ALTERNATIVE LINEAR STRUCTURES FOR CLASSICAL AND QUANTUM SYSTEMS

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The possibility of deforming the (associative or Lie) product to obtain alternative descriptions for a given classical or quantum system has been considered in many papers. Here we discuss the possibility of obtaining some novel alternative descriptions by changing the linear structure instead. In particular we show how it is possible to construct alternative linear structures on the tangent bundle TQ of some classical configuration space Q that can be considered as "adapted" to the given dynamical system. This fact opens the possibility to use the Weyl scheme to quantize the system in different nonequivalent ways, "evading," so to speak, the von Neumann uniqueness theorem.

Keywords: Alternative description; Weyl quantization.

1. Introduction

Since the seminal paper of Wigner,[1] much attention has been devoted to the question of uniqueness of commutation relations and/or of associative products compatible with the dynamics of a given quantum system (the harmonic oscillator in the cited Wigner's paper). It is well known that alternative and compatible Poisson

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brackets appear in connection with the problem of complete integrability within a classical framework.[2] The problem of which alternative quantum structures, after taking the appropriate classical limit, could reproduce the alternative known Hamiltonian descriptions has also been considered in many papers (see for example Ref. 3 and references therein).

The main purpose of this paper is to discuss how one can obtain some novel alternative descriptions, both in the classical and in the quantum context, by "deforming" the *linear structure* instead of the (associative or Lie) product. More explicitly, we will see under what circumstances (for instance the existence of a regular Lagrangian description $\mathcal{L}$ on the tangent bundle TQ of some configuration space Q) one can construct a linear structure on TQ that can be considered as "adapted" to the given dynamical system. If and when this is possible, one obtains a new action of the group $\mathbb{R}^{2n}$ ($n = \dim Q$) on TQ and, as will be shown, the Lagrangian two-form $\omega_{\mathcal{L}}$ can be put explicitly in canonical Darboux form. One can then follow the Weyl procedure[4] to quantize the dynamics, by realizing the associated Weyl system on the Hilbert space of square-integrable functions on a suitable Lagrangian submanifold of TQ.

The fact that many dynamical systems admit genuinely alternative descriptions[5] poses an interesting question, namely: assume that a given dynamical system admits alternative descriptions with more than one linear structure. According to what has been outlined above, one will possibly obtain different actions (realizations) of the group $\mathbb{R}^{2n}$ on TQ that in general will not be linearly related. Then, it will be possible to quantize "*à la*" Weyl the system in two different ways, thereby obtaining different Hilbert space structures on spaces of square-integrable functions on different Lagrangian submanifolds. (Actually what appears as a Lagrangian submanifold in one scheme need not be such in the other. Moreover, the Lebesgue measures will be different in the two cases). The occurrence of this situation seems then to offer the possibility of, so-to-speak, "evading" the von Neumann theorem[6] and this is one of the topics to be discussed in this paper.

As a simple example, consider three Lorentz frames, S, S' and S'', moving relative to each other with constant relative velocities all along the same direction (along the x-axis, say). Let u be the velocity of S' with respect to S and u' the velocity of S'' with respect to S', all in units of the speed of light.[a] Then S'' will have, in the same units, a relative velocity

$$u'' = \frac{u' + u}{1 + u'u} \tag{1.1}$$

with respect to S. The velocity v'' in S of a point-particle moving with respect to S'' with a velocity (again along the x-axis) v can be computed in two different ways, namely:

[a] All the velocities will lie then in the interval $(-1, 1)$.

(1) First we compute the velocity of the point-particle with respect to S' as $v' = (u' + v)/(1 + u'v)$ and then the final velocity as

$$v'' = \frac{u + v'}{1 + uv'} \,. \tag{1.2}$$

In this way we have first "composed" u' and v according to the law (1.1) and then the result has been "composed" with u.

Alternatively we can:

(2) First evaluate u'', according to Eq. (1.1), i.e. first "composing" u and u', and then the result with v, obtaining

$$v'' = \frac{v + u''}{1 + vu''} \,. \tag{1.3}$$

It is obvious that (1.2) and (1.3) yield the same result, namely

$$v'' = \frac{v + u + u' + vu'u}{1 + u'u + uv + u'v} \,. \tag{1.4}$$

All this is elementary, but shows that already the familiar (one-dimensional) relativistic law of addition of the velocities provides us with a composition law for points in the open interval $(-1, 1)$ that has the same associative property as the standard law of addition of (real or complex) numbers. This example, whose discussion will be completed in App. A, serves as a partial motivation for the study of linear structures nonlinearly related to other similar structures. In the next section we will give some more complete definitions and examples, before proceeding to the main subject of the present paper.

2. Alternative Linear Structures

2.1. *Linear structures*

It is well known that all finite-dimensional linear spaces are linearly isomorphic. The same is true for infinite-dimensional Hilbert spaces (even more, the isomorphism can be chosen to be an isometry). However, alternative (i.e. not linearly related) linear structures can be constructed easily on a given set. For instance consider a linear space E with addition $+$ and multiplication by scalars $\cdot$, and a nonlinear diffeomorphism $\phi\colon E \to E$. Now we can define a new addition $+_{(\phi)}$ and a new multiplication by scalar $\cdot_{(\phi)}$ by setting

$$u +_{(\phi)} v =: \phi(\phi^{-1}(u) + \phi^{-1}(v)) \tag{2.1}$$

and

$$\lambda \cdot_{(\phi)} u =: \phi(\lambda \phi^{-1}(u)) \,. \tag{2.2}$$

These operations have all the usual properties of addition and multiplication by a scalar. In particular,

$$(\lambda\lambda') \cdot_{(\phi)} u = \lambda \cdot_{(\phi)} (\lambda' \cdot_{(\phi)} u) \tag{2.3}$$

and

$$\left(u +_{(\phi)} v\right) +_{(\phi)} w = u +_{(\phi)} \left(v +_{(\phi)} w\right). \tag{2.4}$$

Indeed, e.g.

$$\lambda \cdot_{(\phi)} \left(\lambda' \cdot_{(\phi)} u\right) = \phi\big(\lambda\phi^{-1}\big(\lambda' \cdot_{(\phi)} u\big)\big) = \phi\big(\lambda\lambda'\phi^{-1}(u)\big) = (\lambda\lambda') \cdot_{(\phi)} u \tag{2.5}$$

which proves (2.3), and similarly for (2.4).

Obviously, the two linear spaces $(E, +, \cdot)$ and $\left(E, +_{(\phi)}, \cdot_{(\phi)}\right)$ are finite-dimensional vector spaces of the same dimension and hence are isomorphic. However, the change of coordinates defined by ϕ that we are using to "deform" the linear structure is a nonlinear diffeomorphism. In other words, we are using two different (diffeomorphic but not linearly related) global charts to describe the same manifold space E.

As a simple (but significant) example of this idea consider the linear space $\mathbb{R}^2$. This can also be viewed as a Hilbert space of complex dimension 1 that can be identified with $\mathbb{C}$.

We shall denote its coordinates as (q, p) and we choose the nonlinear transformation:[7,8]

$$q = Q(1 + \lambda R^2)\,, \qquad p = P(1 + \lambda R^2)\,, \tag{2.6}$$

with $R^2 = P^2 + Q^2$, which can be inverted as

$$Q = qK(r)\,, \qquad P = pK(r)\,, \tag{2.7}$$

where $r^2 = p^2 + q^2$, and the positive function $K(r)$ is given by the relation $R = rK(r)$ and satisfies the equation

$$\lambda r^2 K^3 + K - 1 = 0 \tag{2.8}$$

(hence, actually, $K = K(r^2)$ as well as $\lambda = 0 \leftrightarrow K \equiv 1$). Using this transformation we construct an alternative linear structure on $\mathbb{C}$ by using formulas (2.1) and (2.2). Let us denote by $+_K$ and $\cdot_K$ the new addition and multiplication by scalars. Then, with

$$\phi: \quad (Q, P) \to (q, p) = (Q(1 + \lambda R^2), P(1 + \lambda R^2))\,, \tag{2.9}$$

$$\phi^{-1}: \quad (q, p) \to (Q, P) = (qK(r), pK(r)) \tag{2.10}$$

one finds

$$\begin{aligned} (q, p) +_{(K)} (q', p') &= \phi(\phi^{-1}(q, p) + \phi^{-1}(q', p')) \\ &= \phi((Q + Q', P + P')) \\ &= \phi(qK + q'K', pK + p'K')\,, \\ K = K(r)\,, \qquad &K' = K(r')\,, \end{aligned} \tag{2.11}$$

i.e.

$$(q, p) +_{(K)} (q', p') = S(r, r')((qK + q'K'), (pK + p'K'))\,, \tag{2.12}$$

where

$$S(r, r') = 1 + \lambda((qK + q'K')^2 + (pK + p'K')^2) . \tag{2.13}$$

Quite similarly

$$\begin{aligned} a \cdot_{(K)} (q, p) &= \phi(a\phi^{-1}(q, p)) \\ &= \phi((aqK(r), apK(r))) \\ &= S'(r)(aK(r)q, aK(r)p) , \end{aligned} \tag{2.14}$$

where

$$S'(r) = 1 + \lambda a^2 r^2 K^2(r) . \tag{2.15}$$

The two different realizations of the translation group in $\mathbb{R}^2$ are associated with the vector fields $(\partial/\partial q, \partial/\partial p)$ and $(\partial/\partial Q, \partial/\partial P)$ respectively. The two are connected by

$$\left| \begin{matrix} \dfrac{\partial}{\partial Q} \\ \dfrac{\partial}{\partial P} \end{matrix} \right| = A \left| \begin{matrix} \dfrac{\partial}{\partial q} \\ \dfrac{\partial}{\partial p} \end{matrix} \right| , \tag{2.16}$$

where A is the Jacobian matrix

$$\begin{aligned} A = \frac{\partial(q,p)}{\partial(Q,P)} &\equiv \left| \begin{matrix} 1 + \lambda(3Q^2 + P^2) & 2\lambda PQ \\ 2\lambda PQ & 1 + \lambda(Q^2 + 3P^2) \end{matrix} \right| \\ &= \left| \begin{matrix} 1 + \lambda K(r)^2(3q^2 + p^2) & 2\lambda K(r)^2 pq \\ 2\lambda K(r)^2 pq & 1 + \lambda K(r)^2(q^2 + 3p^2) \end{matrix} \right| . \end{aligned} \tag{2.17}$$

In the sequel we will write simply A as

$$A = \left| \begin{matrix} a & b \\ d & c \end{matrix} \right| , \tag{2.18}$$

with an obvious identification of the entries. Then, also

$$A^{-1} = \frac{\partial(Q,P)}{\partial(q,p)} = D^{-1} \left| \begin{matrix} c & -b \\ -d & a \end{matrix} \right| , \qquad D = ac - bd . \tag{2.19}$$

The integral curves in the plane (q, p) of the vector fields $\partial/\partial Q$ and $\partial/\partial P$ are shown in Fig. 1. They should be compared with the straight lines associated with $\partial/\partial q$ and $\partial/\partial p$.

Thus the 2D translation group $\mathbb{R}^2$ is realized in two different ways. One interesting consequence of this is that one obtains two different ways of defining the Fourier transform. Also, when considering square-integrable functions in $L_2(\mathbb{R}^2)$, functions that are square-integrable with respect to the unique Lebesgue measure which is invariant with respect to translation defining one linear structure need not be so with respect to the Lebesgue measure defined by the other linear structure. This

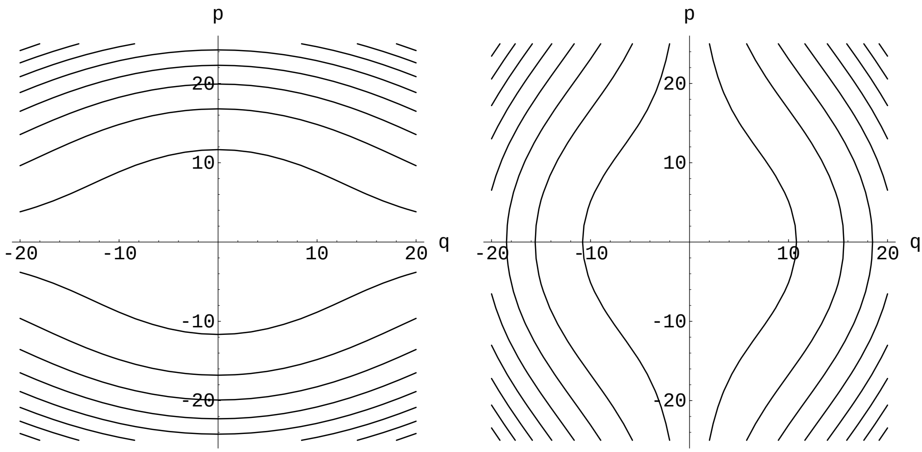

Fig. 1. The integral curves in the plane (q, p) of the vector fields $\frac{\partial}{\partial Q}$, $\frac{\partial}{\partial P}$.

will become important when considering the quantum case and we will come back to this point later on.

The above scheme can be generalized to the case of a diffeomorphism:

$$\phi : E \to M \tag{2.20}$$

between a vector space E and a manifold M possessing *a priori* no linear structures whatsoever. This will require, of course, that M be such that it can be equipped with a one-chart atlas. Then it is immediate to see that Eqs. (2.1) and (2.2) (with u, $v \in M$, now) apply to this slightly more general case as well. Some specific examples (with, e.g. M an open interval of a punctured sphere) will be discussed in App. A while, in App. B, we will discuss briefly how a superposition rule (not a linear one, though) can also be defined in the case, which is relevant for quantum mechanics, of the space of pure states of a quantum system, i.e. on the projective Hilbert space $\mathcal{PH}$ of a (complex linear) Hilbert space $\mathcal{H}$.

2.2. *A geometrical description of linear structures*

To every linear structure there is associated in a canonical way a *dilation* (or Liouville) field Δ which is the infinitesimal generator of dilations (and in fact it can be shown that uniquely characterizes it, see for instance Refs. 9 and 10). Therefore, in the framework of the new linear structure, it makes sense to consider the mapping

$$\Psi : E \times \mathbb{R} \to E \tag{2.21}$$

via

$$\Psi(u, t) =: e^t \cdot_{(\phi)} u =: u(t)\,, \tag{2.22}$$

where again, we are considering a transformation $\phi\colon E \to E$. The transformed flow takes the explicit form

$$u(t) = \phi(e^t \phi^{-1}(u)) \,. \tag{2.23}$$

Property (2.3) ensures that

$$\Psi(u(t'), t) = \Psi(u, t + t') \,, \tag{2.24}$$

i.e. that (2.22) is indeed a one-parameter group. Then, the infinitesimal generator of the group is defined as

$$\Delta(u) = \left[\frac{d}{dt} u(t)\right]_{t=0} = \left[\frac{d}{dt} \phi(e^t \phi^{-1}(u))\right]_{t=0} \,, \tag{2.25}$$

or, explicitly, in components:

$$\Delta = \Delta^i \frac{\partial}{\partial u^i} \,, \tag{2.26}$$

$$\Delta^i = \left[\frac{\partial \phi^i(w)}{\partial w^j} w^j\right]_{w=\phi^{-1}(u)} \,. \tag{2.27}$$

In other words, if we denote by $\Delta_0 = w^i \partial / \partial w^i$ the Liouville field associated with the linear structure $(+, \cdot)$ on E:

$$\Delta = \phi_* \Delta_0 \,, \tag{2.28}$$

where ϕ_* denotes, as usual, the push-forward.

It is clear that, if ϕ is a linear (and invertible) map, then (2.27) yields: $\Delta^i = u^i$, i.e.

$$\phi_* \Delta_0 = \Delta_0 \,. \tag{2.29}$$

Conversely it is simple to see that if a map ϕ satisfies (2.29) then it is linear with respect to the linear structure defined by Δ_0.

Let us go back to the example in $\mathbb{R}^2$ considered in the previous section. First, notice that we have the identification $T^*\mathbb{R} \approx \mathbb{R}^2$ so that the dilation (Liouville) field

$$\Delta = q \frac{\partial}{\partial q} + p \frac{\partial}{\partial p} \tag{2.30}$$

is such that

$$i_\Delta \omega = q dp - p dq \,, \tag{2.31}$$

where $\omega = dq \wedge dp$ is the standard symplectic form.

Another relevant structure that can be constructed is the complex structure, that is defined by the $(1, 1)$ tensor field

$$J = dp \otimes \frac{\partial}{\partial q} - dq \otimes \frac{\partial}{\partial p} \,, \tag{2.32}$$

which satisfies $J^2 = -\mathbb{I}$ (the identity) and, being constant, has a vanishing Nijenhuis tensor:[11,12] $N_J = 0$. Notice that:

$$J \circ \omega = g\,, \tag{2.33}$$

where g is the $(2,0)$ tensor:

$$g = dq \otimes dq + dp \otimes dp\,, \tag{2.34}$$

i.e. a (Euclidean) metric tensor, and $g(\cdot\,,\cdot) = \omega(J\,\cdot\,,\cdot)$.

In this way we have defined three structures on a cotangent bundle (actually on the cotangent bundle of a vector space), namely a symplectic structure, a complex structure and a metric tensor. It should be clear from, e.g. Eq. (2.33) that these three structures are not independent: given any two of them the third one is defined in terms of the previous ones.[13–16]

Consider now the nonlinear change of coordinates (2.6). Just as Δ and the tensors ω, J and g are associated with the linear structure $(+\,,\cdot)$ in the (q,p) coordinates, in the (Q,P) coordinates and again with the $(+\,,\cdot)$ addition and multiplication rules there will be associated the Liouville field:

$$\Delta' = Q\frac{\partial}{\partial Q} + P\frac{\partial}{\partial P}\,, \tag{2.35}$$

the (standard) symplectic form:

$$\omega' = dQ \wedge dP\,, \tag{2.36}$$

the complex structure:

$$J' = dP \otimes \frac{\partial}{\partial Q} - dQ \otimes \frac{\partial}{\partial P}\,, \tag{2.37}$$

as well as the metric tensor:

$$g' = dQ \otimes dQ + dP \otimes dP\,. \tag{2.38}$$

Remark. In, say, the (q,p) coordinates, the dynamics of the 1D harmonic oscillator:

$$\frac{dq}{dt} = p\,, \qquad \frac{dp}{dt} = -q \tag{2.39}$$

is described by the vector field

$$\Gamma = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p} \tag{2.40}$$

and

$$\Gamma = J(\Delta)\,. \tag{2.41}$$

The fact that the nonlinear transformation (2.6) is constructed using constants of the motion for the dynamics implies then

$$\frac{dQ}{dt} = P\,, \qquad \frac{dP}{dt} = -Q\,, \tag{2.42}$$

i.e.

$$\Gamma = P\frac{\partial}{\partial Q} - Q\frac{\partial}{\partial P} \tag{2.43}$$

as well as

$$J(\Delta) = J'(\Delta')\,. \tag{2.44}$$

When transformed back to the (q, p) coordinates, Eqs. (2.35)–(2.38) will provide all the relevant tensorial quantities that are associated, now, with the new linear structure that we have denoted as $\left(+_{(K)}, \cdot_{(K)}\right)$ in the previous subsection (see Eqs. (2.12) and (2.14)). Explicitly, and again in the shorthand notation introduced in (2.19),

$$\Delta' = (aQ + bP)(q,p)\frac{\partial}{\partial q} + (dQ + cP)(q,p)\frac{\partial}{\partial q}\,, \tag{2.45}$$

$$\omega' = \left\{\det\frac{\partial(Q,P)}{\partial(q,p)}\right\}\omega \equiv D^{-1}\omega\,, \tag{2.46}$$

$$J' = -\frac{ad+bc}{D}\left[dq\otimes\frac{\partial}{\partial q} - dp\otimes\frac{\partial}{\partial p}\right] + \frac{a^2+b^2}{D}dp\otimes\frac{\partial}{\partial q} - \frac{c^2+d^2}{D}dq\otimes\frac{\partial}{\partial p}\,, \tag{2.47}$$

as well as

$$g' = \frac{c^2+d^2}{D^2}dq\otimes dq - \frac{ad+bc}{D^2}(dq\otimes dp + dp\otimes dq) + \frac{a^2+b^2}{D^2}dp\otimes dp\,. \tag{2.48}$$

Denoting collectively as $u = (u^1, u^2) \equiv (q, p)$ and $w = (w^1, w^2) \equiv (Q, P)$ the "old" and "new" coordinates, then

$$J = J^i{}_k\, du^k \otimes \frac{\partial}{\partial u^i}\,, \qquad J' = J^i{}_k\, dw^k \otimes \frac{\partial}{\partial w^i} \tag{2.49}$$

with

$$J = |J^i{}_k| = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}\,, \tag{2.50}$$

so that

$$J' = J'^i{}_k\, du^k \otimes \frac{\partial}{\partial u^i}\,, \tag{2.51}$$

where, now

$$J' = A \circ J \circ A^{-1}\,. \tag{2.52}$$

Quite similarly, with

$$g = g_{ij}\, du^i \otimes du^j\,, \qquad g' = g_{ij}\, dw^i \otimes dw^j\,, \qquad g_{ij} = \delta_{ij}\,, \tag{2.53}$$

one finds

$$g' = g'_{ij}\, du^i \otimes du^j \,, \tag{2.54}$$

where the matrix $g' = |g'_{ij}|$ is given by

$$g' = (A^{-1})^t \cdot A^{-1} \,. \tag{2.55}$$

The symplectic form (2.46) can be written as

$$\omega' = \frac{1}{2}\omega'_{ij}\, du^i \wedge du^j \tag{2.56}$$

with the representative matrix

$$\omega' =: \left|\omega'_{ij}\right| = D^{-1} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \,. \tag{2.57}$$

The compatibility condition[13–16] between ω', g' and J' in the $\{u^i\}$ coordinates:

$$\omega'(u_1, u_2) = g'(u_1, J'u_2)\,, \quad \forall u_1,\ u_2 \tag{2.58}$$

is easily seen to imply, in terms of the representative matrices:

$$g' \cdot J' = \omega' \,, \tag{2.59}$$

i.e.

$$\omega' = (A^{-1})^t \cdot J \cdot A^{-1} \tag{2.60}$$

and direct calculation shows that this is indeed the case.

Remark. The Poisson tensors (and hence the Poisson brackets) associated with the symplectic structures ω and ω' are

$$\Lambda = \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p} \tag{2.61}$$

and

$$\Lambda' = \frac{\partial}{\partial Q} \wedge \frac{\partial}{\partial P} \tag{2.62}$$

respectively, and

$$\Lambda' = D\Lambda \tag{2.63}$$

which is, consistently, the same result that obtains by inverting Eq. (2.46). Hence, one obtains the new fundamental Poisson bracket

$$\{q, p\}_{\omega'} = D\{q, p\}_{\omega} = D \,, \tag{2.64}$$

where $\{\cdot\,,\cdot\}_{\omega}$ and $\{\cdot\,,\cdot\}_{\omega'}$ are the Poisson brackets defined by the Poisson tensors Λ and Λ' respectively, and hence, in general

$$\{f, g\}_{\omega'} = D\{f, g\}_{\omega} \,. \tag{2.65}$$

On $\mathbb{R}^2$ we can also introduce complex coordinates

$$z = q + ip\,, \qquad \bar{z} = q - ip\,, \tag{2.66}$$

$$Z = Q + iP\,, \qquad \bar{Z} = Q - iP\,, \tag{2.67}$$

where the imaginary unit i is defined by the complex structures J and J' respectively: $J(u) =: iu$, $J'(w) =: iw$ for any $v = (q,p) \in \mathbb{R}^2$. Finally, starting from (g,ω) and (g',ω'), we construct two Hermitian structure on $\mathbb{R}^2$ which makes it into a Hilbert space of complex dimension 1, namely

$$h(\cdot\,,\cdot) =: g(\cdot\,,\cdot) + i\omega(\cdot\,,\cdot)\,, \tag{2.68}$$

$$h'(\cdot\,,\cdot) =: g'(\cdot\,,\cdot) + i\omega'(\cdot\,,\cdot)\,. \tag{2.69}$$

Using complex coordinates, one has

$$h(z,z') = \bar{z}z'\,, \qquad h'(Z,Z') = \bar{Z}Z'\,. \tag{2.70}$$

It is then clear that the two scalar products, when compared in the same coordinate system, are *not* proportional trough a constant, thus defining two genuinely different Hilbert space structures on the same underlying set.

It is worth pointing out that the construction outlined in this paragraph can be read backwards, showing that starting with a symplectic structure, say ω' in the example above, we can construct a Darboux chart that induces an "adapted" linear structure on the underlying space such that the form is constant with respect to it. We will use this fact on a more general basis shortly below.

2.3. *Linear structures associated with regular Lagrangians*

Now we will exploit the idea pointed out at the end of the previous section in the particular case when our symplectic structures arise from Lagrangian functions. Let us recall that a regular Lagrangian function $\mathcal{L}$ will define the symplectic structure on the velocity phase space of a classical system TQ:

$$\omega_{\mathcal{L}} = d\theta_{\mathcal{L}} = d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right) \wedge dq^i\,, \qquad \theta_{\mathcal{L}} = \left(\frac{\partial \mathcal{L}}{\partial u^i}\right) dq^i\,. \tag{2.71}$$

We look now[17] for Hamiltonian vector fields X_j, Y^j such that

$$i_{X_j}\omega_{\mathcal{L}} = -d\left(\frac{\partial \mathcal{L}}{\partial u^j}\right), \qquad i_{Y^j}\omega_{\mathcal{L}} = dq^j \tag{2.72}$$

which implies, of course,

$$L_{X_j}\omega_{\mathcal{L}} = L_{Y^j}\omega_{\mathcal{L}} = 0\,. \tag{2.73}$$

More explicitly

$$i_{X_j}\omega_{\mathcal{L}} = \left(L_{X_j}\frac{\partial \mathcal{L}}{\partial u^i}\right) dq^i - d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right)(L_{X_j}q^i) \tag{2.74}$$

and this implies

$$L_{X_j} q^i = \delta^i_j\,, \qquad L_{X_j} \frac{\partial \mathcal{L}}{\partial u^i} = 0\,. \tag{2.75}$$

Similarly

$$i_{Y^j}\omega_{\mathcal{L}} = \left(L_{Y^j} \frac{\partial \mathcal{L}}{\partial u^i}\right) dq^i - d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right)(L_{Y^j} q^i) \tag{2.76}$$

and this implies in turn

$$L_{Y^j} q^i = 0\,, \qquad L_{Y^j} \frac{\partial \mathcal{L}}{\partial u^i} = \delta^j_i\,. \tag{2.77}$$

Then using the identity

$$i_{[Z,W]} = L_Z \circ i_W - i_W \circ L_Z\,, \tag{2.78}$$

we obtain, whenever both Z and W are Hamiltonian ($i_Z\omega_{\mathcal{L}} = dg_Z$ and similarly for W):

$$i_{[Z,W]}\omega_{\mathcal{L}} = d(L_Z g_W)\,. \tag{2.79}$$

Taking now $(Z, W) = (X_i, X_j)$, (X_i, Y^j) or (Y^i, Y^j), the Lie derivative of the Hamiltonian of every field with respect to any other field is either zero or a constant (actually unity). Therefore

$$i_{[Z,W]}\omega_{\mathcal{L}} = 0\,, \tag{2.80}$$

whenever $[Z, W] = [X_i, X_j]$, $[X_i, Y^j]$, $[Y^i, Y^j]$, which proves that

$$[X_i, X_j] = [X_i, Y^j] = [Y^i, Y^j] = 0\,. \tag{2.81}$$

Thus defining an infinitesimal action of a $2n$-dimensional Abelian Lie group on TQ. If this action integrates to a free and transitive action of the group $\mathbb{R}^{2n}$ ($\dim Q = n$), this will define a new vector space structure on TQ that by construction is "adapted" to the Lagrangian 2-form $\omega_{\mathcal{L}}$.

Spelling now explicitly Eqs. (2.75) and (2.77) we find that X_j and Y^j have the form

$$X_j = \frac{\partial}{\partial q^j} + (X_j)^k \frac{\partial}{\partial u^k}\,, \quad Y^j = (Y^j)^k \frac{\partial}{\partial u^k}\,; \qquad (X_j)^k\,, \quad (Y^j)^k \in \mathcal{F}(TQ) \tag{2.82}$$

and that

$$L_{X_j} \frac{\partial \mathcal{L}}{\partial u^i} = 0 \Rightarrow \frac{\partial^2 \mathcal{L}}{\partial u^i \partial q^j} + (X_j)^k \frac{\partial^2 \mathcal{L}}{\partial u^i \partial u^k} = 0\,, \tag{2.83}$$

$$L_{Y^j} \frac{\partial \mathcal{L}}{\partial u^i} = \delta^i_j \Rightarrow (Y^j)^k \frac{\partial^2 \mathcal{L}}{\partial u^i \partial u^k} = \delta^j_i\,. \tag{2.84}$$

Therefore, the Hessian being not singular by assumption, $(Y^j)^k$ is the inverse of the Hessian matrix, while $(X_j)^k$ can be obtained algebraically from Eq. (2.83). We can then define the dual forms (α^i, β_i) via

$$\alpha^i(X_j) = \delta^i_j\,, \qquad \alpha^i(Y^j) = 0\,, \tag{2.85}$$

$$\beta_i(Y^j) = \delta^j_i\,, \qquad \beta_i(X_j) = 0\,, \tag{2.86}$$

which can be proven immediately to be closed by testing then the identity

$$d\theta(Z,W) = L_Z(\theta(W)) - L_W(\theta(Z)) - \theta([Z,W]) \tag{2.87}$$

on the pairs $(Z,W) = (X_i, X_j)$, (X_i, Y^j), (Y^i, Y^j). Moreover, it is also immediate to see that

$$\alpha^i = dq^i \tag{2.88}$$

and

$$\beta_i = d\left(\frac{\partial \mathcal{L}}{\partial u^i}\right) \tag{2.89}$$

and that the symplectic form can be written as

$$\omega_{\mathcal{L}} = \beta_i \wedge \alpha^i\,. \tag{2.90}$$

Basically, what this means is that, to the extent that the definition of vector fields and dual forms is global, we have found in this way a global Darboux chart.

As a nontrivial example we can compute the adapted linear structure defined by the Lagrangian of a particle on a time-independent magnetic field $\vec{B} = \nabla \times \vec{A}$. The particular instance of a constant magnetic field will be worked out explicitly in App. C.

The dynamics is given by the second-order vector field ($e = m = c = 1$):

$$\Gamma = u^i \frac{\partial}{\partial q^i} + \delta^{is}\epsilon_{ijk} u^j B^k \frac{\partial}{\partial u^s} \tag{2.91}$$

and the equations of motion are

$$\frac{dq^i}{dt} = u^i\,, \qquad \frac{du^i}{dt} = \delta^{ir}\epsilon_{rjk} u^j B^k\,, \quad i = 1,2,3\,. \tag{2.92}$$

The Lagrangian is given in turn by

$$\mathcal{L} = \frac{1}{2}\delta_{ij} u^i u^j + u^i A_i\,. \tag{2.93}$$

Hence

$$\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial u^i} dq^i = (\delta_{ij} u^j + A_i) dq^i\,. \tag{2.94}$$

The symplectic form is

$$\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}} = \delta_{ij}\, dq^i \wedge du^j - \frac{1}{2}\,\varepsilon_{ijk} B^i\, dq^j \wedge dq^k\,. \tag{2.95}$$

Notice that $\theta_{\mathcal{L}} = \theta_{\mathcal{L}}^{(0)} + A$, $\theta_{\mathcal{L}}^{(0)} = \delta_{ij} u^j \, dq^i$, $A = A_i \, dq^i$, then $dA =: B = \frac{1}{2}\varepsilon_{ijk} B^i \, dq^j \wedge dq^k$, and $\omega_{\mathcal{L}} = \omega_0 - B$.

The field Γ satisfies

$$i_\Gamma \omega_{\mathcal{L}} = dH \,, \tag{2.96}$$

with the Hamiltonian

$$H = \frac{1}{2}\delta_{ij} u^i u^j \,. \tag{2.97}$$

Now it is easy to see that

$$X_j = \frac{\partial}{\partial q^j} - \delta^{ik} \frac{\partial A_k}{\partial q^j} \frac{\partial}{\partial u^i} \,, \tag{2.98}$$

while

$$Y^j = \delta^{jk} \frac{\partial}{\partial u^k} \,. \tag{2.99}$$

Dual forms α^i, β_i, $i = 1, \ldots, n = \dim Q$, (2.85) and (2.86), are easily found:

$$\alpha^i = dq^i \,, \qquad \beta_i = \delta_{ij} dU^j \,, \qquad U^j =: u^j + \delta^{jk} A_k \,. \tag{2.100}$$

Notice that in this way the Cartan form (2.94) is

$$\theta_{\mathcal{L}} = \pi_i \, dq^i \,, \tag{2.101}$$

where

$$\pi_i = \delta_{ij} u^j + A_i \,, \tag{2.102}$$

and the symplectic form becomes

$$\omega_{\mathcal{L}} = dq^i \wedge d\pi_i \,. \tag{2.103}$$

It appears therefore that the mapping

$$\phi : (q, u) \to (Q, U) \,, \tag{2.104}$$

with

$$Q^i = q^i \,, \qquad U^i = u^i + \delta^{ik} A_k \,, \tag{2.105}$$

(hence $\pi_i = \delta_{ij} U^j$) provides us with a symplectomorphism that reduces $\omega_{\mathcal{L}}$ to the canonical form, i.e. that the chart (Q, U) is a Darboux chart "adapted" to the vector potential $\vec{A}$.

The mapping (2.105) is clearly invertible, and

$$\frac{\partial q^i}{\partial Q^j} = \delta^i_j \,, \qquad \frac{\partial q^i}{\partial U^j} = 0 \,, \tag{2.106}$$

while

$$\frac{\partial u^i}{\partial U^j} = \delta^i_j \,, \qquad \frac{\partial u^i}{\partial Q^j} = -\delta^{ik} \frac{\partial A_k}{\partial Q^j} \,, \tag{2.107}$$

$A_k(q) \equiv A_k(Q)$. But then

$$X_j = \frac{\partial}{\partial Q^j}\,, \qquad Y^j = \delta^{jk}\frac{\partial}{\partial U^k}\,, \tag{2.108}$$

as well as

$$\alpha^i = dQ^i\,, \qquad \beta_i = d\pi_i = \delta_{ij}\, dU^j\,. \tag{2.109}$$

The push-forward of the Liouville field: $\Delta_0 = q^i\partial/\partial q^i + u^i\partial/\partial u^i$ will be then

$$\Delta = \phi_*\Delta_0 = Q^i\frac{\partial}{\partial Q^i} + \left[U^i + \delta^{ik}\left(Q^j\frac{\partial A_k}{\partial Q^j} - A_k\right)\right]\frac{\partial}{\partial U^i}\,. \tag{2.110}$$

If we work with the standard Euclidean metric, there is actually no need to distinguish between uppercase and lowercase indices ($Q_i =: \delta_{ij}Q^j = Q^i$, etc.). Then, the push-forward of the dynamical vector field is

$$\tilde{\Gamma} = \phi_*\Gamma = (U^i - A^i)\,\frac{\partial}{\partial Q^i} + (U^k - A^k)\,\frac{\partial A^k}{\partial Q^i}\,\frac{\partial}{\partial U^i} \tag{2.111}$$

and is Hamiltonian with respect to the symplectic form (2.103) with the Hamiltonian

$$\tilde{H} = \phi^* H = \frac{1}{2}\delta_{ij}(U^i - A^i)(U^j - A^j)\,. \tag{2.112}$$

To conclude, a few remarks are in order:

(1) As remarked previously $\phi_*\Delta_0 = \Delta_0$ whenever the vector potential is homogeneous of degree one in the coordinates (constant magnetic field) and hence the mapping (2.105) is linear.
(2) For an arbitrary vector potential the linear structure Δ depends on the gauge choice. This is a consequence of the mapping (2.105) being also gauge-dependent, which means in turn that every choice of gauge will define a *different* linear structure. The symplectic form (2.103) will be however gauge-independent.
(3) Denoting collectively the old and new coordinates as (q, u) and (Q, U) respectively, Eq. (2.105) defines a mapping

$$(q, u) \stackrel{\phi}{\to} (Q, U)\,. \tag{2.113}$$

It is then a straightforward application of the definitions (2.1) and (2.2) to show that the rules of addition and multiplication by a constant become, in this specific case:

$$\begin{aligned}(Q, U) &+_{(\phi)} (Q', U') \\ &= (Q + Q', U + U' + [A(Q + Q') - (A(Q) + A(Q'))])\,,\end{aligned} \tag{2.114}$$

and

$$\lambda \cdot_{(\phi)} (Q, U) = (\lambda Q, \lambda U + [A(\lambda Q) - \lambda A(Q)])\,. \tag{2.115}$$

In particular, with $\lambda = e^t$, the infinitesimal version of (2.115) yields precisely the infinitesimal generator (2.110) and, if the vector potential is, as in the case of a constant magnetic field, homogeneous of degree one in the coordinates, all the terms in square brackets in Eqs. (2.114) and (2.115) vanish identically, as expected.

(4) Notice that the origin of the new linear structure is given by $\phi(0,0) = (0, A(0))$ and, correctly $0 \cdot_{(\phi)} (Q,U) = (0, A(0))\ \forall (Q,U)$ as well as $\lambda \cdot_{(\phi)} (0, A(0)) = (0, A(0))\ \forall \lambda$. Moreover, $(Q,U) + (0, A(0)) = (Q,U)\ \forall (Q,U)$. Finally, the difference between any two points (Q,U) and (Q',U') must be understood as

$$(Q,U) -_{(\phi)} (Q',U') =: (Q,U) +_{(\phi)} ((-1) \cdot_{(\phi)} (Q',U')) \tag{2.116}$$

and, because of $(-1) \cdot_{(\phi)} (Q',U') = (-Q', -U' + A(Q') + A(-Q'))$, we finally get

$$(Q,U) -_{(\phi)} (Q',U') = (Q - Q', U - U' + A(Q - Q') + A(Q') - A(Q))\,. \tag{2.117}$$

Again, if $Q' = Q$, $U' = U$, $(Q,U) -_{(\phi)} (Q,U) = (0, A(0))$.

3. Weyl Systems, Quantization and the von Neumann Uniqueness Theorem

We recall here briefly how Weyl systems are defined and how the Weyl–Wigner–von Neumann quantization programme can be implemented. Let (E, ω) be a symplectic vector space with ω a constant symplectic form. A *Weyl system*[4] is a strongly continuous map: $\mathcal{W} : E \to \mathcal{U}(\mathcal{H})$ from E to the set of unitary operators on some Hilbert space $\mathcal{H}$ satisfying (we set here $\hbar = 1$ for simplicity):

$$\mathcal{W}(e_1)\mathcal{W}(e_2) = e^{\frac{i}{2}\omega(e_1,e_2)}\mathcal{W}(e_1 + e_2)\,, \qquad e_1,\ e_2 \in \mathcal{H} \tag{3.1}$$

or

$$\mathcal{W}(e_1)\mathcal{W}(e_2) = e^{i\omega(e_1,e_2)}\mathcal{W}(e_2)\mathcal{W}(e_1)\,. \tag{3.2}$$

It is clear that operators associated with vectors on a Lagrangian subspace will commute pairwise and can then be diagonalized simultaneously. von Neumann's theorem states then that (a) Weyl systems do exist for any finite-dimensional symplectic vector space and (b) the Hilbert space $\mathcal{H}$ can be realized as the space of square-integrable complex functions with respect to the translationally-invariant Lebesgue measure on a Lagrangian subspace $L \subset E$. Decomposing then E as $L \oplus L^*$, one can define $\mathcal{U} =: \mathcal{W}|_{L^*}$ and $\mathcal{V} =: \mathcal{W}|_L$ and realize their action on $\mathcal{H} = L^2(L, d^n x)$ ($\dim E = 2n$) as

$$(\mathcal{V}(x)\psi)(y) = \psi(x + y)\,, \tag{3.3}$$

$$(\mathcal{U}(\alpha)\psi)(y) = e^{i\alpha(y)}\psi(y)\,, \quad x, y \in L\,,\ \alpha \in L^*\,. \tag{3.4}$$

As a consequence of the strong continuity of the mapping $\mathcal{W}$ one can write, using Stone's theorem:[18]

$$\mathcal{W}(e) = \exp\{i\mathcal{R}(e)\} \quad \forall e \in E\,, \tag{3.5}$$

where $\mathcal{R}(e)$, which depends linearly on e, is the self-adjoint generator of the one-parameter unitary group $\mathcal{W}(te)$, $t \in \mathbb{R}$.

If $\{\mathbb{T}(t)\}_{t\in\mathbb{R}}$ is a one-parameter group of symplectomorphisms (i.e. $\mathbb{T}(t)\mathbb{T}(t') = \mathbb{T}(t+t')\ \forall t,\, t'$ and $\mathbb{T}^t(t)\omega\mathbb{T}(t) = \omega\ \forall t$), then we can define

$$\mathcal{W}_t(e) =: \mathcal{W}(\mathbb{T}(t)e)\,. \tag{3.6}$$

This being an automorphism of the unitary group will be inner and will be therefore represented as a conjugation with a unitary transformation belonging to a one-parameter unitary group associated with the group $\{\mathbb{T}(t)\}$. If $\mathbb{T}(t)$ represents the dynamical evolution associated with a linear vector field, then we can write

$$\mathcal{W}_t(e) = e^{it\hat{H}}\mathcal{W}(e)e^{-it\hat{H}} \tag{3.7}$$

and $\hat{H}$ will be (again in units $\hbar = 1$) the quantum Hamiltonian of the system.

The uniqueness part of von Neumann's theorem states that different realizations of a Weyl system on Hilbert spaces of square-integrable functions on different Lagrangian subspaces of the same symplectic vector space are unitarily related. Generally speaking, any $\phi\colon E \to E$ which is a linear symplectic map of E into itself induces a unitary mapping between the two corresponding Weyl systems. A conspicuous and well known example is the realization, in the case of $T^*\mathbb{R}^n$ with coordinates (q^i, p_i) and with the standard symplectic form, of the associated Weyl system on square-integrable functions of the q's or, alternatively, of the p's. In this case the equivalence is given by the Fourier transform. In this sense the theorem is a *uniqueness* (up to unitary equivalence) theorem. We would like to stress here that it is such if the linear structure (and the symplectic form) are assumed to be given once and for all.

In the general case, if two nonlinearly related linear structures (and associated symplectic forms) are available on E, then one can set up two different Weyl systems $\mathcal{W}$ and $\mathcal{W}'$ realized on two different Hilbert space structures made of functions defined on the same Lagrangian subspace. However, the two measures on this function space that help defining the Hilbert space structures are not linearly related and functions that are square-integrable in one setting need not be such in the other. Moreover, a necessary ingredient in the Weyl quantization program is the use of the (standard or symplectic) Fourier transform. For the same reasons as outlined above, it is clear then the two different linear structures will define genuinely different Fourier transforms.

In this way one can "evade" the uniqueness part of von Neumann's theorem. What the present discussion is actually meant at showing is that there are assumptions, namely that the linear structure (and symplectic form) are given once and for all and are unique, that are implicitly assumed but not explicitly stated in the usual formulations of the theorem, and that, whenever alternative structures are available at the same time, the situation can be much richer and lead to genuinely and nonequivalent (in the unitary sense) formulations of quantum mechanics.

Let us illustrate these considerations by going back to the example of the geometry of the 1D harmonic oscillator that was discussed in Subsec. 2.2. To quantize this system according to the Weyl scheme we have first of all to select a Lagrangian subspace $\mathcal{L}$ of $\mathbb{R}^2$ and a Lebesgue measure $d\mu$ on it defining then $L^2(\mathcal{L}, d\mu)$. When we endow $\mathbb{R}^2$ with the standard linear structure we choose $\mathcal{L} = \{(q, 0)\}$ and $d\mu = dq$. Alternatively, when we use the linear structure (2.12), we take $\mathcal{L}' = \{(Q, 0)\}$ and $d\mu = dQ$. Notice that $\mathcal{L}$ and $\mathcal{L}'$ are the same subset of $\mathbb{R}^2$, defined by the conditions $P = p = 0$ and with coordinates related by: $Q = qK(r = |q|)$. Nevertheless the two Hilbert spaces $L^2(\mathcal{L}, d\mu)$ and $L^2(\mathcal{L}', d\mu')$ are not related via a unitary map.

As a second step in the Weyl scheme, we construct in $L^2(\mathcal{L}, d\mu)$ the operator $\hat{U}(\alpha)$:

$$(\hat{U}(\alpha)\psi)(q) = e^{i\alpha q/\hbar}\psi(q)\,, \qquad \psi(q) \in L^2(\mathcal{L}, d\mu)\,, \tag{3.8}$$

whose generator is $\hat{x} = q$, and the operator $\hat{V}(h)$:

$$(\hat{V}(h)\psi)(q) = \psi(q + h)\psi(q) \in L^2(\mathcal{L}, d\mu)\,, \tag{3.9}$$

which is generated by $\hat{\pi} = -i\hbar\partial/\partial q$, and implements the translations defined by the standard linear structure. The quantum Hamiltonian can be written as $H = \hbar\big(a^\dagger a + \frac{1}{2}\big)$ where $a = (\hat{x} + i\hat{\pi})/\sqrt{2}\hbar$ (here the adjoint is taken with respect to the Hermitian structure defined with the Lebesgue measure dq).

Similar expressions hold in $L^2(\mathcal{L}', d\mu')$ for $\hat{x}'$, $\hat{\pi}'$ and $\hat{U}'(\alpha)$, $\hat{V}'(h)$. Notice that, when seen as operators in the previous Hilbert space, $\hat{V}'(h)$ implements translations with respect to the linear structure (2.12):

$$(\hat{V}'(h)\psi)(q) = \psi(q +_{(K)} h)\,. \tag{3.10}$$

Now the quantum Hamiltonian is $H' = \hbar\big(A^{\dagger\prime} A + \frac{1}{2}\big)$ with $A = (\hat{x}' + i\hat{\pi}')/\sqrt{2}\hbar$, where now the adjoint is taken with respect to the Hermitian structure defined with the Lebesgue measure dQ. Put it in a slightly different way, we may define the creation/annihilation operators $a^\dagger$, a and $A^{\dagger\prime}$, A through Eq. (3.5) as those operators such that

$$a(v) =: [\mathcal{R}(v) + i\mathcal{R}(Jv)]/\sqrt{2}\,, \qquad a^\dagger(v) =: [\mathcal{R}(v) - i\mathcal{R}(Jv)]/\sqrt{2} \tag{3.11}$$

and

$$A(v) =: [\mathcal{R}'(v) + i\mathcal{R}'(J'v)]/\sqrt{2}\,, \qquad A^{\dagger\prime}(v) =: [\mathcal{R}'(v) - i\mathcal{R}'(J'v)]/\sqrt{2} \tag{3.12}$$

for any $v \in \mathbb{R}^2$. (Here i represents the imaginary unit of the complex numbers $\mathbb{C}$, target space of $L^2(\mathcal{L}, d\mu)$ and $L^2(\mathcal{L}', d\mu')$.)

It is interesting to notice that, in the respective Hilbert spaces,

$$[a, a^\dagger] = \mathbb{I}\,, \tag{3.13}$$

$$[A, A^{\dagger\prime}] = \mathbb{I}\,, \tag{3.14}$$

so that we get different realizations of the algebra of the 1D harmonic oscillator. To be more explicit, we notice that, from Eqs. (2.16) and (2.17), one can easily find, after having chosen the Lagrangian submanifolds defined by $p = P = 0$:

$$\hat{x} = q = Q(1 + \lambda Q^2) = \hat{x}'[1 + \lambda(\hat{x}')^2]\,, \tag{3.15}$$

$$\hat{\pi} = -i\hbar\partial_q = -i\hbar(1 + 3\lambda Q^2)^{-1}\partial_Q = [1 + 3\lambda(\hat{x}')^2]^{-1}\hat{\pi}'\,, \tag{3.16}$$

so that

$$a = \frac{\hat{x} + i\hat{\pi}}{\sqrt{2\hbar}} = \frac{1}{\sqrt{2\hbar}}[1 + \lambda(\hat{x}')^2]\hat{x}' + i[1 + 3\lambda(\hat{x}')^2]^{-1}\hat{\pi}'\,, \tag{3.17}$$

$$a^\dagger = \frac{\hat{x} - i\hat{\pi}}{\sqrt{2\hbar}} = \frac{1}{\sqrt{2\hbar}}[1 + \lambda(\hat{x}')^2]\hat{x}' - i[1 + 3\lambda(\hat{x}')^2]^{-1}\hat{\pi}'\,. \tag{3.18}$$

Clearly $\hat{x}$ and $\hat{\pi}$ are self-adjoint with respect to the measure $d\mu = dq$, while the latter is not when considering $d\mu' = dQ$:

$$\hat{x}^\dagger = \hat{x}\,, \qquad \hat{x}^{\dagger\prime} = \hat{x}\,, \tag{3.19}$$

$$\hat{\pi}^\dagger = \hat{\pi}\,, \qquad \hat{\pi}^{\dagger\prime} = \hat{\pi} - (6i\lambda\hat{x}')[1 + 3\lambda(\hat{x}')^2]^{-2}\,. \tag{3.20}$$

This means that $a^\dagger$ is not the adjoint of a if one uses this measure. Thus, the (C^*) algebra generated by $\hat{x}$, $\hat{\pi}$, $\mathbf{I}$ seen as operators acting on $L^2(\mathcal{L}, d\mu)$ is closed, whereas the one generated by $\hat{x}$, $\hat{\pi}$, $\mathbf{I}$ and their adjoints $\hat{x}^{\dagger\prime}$, $\hat{\pi}^{\dagger\prime}$, $\mathbf{I}^{\dagger\prime}$ acting on $L^2(\mathcal{L}', d\mu')$ does not close because we generate new operators whenever we consider the commutator between $\hat{\pi}$ and $\hat{\pi}^{\dagger\prime}$. As a consequence, the operators $\hat{x}$, $\hat{\pi}$ and $\hat{x}'$, $\hat{\pi}'$ close the Heisenberg algebra only if we let them act on two different Hilbert spaces generated, respectively, by the sets of the Fock states:[b]

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\,, \tag{3.21}$$

$$|N\rangle = \frac{1}{\sqrt{N!}}(A^{\dagger\prime})^N|0\rangle\,. \tag{3.22}$$

A further example is provided by the case of a charged particle in a constant magnetic field[19] (and in the symmetric gauge) as described in the previous section and in App. C (in the following we reinstate Planck's constant in the appropriate places). We can choose as Hilbert space that of the square-integrable functions on

[b]In this example we have obtained two different realizations of the quantum 1D harmonic oscillator starting from two alternative linear structures on the classical phase space. One can also think of changing the (real) linear structure, and the corresponding additional geometric structures, on the target space $\mathbb{C}$ of the L^2 space. In this way one can get even other realizations (details may be found in Refs. 7 and 8).

the Lagrangian subspace defined by: $U^i = 0, i = 1, 2$ (i.e. the subspace: $u^i = -A^i(q)$ in the original coordinates). Square-integrable wave functions will be denoted as $\psi(Q^1, Q^2)$ or $\psi(Q)$ for short. Then we can define the Weyl operators

$$\begin{aligned} \hat{\mathcal{W}}(x, \pi) &= \exp\left\{\frac{i}{\hbar}[x\hat{U} - \pi\hat{Q}]\right\} \\ &=: \exp\left\{\frac{i}{\hbar}\left[x_1\hat{U}^1 + x_2\hat{U}^2 - \pi_1\hat{Q}^1 - \pi_2\hat{Q}^2\right]\right\} \end{aligned} \tag{3.23}$$

acting on wave functions as

$$(\hat{\mathcal{W}}(x, \pi)\psi)(Q) = \exp\left\{-\frac{i}{\hbar}\pi\left(Q + \frac{x}{2}\right)\right\}\psi(Q + x)\,. \tag{3.24}$$

Then $\hat{U} = -i\hbar\nabla_Q$ while $\hat{Q}$ acts as the usual multiplication operator, i.e. $(\hat{Q}^i\psi)(Q) = Q^i\psi(Q)$. Equation (3.23) can be rewritten in a compact way as

$$\hat{\mathcal{W}}(x, \pi) = \exp\left\{\frac{i}{\hbar}\xi^T\mathbf{g}\hat{X}\right\}, \tag{3.25}$$

where

$$\xi = \begin{vmatrix} x \\ \pi \end{vmatrix}, \qquad \hat{X} = \begin{vmatrix} \hat{U} \\ \hat{Q} \end{vmatrix} \tag{3.26}$$

and

$$\mathbf{g} = \begin{vmatrix} \mathbb{I}_{2\times 2} & \mathbf{0} \\ \mathbf{0} & -\mathbb{I}_{2\times 2} \end{vmatrix}. \tag{3.27}$$

The dynamical evolution defines then the one-parameter family of Weyl operators:

$$\begin{aligned} \hat{\mathcal{W}}_t(x, \pi) &= \hat{\mathcal{W}}(x(t), \pi(t)) \\ &= \exp\left\{\frac{i}{\hbar}\left[x(t)\hat{U} - \pi(t)\hat{Q}\right]\right\} \\ &\equiv \exp\left\{\frac{i}{\hbar}\xi^T(t)\mathbf{g}\hat{X}\right\}, \end{aligned} \tag{3.28}$$

where

$$\xi(t) = \mathbb{F}(t)\xi\,. \tag{3.29}$$

According to the standard procedure, this can be rewritten as

$$\hat{\mathcal{W}}_t(x, \pi) = \exp\left\{\frac{i}{\hbar}\left[x\hat{U}(t) - \pi\hat{Q}(t)\right]\right\} = \exp\left\{\frac{i}{\hbar}\xi^T\mathbf{g}\hat{X}(t)\right\}, \tag{3.30}$$

where

$$\hat{X}(t) = \tilde{\mathbb{F}}(t)\hat{X}\,, \qquad \tilde{\mathbb{F}}(t) = \mathbf{g}\mathbb{F}(t)^T\mathbf{g} \tag{3.31}$$

and $\mathbb{F}(t)^T$ denotes the transpose of the matrix $\mathbb{F}(t)$. Explicitly,

$$\hat{U}^1(t) = \frac{1}{2}\hat{U}^1(1+\cos(Bt)) - \frac{1}{2}\hat{U}^2\sin(Bt) + \frac{B}{4}\hat{Q}^1\sin(Bt) - \frac{B}{4}\hat{Q}^2(1-\cos(Bt))\,, \tag{3.32}$$

$$\hat{U}^2(t) = \frac{1}{2}\hat{U}^1\sin(Bt) + \frac{1}{2}\hat{U}^2(1+\cos(Bt)) - \frac{B}{4}\hat{Q}^1(\cos(Bt)-1) + \frac{B}{4}\hat{Q}^2\sin(Bt)\,, \tag{3.33}$$

and

$$\hat{Q}^1(t) = \frac{1}{B}\hat{U}^1\sin(Bt) + \frac{1}{B}\hat{U}^2(\cos(Bt)-1) - \frac{1}{2}\hat{Q}^1(1+\cos(Bt)) + \frac{1}{2}\hat{Q}^2\sin(Bt)\,, \tag{3.34}$$

$$\hat{Q}^2(t) = \frac{1}{B}\hat{U}^1(1-\cos(Bt)) + \frac{1}{B}\hat{U}^2\sin(Bt) - \frac{1}{2}\hat{Q}^1\sin(Bt) - \frac{1}{2}\hat{Q}^2(1+\cos(Bt))\,. \tag{3.35}$$

Now

$$\hat{\mathcal{W}}_t(x,\pi) = \hat{\mathcal{U}}(t)^\dagger\hat{\mathcal{W}}(x,\pi)\hat{\mathcal{U}}(t)\,, \qquad \hat{\mathcal{U}}(t) = \exp\left\{-\frac{it}{\hbar}\hat{\mathcal{H}}\right\} \tag{3.36}$$

and hence

$$\hat{Q}^i(t) = \hat{\mathcal{U}}(t)^\dagger\hat{Q}^i\hat{\mathcal{U}}(t) \tag{3.37}$$

and similarly for the $\hat{U}^i$'s. Expanding in t we find the commutation relations

$$\frac{i}{\hbar}[\hat{U}^1,\hat{\mathcal{H}}] = \frac{B}{2}\left(\hat{U}^2 - \frac{B}{2}\hat{Q}^1\right), \tag{3.38}$$

$$\frac{i}{\hbar}[\hat{U}^2,\hat{\mathcal{H}}] = -\frac{B}{2}\left(\hat{U}^1 + \frac{B}{2}\hat{Q}^2\right). \tag{3.39}$$

One also has the relations

$$\frac{i}{\hbar}[\hat{Q}^1,\hat{\mathcal{H}}] = -\left(\hat{U}^1 + \frac{B}{2}\hat{Q}^2\right), \tag{3.40}$$

$$\frac{i}{\hbar}[\hat{Q}^2,\hat{\mathcal{H}}] = -\left(\hat{U}^2 - \frac{B}{2}\hat{Q}^1\right) \tag{3.41}$$

that, by using the commutation relations $[\hat{Q}^i, \hat{U}^j] = i\hbar\delta^{ij}$, can be easily proven to be consistent with the Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2}\left\{\left(\hat{U}^1 + \frac{B}{2}\hat{Q}^2\right)^2 + \left(\hat{U}^2 - \frac{B}{2}\hat{Q}^1\right)^2\right\}, \tag{3.42}$$

which is the quantum version of (2.112).

Finally we recall[c] that, following the Weyl–Wigner–Moyal program,[20,21] one can define an inverse mapping (the Wigner map[20]) of (actually Hilbert–Schmidt[18]) operators onto square-integrable functions in phase space endowed with a noncommutative "$*$-product," the Moyal product[21] which is defined in general (i.e. for, say, $\mathbf{q}$, $\mathbf{p} \in \mathbb{R}^n$) as

$$(f * g)(\mathbf{q}, \mathbf{p}) = f(\mathbf{q}, \mathbf{q}) \exp\left\{\frac{i\hbar}{2}\left[\frac{\overleftarrow{\partial}}{\partial \mathbf{q}} \cdot \frac{\overrightarrow{\partial}}{\partial \mathbf{p}} - \frac{\overleftarrow{\partial}}{\partial \mathbf{p}} \cdot \frac{\overrightarrow{\partial}}{\partial \mathbf{q}}\right]\right\} g(\mathbf{q}, \mathbf{p}) \tag{3.43}$$

and with the standard symplectic form ω. The Moyal product defines in turn the Moyal bracket

$$\{f, g\}_M =: \frac{1}{i\hbar}(f * g - g * f) \tag{3.44}$$

and it is well known[20,21] that

$$\{f, g\}_M = \{f, g\}_\omega + \mathcal{O}(\hbar^2)\,. \tag{3.45}$$

Different (and not unitarily equivalent) Weyl systems will lead to different Moyal products and brackets, and to different (and not canonically related) Poisson brackets in the classical limit.

For example, in the 2D case analyzed in the previous sections one has Eq. (3.43) for the ordinary Moyal product and,

$$(f *_K g)(Q, P) = f(Q, P) \exp\left\{\frac{i\hbar}{2}\left[\frac{\overleftarrow{\partial}}{\partial Q}\frac{\overrightarrow{\partial}}{\partial P} - \frac{\overleftarrow{\partial}}{\partial P}\frac{\overrightarrow{\partial}}{\partial Q}\right]\right\} g(Q, P)\,, \tag{3.46}$$

which define the corresponding Moyal brackets $\{f, g\}_M$ and $\{f, g\}_{M_K}$. It is then not difficult to check that the Moyal products (and brackets) (3.43) and (3.46) reproduce, in the limit $\hbar \to 0$, the Poisson brackets $\{\cdot\,, \cdot\}_\omega$ and $\{\cdot\,, \cdot\}_{\omega'}$ respectively (cf. Eqs. (2.64) and (2.65)).

Thus, in addition to the possibility[2,3] of deforming the product, one can change the linear structure (of the classical phase space or of the quantum Hilbert space) in such a way to obtain novel descriptions still compatible with the dynamics of the given system.

[c]For reviews, see Refs. 23–25.

Appendix A. The Relativistic Law of Addition Again

The example discussed in the Introduction can be completed as follows. Let $E = \mathbb{R}$, $M = (-1, 1)$ and

$$\phi : E \to M\,, \qquad x \to X =: \tanh x\,. \tag{A.1}$$

Then

$$\lambda \cdot_{(\phi)} X = \tanh(\lambda \tanh^{-1}(X)) \tag{A.2}$$

and

$$\begin{aligned} \lambda \cdot_{(\phi)} (\lambda' \cdot_{(\phi)} X) &= \lambda \cdot_{(\phi)} \tanh(\lambda' \tanh^{-1}(X)) \\ &= \tanh(\lambda\lambda' \tanh^{-1}(X)) = (\lambda\lambda') \cdot_{(\phi)} X\,, \end{aligned} \tag{A.3}$$

while

$$X +_{(\phi)} Y = \tanh(\tanh^{-1}(X) + \tanh^{-1}(Y)) = \frac{X+Y}{1+XY}\,, \tag{A.4}$$

which is nothing but the one-dimensional relativistic law (in appropriate units) for the addition of velocities. It is also simple to prove that

$$\begin{aligned} (X +_{(\phi)} Y) +_{(\phi)} Z &= \tanh(\tanh^{-1}(X +_{(\phi)} Y) + \tanh^{-1}(Z)) \\ &= \tanh(\tanh^{-1} X + \tanh^{-1}(Y) + \tanh^{-1}(Z)) \end{aligned} \tag{A.5}$$

i.e.

$$\big(X +_{(\phi)} Y\big) +_{(\phi)} Z = X +_{(\phi)} (Y +_{(\phi)} Z)\,. \tag{A.6}$$

Explicitly,

$$X +_{(\phi)} Y +_{(\phi)} Z = \frac{X+Y+Z+XYZ}{1+XY+XZ+YZ}\,. \tag{A.7}$$

The mapping (2.23) is now

$$X(t) = \tanh\left(e^t \tanh^{-1}(X)\right) \tag{A.8}$$

and we obtain, for the Liouville field on $(-1, 1)$:

$$\Delta(X) = (1 - X^2) \tanh^{-1}(X) \frac{\partial}{\partial X} \tag{A.9}$$

and $\Delta(X) = 0$ for $X = 0$.

Appendix B. Constant Magnetic Field

We can compute explicitly the example of a particle in a magnetic discussed in Subsec. 2.3, for the particular case of a constant magnetic field $B = (0, 0, B)$ with, e.g. the vector potential in the symmetric gauge:

$$\vec{A} = \frac{B}{2}(-q^2, q^1, 0) = \frac{1}{2}\vec{B} \times \vec{r}\,, \qquad \vec{B} = B\hat{k} \Rightarrow A_i = \frac{1}{2}\varepsilon_{ijk}B^j q^k\,, \tag{B.1}$$

for which

$$X_1 = \frac{\partial}{\partial q^1} - \frac{B}{2}\frac{\partial}{\partial u^2}\,, \quad X_2 = \frac{\partial}{\partial q^2} + \frac{B}{2}\frac{\partial}{\partial u^1}\,, \quad X_3 = \frac{\partial}{\partial q^3} \tag{B.2}$$

and

$$\alpha^i = dq^i\,, \tag{B.3}$$

$$\begin{aligned} \beta_1 &= du^1 - \frac{B}{2}dq^2\,, \\ \beta_2 &= du^2 + \frac{B}{2}dq^1\,, \\ \beta_3 &= du^3\,, \end{aligned} \tag{B.4}$$

while $\Delta = \Delta_0$, as expected.

According to Eqs. (2.105) and (2.92), the equations of motion in the new coordinates are given by

$$\frac{d}{dt}\begin{vmatrix} Q^1 \\ Q^2 \\ U^1 \\ U^2 \end{vmatrix} = \mathbb{G}\begin{vmatrix} Q^1 \\ Q^2 \\ U^1 \\ U^2 \end{vmatrix}\,, \tag{B.5}$$

where

$$\mathbb{G} = \|G^i{}_j\| = \begin{vmatrix} 0 & B/2 & 1 & 0 \\ -B/2 & 0 & 0 & 1 \\ -B^2/4 & 0 & 0 & B/2 \\ 0 & -B^2/4 & -B/2 & 0 \end{vmatrix}\,. \tag{B.6}$$

In other words (cf. Eq. (2.104)),

$$\begin{aligned} \phi_*\Gamma = {} & \left(U^1 + \frac{B}{2}Q^2\right)\frac{\partial}{\partial Q^1} + \left(U^2 - \frac{B}{2}Q^1\right)\frac{\partial}{\partial Q^2} \\ & + \frac{B}{2}\left(U^2 - \frac{B}{2}Q^1\right)\frac{\partial}{\partial U^1} - \frac{B}{2}\left(U^1 + \frac{B}{2}Q^2\right)\frac{\partial}{\partial U^2}\,. \end{aligned} \tag{B.7}$$

As the transformation (2.105) is not a point-transformation (i.e. it is the identity on the base and acts only along the fibers), it comes to no surprise that the transformed vector field is no more a second-order field in the new coordinates. However, $\phi_*\Gamma$ is still Hamiltonian with respect to the symplectic form $\phi^*\omega_{\mathcal{L}} = dQ^i \wedge dU_i$ with Hamiltonian:

$$\phi^* H = \frac{1}{2}\delta_{ij}(U^i - \delta^{ik}A_k)(U^j - \delta^{jk}A_k)\,. \tag{B.8}$$

Spelled out explicitly, the equations of motion in the (Q, U) coordinates are

$$\frac{dQ^1}{dt} = U^1 + \frac{B}{2}Q^2\,, \qquad \frac{dQ^2}{dt} = U^2 - \frac{B}{2}Q^1\,, \tag{B.9}$$

$$\frac{dU^1}{dt} = \frac{B}{2}\left(U^2 - \frac{B}{2}Q^1\right), \quad \frac{dU^2}{dt} = -\frac{B}{2}\left(U^1 + \frac{B}{2}Q^2\right). \tag{B.10}$$

Hence

$$\frac{dU^1}{dt} = \frac{B}{2}\frac{dQ^2}{dt}\,, \tag{B.11}$$

$$\frac{dU^2}{dt} = -\frac{B}{2}\frac{dQ^1}{dt}\,. \tag{B.12}$$

Therefore

$$\chi_1 =: U^1 - \frac{B}{2}Q^2 \quad \text{and} \quad \chi_2 =: U^2 + \frac{B}{2}Q^1 \tag{B.13}$$

are constants of the motion (they are proportional to the coordinates of the center of the Larmor orbit,[22] see also Eqs. (B.16) and (B.17) below), and this allows an easy integration of the equations of motion. Indeed, using (B.13) one finds at once

$$\frac{dQ^1}{dt} = \chi_1 + BQ^2\,, \tag{B.14}$$

$$\frac{dQ^2}{dt} = \chi_2 - BQ^1\,. \tag{B.15}$$

We can define the quantities

$$Q^1(t) = \frac{\chi_2}{B} + \tilde{Q}^1(t)\,, \qquad Q^2(t) = -\frac{\chi_1}{B} + \tilde{Q}^2(t) \tag{B.16}$$

that obey the equations

$$\frac{d\tilde{Q}^1}{dt} = B\tilde{Q}^2\,, \qquad \frac{d\tilde{Q}^2}{dt} = -B\tilde{Q}^1 \Rightarrow \frac{d^2\tilde{Q}^i}{dt^2} + B^2\tilde{Q}^i = 0\,, \quad i = 1, 2\,. \tag{B.17}$$

These integrate easily and, using again Eqs. (B.10), the final result is

$$\begin{vmatrix} Q^1(t) \\ Q^2(t) \\ U^1(t) \\ U^2(t) \end{vmatrix} = \mathbb{F}(t) \begin{vmatrix} Q^1 \\ Q^2 \\ U^1 \\ U^2 \end{vmatrix}\,, \tag{B.18}$$

where $Q^i = Q^i(0)$, $U^i = U^i(0)$ and $\mathbb{F}(t) =: \exp\{t\mathbb{G}\}$ is given explicitly by

$$\mathbb{F}(t) = \left|\begin{array}{cccc} \frac{1+\cos(Bt)}{2} & \frac{\sin(Bt)}{2} & \frac{\sin(Bt)}{B} & \frac{1-\cos(Bt)}{B} \\ -\frac{\sin(Bt)}{2} & \frac{1+\cos(Bt)}{2} & \frac{\cos(Bt)-1}{B} & \frac{\sin(Bt)}{B} \\ -\frac{B\sin(Bt)}{4} & \frac{B(\cos(Bt)-1)}{4} & \frac{1+\cos(Bt)}{2} & \frac{\sin(Bt)}{2} \\ \frac{B(1-\cos(Bt))}{4} & -\frac{B\sin(Bt)}{4} & -\frac{\sin(Bt)}{2} & \frac{1+\cos(Bt)}{2} \end{array}\right| . \quad \text{(B.19)}$$

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