Approximation of Transfer Functions of Infinite Dimensional Dynamical Systems by Rational Interpolants with Prescribed Poles

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Abstract

Rational interpolants with prescribed poles are used to approximate holomorphic functions on the closure of their region of analyticity under natural assumptions of their analytic properties on the boundary. The transfer functions of some general infinite dimensional dynamical systems of interest in applications satisfy the restrictions we impose. This is the case of discrete-time fractional filters, time-delay systems, and heat transfer control systems. We give a general theoretical result by which, in particular, the transfer functions which arise in such dynamical systems may be approximated. Estimates for the rate of convergence are given. We also include some numerical examples which compare the performance of the method we propose with others commonly used in systems theory.

1 Introduction

The approximation of analytic functions by rational interpolants with prescribed poles (RIPP), recently called Padé-type approximants by some authors, plays a major role in approximation theory. The book of J. L. Walsh [?] is to a great extend dedicated to this subject. In the last few years, there is a renewed interest in such approximants. See, for example, [?, ?, ?]. This is partially due to the following facts discussed in [?]. When compared with Padé approximants, which for some time have been more popular:

1. The class of functions for which convergence can be guaranteed is larger.

2. RIPP are easier to compute numerically.

3. If the interpolation points are conveniently chosen, the rate of convergence of RIPP is usually not worse than that of Padé approximants.

We can add another reason, which is very important in Systems Theory. A proper selection of the prescribed poles of the approximants ensures that the approximants of a stable systems is again stable. This is not the case with Padé approximants where the poles are left free and may fall in the region where we wish to approximate the function.

Moreover, it is well known (see e.g. [?]) that there exist holomorphic functions in a given region which contains the origin such that the set of poles of its Padé approximants at 0 is everywhere dense in the region.

In spite of these arguments, the theory of rational interpolants with prescribed poles has not found its way in Systems Theory to its full extent. One reason may be that frequently the transfer function of a dynamical system has singularities on the boundary of the region where approximation is required, whereas most known results on convergence of RIPP only give uniform convergence of the approximants on compact subsets of the region of analyticity (see, for example, [?, ?, ?]). We illustrate this with some examples.

Example 1.1 *The transfer function of a discrete time fractional filter (see e.g. [?]) has the form* $f(z) = \prod_{j=1}^{J}(1 - a_j z)^{d_j}, |a_j| \leq 1$. *Ocasionally, it may occur that for some* j, $|a_j| = 1$ *and* $d_j > 0$ *is not an integer. Therefore, though* f *is analytic in the unit disk* D *and is continuous on its closure* $\overline{D}$ *if all the exponents are positive, it may have branch points on the boundary. A direct application of known results on convergence of RIPP normally gives convergence of the approximants to* f *on compact subsets of* $\overline{D} \setminus \{a_j : |a_j| = 1 \text{ and } d_j \notin \mathcal{N}\}$ *and not on all* $\overline{D}$ *as needed.*

Example 1.2 *A typical transfer function corresponding to a time-delay system is*

$$G(s) = \frac{\sum_1^Q q_j(s)e^{-\alpha_j s}}{p_0(s) + \sum_1^P p_i(s)e^{-\gamma_i s}},$$

where p_i, q_j *are polynomials with* $\deg p_0 > \deg p_i, \deg q_j$ *for all* $i \neq 0$ *and all* j, *and* $\gamma_i, \alpha_j \geq 0$. *If the system is stable then all its poles lie in the left half plane. It is required to approximate this function on* $\{\Re z \geq 0\}$ *but it has an essential singularity at infinity which is a boundary point of the right half plane. In section 4, we show how such functions may be uniformly approximated on* $\{\Re z \geq 0\}$ *despite the existence of an essential singularity at infinity.*

Example 1.3 *Let*

$$G(s) = \frac{\cosh(\sqrt{s}x_1)}{\sqrt{s}\sinh(\sqrt{s})} = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{2(-1)^n \cos n\pi x_1}{s + (n\pi)^2}, \qquad 0 \leq x_1 \leq 1,$$

be the transfer function of a heat transfer control systems (see e.g [?], Example 4.3.12*). A natural selection of rational approximants for this function is to take the partial sums of the series. Unfortunately, it is well known that this series converges very slowly, so it is in order to look for other types of approximants. This function also has an essential singularity at infinity which creates some difficulty in finding proper rational approximants.*

In the present paper, we consider the problem of giving general sufficient conditions for the uniform approximation by means of rational interpolants with prescribed poles of functions which are analytic on a region and continuous up to the boundary. Indications are given as to how to select the poles and the interpolation points. Estimates of the rate of convergence are provided. The functions in the examples above are included in the results obtained.

The paper is organized as follows. In the next section, for convenience of the reader, we give the formal definition of RIPP and state some classical results. In Sections 3 and 4, we extend these results in order to cover the situations we have encountered in the examples above between others. Section 3 is devoted to the approximation on the closed unit disk, and Section 4 to the closed right half plane. In Section 5, we prove some preliminary results on the rate of convergence. In the final section, we conclude with some numerical results which illustrate the behavior of the method.

2 Preliminaries

We begin stating some known facts about rational interpolants with prescribed poles. These results can be found in [?], §§ 8.1-8.3.

Theorem 2.1 *Let $Q(z) = \prod_{i=1}^{n}(z-\alpha_i)$ be a given polynomial of degree n. Let the points $\beta_1, \dots, \beta_n \in \mathcal{C}$ be distinct from the α_i, and consider the set of complex numbers $\mu_1, \dots, \mu_n$. Then there exists a unique rational function $r(z)$ of the form*

$$r(z) = \frac{b_{n-1}z^{n-1} + \cdots + b_0}{Q(z)}$$

which takes the value μ_k at the point β_k, $k = 1, \dots, n$.

The points $\beta_1, \dots, \beta_n$ need not be all distinct. In case of repetition the interpolation is understood in Hermite's sense. More precisely, if $\beta_k = \ldots = \beta_l$, $1 \le k < l \le n$, then $r^{(s-k)}(\beta_k) = \mu_s$, $s = k, \dots, l$.

Definition 2.2 *If $\mu_i = f(\beta_i)$ for some function f, then the rational function r is called a rational interpolant with prescribed poles (RIPP) of f.*

The problem we wish to investigate can be formulated as follows. Given a function f analytic on a region G and continuous on $\overline{G}$, a sequence of polynomials $\{Q_n\}$, $n \in \mathcal{N}, \deg Q_n = n$), and a triangular table of points $\{\beta_{n,k}\}$, $n \in \mathcal{N}$, $1 \le k \le n$, find conditions on $(f, \{Q_n\}, \{\beta_{n,k}\})$ so that the sequence $\{r_n\}, n \in \mathcal{N}$, where r_n is the RIPP obtained interpolating f at the points $\beta_{n,k}$, with poles at the zeros of Q_n, converges to f on $\overline{G}$.

We use the following notation. If Γ is a closed Jordan curve in $\mathcal{C}$, we call the interior of Γ, and denote it by $\texttt{int}(\Gamma)$, the bounded connected component of $\mathcal{C} \setminus \Gamma$.

The following well known integral representation of the error of the approximation is very useful ([?], § 8.1). It follows directly from Cauchy's integral formula.

Theorem 2.3 *Let Γ be a closed rectifiable Jordan curve, let f be a function analytic in $\texttt{int}(\Gamma)$ and continuous on $\overline{\texttt{int}(\Gamma)}$, and let $Q(z) = \prod_{k=1}^{n}(z-\alpha_k)$ be a given polynomial with no zeros on Γ. Let the points $\beta_1, \dots, \beta_n$ lie in $\texttt{int}(\Gamma)$, and denote by B the (monic) polynomial whose zeros are the points β_i. If r is the corresponding RIPP of f, then*

$$f(z) - r(z) = \frac{1}{2\pi i}\frac{B(z)}{Q(z)}\int_\Gamma \frac{Q(t)}{B(t)}\frac{f(t)dt}{t-z}, \qquad z \in \texttt{int}(\Gamma),\ z \neq \alpha_k. \tag{1}$$

A straightforward consequence of this integral formula is the following criterium for convergence of RIPP (see citeWalsh65,§8.3).

Theorem 2.4 *Let Γ and f be as in Theorem 2.3, and let Q_n, $n \in \mathcal{N}$, be a sequence of polynomials with no zeros on Γ. Let $\beta_{n,k}$, $n \in \mathcal{N}$, $1 \le k \le n$ be contained in* $\mathtt{int}(\Gamma)$. *If*

$$\lim_{n\to\infty} \frac{B_n(z)}{B_n(t)} \frac{Q_n(t)}{Q_n(z)} = 0$$

uniformly for t on Γ and z on the boundary of a compact subset K of $\mathtt{int}(\Gamma)$ *which contains no zeros of Q_n, $n \in \mathcal{N}$, then*

$$\lim_n r_n(z) = f(z) \qquad \text{uniformly on } K.$$

This theorem does not suffice to deal with the examples given in the introduction because of the existence of singularities on the boundary of the set K where we need to approximate. In the next sections we obtain adequate modifications of this result which fit our purpose.

3 Convergence on the disk

Here, we concentrate on the case when K is the closed unit disk and f has a finite number of algebraic singularities on the boundary of the unit disk (as in Example 1.1). For simplicity, we state our results for the case when there is only one singular point on the boundary of the unit disk, but from the proof it is easy to see that analogous results are valid when the number of algebraic singularities on the boundary is finite.

Assumptions: Throughout this section $D = \{z : |z| < 1\}, D^* = \overline{D} \setminus \{-1\}$, and f is a holomorphic function in a region $G \supset D^*$ which is continuous on $\overline{D}$ (with a possible singular point at $z = -1$). Let Γ be a closed rectifiable Jordan curve which is contained in $G \setminus D^*$ and is non-tangential to D at $z = -1$; more precisely, there exists a neighborhood V of $\{-1\}$ such that

$$\Gamma \cap V \subset \{z : |\arg(-(z+1))| \le \theta\} := A_\theta \tag{2}$$

for some $\theta \in [0, \frac{\pi}{2})$. Furthermore, we assume that $\frac{f(z)}{z+1}$ is integrable on Γ (with respect to the arc length).

Remark 3.1 *It is easy to verify that* (2) *implies that*

$$\left|\frac{t+1}{t-z}\right| \le \frac{1}{\cos\theta} \qquad \text{for all } t \in \Gamma \cap V,\ z \in D^*.$$

We are ready for the proof of

Theorem 3.2 *With the assumptions above, let Q_n be a sequence of polynomials with no zeros in $\overline{D} \cup \Gamma$, and let the points $\{\beta_{n,k}\}$ lie in $\overline{D}$. Assume that the sequence*

$$A_n(t, z) := \frac{B_n(z)}{B_n(t)} \frac{Q_n(t)}{Q_n(z)}$$

is uniformly bounded in $\Gamma \times \partial D$ and converges uniformly to zero in each set of the form $\Gamma' \times \partial D$, with Γ' a closed subset of $\Gamma \setminus \{-1\}$. Then the corresponding sequence of RIPP converges uniformly to f on $\overline{D}$.

Proof: Let $\epsilon > 0$ be arbitrary. Take a neighborhood V of $\{-1\}$ such that $\Gamma \cap V \subset A_\theta$ and

$$\int_{\Gamma\cap V} \left|\frac{f(t)}{t+1}\right| |dt| < \epsilon \frac{\pi \cos\theta}{M},$$

where M is a uniform bound of $|A_n|$ on $\Gamma \times \partial D$. Let n_ϵ be such that

$$|A_n(t,z)| < \epsilon \frac{\pi \inf\{|t-z|,\ t \in \Gamma\setminus V,\ z \in \partial D\}}{\|f\|_\Gamma \mathtt{length}(\Gamma)}$$

for all $(t,z) \in \Gamma \setminus V \times \partial D$ and all $n > n_\epsilon$. Then

$$\begin{aligned} |f(z) - r_n(z)| &\leq \frac{1}{2\pi}\int_{\Gamma\cap V} |A_n(t,z)| \left|\frac{t+1}{t-z}\right| \left|\frac{f(t)}{t+1}\right| |dt| + \frac{1}{2\pi}\int_{\Gamma\setminus V} |A_n(t,z)| \left|\frac{f(t)}{t-z}\right| |dt| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \qquad z \in \partial D \setminus \{-1\},\ n > n_\epsilon, \end{aligned}$$

(we have used Theorem 2.3 and Remark 3.1). Since f and r_n are continuous on ∂D, from the inequality above, it follows that

$$|f(z) - r_n(z)| \leq \epsilon, \qquad z \in \partial D,\ n > n_\epsilon. \tag{3}$$

From the Maximum Principle for holomorphic functions, we obtain that (3) is valid for all $z \in \overline{D}$ and $n > n_\epsilon$, and the theorem follows from the arbitrariness of ϵ. ♠

Let us see how the polynomials B_n and Q_n may be chosen in order to guarantee the conditions of Theorem 3.2:

Example 3.3 *Let $b \in \overline{D}$ and $a \in \overline{D}^c$ be given. Take Γ such that $\max\{|t| : t \in \Gamma\} < |a|$, and let $B_n(z) = z^n - b^n$, $Q_n(z) = z^n - a^n$. Then*

$$A_n(t,z) = \frac{\left(\frac{t}{a}\right)^n - 1}{\left(\frac{z}{a}\right)^n - 1} \cdot \frac{z^n - b^n}{t^n - b^n}.$$

We have that $\left|\frac{\left(\frac{t}{a}\right)^n - 1}{\left(\frac{z}{a}\right)^n - 1}(z^n - b^n)\right|$ converges uniformly to 1 on $\Gamma \times \partial D$, and $|t^n - b^n| \geq \frac{1}{2}$ for all sufficiently large n and all $t \in \Gamma$. Therefore, A_n is uniformly bounded on $\Gamma \times \partial D$. Furthermore, if $\Gamma' \subset \Gamma \setminus \{-1\}$ is closed, there exists $q > 1$ such that $|t| > q$ for all $t \in \Gamma'$. This implies that $|A_n(t,z)| < q^{-n}$ for all $(t,z) \in \Gamma' \times \partial D$ and all sufficiently large n. Thus, the conditions on A_n in Theorem 3.2 are satisfied.

The way of selecting the points in the previous example is very particular. Normally, one wants to have more freedom in choosing the poles and interpolation points of the approximating functions. The next result will help us in this regard.

Corollary 3.4 *Let f, Γ and $\{Q_n\}$ be as in Theorem 3.2. Let $\{\alpha_{n,i}\}$, $1 \leq i \leq n$, be the zeros of Q_n. Take $\beta_{n,i} = \frac{1}{\overline{\alpha}_{n,i}}$, $1 \leq i \leq n$, and let r_n be the corresponding RIPP of f. If for all $T > 1$,*

$$\lim_n \left|\prod \frac{t - \alpha_{n,i}}{\overline{\alpha}_{n,i} t - 1}\right| = 0 \tag{4}$$

uniformly on $|t| \geq T$, then $\{r_n\}$ converges uniformly to f on $\overline{D}$.

Proof: We have that $\left|\frac{\overline{\alpha}_{n,i}z-1}{z-\alpha_{n,i}}\right| = 1$ for all $z \in \partial D$, $\left|\frac{t-\alpha_{n,i}}{\overline{\alpha}_{n,i}t-1}\right| \leq 1$ for all $t \in \Gamma$, and

$$A_n(t,z) = \frac{B_n(z)}{Q_n(z)}\frac{Q_n(t)}{B_n(t)} = \prod \frac{\overline{\alpha}_{n,i}z-1}{z-\alpha_{n,i}} \prod \frac{t-\alpha_{n,i}}{\overline{\alpha}_{n,i}t-1},$$

thus the result follows from Theorem 3.2. ♠

Remark 3.5 *In [?], §9.6, it is shown that*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{|\alpha_{n,k}|-1}{|\alpha_{n,k}|} = \infty$$

is a sufficient condition for (4) to take place. This in turn is implied by any one of the following two (equivalent) conditions:

$$\lim_{n\to\infty} n\left(\min_k\{|\alpha_{n,k}|\}-1\right) = \infty,$$
$$\lim_{n\to\infty} \left(\min_k\{|\alpha_{n,k}|\}\right)^n = \infty.$$

If $\alpha_{n,k} = \alpha_k$ is independent of n, then (4) is equivalent to the divergence of the series $\sum(1-|\beta_k|)$, $\beta_k = \frac{1}{\overline{\alpha}_k}$ (see e.g [?], § 10.1).

4 Convergence on the half plane

Let $\mathcal{C}_+$ denote the right half plane. Now, we turn to the the case when approximation is required on the closure of $\mathcal{C}_+$. Our main result here is Theorem 4.2. As a consequence, we can approximate (among others) all stable time delay systems (i.e. whose poles lie in the left half plane), with transfer function of the form

$$G(s) = \frac{\sum_1^Q q_j(s)e^{-\alpha_j s}}{p_0(s) + \sum_1^P p_i(s)e^{-\gamma_i s}}, \tag{5}$$

where p_i, q_j are polynomials with $\deg p_0 > \deg p_i$, $\deg q_j$ for all $i \neq 0$ and all j, and $\gamma_i, \alpha_j \geq 0$.

In this section, we use $^{-}$ to denote the closure in the complex plane $\mathcal{C}$, and $\hat{}$ to denote the closure in the extended complex plane $\mathcal{C} \cup \{\infty\}$. If Γ is a closed Jordan curve in $\hat{\mathcal{C}} \setminus \overline{\mathcal{C}}_+$, we call exterior of Γ, and denote it by $\texttt{ext}(\Gamma)$, the connected component of $\mathcal{C} \setminus \Gamma$ that contains $\mathcal{C}_+$. When $\infty \notin \Gamma$, the above convention coincides with the usual definition of exterior of Γ. Nevertheless, this case reduces to Theorem 2.4, thus we will always assume that $\infty \in \Gamma$. We say that an unbounded Jordan curve Γ is *rectifiable* if for all $R > 0$, $\Gamma_R := \Gamma \cap \{w : |w| \leq R\}$ is the union of finitely many rectifiable Jordan arcs.

Assumptions: In this section, f denotes a holomorphic function in a region $G \supset \overline{\mathcal{C}}_+$. Let Γ be a rectifiable Jordan curve in $G \cup \{\infty\} \setminus \overline{\mathcal{C}}_+$. We assume that $|f(z)| \leq \frac{C}{|z|^\alpha}$, $z \in \overline{\texttt{ext}(\Gamma)}$, for some C, $\alpha > 0$.

Let $C_R = \{w : |w| = R\} \cap \texttt{ext}(\Gamma)$. We say that Γ_R is positively oriented when $\Gamma_R \cup C_R$ is positively oriented as a closed Jordan curve in $\mathcal{C}$. In the following, Γ_R is positively oriented, and

$$\int_\Gamma f(z)dz := \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz,$$

assuming that the limit on the right hand exists.

Our first goal is to extend the validity of (1).

Theorem 4.1 *Assume that f and Γ satisfy the assumptions above, and let $Q(z) = \prod_1^n (z-\alpha_k)$ be a polynomial with no zeros on Γ. Let $B(z) = \prod_1^n (z-\beta_i)$, where the points β_i lie in $\mathtt{ext}(\Gamma)$. If r is the corresponding RIPP of f, then*

$$f(z) - r(z) = \frac{1}{2\pi i}\frac{B(z)}{Q(z)}\int_\Gamma \frac{Q(t)}{B(t)}\frac{f(t)dt}{t-z}, \qquad z \in \mathtt{ext}(\Gamma),\ z \neq \alpha_k. \tag{6}$$

Proof: Fix $z \in \mathtt{ext}(\Gamma)$ and $R > 0$. Take $\Gamma_R = \Gamma \cap \{w : |w| \leq R\}$ and $C_R = \{|w| = R\} \cap \mathtt{ext}(\Gamma)$. If $R > |z|$ and $|\alpha_i| \neq R,\ 1 \leq i \leq n$, Theorem 2.3 gives us that

$$f(z) - r(z) = \frac{1}{2\pi i}\frac{B(z)}{Q(z)}\int_{\Gamma_R \cup C_R} \frac{Q(t)}{B(t)}\frac{f(t)dt}{t-z}.$$

We have that $\frac{Q(t)}{B(t)}$ is bounded in some neighborhood of infinity; that is, there exists $M > 0$ such that $\left|\frac{Q(t)}{B(t)}\right| < M$ if $|t| > M$. Furthermore, for $R > 2|z|$,

$$\int_{C_R}\left|\frac{f(t)}{t-z}\right||dt| \leq 2C\int_{C_R}\frac{|dt|}{R^{1+\alpha}} \leq \frac{4\pi C}{R^\alpha}.$$

Making R tend to infinity, we conclude the proof. ♠

We are ready for the proof of the main result.

Theorem 4.2 *Assume that f and Γ satisfy the assumptions above, where $\Gamma = \{z : \Re z = -a\}$ for some $a > 0$. Let $\{Q_n\}$ be a sequence of polynomials with no zeros in $\{\Re z \geq -a\}$, and let the points $\{\beta_{n,k}\}$ lie in $\overline{\mathcal{C}}_+$. Assume that the sequence*

$$A_n(t,z) := \frac{B_n(z)}{B_n(t)}\frac{Q_n(t)}{Q_n(z)} \tag{7}$$

converges uniformly to zero on bounded subsets of $\Gamma \times i\mathcal{R}$, and is uniformly bounded on this set. Then the corresponding sequence $\{r_n\}$ of RIPP converges uniformly to f on $i\mathcal{R}$.

Remark 4.3 *It is known (see [?]) that the functions of the form* (5) *have a finite number of poles in any set of the form $\{\Re z > l,\ l \in \mathcal{R}\}$, thus the stable functions of this form satisfy the hypothesis of the Theorem.*

Before proving this theorem it is convenient to establish the following auxiliary result.

Lemma 4.4 *Let f and Γ be as in Theorem 4.2. Then*

$$\sup_{y\in\mathcal{R}}\int_{im-a}^{i\infty-a}\left|\frac{f(t)}{t-iy}\right||dt| = O\left(\frac{\log m}{m^\alpha}\right).$$

Proof: Fix $y \in \mathcal{R}$ and $m \in \mathcal{N}$ sufficiently large. The proof consists in showing that if either $y \leq m-1$ or $y \geq m-1$ the integral in the expression is bounded by the quantity indicated.

Let $y \leq m-1$, we have that

$$\begin{aligned}\int_{im-a}^{i\infty-a} \left|\frac{f(t)}{t-iy}\right| |dt| &\leq \sum_{k=0}^{\infty} \frac{C}{(m+k)^\alpha (k+1)} \leq \frac{C}{m^\alpha} \sum_{k=1}^{m} \frac{1}{k} + 2C \sum_{k=2m}^{\infty} \frac{1}{k^{\alpha+1}} \\ &\leq C \frac{1+\log m}{m^\alpha} + \frac{2C}{\alpha(2m-1)^\alpha} \leq C_1 \frac{\log m}{m^\alpha}, \qquad (8)\end{aligned}$$

where C is the constant of the assumptions. We have used here that

$$\sum_{k=2}^{m} \frac{1}{k} \leq \int_1^m \frac{dx}{x} = \log m.$$

If $y \geq m-1$, then

$$\int_{im-a}^{i\infty-a} \left|\frac{f(t)}{t-iy}\right| |dt| \leq \int_{im-a}^{i(y-1)-a} \left|\frac{f(t)}{t-iy}\right| |dt| + \int_{i(y-1)-a}^{i(y+1)-a} \left|\frac{f(t)}{t-iy}\right| |dt| + \int_{i(y+1)-a}^{i\infty-a} \left|\frac{f(t)}{t-iy}\right| |dt|.$$

As in (8), we obtain that

$$\int_{i(y+1)-a}^{i\infty-a} \left|\frac{f(t)}{t-iy}\right| |dt| \leq \sum_{k=1}^{\infty} \frac{C}{(y+k)^\alpha k} \leq C_1 \frac{\log(y+1)}{(y+1)^\alpha} \leq C_1 \frac{\log m}{m^\alpha}.$$

Using the change of variables $ix - a = t$ and taking into account that $|t - iy| \geq a$ along the path of integration, we also find that

$$\int_{i(y-1)-a}^{i(y+1)-a} \left|\frac{f(t)}{t-iy}\right| |dt| \leq \frac{C}{a} \int_{y-1}^{y+1} \frac{dx}{x^\alpha} \leq \frac{2C}{a(y-1)^\alpha} \leq \frac{2C}{a(m-2)^\alpha} \leq \frac{C_2}{m^\alpha}.$$

Moreover,

$$\int_{im-a}^{i(y-1)-a} \left|\frac{f(t)}{t-iy}\right| |dt| \leq \sum_{k=1}^{p} \frac{C}{(y-k-1)^\alpha k},$$

where $p \in \mathcal{N}$ is such that $m-1 < y-p-1 \leq m$.

It rests to estimate the sum in the right hand of the last inequality. To this end, we consider two cases. For $m-1 \leq y \leq 2m$, from the conditions on p it follows that $y-k-1 > m-1$ and $p < m$. Thus

$$\sum_{k=1}^{p} \frac{C}{(y-k-1)^\alpha k} \leq \frac{C}{(m-1)^\alpha} \sum_{k=1}^{m} \frac{1}{k} \leq C_3 \frac{\log m}{m^\alpha}.$$

On the other hand, if $y > 2m$, then

$$\begin{aligned}\sum_{k=1}^{p} \frac{C}{(y-k-1)^\alpha k} &= \sum_{k=1}^{\left[\frac{y}{2}\right]-1} \frac{C}{(y-k-1)^\alpha k} + \sum_{k=\left[\frac{y}{2}\right]}^{p} \frac{C}{(y-k-1)^\alpha k} \\ &\leq \sum_{k=1}^{\left[\frac{y}{2}\right]-1} \frac{C}{\left[\frac{y}{2}\right]^\alpha k} + C \frac{p - \left[\frac{y}{2}\right] + 1}{\left[\frac{y}{2}\right](y-p-1)^\alpha} \\ &\leq C_1 \frac{\log\left[\frac{y}{2}\right]}{\left[\frac{y}{2}\right]^\alpha} + C \frac{2\left[\frac{y}{2}\right] + 3}{\left[\frac{y}{2}\right](m-1)^\alpha} \leq C_4 \frac{\log m}{m^\alpha}.\end{aligned}$$

In the last bound we use the fact that the function $x^{-\alpha}\log x$ is strictly decreasing for $x = \left[\frac{y}{2}\right] > e^{1/\alpha}$ and this inequality holds for all $y > 2m$ for sufficiently large m. Since the constants C_1 to C_4 may be taken independent of m and y the proof is complete. ♠

Let us prove Theorem 4.2.

Proof: Let $\epsilon > 0$ be given. According to Lemma 4.4, there exists $m \in \mathcal{N}$ such that

$$\frac{1}{\pi}\left|\int_{\Gamma\setminus\Gamma_m} A_n(t,z)\frac{f(t)dt}{t-z}\right| \le \frac{M}{\pi}\int_{\Gamma\setminus\Gamma_m}\left|\frac{f(t)}{t-z}\right||dt| < \epsilon, \qquad z \in i\mathcal{R}, \tag{9}$$

where M is a uniform bound for A_n, and $\Gamma_m := \{t : t = ix - a,\ -m \le x \le m\}$.

Since Γ_m is a compact set, there exists a compact set $K \subset i\mathcal{R}$ such that

$$|t-z| \ge \frac{2M}{\pi\epsilon}\|f\|_\Gamma \mathtt{length}(\Gamma_m), \qquad z \in i\mathcal{R}\setminus K, \qquad t \in \Gamma_m.$$

That is,

$$\left|\frac{1}{2\pi i}\int_{\Gamma_m} A_n(t,z)\frac{f(t)dt}{t-z}\right| \le \frac{\epsilon}{2},\ z \in i\mathcal{R}\setminus K. \tag{10}$$

Furthermore, from the unifrom convergence to zero of $A_n(t,z)$ on compact subsets of $\Gamma \times i\mathcal{R}$, there exists $n_\epsilon \in \mathcal{N}$ such that for all $n > n_\epsilon$ and all $(t,z) \in \Gamma_m \times K$, $|A_n(t,z)| \le \frac{\pi a \epsilon}{\|f\|_\Gamma \mathtt{length}(\Gamma_m)}$, thus

$$\left|\frac{1}{2\pi i}\int_{\Gamma_m} A_n(t,z)\frac{f(t)dt}{t-z}\right| \le \frac{\epsilon}{2}, \qquad z \in K,\ n > n_\epsilon. \tag{11}$$

Combining (9), (10) and (11) with the integral expression for the error given by Theorem 4.1, we conclude the proof. ♠

Remark 4.5 *Let $T(z) = \frac{pz+q}{rz+s}$ be an arbitrary bilinear transformation. It is well known that*

$$\frac{T(z)-T(b)}{T(z)-T(a)}\cdot\frac{T(t)-T(a)}{T(t)-T(b)} = \frac{z-b}{z-a}\cdot\frac{t-a}{t-b}$$

for any complex numbers z, t, a, b. Thus the problem of finding a sequence A_n satisfying the conditions of Theorem 4.2 may be reduced to an equivalent problem on the unit disk by taking a convenient T which transforms the right plane into the unit circle. With the aid of Example 3.3 and Remark 3.5, we can find ways of constructing such points. For example, taking $T(z) = \frac{1-z}{1+z}$, from Remark 3.5 it follows that a sufficient condition on the zeros $\{\alpha_{n,k}\}$ of Q_n in order that the conditions on A_n be satisfied is that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(1 - \left|\frac{1+\alpha_{n,k}}{1-\alpha_{n,k}}\right|\right) = \infty.$$

Notice that the main condition on A_n has an asymptotic character. Therefore, some of the poles and interpolation points may be selected in a fairly arbitrary manner. This is of particular interest if one knows the location of some (or all) of the zeros and poles of the function we wish to approximate.

The asymptotic condition on the zeros of the polynomials Q_n may be substituted by the knowledge that a certain sequence of rational functions whose denominators are precisely the Q_n converges to the function to be approximated. This is useful to know when one wishes to try out other interpolation schemes in order to accelerate the convergence process.

Theorem 4.6 *Let f and Γ be as in Theorem 4.2. Let $R_n = \frac{P_n}{Q_n}$, $\deg P_n < \deg Q_n = n$, be a sequence of rational functions with no poles on $\overline{\mathcal{C}}_+ \cup \Gamma$ such that $\{R_n\}$ converges uniformly to f on compact subsets of Γ, and $|R_n(z)| \leq \frac{c_1}{|z|^\gamma}$ for some $c_1, \gamma > 0$ and all $z \in \Gamma$, $n \in \mathcal{N}$. Let the points $\{\beta_{n,k}\}$, $1 \leq k \leq n$, lie in $\overline{\mathcal{C}}_+$. If A_n is uniformly bounded on $\Gamma \times i\mathcal{R}$, then the corresponding sequence of RIPP converges uniformly to f on $\overline{\mathcal{C}}_+$.*

Proof: From Theorem 4.1 (applied to R_n) and the uniqueness of the interpolating rational function established in Theorem 2.1, we have that

$$R_n(z) - R_n(z) = \frac{1}{2\pi i} \int_\Gamma A_n(t,z) \frac{R_n(t)dt}{t-z} = 0.$$

Therefore,

$$f(z) - r_n(z) = \frac{1}{2\pi i} \int_\Gamma A_n(t,z) \frac{f(t) - R_n(t)}{t-z} dt.$$

From this point on, the proof runs as the one given for Theorem 4.2. In proving an inequality similar to (11), we use that $\{R_n\}$ converges uniformly to f on Γ_m taking advantage of the fact that $f(t) - R_n(t)$ stands under the integral sign in the previous formula. ♠

As an application, we have the following

Corollary 4.7 *Let $a_n > cn^{1+\alpha}$ for some $c, \alpha > 0$ and all $n \in \mathcal{N}$, and let $\{b_n\}$ be a bounded sequence of numbers. Let $f(z) = \sum_{n=1}^\infty \frac{b_n}{z+a_n}$, and let $B_n(z) = (-1)^n Q_n(-z)$, $Q_n(z) = \prod_{k=1}^n (z + a_n)$. Then the corresponding sequence of RIPP converges uniformly to f on $\overline{\mathcal{C}}_+$.*

Proof: From the conditions on a_n and b_n, it is obvious that $f(z)$ defines a meromorphic function in $\mathcal{C} \setminus \{a_n\}$ which is analytic in $\Re z \geq -c$. Let $\Gamma = \{z : \Re z = -\frac{c}{2}\}$, where c is the constant given in the statement. We have that $\left|\frac{B_n(z)}{Q_n(z)}\right| = 1$ on $i\mathcal{R}$, and $\left|\frac{Q_n(z)}{B_n(z)}\right| < 1$ on Γ. Therefore, A_n is uniformly bounded on $\Gamma \times i\mathcal{R}$.

Now, we want to show that $|f(z)| \leq \frac{c_1}{|z|^\beta}$ for some $c_1, \beta > 0$ and all $z \in \mathtt{ext}(\Gamma)$, $|z| > 1$. We have that $|z + a_n| \geq |z|$ for all $z \in \mathtt{ext}(\Gamma)$ and all $n > 0$. Moreover, $a_n \leq |z| + |z + a_n|$, thus $3|z + a_n| \geq |z| + a_n, z \in \mathtt{ext}(\Gamma)$. Let $|b_n| \leq M, n \in \mathcal{N}$. It follows that for $\{|z| = R\} \cap \mathtt{ext}(\Gamma)$,

$$\begin{aligned} |f(z)| &\leq M \sum_{n=1}^\infty \frac{1}{|z+a_n|} \leq 3M \sum_{n=1}^\infty \frac{1}{R+a_n} = 3M \left(\sum_{n=1}^k \frac{1}{R+a_n} + \sum_{n=k+1}^\infty \frac{1}{R+a_n} \right) \\ &\leq 3M \left(\sum_{n=1}^k \frac{1}{R} + \sum_{n=k+1}^\infty \frac{1}{a_n} \right) \leq 3M \left(\frac{k}{R} + \frac{1}{c} \int_k^\infty \frac{dt}{t^{1+\alpha}} \right) = 3M \left(\frac{k}{R} + \frac{1}{c\alpha k^\alpha} \right). \end{aligned}$$

Let $R \geq 1$. Take $k \in \mathcal{N}$ such that $(k-1)^{1+\alpha} \leq R < k^{1+\alpha}$. Then

$$(k-1)^{1+\alpha} \leq R \Rightarrow k \leq R^{\frac{1}{1+\alpha}} + 1 < 2R^{\frac{1}{1+\alpha}} \Rightarrow \frac{k}{R} \leq \frac{2}{R^{\frac{\alpha}{1+\alpha}}}.$$

Moreover, $R < k^{1+\alpha}$ implies that $\frac{1}{k^\alpha} < \frac{k}{R} \leq \frac{2}{R^{\frac{\alpha}{1+\alpha}}}$. Therefore, making $R = |z|$ vary from 1 to infinity, from the estimate above, it follows that

$$|f(z)| \leq 3M\left(\frac{k}{|z|} + \frac{1}{c\alpha}\frac{1}{k^\alpha}\right) < \frac{c_1}{|z|^{\frac{\alpha}{1+\alpha}}}, \qquad \{|z| \geq 1\} \cap \mathtt{ext}(\Gamma),$$

with $c_1 = 3M(2 + \frac{2}{c\alpha})$.

Obviously, we also have that

$$|R_n(z)| \leq M\sum_{n=1}^{\infty}\frac{1}{|z+a_n|} < \frac{c_1}{|z|^{\frac{\alpha}{1+\alpha}}}, \qquad \{|z| > 1\} \cap \mathtt{ext}(\Gamma),$$

for all $n \in \mathcal{N}$, where $R_n(z) = \sum_{k=1}^{n}\frac{b_k}{z+a_k}$. Clearly, $\{R_n\}$ converges unifomly to f on compact subsets of Γ, thus all the hypothesis of Theorem 4.6 are satisfied. ♠

5 Convergence rates

From the proof of Theorem 3.2 (resp. Theorem 4.2) it follows that in order to estimate the speed of convergence of the RIPP we need to know more precisely the asymptotic behavior of A_n on compact subsets of $\Gamma \setminus \{-1\} \times \partial D$ (resp. $\Gamma \times i\mathcal{R}$), and bounds on f on a neighborhood of $\{-1\}$ (resp. $\{\infty\}$). The results of this section show several ways as to how to carry on. We consider mainly the case of the unit circle developed in section 3 and state the corresponding results for the right half plane studied in section 4. First, let us give a technical result which provides some bounds on the convergence to zero of the functions A_n.

Lemma 5.1 *We have:*

1. *Let* $Q_n = z^n - a^n,\ B_n = z^n - b^n$, *with* $|b| < 1 < |a|$. *Let* $\Gamma_n \subset \{z : 1 + \frac{1}{n^r} \leq |z| \leq M\}$ *for some* $r < 1,\ M < |a|$. *Then* $\|A_n\|_{\Gamma_n \times \partial D} = O\left(e^{-n^{1-r}}\right)$.
2. *More generally, let*

$$A_n(t,z) = \prod \frac{\overline{\alpha}_{n,i}z - 1}{z - \alpha_{n,i}} \prod \frac{t - \alpha_{n,i}}{\overline{\alpha}_{n,i}t - 1},$$

where $\{\alpha_{n,k}\}$, $1 \leq k \leq n$, *satisfies* $|\alpha_{n,k}| \geq a > 1$ *for some* a. *Let* $\Gamma_n \subset \{z : \sqrt{1 + \frac{1}{n^r}} \leq |z| \leq M\}$ *for some* $r < 1,\ M < a$. *Then, there exists* $q \in (0,1)$ *such that* $\|A_n\|_{\Gamma_n \times \partial D} = O\left(q^{n^{1-r}}\right)$.

Proof: First we consider the rather particular choise of A_n proposed in the first part of the Lemma and then look at the more general case.

1. For all sufficiently large n and all $(t,z) \in \Gamma_n \times \partial D$, we have that

$$|A_n(t,z)| = \left| \frac{\left(\frac{t}{a}\right)^n - 1}{\left(\frac{z}{a}\right)^n - 1} \cdot \frac{z^n - b^n}{t^n - b^n} \right| < \frac{2}{t^n} \leq \frac{2}{(1+\frac{1}{n^r})^n} < \frac{4}{e^{n^{1-r}}}.$$

   Thus the estimate holds.

2. It is easy to verify that

$$\left| \frac{t-\alpha}{\overline{\alpha}t - 1} \right|^2 = 1 - \frac{|\alpha|^2 - 1}{|\overline{\alpha}t-1|^2}(|t|^2 - 1).$$

   The function $(t,\alpha) \to \frac{|\alpha|^2-1}{|\overline{\alpha}t-1|^2}$ is continuous and different from zero on the compact set $\{t : 1 \leq |t| \leq M\} \times \{\alpha \in \hat{\mathcal{C}} : |\alpha| \geq a\}$. Therefore, there exists $m > 0$ such that $\frac{|\alpha|^2-1}{|\overline{\alpha}t-1|^2} \geq m$ for all (t,α) on the indicated set. It follows that for all sufficiently large n and $(t,z) \in \Gamma_n \times \partial D$

$$|A_n(t,z)|^2 \leq \left(1 - m(|t|^2-1)\right)^n \leq \left(1 - \frac{m}{n^r}\right)^{n^r n^{1-r}} < 2\left(e^{-m}\right)^{n^{1-r}}.$$

With this we conclude the proof. ♠

Corollary 5.2 *Under the assumptions of Section 3, let $f(z) = O(z+1)^\alpha$ in a neighborhood of $\{-1\}$, with $\alpha > 0$. Take the polynomials Q_n and B_n as indicated in the first part of Lemma 5.1. The corresponding sequence of RIPP satisfies*

$$\|f - r_n\|_{\overline{D}} = o(n^{-\beta})$$

for all $\beta < \alpha$.

Proof: Let Γ be as in the Assumptions of Section 3. Take Γ so that $\Gamma \cap V$ are line segments for some neighborhood V of $\{-1\}$. Fix $r \in (0,1)$, and let $\Gamma_n = \Gamma \cap \{z : |z| \geq 1 + \frac{1}{n^k}\}$. Since Q_n and B_n are as in Lemma 5.1 (1), then

$$\left| \int_{\Gamma_n} A_n(t,z) \frac{f(t)dt}{t-z} \right| = O\left(e^{-n^{1-r}}\right) = o(n^{-\alpha}).$$

Furthermore, there exists $M > 0$ such that for all sufficiently large n

$$\int_{\Gamma \setminus \Gamma_n} \left| \frac{f(t)}{t+1} \right| |dt| \leq M \int_0^{\frac{1}{n^k}} t^{\alpha-1} dt = \frac{M}{\alpha} n^{-\alpha k}.$$

Since $r \in (0,1)$ is arbitrary, this completes the proof. ♠

Remark 5.3 *The same results can be obtained with B_n and Q_n as in Lemma 5.1 (2), taking $\Gamma_n = \Gamma \cap \{z : |z| \geq \sqrt{1 + \frac{1}{n^k}}\}$.*

Making use of the linear fractional transformation $T(z) = \frac{1-z}{1+z}$ (see Remark 4.5), we obtain the following

Corollary 5.4 *Under the assumptions of Theorem 4.2, the polynomials Q_n and B_n may be chosen so that*

$$\|f - r_n\|_{\overline{\mathcal{C}}_+} = o(n^{-\beta})$$

for all $\beta < \alpha$.

In addition to Corollary 5.2 (resp. Corollary 5.4), we can also obtain geometric rate of convergence on compact subsets of D (resp. $\mathcal{C}_+$). To this end, denote by β_n the *normalized counting measure* of the points $\beta_{n,k}$. More precisely, $\beta_n := \frac{1}{n}\sum_{k=1}^{n} \delta_{\beta_{n,k}}$, where δ_w is the measure of mass 1 concentrated at $\{w\}$. Then, we have

Theorem 5.5 *Assume that the conditions of Theorem 3.2 are satisfied. Let the sequence of normalized counting measures $\{\beta_n\}$ have a finite number of weak* limit points, and assume that the support of each limit point is not entirely contained in ∂D. Then*

$$\limsup_n \|f - r_n\|_K^{\frac{1}{n}} < 1, \qquad K \subset D \text{ compact.}$$

Proof: Let $g_D(z,w)$ be the Green function of D with singularity at $w \in D$. It is well known and easy to verify that $g_D(z,w) = \log\left|\frac{1-\overline{w}z}{z-w}\right|$. For $z \in \partial D$, we have

$$\log|(f - r_n)(z)| + \sum_{k=1}^{n} g_D(z, \beta_{n,k}) \le \log \|f - r_n\|_{\partial D}. \qquad (12)$$

The left hand side of (12) is a subharmonic function in D, thus by the Maximum Principle (12) is valid for all $z \in \overline{D}$. It follows that

$$|(f - r_n)(z)| \le \|f - r_n\|_{\partial D}\, e^{-\sum_{k=1}^{n} g_D(z,\beta_{n,k})}, \qquad z \in \overline{D},$$

that is,

$$|(f - r_n)(z)|^{\frac{1}{n}} \le \|f - r_n\|_{\partial D}^{\frac{1}{n}}\, e^{-\int g_D(z,w) d\beta_n(w)}, \qquad z \in \overline{D}.$$

Let β be a weak* limit point of $\{\beta_n\}$. That is, $\beta_{n_j} \xrightarrow{*} \beta$ for some sequence $\{n_j\} \subset \mathcal{N}$. Then

$$\limsup_j |(f - r_{n_j})(z)|^{\frac{1}{n_j}} \le e^{-\int g_D(z,w) d\beta(w)}, \qquad z \in \overline{D}.$$

Thus, if $K \subset D$ is a compact set, then

$$\limsup_j \|(f - r_{n_j})(z)\|_K^{\frac{1}{n_j}} \le e^{-\inf_{z\in K}\int g_D(z,w) d\beta(w)} < 1,$$

and the result follows. ♠

Analogously, we can obtain the following

Theorem 5.6 *Assume that the conditions of Theorem 4.2 are satisfied. Let the sequence β_n have a finite number of weak* limit points, and assume that the support of each limit points is not a subset of $i\mathcal{R} \cup \{\infty\}$. Then*

$$\limsup_n \|f - r_n\|_K^{\frac{1}{n}} < 1, \qquad K \subset \mathcal{C}_+ \text{ compact.}$$

Remark 5.7 *Under the Assumptions of Section 3 we can find $c > 0$ sufficiently small such that $D_c^* := \{z : |z - c| \le c + 1\}$ is entirely contained in* $\mathtt{int}(\Gamma)$. *Now applying the results and methods of Section 3 and Theorem 5.5 to this set in place of D, we can achieve geometric rate of convergence of the RIPP on compact subsets of $\overline{D} \setminus \{-1\}$.*

Similarly, applying the results and methods of Section 4 and Theorem 5.6 to a region $\{z : \Re z > -l\}$, with $l > 0$ sufficiently small, we obtain a geometric rate of convergence of the RIPP on compact subsets of $\overline{\mathcal{C}}_+$.

6 Numerical examples

In this section, we illustrate with some numerical results obtained applying the method indicated in the previous sections to the examples given in the Introduction.

6.1 Fractional Filters

Let $f(z) = (1+z)^{\frac{3}{4}}$. This function was used by Ward and Partington in [?] to illustrate their method of approximation based on rational wavelets.

For the denominator of our RIPP, we take $Q_n(z) = L_n(1-z)$, where L_n is the n^{th} monic Laguerre polynomial. The interpolation points $\{\beta_{n,k}\}$ are taken according to Corollary 3.4. It is known that there exists $c > 0$ such that $x_{n,v} > \frac{c}{n}$ for the zeros $x_{n,v}$ of L_n (see e.g. [?]). From Remark 3.5, it follows that the zeros of Q_n satisfy (4). The results are given in Figure 1 (the scale in the Y-axis is logarithmic). For comparison, we include the numerical results given by Ward and Partington by their method, as well as the $\mathcal{H}_\infty$ errors of the truncated power series.

Figure 1: $\mathcal{H}_\infty$-errors of approximations of the fractional filter $f(z) = (1+z)^{\frac{3}{4}}$.

As can be seen, for $n \ge 4$ the RIPP whose denominators are the analogous of the Laguerre polynomials in $(-\infty, -1)$ give better approximations in this example.

6.2 Heat transfer systems

Now, we apply the method proposed to the function G_s given in Example 1.3. Notice that in this case the hypothesis of Corollary 4.7 are satisfied.

We compute the RIPP for $x_1 = 1$; the results are given in Figure 2. For comparison, we also include the $\mathcal{H}_\infty$ error given by the partial sums of the series which defines G_s.

Figure 2: $\mathcal{H}_\infty$ errors of the approximations of $G_s(s) = 2\sum_{n=1}^{\infty} \frac{1}{s+(n\pi)^2}$ using Padé-type approximants and partial sums.

As can be observed, the RIPP give better estimates.

6.3 Delay systems

Now let $G(s) = \frac{e^{-s}}{s+1}$. This function was used in [?] to compare various methods of approximation.

For the denominator of our RIPP, we take $Q_n(z) = (z+1)P_{n-1}(z)$, where P_n is the denominator of the $[n/n]$ Padé approximant of e^{-z}. As before, we take $B_n(z) = Q_n(-z)$. We know (see [?]) that $\Re x_{n,v} < -2$ and $|x_{n,v}| < 2n$ for all $n > 3$; thus, (see Remark 4.5) the conditions of Theorem 4.2 are satisfied.

The results are given in Figure 3. We have included the numerical results given in [?] for Hankel norm approximants and the method based on classical Padé approximations of the exponential.

In this example, optimal Hankel-norm approximation gives better results. Nevertheless, the numerical complexity of their computation is high. Therefore, we think that RIPP give a valid alternative.

Figure 3: $\mathcal{H}_\infty$ errors of the approximations of the delay system $f(z) = \frac{e^{-s}}{s+1}$.