], w)$ for each n. Now we proceed to paste them. If we have either (a) $1/w_k \in L^\infty([a_n, b_n])$ and $1/w_k \in L^\infty([a_{n+1}, b_{n+1}])$, or (b) $\operatorname{ess\,lim\,sup}_{t\to x} w_k(t) = \infty$ for every $x \in [a_{n+1}, b_n]$, it is trivial to paste the approximations to f in $W^{k,\infty}([a_n, b_n], w)$ and in $W^{k,\infty}([a_{n+1}, b_{n+1}], w)$, since both are equal to f in $[a_{n+1}, b_n]$.

Therefore, we only need to paste functions on $[a_{n+1}, b_n]$ with w of type 1 in $[a_{n+1}, b_n]$ such that $w_k \in L^\infty(I_n)$ for some interval $I_n \subset [a_{n+1}, b_n]$. Then we have $w \in L^\infty(I_n)$, $\int_{I_n} w_0 > 0$ and $1/w_k \in L^1(I_n)$. Without loss of generality we can assume that this fact holds for every n, since if we have either (a) or (b), we can join $[a_n, b_n]$ and $[a_{n+1}, b_{n+1}]$ in a single interval. Then theorems 4.1 and 5.3 imply the conclusion (the intervals $\{I_n\}_n$ satisfy the technical hypotheses of Theorem 5.3, by the remark to Theorem 5.3).

□

We can deduce the following consequence.

Theorem 4.4. *Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ in the interval I, with $w_0, \dots, w_{k-1} \in L^\infty_{loc}(I)$ and $1/w_k \in L^1_{loc}(I)$. Then the closure of $C^\infty(I) \cap W^{k,\infty}(I, w)$ and $C^k(I) \cap W^{k,\infty}(I, w)$ in $W^{k,\infty}(I, w)$ are, respectively,*

$$\begin{aligned}
H_2 &:= \left\{ f \in W^{k,\infty}(I, w) :\ f^{(k)} \in \overline{C^\infty(I) \cap L^\infty(I, w_k)}^{L^\infty(I,w_k)} \right\}, \\
H_3 &:= \left\{ f \in W^{k,\infty}(I, w) :\ f^{(k)} \in \overline{C(I) \cap L^\infty(I, w_k)}^{L^\infty(I,w_k)} \right\}.
\end{aligned}$$

Proof. The second equality is a direct consequence of Theorem 4.3. It is enough to split I as a union of compact intervals $[a_n, b_n]$ (n belonging to a finite set, to $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$), with $b_{n-1} < a_{n+1} < b_n$ for every n. We have that w is of type 1 in each $[a_n, b_n]$, since $w \in L^\infty([a_n, b_n])$ and $1/w_k \in L^1([a_n, b_n])$ for every n.

The first equality is similar. We only need to change C and C^k by C^∞ everywhere in the proof of Theorem 4.3 (in this case, w is of type 1 in every interval).

□

5. Some more technical results.

We collect in this section some complementary results, which require more background. We refer to [28] for the precise definitions that we need; we do not explain these definitions in a rigorous way here since it would require several pages with many technical details, and the results in this section are not the central theorems of the paper. However, we present here an heuristic explanation of the more important concepts that we need.

A point $a \in I$ is right (respectively, left) m-regular if every function f in $W^{k,\infty}(I,w)$ verifies that $f^{(m)}$ is absolutely continuous in a right (respectively, left) neighborhood of a (it can be granted by the iterated use of Muckenhoupt inequality). A point is m-regular if it is right m-regular and left m-regular. We denote by $\Omega^{(m)}$ the set of m-regular points (or half-points). (If $[a,b] \subseteq \Omega^{(m)}$, then $f^{(m)} \in AC([a,b])$ for every function $f \in W^{k,\infty}(I,w)$.) It is clear that $\Omega_{m+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(m)}$ (see the definition of Ω_j at the end of Section 2).

We denote by $K(I,w)$ the set of functions f in $W^{k,\infty}(I,w)$ with $\|f\|_{W^{k,\infty}(I,w)} = 0$. It is convenient that $K(I,w) = \{0\}$, but there are vectorial weights, as $(w_0,w_1) = (0,1)$, that do not satisfy this property. The condition $(I,w) \in C_0$ is a technical requirement a little stronger than $K(I,w) = \{0\}$; it is satisfied if, for example, $K(I,w) = \{0\}$ and $\Omega^{(0)} \setminus (\Omega_1 \cup \cdots \cup \Omega_k)$ has only a finite number of points in each connected component of $\Omega^{(0)}$ (see Remark 1 to Definition 3.10 in [28], or the proof of [26], Theorem 4.3). This is a weak condition, since $\Omega_{m+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(m)} \subseteq \overline{\Omega_{m+1} \cup \cdots \cup \Omega_k}$ (see the remark before Definition 3.7 in [28] or the remark before Definition 7 in [26]).

If $(I,w_m,\dots,w_k) \in C_0$ and J is a compact interval contained in $\Omega^{(m-1)}$, we have that there exists a constant $c = c(J,w_m,\dots,w_k)$ with

$$\|f^{(m)}\|_{L^1(J)} \le c\,\|f^{(m)}\|_{W^{k-m,\infty}(I,w_m,\dots,w_k)}\,,$$

for every $f \in W^{k-m,\infty}(I,w_m,\dots,w_k)$ which can be approximated by functions in $C^{k-m}(I) \cap W^{k-m,\infty}(I,w_m,\dots,w_k)$ with the norm of $W^{k-m,\infty}(I,w_m,\dots,w_k)$ (see Corollary B in [28] or Corollary 4.3 in [26]). In fact, these corollaries are stronger, but this statement is good enough for our applications in this section.

We need a specific definition.

Definition 5.1. *We say that a vectorial weight $w = (w_0,\dots,w_k)$ in $[a,b]$ is of* type 3 *if there exist real numbers $a \le a_1 < a_2 < a_3 < a_4 \le b$ and integers $k_1,k_2 \ge 0$ such that*

1. *$1/w_k \in L^1([a_1,a_4])$, and $w_0,\dots,w_{k-1} \in L^\infty([a,b])$,*
2. *if $a < a_1$, then w_j is comparable to a finite non-decreasing weight in $[a,a_2]$, for $k_1 \le j \le k$, and a is right (k_1-1)-regular if $k_1 > 0$,*
3. *if $a_4 < b$, then w_j is comparable to a finite non-increasing weight in $[a_3,b]$, for $k_2 \le j \le k$, and b is left (k_2-1)-regular if $k_2 > 0$.*

Observe that the weights of type 1 or 2 are also of type 3.

Theorem 5.1. *Let us consider a vectorial weight $w = (w_0,\dots,w_k)$ of type 3 in a compact interval $I = [a,b]$. Then the closure of $C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$ is*

$$H_4 := \left\{ f \in W^{k,\infty}(I,w) : \ f^{(j)} \in \overline{C(I) \cap L^\infty(I,w_j)}^{L^\infty(I,w_j)} \ \text{ for } \ 0 \le j \le k \right\}.$$

Proof. Consider $f \in H_4$ and $f_i = f\psi_i$ for $i = 1,2,3$, as in the proof of Theorem 4.2. It is enough to show that each f_i can be approximated by functions in $C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$ with the norm of $W^{k,\infty}(I,w)$.

(1) Approximation of f_1.

If $k_1 = 0$, we can approximate f_1 as in the case of weights of type 2. Assume now $k_1 > 0$.

Let us define $\tilde w_j = w_j + \chi_{[a_2,b]}$ for $0 \le j \le k$, and $\tilde w = (\tilde w_0, \tilde w_1, \dots, \tilde w_k)$, which is also a weight of type 3. Then f_1 belongs to $W^{k,\infty}(I,\tilde w)$, since $f_1 = 0$ in $[a_2,b]$. It is obvious that it is more complicated to approximate f_1 in $W^{k,\infty}(I,\tilde w)$ than in $W^{k,\infty}(I,w)$. Let us observe that $K(I,\tilde w_{k_1},\dots,\tilde w_k) = \{0\}$. We have that $[a,a_1] \subset \operatorname{supp} w_{k_1} \cup \cdots \cup \operatorname{supp} w_k$, since w_j is comparable to a finite non-decreasing weight in $[a,a_2]$, for $k_1 \le j \le k$, and a is right (k_1-1)-regular. Then we conclude that $(a,b] \subseteq \Omega_{k_1} \cup \cdots \cup \Omega_k$. This implies that $(a,b] \subseteq \Omega^{(k_1-1)} = [a,b] = I$, since a is right (k_1-1)-regular; consequently, $\Omega^{(k_1-1)} \setminus (\Omega_{k_1} \cup \cdots \cup \Omega_k) \subseteq \{a\}$. This fact and $K(I,\tilde w_{k_1},\dots,\tilde w_k) = \{0\}$ allows to deduce that $(I,\tilde w_{k_1},\dots,\tilde w_k) \in C_0$.

Therefore, without loss of generality we can assume that $(I,w_{k_1},\dots,w_k) \in C_0$ in order to approximate f_1 by functions in $C^k(I)$.

By Theorem 4.2, it is possible to approximate $f_1^{(k_1)}$ by functions in $C^{k-k_1}(\mathbb{R})$ in the norm of $W^{k-k_1,\infty}(I,w_{k_1},\dots,w_k)$.

If $g \in C^{k-k_1}(\mathbb{R})$ approximates $f_1^{(k_1)}$ in $W^{k-k_1,\infty}(I, w_{k_1}, \dots, w_k)$, we can consider the function

$$h(x) := \sum_{j=0}^{k_1-1} f_1^{(j)}(a) \frac{(x-a)^j}{j!} + \int_a^x g(t) \frac{(x-t)^{k_1-1}}{(k_1-1)!}\, dt\,,$$

since there exists $f_1^{(k_1-1)}(a)$, because a is right (k_1-1)-regular. Then we have

$$f_1^{(j)}(x) - h^{(j)}(x) = \int_a^x \left(f_1^{(k_1)}(t) - g(t) \right) \frac{(x-t)^{k_1-j-1}}{(k_1-j-1)!}\, dt, \quad \text{for} \quad 0 \le j < k_1\,.$$

Now, by Corollary B in [28], we have for $0 \le j < k_1$,

$$\left\| f_1^{(j)} - h^{(j)} \right\|_{L^\infty(I)} \le c \left\| f_1^{(k_1)} - g \right\|_{L^1(I)} \le c \left\| f^{(k_1)} - g \right\|_{W^{k-k_1,\infty}(I, w_{k_1}, \dots, w_k)}\,,$$

since $(I, w_{k_1}, \dots, w_k) \in C_0$ and $I = \Omega^{(k_1-1)}$. Hence, we have for $0 \le j < k_1$,

$$\left\| f_1^{(j)} - h^{(j)} \right\|_{L^\infty(I, w_j)} \le c \left\| f^{(k_1)} - g \right\|_{W^{k-k_1,\infty}(I, w_{k_1}, \dots, w_k)}\,,$$

since $w_0, \dots, w_{k_1-1} \in L^\infty(I)$.

(2) Approximation of f_2.

We use the same proof with the appropriate symmetry.

(3) Approximation of f_3.

We proceed as in the proof of Theorem 4.2.

This finishes the proof of Theorem 5.1.

□

The ideas in the proof of Theorem 5.1 can be generalized in order to obtain the following result, which results very useful, since in [25] there are theorems which characterize the closure of $C^1(\mathbb{R})$ in $W^{1,\infty}(I, w_0, w_1)$, for very general weights w_0, w_1.

THEOREM 5.2. *Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ in a compact interval $I = [a, b]$, verifying $I = \Omega^{(m-1)}$ and $w_0, \dots, w_{m-1} \in L^\infty(I)$, for some $0 < m \le k$. Let us assume that $(I, w_m, \dots, w_k) \in \mathrm{C}_0$. If the closure of $C^{k-m}(\mathbb{R}) \cap W^{k-m,\infty}(I, w_m, \dots, w_k)$ in $W^{k-m,\infty}(I, w_m, \dots, w_k)$ is H, then the closure of $C^k(\mathbb{R}) \cap W^{k,\infty}(I, w)$ in $W^{k,\infty}(I, w)$ is*

$$H_5 := \left\{ f \in W^{k,\infty}(I, w) :\ f^{(m)} \in H \right\}.$$

Proof. If $g \in C^{k-m}(\mathbb{R})$ approximates $f^{(m)}$ in $W^{k-m,\infty}(I, w_m, \dots, w_k)$, we can consider the function

$$h(x) := \sum_{j=0}^{m-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \int_a^x g(t) \frac{(x-t)^{m-1}}{(m-1)!}\, dt\,,$$

since there exists $f^{(m-1)}(a)$, because $a \in I = \Omega^{(m-1)}$. Then we have

$$f^{(j)}(x) - h^{(j)}(x) = \int_a^x \left(f^{(m)}(t) - g(t) \right) \frac{(x-t)^{m-j-1}}{(m-j-1)!}\, dt, \quad \text{for} \quad 0 \le j < m\,.$$

Now, by Corollary B in [28], we have for $0 \le j < m$,

$$\left\| f^{(j)} - h^{(j)} \right\|_{L^\infty(I)} \le c \left\| f^{(m)} - g \right\|_{L^1(I)} \le c \left\| f^{(m)} - g \right\|_{W^{k-m,\infty}(I, w_m, \dots, w_k)}\,,$$

since $I = \Omega^{(m-1)}$, and $(I, w_m, \dots, w_k) \in C_0$. Hence, we have for $0 \le j < m$,

$$\left\| f^{(j)} - h^{(j)} \right\|_{L^\infty(I, w_j)} \le c \left\| f^{(m)} - g \right\|_{W^{k-m,\infty}(I, w_m, \dots, w_k)}\,,$$

since $w_0, \dots, w_{m-1} \in L^\infty(I)$.

□

The results of this paper are more valuable thanks to the following theorem. It allows to deal with weights which can be obtained by "gluing" simpler ones. Consequently, the theorems in this paper can be used together with the results in [28] and [25].

Theorem 5.3. *([28], Theorem 5.2) Let us consider strictly increasing sequences of real numbers $\{a_n\}$, $\{b_n\}$ (n belonging to a finite set, to $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$) with $a_{n+1} < b_n$ for every n. Let $w = (w_0, \dots, w_k)$ be a vectorial weight in the interval $I := \cup_n [a_n, b_n]$. Assume that for each n there exists an interval $I_n \subset [a_{n+1}, b_n]$ with $w \in L^\infty(I_n)$ and $(I_n, w) \in \mathrm{C}_0$. Then f can be approximated by functions of $C^\infty(I)$ in $W^{k,\infty}(I, w)$ if and only if it can be approximated by functions of $C^\infty([a_n, b_n])$ in $W^{k,\infty}([a_n, b_n], w)$ for each n. The same result is true if we replace C^∞ by C^k in both cases.*

Remark 5.1. *Condition $(I_n, w) \in \mathrm{C}_0$ is satisfied in many cases; it holds, for example, if $\int_{I_n} w_0 > 0$ and $1/w_k \in L^1(I_n)$.*

References

[1] V. Alvarez, D. Pestana, J. M. Rodríguez, E. Romera, Weighted Sobolev spaces on curves, *J. Approx. Theory* **119** (2002) 41-85.

[2] A. Branquinho, A. Foulquié, F. Marcellán, Asymptotic behavior of Sobolev type orthogonal polynomials on a rectifiable Jordan curve or arc, *Constr. Approx.* **18** (2002), 161-182.

[3] R. C. Brown, B. Opic, Embeddings of weighted Sobolev spaces into spaces of continuous functions, *Proc. R. Soc. Lond. A* **439** (1992), 279-296.

[4] A. Cachafeiro, F. Marcellán, Orthogonal polynomials of Sobolev type on the unit circle, *J. Approx. Theory* **78** (1994), 127-146.

[5] P. J. Davis, "Interpolation and Approximation", Dover, New York, 1975.

[6] B. Della Vecchia, G. Mastroianni, J. Szabados, Approximation with exponential weights in $[-1, 1]$, *J. Math. Anal. Appl.* **272** (2002), 1-18.

[7] W. N. Everitt, L. L. Littlejohn, The density of polynomials in a weighted Sobolev space, *Rendiconti di Matematica*, Serie VII, **10** (1990), 835-852.

[8] W. N. Everitt, L. L. Littlejohn, S. C. Williams, Orthogonal polynomials in weighted Sobolev spaces, Lecture Notes in Pure and Applied Mathematics, 117 (1989), Marcel Dekker, 53-72.

[9] W. N. Everitt, L. L. Littlejohn, S. C. Williams, Orthogonal polynomials and approximation in Sobolev spaces, *J. Comput. Appl. Math.* **48** (1993), 69-90.

[10] A. Foulquié, F. Marcellán, K. Pan, Asymptotic behavior of Sobolev-type orthogonal polynomials on the unit circle, *J. Approx. Theory* **100** (1999), 345-363.

[11] J. Heinonen, T. Kilpeläinen, O. Martio, "Nonlinear Potential Theory of degenerate elliptic equations", Oxford Science Publ., Clarendon Press, 1993.

[12] A. Iserles, P. E. Koch, S. P. Norsett, J. M. Sanz-Serna, "Orthogonality and approximation in a Sobolev space" in "Algorithms for Approximation". J. C. Mason and M. G. Cox, Chapman & Hall, London, 1990.

[13] A. Iserles, P. E. Koch, S. P. Norsett, J. M. Sanz-Serna, Weighted Sobolev spaces and capacity, *J. Approx. Theory* **65** (1991), 151-175.

[14] T. Kilpeläinen, Weighted Sobolev spaces and capacity, *Ann. Acad. Sci. Fennicae*, Series A. I. Math. **19** (1994), 95-113.

[15] A. Kufner, "Weighted Sobolev Spaces", Teubner Verlagsgesellschaft, Teubner-Texte zur Mathematik (Band 31), Leipzig, 1980. Also published by John Wiley & Sons, New York, 1985.

[16] A. Kufner, B. Opic, How to define reasonably Weighted Sobolev Spaces, *Commentationes Mathematicae Universitatis Caroline* **25**(3) (1984), 537-554.

[17] A. Kufner, A. M. Sändig, "Some Applications of Weighted Sobolev Spaces", Teubner Verlagsgesellschaft, Teubner-Texte zur Mathematik (Band 100), Leipzig, 1987.

[18] D. S. Lubinsky, Weierstrass' Theorem in the twentieth century: a selection, *Quaestiones Mathematicae* **18** (1995), 91-130.

[19] G. López Lagomasino, H. Pijeira, Zero location and n-th root asymptotics of Sobolev orthogonal polynomials, *J. Approx. Theory* **99** (1999), 30-43.

[20] G. López Lagomasino, H. Pijeira, I. Pérez, Sobolev orthogonal polynomials in the complex plane, *J. Comp. Appl. Math.* **127** (2001), 219-230.

[21] V. G. Maz'ja, "Sobolev spaces", Springer-Verlag, New York, 1985.

[22] B. Muckenhoupt, Hardy's inequality with weights, *Studia Math.* **44** (1972), 31-38.

[23] A. Pinkus, Weierstrass and Approximation Theory, *J. Approx. Theory* **107** (2000), 1-66.

[24] A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís, Weierstrass' Theorem with weights, to appear in *J. Approx. Theory.*

[25] A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís, Weighted Weierstrass' Theorem with first derivatives. Preprint.

[26] J. M. Rodríguez, V. Alvarez, E. Romera, D. Pestana, Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials I, to appear in *Acta Appl. Math.*
[27] J. M. Rodríguez, V. Alvarez, E. Romera, D. Pestana, Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials II, *Approx. Theory and its Appl.* **18:2** (2002), 1-32.
[28] J. M. Rodríguez, Weierstrass' Theorem in weighted Sobolev spaces, *J. Approx. Theory* **108** (2001), 119-160.
[29] J. M. Rodríguez, The multiplication operator in Sobolev spaces with respect to measures, *J. Approx. Theory* **109** (2001), 157-197.
[30] J. M. Rodríguez, Approximation by polynomials and smooth functions in Sobolev spaces with respect to measures, *J. Approx. Theory* **120** (2003), 185-216.
[31] J. M. Rodríguez, V. A. Yakubovich, A Kolmogorov-Szegö-Krein type condition for weighted Sobolev spaces, to appear in *Indiana U. Math. J.*
[32] W. Rudin, "Real and Complex Analysis" (Second Edition), McGraw-Hill, 1974.

Ana Portilla, José M. Rodríguez, Eva Tourís
Departamento de Matemáticas
Escuela Politécnica Superior
Universidad Carlos III de Madrid
Avenida de la Universidad, 30
28911 Leganés (Madrid)
SPAIN

Yamilet Quintana
Departamento de Matemáticas
Puras y Aplicadas
Edificio Matemáticas y Sistemas (MYS)
Apartado Postal: 89000, Caracas 1080 A
Universidad Simón Bolívar
Caracas
VENEZUELA

E-mail address: apferrei@math.uc3m.es, jomaro@math.uc3m.es, etouris@math.uc3m.es, yquintana@usb.ve