Multipoint Rational Approximants with Preassigned Poles

Francisco Cala Rodríguez(*) and Guillermo López Lagomasino(+)

Dpto. de Análisis Matemático
Universidad de La Laguna
38271 La Laguna, Tenerife
Islas Canarias, Spain

Departamento de Matemáticas
Universidad Carlos III de Madrid
c/ Butarque, 28911 Leganés
Madrid, Spain

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Galley proofs should be sent to:

Guillermo López Lagomasino
Departamento de Matemáticas
Universidad Carlos III de Madrid
c/ Butarque 15, 28911 Leganés
Madrid, Spain
e-mail: lago@math.uc3m.es

Abstract

Let μ be a finite positive Borel measure whose support $S(\mu)$ is a compact regular set contained in $\mathbb{R}$. For a function of Markov type

$$\hat{\mu}(z) = \int\limits_{S(\mu)} \frac{d\mu(x)}{z-x} \ , \quad z \in \mathbb{C} \setminus S(\mu),$$

we consider Multipoint Padé-type Approximants (MPTAs) , where some poles are preassigned and interpolation is carried out along a table of points contained in $\overline{\mathbb{C}} \setminus Co(S(\mu))$ which is symmetrical with respect to the real line. The main purpose of this paper is the study of the "exact rate of convergence" of the MPTAs to the function $\hat{\mu}$.

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1. Introduction

1.1 - Some definitions. Let μ be a finite positive Borel measure whose support $S(\mu)$ is a compact set contained in $\mathbb{R}$. Set

$$\hat{\mu}(z) = \int\limits_{S(\mu)} \frac{d\mu(x)}{z-x} \ , \quad z \in \mathbb{C} \setminus S(\mu). \tag{1}$$

Let $\{L_n\}, \ n \in \mathbb{N}$, be a sequence of monic polynomials whose zeros lie in $Co(S(\mu))$, the convex hull of $S(\mu)$ such that $deg \ L_n = k(n) \leq n$. We fix another family of polynomials

$$A_{2n}(z) = \prod_{i=1}^{2n} (z - \alpha_{n,i}) \ , \quad n \in \mathbb{N}, \tag{2}$$

whose zeros

$$\{\alpha_{n,i}\} \quad , \quad i = 1, 2, ..., 2n \ , \quad n \in \mathbb{N}, \tag{3}$$

are contained in a compact set $A \subset \overline{\mathbb{C}} \setminus Co(S(\mu))$ and lie symmetrically with respect to the real line (counting multiplicities). In case that for some $i, \quad \alpha_{n,i} = \infty$, the corresponding factor must be omitted.

We also assume the existence of a measure α, (supp $\alpha \subset A$), such that

$$\frac{1}{2n}\sigma(A_{2n}) \overset{*}{\longrightarrow} \alpha \quad , \qquad as \qquad n \longrightarrow \infty \tag{4}$$

(in the weak star topology), where $\sigma(A_{2n})$ denotes the zero counting measure of the polynomial A_{2n}. That is

$$\sigma(A_{2n}) = \sum_{i=1}^{2n} \delta_{\alpha_{n,i}},$$

where $\delta_{\alpha_{n,i}}$ denotes the Dirac measure with mass 1 at $\alpha_{n,i}$. A sequence of measures $\{\nu_n\}$ is said to converge *weakly* (or in the weak star topology) to a measure ν (in $\overline{\mathbb{C}}$), if for every function f continuous in $\overline{\mathbb{C}}$, we have

$$\int f d\nu_n \longrightarrow \int f d\nu \quad , \qquad as \quad n \to \infty$$

(see, for instance, [7], Chapter 1 , page 7).

In the following, we restrict our attention to the type of interpolating rational functions whose asymptotics we shall study. For the definition of Multipoint Padé-type approximants, some of the restrictions we impose are not necessary, but they arise naturally in the location of the zeros of the orthogonal polynomials with respect to varying measures connected with Padé-type approximation.

Definition 1. The Multipoint Padé-type Approximant (MPTA) of $\hat{\mu}$ with preassigned poles at the zeros of a given polynomial L_n^2, $deg\ L_n = k(n) \leq n$, $L_n \not\equiv 0$, which interpolates the function $\hat{\mu}$ at the zeros of the polynomial A_{2n} with $deg\ A_{2n} \leq 2n$, given by (2), is the unique rational function

$$R_n = \frac{P_n}{Q_n L_n^2}, \tag{5}$$

where P_n and Q_n are polynomials satisfying:

(i) $deg\ P_n \leq n + k(n) - 1$, $deg\ Q_n \leq n - k(n)$, and $Q_n \not\equiv 0$.

(ii) $\dfrac{Q_n L_n^2 \hat{\mu} - P_n}{A_{2n}} \in H(\mathbb{C} \setminus S(\mu))$, where $H(\mathbb{C} \setminus S(\mu))$ denotes the set of all holomorphic functions defined on $\mathbb{C} \setminus S(\mu)$.

(iii) $\dfrac{Q_n L_n^2 \hat{\mu} - P_n}{A_{2n}}\ (z) = \mathcal{O}\left(\dfrac{1}{z^{n-k(n)+1}}\right), \qquad as \quad z \longrightarrow \infty.$

The MPTA given by (5) has order $n+k(n)$ and the error of approximation to the function $\hat{\mu}$ satisfies

$$r_n(z) = (\hat{\mu} - R_n)(z) = \frac{A_{2n}(z)}{(L_nQ_n)^2(z)} \int \frac{(L_nQ_n)^2(x)}{A_{2n}(x)} \frac{d\mu(x)}{z-x}, \qquad z \in \overline{\mathbb{C}} \setminus S(\mu). \quad (6)$$

This formula is easy to prove using (iii) and Cauchy's integral formula (see,e.g. [3], Lemma 1).

1.2 - The contents of this paper. It is known that a polynomial p_n of degree n is orthonormal with respect to μ if

$$1 = ||p_n||_2 \leq ||p_n + v||_2 \quad (7)$$

for all polynomials v satisfying $deg\ v \leq n-1$, where $||\ .\ ||_2$ denotes the L^2_μ-norm

$$||\ .\ ||_2 := \left(\int |\ .\ |^2 d\mu \right)^{\frac{1}{2}} . \quad (8)$$

It is well-known (see e.g. [7], Theorem 3.1.1, Chap.4) that for a very general class of measures μ , the orthonormal polynomials p_n satisfy the following limit relation

$$\lim_{n\to\infty} |p_n(z)|^{\frac{1}{n}} = exp\ \{g_\Omega(z;\infty)\}\ , \quad (9)$$

uniformly on compact subsets of $\mathbb{C} \setminus Co(S(\mu))$, where $g_\Omega(z;\infty)$ denotes the Green function of $\Omega = \overline{\mathbb{C}} \setminus S(\mu)$ with pole at infinity. In this case μ is called a regular measure and it is denoted by $\mu \in Reg$ (for other equivalent forms of defining the class of measures Reg, see [7], Chapter 3, page 61). Furthermore, for this class of measures, the zeros of p_n have regular asymptotic behavior in the sense that, as n tends to infinity, the normalized zero counting measure of p_n converges in the weak star topology to the equilibrium measure of $S(\mu)$ for the logarithmic potentials (see [7], Theorem 3.1.4).

Recently, A. Ambroladze and H. Wallin extended this result to the case when $\mu \in Reg$ and the zeros of p_n are partially fixed. More precisely, instead of p_n they considered polynomials of the form $\tilde{p}_n := \tilde{q}_n T_n$ where T_n is a monic polynomial with all its zeros in $Co(S(\mu))$, $deg\ T_n = k(n)$, $k(n) \leq n$, and $\tilde{q}_n$ is a polynomial of degree $n - s(n)$ satisfying the following condition, analogous to (7)

$$1 = ||\tilde{q}_n T_n||_2 \leq ||(\tilde{q}_n + v)\ T_n||_2 \quad (10)$$

for all polynomials v satisfying $deg\ v < deg\ \tilde{q}_n$ (If $deg\ \tilde{q}_n = 0$, then $v = 0$ in (10)). This means that $\tilde{q}_n$ is the orthonormal polynomial of degree $n - k(n)$ with respect to the measure $|T_n|^2 d\mu, \mu \in Reg$. They showed that the polynomials $\tilde{p}_n = \tilde{q}_n T_n$ have the same n^{th} root asymptotic behavior as p_n defined by (7), under fairly weak conditions on the sequence $\{T_n\}$ (under more restrictive conditions on μ this result was previously obtained in [2]).

Here, we consider the polynomial $\hat{p}_n = q_n L_n$ where L_n is a monic polynomial with all its zeros in $Co(S(\mu))$, such that $deg\ L_n = k(n) \leq n$, and q_n is a polynomial of degree $n - k(n)$ satisfying the following conditions, analogous to (10):

$$1 = \left\| \frac{q_n L_n}{|A_{2n}|^{\frac{1}{2}}} \right\|_2 \leq \left\| \frac{(q_n + v) L_n}{|A_{2n}|^{\frac{1}{2}}} \right\|_2 \tag{11}$$

for all polynomials v satisfying $deg\ v < deg\ q_n$ (if $deg\ q_n = 0$, then $v \equiv 0$). This means that q_n is the orthonormal polynomial of degree $n - k(n)$ with respect to the varying measure

$$\frac{|L_n|^2}{|A_{2n}|} d\mu. \tag{12}$$

This paper deals with the problem of finding sufficient conditions on the measure μ, on the preassigned polynomials L_n, and on the choice of the interpolation points ($\alpha_{n,i}$, $i = 1, 2, ..., 2n$, $n \in \mathbb{N}$) in order to have exact rate for the asymptotic behavior of $\hat{p}_n$ and r_n.

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2. Statement of Results

2.1 - Notation. The following notation will be used throughout this paper.

μ	A finite positive Borel measure on $\mathbb{R}$ with non-empty compact and regular support.
$S(\mu)$	Support of μ.
$Co(S(\mu))$	The convex hull of $S(\mu)$.

$\Omega = \Omega(\mu)$	$\Omega := \overline{\mathbb{C}} \setminus S(\mu)$.
L_n	A given polynomial with $deg\ L_n = k(n) \leq n$, whose zeros lie in $Co(S(\mu))$.
A_{2n}	A given polynomial with $deg\ A_{2n} \leq 2n$, whose zeros are contained in $\overline{\mathbb{C}} \setminus Co(S(\mu))$ and are symmetric with respect to the real line. Moreover, we assume, without loss of generality, that $A_{2n}(x) > 0 \quad \text{if} \quad x \in Co(S(\mu))$.
q_n	The orthonormal polynomial (with positive leading coefficient λ_n) of degree $n - k(n)$, with respect to the measure $(\lvert L_n\rvert^2/\lvert A_{2n}\rvert)d\mu$.
$\hat{p}_n$	A polynomial of degree n , defined by $\hat{p}_n := q_n L_n$.
$Z(.)$	The set of zeros of the polynomial ".".
$\sigma(.)$	The zero counting measure of the polynomial "." which puts mass 1 at each zero of the polynomial "."(counting multiplicities).
$\lvert\lvert.\rvert\rvert_2$	The L^2_μ-norm (see (8) above).
$\lvert\lvert.\rvert\rvert_\odot$	The supremum norm of "." on the set "$\odot$".
diam(.)	The diameter of the set ".".
d(.; $\diamond$)	The distance between "." and "$\diamond$".
cap(.)	The logarithmic capacity of the set ".".
$p(.;\ \oplus)$	The logarithmic potential of the measure "." at the point "$\oplus$".
$g_\triangleleft(z;.)$	The (generalized) Green function of "$\triangleleft$" with pole at the point ".".
$g_\triangleleft(z)$	$g_\triangleleft(z) := g_\triangleleft(z;\infty)$.
$G_\ominus(\nu;.)$	The Green potential of the measure "ν" in "$\ominus$"$\subseteq \overline{\mathbb{C}}$ at the point "." which is defined by $$G_\ominus(\nu;.) := \int_{S(\nu)} g_\ominus(.\,;\zeta)\ d\nu(\zeta).$$

$\nu_n \xrightarrow{*} \nu$ The sequence of measures $\{\nu_n\}$ converges "weakly" (or in the "*weak star topology*") to the measure ν (in $\overline{\mathbb{C}}$).

2.2 - Results. We denote by λ, the unit equilibrium measure of $S(\mu)$ in the presence of the external field $p_\alpha(x) = -p(\alpha; x)$. Since $S(\mu)$ is a regular compact set, then $cap\ S(\mu) > 0$ and the measure λ is characterized by the conditions

$$\begin{aligned} p(\lambda; z) - p(\alpha; z) &= \omega\ , \qquad z \in S(\lambda), \\ &\geq \omega\ , \qquad z \in S(\mu) \setminus S(\lambda), \end{aligned} \tag{13}$$

where α, with $S(\alpha) \subset A$, is the unit measure given by (4). Since $A \subset \overline{\mathbb{C}} \setminus Co(S(\mu))$ it is well known that for this type of external field λ is the balayage of α onto $S(\mu)$ (see, for instance [5], Chapter IV). Thus, $S(\lambda) = S(\mu)$ and equality takes place on all $S(\mu)$. Therefore,

$$p(\lambda; x) - p(\alpha; x) \equiv \omega \qquad \text{on} \qquad S(\mu). \tag{14}$$

The constant "ω" is called "the extremal constant" or "the equilibrium constant" (cf., for instance, [4]).

Let us introduce some restrictions on the fixed zeros. We assume that

$$\limsup_{n\to\infty} \frac{1}{n} \sigma(L_n) \leq \lambda, \tag{15}$$

in the weak star topology. By (15) we mean that for any positive continuous functions h in $\mathbb{C}$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \sigma(L_n)(h) \leq \lambda(h),$$

where $\lambda(h) := \int\limits_{S(\nu)} h d\lambda$. This condition was first introduced in [2] and later used also in [1] and [5]. We also assume that

$$Z(L_n) \subset Co(S(\mu)). \tag{16}$$

This restriction can be replaced by the slightly weaker necessary condition that all the limit points of the zeros of L_n lie in $Co(S(\mu))$.

THEOREM 1. *Let $\mu \in Reg$ be a measure with regular compact support $S(\mu) \subset \mathbb{R}$. Let $\{L_n\}, n \in \mathbb{N}$, be a sequence of polynomials, deg $L_n = k(n) \leq n$,with the* *properties* (15) *and* (16), *and* $\{A_{2n}\}$, $n \in \mathbb{N}$, *be a sequence of polynomials which satisfies* (3) *and* (4). *Let* $\hat{p}_n := q_n L_n$, *where* q_n *is defined as in* (11). *Then*

$$\lim_{n\to\infty} |\hat{p}_n(z)|^{\frac{1}{n}} = \exp\{\omega - p(\lambda; z)\}, \tag{17}$$

uniformly on compact subsets of $\mathbb{C} \setminus Co(S(\mu))$. *Furthermore, the zeros of* $\hat{p}_n$ *have a quasi-regular asymptotic distribution in the sense that*

$$\lim_{n\to\infty} \frac{1}{n}\sigma(\hat{p}_n) = \lambda, \tag{18}$$

where λ *is the unit measure which solves the extremal problem* (13).

From Theorem 1 and the error formula (6), the following theorem is obtained in a standard way (see, for instance [3], §3). Recall that A is a compact set contained in $\overline{\mathbb{C}} \setminus Co(S(\mu))$, symmetric with respect to the real line, and contains the table of points (3), which is also symmetric with respect to the real line and includes all the zeros of the polynomials A_{2n} given by (2).

THEOREM 2. Under the assumptions of Theorem 1, the MPTAs *of the function* $\hat{\mu}$ *with preassigned poles at the* $2k(n)$ *zeros of* L_n^2 *(such that* $n - k(n)$ *poles remain free) which interpolates* $\hat{\mu}$ *at the* $2n$ *zeros of* A_{2n}, *satisfy*

$$\lim_{n\to\infty} |r_n(z)|^{\frac{1}{2n}} = \exp\{-G_\Omega(\alpha; z)\}, \tag{19}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (A \cup Co(S(\mu)))$, *and*

$$\limsup_{n\to\infty} |r_n(z)|^{\frac{1}{2n}} = \exp\{-G_\Omega(\alpha; z)\}, \tag{20}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus Co(S(\mu))$ *of positive capacity.*

Remark 1. If $\alpha_{n;i} = \infty$ for every i=1,2,...,2n, then $G_\Omega(\alpha; z) = g_\Omega(z)$, λ represents the equilibrium measure of $S(\mu)$, and Theorem $1'$ in [1] is regained.

Remark 2. If the fixed poles represent limitwise a fixed proportion with respect to the order of the MPTA; that is, if

$$\lim_{n\to\infty} \frac{2k(n)}{n + k(n)} = \theta \in [0; 1], \tag{21}$$

then from Theorem 2 follows Theorem 1 in [3] under weaker conditions on the measure.

3. Proofs

3.1 - Some lemmas. In the sequel, we will assume without loss of generality that the set A which contains all the zeros of the polynomials A_{2n} is a compact subset of $\mathbb{C} \setminus Co(S(\mu))$. The reduction to this case may be achieved by means of a Möbius transformation of the variable in the initial problem, which transforms $S(\mu)$ into another compact subset of $\mathbb{R}$ and $A \subset \overline{\mathbb{C}} \setminus Co(S(\mu))$ into another compact set contained in $\mathbb{C} \setminus Co(S(\mu'))$ where μ' is the image measure of μ by the Möbius transformation (cf. [7], proof of Theorem 6.1.6).

Let us find the extremal constant ω and the unit equilibrium measure λ on $S(\mu)$ which are uniquely determined by (13) (or, more precisely, (14)). To this end we need some auxiliary results. The following lemma is very useful. It was given in [1], §3. We state it for convenience of the reader.

Lemma 1. *Let μ be a finite positive Borel measure with compact support $S(\mu) \subset \mathbb{R}$ of positive capacity. Let the polynomials L_n, with deg $L_n = k(n) \leq n$, satisfy* (15). *Then, there exist polynomials T_n , deg $T_n = n - k(n)$, $n = 1, 2, ...,$ with zeros in a fixed compact subset of $\mathbb{C}$ such that the zeros of T_nL_n have "quasi-regular" asymptotic distribution, i.e.*

$$\frac{1}{n}\sigma(T_nL_n) \overset{*}{\longrightarrow} \lambda \qquad as \qquad n \to \infty. \tag{22}$$

The following result is well known (see Theorem 5.1, Chap. II of [6]).

Lemma 2. *We have*

$$G_\Omega(\alpha; z) = \omega - p(\lambda; z) + p(\alpha; z), \qquad z \in \Omega, \tag{23}$$

where $G_\Omega(\alpha; z)$ is the Green potential of the measure α in Ω.

Set

$$M_{2n} := \left|\left|\frac{P_n^2}{A_{2n}}\right|\right|_{S(\mu)} \tag{24}$$

where A_{2n} is as described by (2) and P_n is an arbitrary monic polynomial with $deg\ P_n = n$.

Lemma 3. *Assume that* (4) *takes place. Then*

$$\limsup_{n\to\infty} \left| \frac{P_n^2(z)}{M_{2n}A_{2n}(z)} \right|^{\frac{1}{2n}} \leq \ \exp\ \{G_\Omega(\alpha; z)\} \tag{25}$$

uniformly on compact subsets of $\Omega \setminus A$ *and*

$$\liminf_{n\to\infty}\ (M_{2n})^{\frac{1}{2n}} \geq \exp\ \{-\omega\} \tag{26}$$

Proof. Consider the sequence of functions $\{v_{2n}(z)\}$, $n \in \mathbb{N}$, $z \in \Omega = \overline{\mathbb{C}} \setminus S(\mu)$ defined by

$$v_{2n}(z) := \frac{1}{2n} \left\{ \log \left| \frac{P_n^2(z)}{M_{2n}A_{2n}(z)} \right| - \sum_{i=1}^{2n} g_\Omega(z; \alpha_{n,i}) \right\}. \tag{27}$$

For each $n \in \mathbb{N}$, it is easy to verify that v_{2n} is subharmonic in Ω, and

$$\lim_{z\to\infty} \log \left| \frac{P_n^2(z)}{M_{2n}A_{2n}(z)} \right| = \log \frac{1}{M_{2n}}. \tag{28}$$

On the other hand,

$$v_{2n}(z) \leq 0, \qquad z \in S(\mu),$$

and using the Maximum Principle for subharmonic functions, we obtain that

$$v_{2n}(z) \leq 0, \qquad z \in \Omega. \tag{29}$$

Then, from (27) and (29), it follows that

$$\frac{1}{2n} \log \left| \frac{P_n^2(z)}{M_{2n}A_{2n}(z)} \right| \leq \frac{1}{2n} \sum_{i=1}^{2n} g_\Omega(z; \alpha_{n,i}), \qquad z \in \Omega, \tag{30}$$

and from (28), we have

$$\frac{1}{2n} \log \frac{1}{M_{2n}} \leq \frac{1}{2n} \sum_{i=1}^{2n} g_\Omega(\infty; \alpha_{n,i}) = \frac{1}{2n} \sum_{i=1}^{2n} g_\Omega(\alpha_{n,i}; \infty). \tag{31}$$

Notice that

$$\frac{1}{2n} \sum_{i=1}^{2n} g_\Omega(z; \alpha_{n,i}) = \int_A g_\Omega(z; \zeta) \left[\frac{1}{2n} d\sigma(A_{2n}) \right] (\zeta). \tag{32}$$

If $z \in \overline{\mathbb{C}} \setminus A$ the function $g_\Omega(z;\zeta)$ is continuous in ζ on A (for $z \in S(\mu)$, $g_\Omega(z;\zeta)$ is extended by continuity, which is possible because we have assumed that $S(\mu)$ is a regular set). Moreover, if $z \in K$, where K is a compact set contained in $\overline{\mathbb{C}} \setminus A$ and $\zeta \in A$, then $g_\Omega(z;\zeta)$ is continuous in both variables and

$$\lim_{n\to\infty} \int_A g_\Omega(z;\zeta) \left[\frac{1}{2n} d\sigma(A_{2n})\right](\zeta) = \int_A g_\Omega(z;\zeta) d\alpha(\zeta) = G_\Omega(\alpha;z), \tag{33}$$

uniformly with respect to $z \in K \subset \overline{\mathbb{C}} \setminus A$.

From (30) and (33) follows (25). Using (31) and (33), we obtain

$$\limsup_{n\to\infty} \left(\frac{1}{M_{2n}}\right)^{\frac{1}{2n}} \leq \exp\{G_\Omega(\alpha;\infty)\},$$

or what is the same,

$$\liminf_{n\to\infty} (M_{2n})^{\frac{1}{2n}} \geq \exp\{-G_\Omega(\alpha;\infty)\}.$$

Since $p(\alpha-\lambda;\infty) = \lim_{z\to\infty} p(\alpha-\lambda;z) = 0$, applying Lemma 2, we obtain (26) and Lemma 3 is proved.■

The following lemma is key in the proof of Theorem 1 and has independent interest.

Lemma 4. *Assume that $Z(P_n) \subset Co(S(\mu))$, and (4) takes place. Then, the following conditions are pairwise equivalent:*

$$\lim_{n\to\infty} \frac{1}{n}\sigma(P_n) = \lambda \tag{34}$$

in the weak star topology.

$$\lim_{n\to\infty} (M_{2n})^{\frac{1}{2n}} = \exp\{-\omega\}. \tag{35}$$

$$\lim_{n\to\infty} \left|\frac{P_n^2(z)}{A_{2n}(z)}\right|^{\frac{1}{2n}} = \exp\{G_\Omega(\alpha;z) - \omega\} \tag{36}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (A \cup Co(S(\mu)))$.

Proof. Let us show that (34) implies (35). From (34) and the "principle of descent" (see, for example, [7]), we have that

$$p(\lambda;z) \leq \liminf_{n\to\infty} p\left(\frac{1}{2n}\sigma(P_n^2);z\right), \tag{37}$$

uniformly on each compact subset of $\mathbb{C}$. Using (4) and the assumption that the table (3) is contained in the set A , we obtain

$$\lim_{n\to\infty} p\left(\frac{1}{2n}\sigma(A_{2n}); z\right) = p(\alpha; z), \tag{38}$$

uniformly on each compact subset of $\mathbb{C}\setminus A$. Take any contour Γ surrounding $S(\mu)$ such that A lies in the exterior of Γ. From (37) and (38), we have that

$$\limsup_{n\to\infty}\left|\frac{P_n^2(z)}{A_{2n}(z)}\right|^{\frac{1}{2n}} = \frac{\limsup_{n\to\infty}\left|P_n^2(z)\right|^{\frac{1}{2n}}}{\lim_{n\to\infty}\left|A_{2n}(z)\right|^{\frac{1}{2n}}} \leq \exp\{-p(\lambda; z) + p(\alpha; z)\}, \tag{39}$$

uniformly on Γ.

Take $\varepsilon > 0$ sufficiently small so that the level curve

$$\Gamma_\varepsilon := \{z : p(\lambda; z) - p(\alpha; z) = \omega - \varepsilon\} \tag{40}$$

has A in its exterior, i.e. $A \subset \mathrm{Ext}(\Gamma_\varepsilon)$. From (14), we know that $p(\lambda; z) - p(\alpha; z) \equiv \omega$ on $S(\mu)$. Hence, $S(\mu)$ is entirely inside Γ_ε, i.e. $S(\mu) \subset \mathrm{Int}(\Gamma_\varepsilon)$. From the maximum principle for analytic functions it follows that (see (24))

$$M_{2n} = \left|\left|\frac{P_n^2}{A_{2n}}\right|\right|_{S(\mu)} \leq \left|\left|\frac{P_n^2}{A_{2n}}\right|\right|_{\Gamma_\varepsilon}. \tag{41}$$

So, from (39), (40) and (41) with $\Gamma = \Gamma_\varepsilon$, we obtain

$$\begin{aligned}\limsup_{n\to\infty} M_{2n}^{\frac{1}{2n}} &\leq \limsup_{n\to\infty}\left|\left|\frac{P_n^2}{A_{2n}}\right|\right|_{\Gamma_\varepsilon} \\ &\leq \exp\{\sup_{z\in\Gamma_\varepsilon}(-p(\lambda; z) + p(\alpha; z))\}. \\ &= \exp\{-\omega + \varepsilon\}.\end{aligned} \tag{42}$$

Since (42) is true for all sufficiently small $\varepsilon > 0$, we obtain

$$\limsup_{n\to\infty} M_{2n}^{\frac{1}{2n}} \leq \exp\{-\omega\}. \tag{43}$$

From (26) in Lemma 3 and (43), we obtain (35) as required.

Now, let us prove that (36) follows from (35). From (25) in Lemma 3 and (35), we have that

$$\limsup_{n\to\infty}\left|\frac{P_n^2(z)}{A_{2n}(z)}\right|^{\frac{1}{2n}} \leq \exp\{G_\Omega(\alpha; z) - \omega\}. \tag{44}$$

Moreover, using (4) (see(38)) and taking logarithms, we find that

$$\limsup_{n\to\infty} \frac{1}{n} \log \ |P_n(z)| \le G_\Omega(\alpha; z) - \omega - p(\alpha; z), \tag{45}$$

uniformly on each compact subset of $\mathbb{C}\setminus(A\cup Co(S(\mu)))$. Because of (23) in Lemma 2, we can write

$$\limsup_{n\to\infty} \frac{1}{n} \log \ |P_n(z)| \le -p(\lambda; z), \tag{46}$$

uniformly on each compact subset of $\mathbb{C}\setminus(A\cup Co(S(\mu)))$.

Since $Z(P_n)\subset Co(S(\mu))$, we have that for each $n\in\mathbb{N}$, the function $\frac{1}{n}$ log $|P_n(z)|$ is harmonic in $\mathbb{C}\setminus Co(S(\mu))$ and the family of functions $\left\{\frac{1}{n} \ \log \ |P_n(z)|\right\}$ is uniformly bounded on each compact subset of $\mathbb{C}\setminus Co(S(\mu))$. Take any convergent subsequence; that is, let $\Lambda\subset\mathbb{N}$ be any sequence of indexes such that

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} \frac{1}{n} \ \log \ |P_n(z)| = u_\Lambda(z), \tag{47}$$

uniformly on compact subsets of $\mathbb{C}\setminus Co(S(\mu))$. Take an arbitrary contour γ that surrounds $Co(S(\mu))$ such that $A\subset \mathrm{Ext}(\gamma)$. From (46) and (47), we get that

$$u_\Lambda(z) + p(\lambda; z) \le 0 \qquad \text{on } \gamma. \tag{48}$$

From (47), we have that $u_\Lambda(z)$ is a harmonic function in $\mathbb{C}\setminus Co(S(\mu))$ and it has a singularity of type $\log\ |z|$ at infinity. On the other hand, $p(\lambda; z)$ is harmonic in $\mathbb{C}\setminus S(\mu)$ and has a singularity of type $\log \dfrac{1}{|z|}$, at infinity. Thus

$$u_\Lambda(z) + p(\lambda; z) \qquad \text{is harmonic in } \overline{\mathbb{C}}\setminus Co(S(\mu))$$

and, in particular, this function is harmonic in $\mathrm{Ext}(\gamma)$, including infinity. Since

$$u_\Lambda(\infty) + p(\lambda; \infty) = 0,$$

and (48) takes place, using the maximum principle, we conclude that

$$u_\Lambda(z) + p(\lambda; z) \equiv 0 \ \ , \ z\in \mathrm{Ext}\ (\gamma). \tag{49}$$

But γ can be taken arbitrarily close to $Co(S(\mu))$, hence

$$u_\Lambda(z) \equiv -p(\lambda; z) \ \ , \ \ z\in\mathbb{C}\setminus Co(S(\mu)).$$

Therefore, the limit $u_\Lambda(z)$ in (47) does not depend on Λ. Using the normality of the family of functions it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log |P_n(z)| = -p(\lambda; z), \tag{50}$$

uniformly on compact subsets of $\mathbb{C} \setminus Co(S(\mu))$. Thus, (36) is a consequence of (4), (50), and (23).

Finally, we deduce (34) from (36). From the assumption that $Z(P_n) \subset Co(S(\mu))$, we have that

$$S\left(\frac{1}{n}\sigma(P_n)\right) \subset Co(S(\mu)), \qquad n \in \mathbb{N}.$$

Moreover, the family

$$\left\{\frac{1}{n}\sigma(P_n)\right\}, \qquad n \in \mathbb{N}, \tag{51}$$

is compact in the weak star topology of measures. Thus, in order to prove this part of the lemma, it is sufficient to show that any weak star convergent subsequence of (51) has for limit the measure λ.

Assume that $\Upsilon \subset \mathbb{N}$ is a set of indexes such that

$$\lim_{\substack{n\to\infty \\ n\in\Upsilon}} \frac{1}{n}\sigma(P_n) = \lambda_\Upsilon, \tag{52}$$

in the weak star topology. Obviously, $S(\lambda_\Upsilon) \subset Co(S(\mu))$. We know that

$$S(\lambda) = S(\mu) \subset Co(S(\mu)).$$

Therefore

$$(\lambda_\Upsilon = \lambda) \Longleftrightarrow \left(\int f d\lambda_\Upsilon = \int f d\lambda\right), \tag{53}$$

for all functions f continuous on $Co(S(\mu))$. Moreover, it is sufficient to check (53) on a dense subset of the space of continuous functions on $Co(S(\mu))$ (with the topology of uniform convergence).

From the assumption (36), using (4) and Lemma 2, we obtain (50). In turn, formulas (50) and (52) imply

$$\int \log \frac{1}{|z-\zeta|}\, d\lambda_\Upsilon(\zeta) = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \int \log \frac{1}{|z-\zeta|}\, d\left(\frac{1}{n}\sigma(P_n)\right)(\zeta) = p(\lambda; z)\ , \tag{54}$$

which is equivalent to

$$\int \log \frac{1}{|z-\zeta|} \; d\lambda_\Upsilon(\zeta) = \int \log \frac{1}{|z-\zeta|} \; d\lambda(\zeta). \tag{55}$$

It is well known that the family of functions (with respect to $\zeta \in Co(S(\mu))$),

$$\left\{ \log \frac{1}{|z-\zeta|} \right\}, \qquad z \in \mathbb{C} \setminus Co(S(\mu)),$$

is total in the space of continuous functions on $Co(S(\mu))$. That is, linear combinations (for different z) are uniformly dense in the space of continuous functions on $Co(S(\mu))$. Therefore, (53) takes place and $\lambda_\Upsilon = \lambda$, independent of Υ. Hence, (34) holds. With this, we conclude the proof of Lemma 4.■

Let us consider the monic polynomial Q_n of degree $\leq n - k(n)$, which satisfies (i)-(iii) of Definition 1, and the monic polynomial A_{2n} defined by (2). The following orthogonality relations are easy to verify (for the proof, see [3]).

Lemma 5. *We have*

$$\int_{S(\mu)} x^j Q_n(x) \frac{L_n^2(x) d\mu(x)}{A_{2n}(x)} = 0, \qquad j = 0, 1, ..., n - k(n) - 1. \tag{56}$$

From Lemma 5, it follows that for each $n \in \mathbb{N}$, Q_n satisfies the following inequality

$$\left\| \frac{Q_n L_n}{A_{2n}^{\frac{1}{2}}} \right\|_2 \leq \left\| \frac{(Q_n + v) L_n}{A_{2n}^{\frac{1}{2}}} \right\|_2 \tag{57}$$

for all polynomials v such that $deg\ v < n - k(n)$. Set

$$q_n(z) = \lambda_n \; Q_n(z), \tag{58}$$

where

$$\lambda_n = \left\| \frac{Q_n L_n}{A_{2n}^{\frac{1}{2}}} \right\|_2^{-1}. \tag{59}$$

Therefore, q_n is the polynomial of degree $n - k(n)$ orthonormal with respect to the (varying) measure $L_n^2 d\mu / A_{2n}$ and (11) takes place.

The following bounds for the error formula given by (6) are straightforward (see (1.34) in Section 6.1 of [7]).

Lemma 6. *There exist two positive continuous functions $d_1(z) > 0$ and $d_2(z) < \infty$ on $\mathbb{C} \setminus Co(S(\mu))$ independent of n such that*

$$d_1(z) \left| \frac{A_{2n}(z)}{(q_n L_n)^2(z)} \right| \leq |r_n(z)| \leq d_2(z) \left| \frac{A_{2n}(z)}{(q_n L_n)^2(z)} \right|. \tag{60}$$

Since $d_1(z)$ and $d_2(z)$ are bounded from above and below on each compact subset of $\mathbb{C} \setminus Co(S(\mu))$, formula (60) guarantees that the $n^{th}-\text{root}$ asymptotic behavior of the error is completely determined by that of the sequence

$$\left\{ \left| \frac{A_{2n}(z)}{(q_n L_n)^2(z)} \right| \right\}_{n \in \mathbb{N}} .$$

3.2 - Proof of Theorem 1. Let $T_n(z) = z^{n-k(n)} + ..., n = 1, 2, ...$ be the monic polynomials given by Lemma 1. From (8) and (57), we have

$$\begin{aligned} \left\| \frac{Q_n L_n}{A_{2n}^{\frac{1}{2}}} \right\|_2^2 &= \int \frac{(Q_n L_n)^2(x)}{A_{2n}(x)} \, d\mu(x) \\ &\leq \int \frac{(T_n L_n)^2(x)}{A_{2n}(x)} \, d\mu(x) \leq \mu(S(\mu)) \left\| \frac{(T_n L_n)^2}{A_{2n}} \right\|_{S(\mu)} . \end{aligned} \tag{61}$$

From (15), we have that (22) takes place, and using Lemma 4, for the sequence $P_n = T_n L_n$, we obtain

$$\lim_{n \to \infty} \left\| \frac{(T_n L_n)^2}{A_{2n}} \right\|_{S(\mu)}^{\frac{1}{2n}} = \exp\{-\omega\}. \tag{62}$$

Therefore, from (61) and (62), we have

$$\limsup_{n \to \infty} \left\| \frac{Q_n L_n}{A_{2n}^{\frac{1}{2}}} \right\|_2^{\frac{1}{n}} \leq \exp\{-\omega\}. \tag{63}$$

On the other hand, $\mu \in Reg$ and $S(\mu)$ is regular with respect to Dirichlet's problem; therefore (cf. [7], Chapter 3, page 68),

$$\lim_{n \to \infty} \frac{\left\| \frac{Q_n L_n}{A_{2n}^{\frac{1}{2}}} \right\|_{S(\mu)}^{\frac{1}{n}}}{\left\| \frac{Q_n L_n}{A_{2n}^{\frac{1}{2}}} \right\|_2^{\frac{1}{n}}} = 1. \tag{64}$$

Thus, from (63) and (64), we get

$$\limsup_{n\to\infty}\left\|\frac{(Q_nL_n)^2}{A_{2n}}\right\|_{S(\mu)}^{\frac{1}{2n}}=\limsup_{n\to\infty}\left\|\frac{Q_nL_n}{A_{2n}^{\frac{1}{2}}}\right\|_{S(\mu)}^{\frac{1}{n}}=$$
$$=\limsup_{n\to\infty}\left\|\frac{Q_nL_n}{A_{2n}^{\frac{1}{2}}}\right\|_{2}^{\frac{1}{n}}\leq\exp\{-\omega\}.$$

This together with (26) gives

$$\lim_{n\to\infty}\left\|\frac{(Q_nL_n)^2}{A_{2n}}\right\|_{S(\mu)}^{\frac{1}{2n}}=\exp\{-\omega\}. \tag{65}$$

Due to Lemma 4, (65) implies (18) and

$$\lim_{n\to\infty}\left|\frac{(Q_nL_n)^2(z)}{A_{2n}(z)}\right|^{\frac{1}{2n}}=\exp\{G_\Omega(\alpha;z)-\omega\}, \tag{66}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus(A\cup Co(S(\mu)))$. From (59), (64), and (65), we also have

$$\lim_{n\to\infty}(\lambda_n)^{\frac{1}{n}}=\exp\{\omega\}. \tag{67}$$

Therefore, using (66) and (67), it follows that

$$\lim_{n\to\infty}\left|\frac{(q_nL_n)(z)}{A_{2n}^{\frac{1}{n}}(z)}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{\hat{p}_n(z)}{A_{2n}^{\frac{1}{2}}(z)}\right|^{\frac{1}{n}}=\exp\{G_\Omega(\alpha;z)\}. \tag{68}$$

Finally, (68),(4) (see (38)), and (23) imply (17). The proof of Theorem 1 is complete.■

Theorem 2 is an immediate consequence of Theorem 1 and Lemma 6 (for more details, see proof of Theorem 1 in [3]).

4. Applications to quadratures

According to (6), $(\hat{\mu}-R_n)/A_{2n}$ is holomorphic in $\overline{\mathbb{C}}\setminus S(\mu)$ and

$$\frac{\hat{\mu}-R_n}{A_{2n}}=\mathcal{O}\left(\frac{1}{z^{2n+1}}\right)\qquad\text{as}\quad z\to\infty. \tag{69}$$

Consider the partial fraction decomposition of R_n

$$R_n(z) = \sum_{i=1}^{N_n} \sum_{j=0}^{M_{n,i}} \frac{j! A_{i,j}^n}{(z - x_{n,i})^{j+1}}, \tag{70}$$

where N_n denotes the total number of distinct poles of R_n. The points $x_{n,i}$ are zeros of $L_n^2 Q_n$ and although the zeros of Q_n are simple (see (56)) they may coincide with zeros of L_n; therefore, for given $x_{n,i}$ any value of $M_{n,i}$ is possible (unless we restrict the multiplicity of the zeros of L_n).

Lemma 7. *For any polynomial p of degree at most $2n-1$, we have*

$$\int \left(\frac{p}{A_{2n}} \right)(x) d\mu(x) = \sum_{i=1}^{N_n} \sum_{j=0}^{M_{n,i}} A_{i,j}^n \left(\frac{p}{A_{2n}} \right)^{(j)} (x_{n,i}). \tag{71}$$

Proof. From (69), for any polynomial p of degree at most $2n-1$,

$$\frac{p(\hat{\mu} - R_n)}{A_{2n}}$$

has a zero at infinity of order at least two and is holomorphic in $\overline{\mathbb{C}} \setminus S(\mu)$. Then, integrating along any contour Γ which surrounds $Co(S(\mu))$, in such a way that A lies outside the closed domain determined by Γ, from Cauchy's Theorem, Fubini's Theorem, Cauchy's integral formula, and using (70), it follows that

$$0 = \int_\Gamma \frac{p(\hat{\mu} - R_n)}{A_{2n}}(z) dz = \int_\Gamma \frac{p\hat{\mu}}{A_{2n}}(z) dz - \int_\Gamma \frac{pR_n}{A_{2n}}(z) dz =$$
$$= 2\pi i \left[\int \left(\frac{p}{A_{2n}} \right)(x) d\mu(x) - \sum_{i=1}^{N_n} \sum_{j=0}^{M_{n,i}} A_{i,j}^n \left(\frac{p}{A_{2n}} \right)^{(j)} (x_{n,i}) \right]$$

which implies (71). Lemma 7 is proved.■

Let f be a continuous function on $Co(S(\mu))$ such that the operations indicated in the following expressions make sense. Denote

$$I(f) = \int f(x) d\mu(x) \qquad \text{and} \qquad I_n(f) = \sum_{i=1}^{N_n} \sum_{j=0}^{M_{n,i}} A_{i,j}^n f^{(j)}(x_{n,i}).$$

Formula (71) indicates that for any polynomial p of degree at most $2n-1$ we have

$$I\left(\frac{p}{A_{2n}}\right) = I_n\left(\frac{p}{A_{2n}}\right).$$

Using arguments similar to those in the proof of Lemma 7, it is easy to verify the following.

Lemma 8. Let $f \in H(V)$ where V is a neighborhood of $Co(S(\mu))$. Then, for any contour Γ contained in V such that $Co(S(\mu)) \subset Int(\Gamma)$ and $A \subset Ext(\Gamma)$, we have

$$I(f) - I_n(f) = \frac{1}{2\pi i}\int_\Gamma f(z)(\hat{\mu} - R_n)(z)dz. \tag{72}$$

Now, we can prove

THEOREM 3. *Let $f \in H(V)$ where V is a neighborhood of $Co(S(\mu))$. Under the conditions of Theorem 2, we have*

$$\limsup_{n\to\infty} |I(f) - I_n(f)|^{\frac{1}{2n}} \le \sup_{z\in\partial V} \exp\{-G_\Omega(\alpha; z)\}. \tag{73}$$

Assume that $\mathrm{cap}(\partial V) > 0$ *and denote*

$$q_n(f) = |I(f) - I_n(f)|,$$

then

$$\sup_{f\in H(V)} \limsup_{n\to\infty} q_n^{\frac{1}{2n}}(f) = \sup_{z\in\partial V} \exp\{-G_\Omega(\alpha; z)\}. \tag{74}$$

Proof. Take Γ as indicated in Lemma 8. From (19), it follows that

$$|I(f) - I_n(f)| \le (2\pi)^{-1} l_\Gamma \|f\|_\Gamma \|\hat{\mu} - R_n\|_\Gamma, \tag{75}$$

where l_Γ denotes the length of Γ. From (75) and (20), we have that

$$\limsup_{n\to\infty} |I(f) - I_n(f)|^{\frac{1}{2n}} \le \sup_{z\in\Gamma} \exp\{-G_\Omega(\alpha; z)\}.$$

We can take Γ as close to ∂V as we please; therefore, (73) follows immediately.

Since the right hand side of (73) does not depend on $f \in H(V)$, it follows that

$$\sup_{f\in H(V)} \limsup_{n\to\infty} q_n^{\frac{1}{2n}}(f) \le \sup_{z\in\partial V} \exp\{-G_\Omega(\alpha; z)\}.$$

For the lower bound, take $f_z(x) = (z-x)^{-1}$, where $z \in \mathbb{C} \setminus V$. This function (with respect to x) belongs to $H(V)$ and

$$q_n(f_z) \ge |I(f_z) - I_n(f_z)| = |\hat{\mu}(z) - R_n(z)|.$$

Therefore,

$$\limsup_{n\to\infty} q_n^{\frac{1}{2n}}(f_z) \ge \limsup_{n\to\infty} |\hat{\mu}(z) - R_n(z)|^{\frac{1}{2n}}. \tag{76}$$

Using (20), it follows from (76) that

$$\begin{aligned} \sup_{f\in H(V)} \limsup_{n\to\infty} q_n^{\frac{1}{2n}}(f) &\ge \sup_{z\in\mathbb{C}\setminus V} \limsup_{n\to\infty} |\hat{\mu}(z) - R_n(z)|^{\frac{1}{2n}} \\ &= \sup_{z\in\partial V} \exp\{-G_\Omega(\alpha; z)\}, \end{aligned} \tag{77}$$

which is what we needed to prove.■

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