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EIGENSTRUCTURE OF NONSTATIONARY FACTOR MODELS

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Abstract

In this paper we present a generalized dynamic factor model for a vector of time series which seems to provide a general framework to incorporate all the common information included in a collection of variables. The common dynamic structure is explained through a set of common factors, which may be stationary or nonstationary, as in the case of common trends. Also, it may exist a specific structure for each variable. Identification of the nonstationary *I(d)* factors is made through the common eigenstructure of the generalized covariance matrices, properly normalized. The number of common trends, or in general *I(d)* factors, is the number of nonzero eigenvalues of the above matrices. It is also proved that these nonzero eigenvalues are strictly greater than zero almost sure. Randomness appears in the eigenvalues as well as the eigenvectors, but not on the subspace spanned by the eigenvectors.

Key Words

Cointegration and common factors, eigenvectors and eigenvalues, generalized covariance matrices, factor model, nonstationary *I(d)* factors, vector time series, Wiener processes

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1 Introduction

Factor models are of great importance when dealing with reduction of dimensionality problems. When data is dynamic, this is specially important since for vector ARMA models, as well as for econometric models, the number of parameters to estimate grows rapidly with the number of observed variables. Dynamic factor models have been studied by Anderson (1963), Priestly et al (1974), Box and Tiao (1977), Brillinger (1981), Engle and Watson (1981), Shumway and Stoffer (1982), Watson and Engle (1983), Peña and Box (1987) and Velu et al (1986) among others.

In the nonstationary case, estimating the nonstationary factors is equivalent to testing for cointegration, since as it was formally shown by Escribano and Peña (1994), both concepts are closely related. Engle and Granger (1987) presented a two step estimator based on OLS regressions. Phillips and Ouliaris (1988) proposed a method based in principal component analysis applied to the innovation sequence resulting after taking first differences of the series. Stock and Watson (1988) developed a method to identify the number of common trends for VAR models. Residual based tests for cointegration are discused in Phillips and Ouliaris (1990). Related work on the topic is that of Tiao and Tsay (1989) and Gonzalo and Granger (1995). Johansen (1988, 1991) developed a maximum likelihood approach to estimate the linear space spanned by the cointegration vectors. Reinsel and Ahn (1992) have proposed a reduced rank model to deal with this problem and Ahn (1997) related it to the scalar component models of Tiao and Tsay (1989). Nonparametric cointegration analysis is considered in Bierens (1997).

In order to model the dynamics of variables that exhibit cointegration, the cointegration relations that make the series stationary should be estimated and interpreted. Apart from problems that can arise due to arbitrary normalizations, when the number of series is moderate to large, the task becomes difficult. Then, the number of cointegration relations can be large, and since the basis from the cointegrating subspace that can be chosen is arbitrary, its interpretation can be very complicated. In this case, it could be better to estimate and interpret a small number of nonstationary factors that characterize the growing behaviour of the series. This can be achieved using dynamic factor models. In this article, we propose a method to identify the factor space by looking at the eigenvalues of the generalized (properly normalized) variance-covariance matrices of the observed series. It is shown that the nonzero eigenvalues converge to random quantities (functionals of Wiener processes) while the eigenvectors for those nonzero eigenvalues converge to a random basis of the vector space spanned by the factor loading matrix. The subspace of stationarity or cointegration is given by the eigenvectors of the zero eigenvalues. As it will be shown it is orthogonal to the subspace of nonstationarity. An important advantage of this approach is that no model is required.

Besides, it constitutes a simple extension to the one applied in the stationary case.

This article is organized as follows. Section 2 presents the generalized dynamic factor model and study its properties. Section 3 presents the main result, which is the basis to separate the nonstationary factors from the stationary ones, and shows how this can be carried out by a generalization of a method proposed by Peña and Box (1987) for stationary factors and, finally, section 4 presents some conclusions.

2 The Factor Model

Let y_t be an m-dimensional vector of observable time series, generated by a set of not observable factors. We assume that each component of the vector of observed series, y_t, can be written as a linear combination of common factors and specific components; that is

$$\underset{m\times 1}{y_t} = \underset{m\times r}{P} \underset{r\times 1}{f_t} + \underset{m\times 1}{n_t} + \underset{m\times 1}{\epsilon_t} \tag{1}$$

where f_t is the r-dimensional vector of **common factors**, P is the factor loading matrix, and n_t is the vector of **specific components** and ϵ_t is white noise $(0, \Sigma_\epsilon)$.Therefore, the common dynamic structure comes through the common factors, f_t, whereas the vector n_t explains the dynamics specific to each time series.We suppose that the vector of common factors follows a VARMA(p, q) model

$$\Phi(B) f_t = \Theta(B) a_t, \tag{2}$$

where $\Phi(B) = I - \Phi(1)B-, \cdots, -\Phi(p)B^p$, and $\Theta(B) = I - \Theta(1)B-, \cdots, -\Theta(q)B^q$, are $r \times r$ polinomial matrices and B is the backshift operator. The sequence of vectors a_t are normally distributed, have zero mean, a full rank covariance matrix Σ_a and are serially uncorrelated, that is

$$\text{E } (a_t a'_{t-h}) = 0 \quad h \neq 0 \tag{3}$$

The vector of common factors, f_t, can include stationary and nonstationary terms. We assume that the specific components, n_t, if they exist, have stationary dynamic structure and follow an ARMA model,

$$\Phi_n(B) n_t = \Theta_n(B) e_t,$$

where Φ_n and Θ_n are $m \times m$ diagonal matrices given by $\Phi_n(B) = I - \Phi_n(1)B-, \cdots, -\Phi_n(p)B^p$, and $\Theta_n(B) = I - \Theta_n(1)B-, \cdots, -\Theta_n(q)B^q$, and therefore each component follows an univariate ARMA(p_i, q_i), $i = 1, 2, \cdots, m$, being p=max(p_i) and q=max(q_i), $i = 1, 2, \cdots, m$.

The sequence of vectors e_t are normally distributed, with zero mean and diagonal covariance matrix Σ_e. We assume that the noises from the common factors and the specific components, are also uncorrelated for all lags,

$$\mathrm{E}\ \ (a_t e'_{t-h}) = 0 \quad \forall h, \tag{4}$$

and both noises are uncorrelated with the noise in model (1), ϵ_t, for all lags

$$\mathrm{E}\ \ (a_t \epsilon'_{t-h}) = 0, \quad \forall h \tag{5}$$

and

$$\mathrm{E}\ \ (e_t \epsilon'_{t-h}) = 0 \quad \forall h. \tag{6}$$

The model as stated is not identified, because for any $r \times r$ non singular matrix H the observed series y_t can be expressed in terms of a new set of factors,

$$y_t = P^* f_t^* + n_t \tag{7}$$

$$\Phi^*(B) f_t^* = \Theta(B)^* a_t^* \tag{8}$$

with $P^{*\prime} P^* = (H^{-1})' P' P H^{-1}$, $f_t^* = H f_t$, $a_t^* = H a_t$, $\Phi^*(B) = H \Phi H^{-1}$, $\Theta^*(B) = H \Theta H^{-1}$, and $\Sigma_a^* = H \Sigma_a H'$. To solve the identification problem, we follow the work by Hannan (1969, 1971, 1976) and Kohn (1979) which has been more recently extended to nonstationary state space models by Wall (1987), and look for parametrizations that are unique in their effect on first and second moments of the observed time series.

Several identifying restrictions appear in the literature. Usually the factors noise covariance matrix Σ_a is considered to be diagonal. Also P can be chosen such that $P'P = I$. Some parameters of the processes followed by the factors may also be restricted: for example, if there is a common trend orthogonal to some stationary factors, the matrix Φ has already some fixed paramenters. Note that if we assume Σ_a diagonal, it is also implied by the model that

$$\mathrm{E}\ \ (f_{i,t} f'_{j,\tau}) = 0 \quad \forall t, \tau; \qquad \text{for} \quad i \neq j. \tag{9}$$

This condition is not restrictive, since the factor model can be rotated for a better interpretation when needed (see for example Harvey (1989) for a brief discussion about it), and helps to make easier the derivation of the asymptotics.

3 Eigenstructure of nonstationary factor models

When n_t is white noise and the factors are stationary model (1) and (2) reduces to the factor model studied by Peña and Box (1987). These authors developed a method of identifying the number of common factors based in the common eigenstructure of the lagged covariances matrices of the vector of time series. Nevertheless, in many cases real time series vectors are nonstationary. Suppose that the vector of time series is $I(d)$. In a general case, some common factors will be stationary, while others will be nonstationary. In particular, a nonstationary factor can be a common trend, in the sense of Stock and Watson (1988), driving all the series.

Assume that y_t is $I(d)$, for $d \geq 1$, we define the **generalized sample covariance matrices** $C_y(k)$ properly normalized as

$$C_y(k) = \frac{1}{T^{2d}} \sum (y_{t-k} - \bar{y})(y_t - \bar{y})' \tag{10}$$

and we will see that these matrices play the role of proper sample covariance matrices for the stationary case.

Suppose that there are r_1 common $I(d)$ factors, $f_{1,t} = (f_{1,t}^1, \cdots, f_{1,t}^{r_1})'$, r_2 common zero mean stationary factors, $f_{2,t} = (f_{2,t}^1, \cdots, f_{2,t}^{r_2})'$, and that the specific components, $n_t = (n_t^1, \cdots, n_t^m)'$, if they exist, are zero mean stationary ones. Divide the vectors of common factors and noise as $f_t' = (f_{1,t}', f_{2,t}')$ and $a_t' = (a_{1,t}', a_{2,t}')$, respectively, and the diagonal variance matrix for a_t as $\Sigma_a = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$.

Assumption 1. Suppose that the equation for the common nonstationary factors is $(d \geq 1)$

$$\begin{aligned} \nabla^d f_{1,t} &= u_t \\ u_t &= \Psi(B) a_{1,t} \end{aligned} \tag{11}$$

with E $(a_{1,t}) = 0$ and var $(a_{1,t}) = \Sigma_1 = diag(\sigma_1^2, \cdots, \sigma_{r_1}^2) > 0$, $f_{1,-(d-1)} = f_{1,-(d-2)} = \ldots = f_{1,0} = 0$, $||\Psi_i|| = [tr(\Psi_i'\Psi)]^{1/2}$ and $\sum i||\Psi_i|| < \infty$. Define matrix $\Psi(1) = \sum_{i=0}^{\infty} \Psi_i$ with rank$(\Psi(1)) = r_1$. Then, the following result will help us to identify the nonstationary factors and to separate them from the stationary ones.

Theorem 1. For the nonstationary factor model presented in sections 2 and 3 and assumption 1, define $C_y(k)$ as in (10) , for $k = 0, 1, \cdots, K$, such that $K/T \to 0$. Then:
(i) The number of common nonstationary $I(d)$ factors, r_1, is the number of nonzero eigenvalues of lim $C_y(k)$, $k = 0, 1, \cdots, K$, where limits are taken as T goes to infinity.
(ii) Define the variability in $\{y_t\}_{t=1}^T$ as $trace(C_y(0))$. In the limit, when $T \to \infty$, the amount

of variability is random (as in the finite sample case), but the subspace spanned by the eigenvectors corresponding to eigenvalues associated with nonstationary factors is constant.

Proof (i) Substituting y_t, expressed as in (1), in equation (10), we have

$$
\begin{aligned}
C_y(k) &= \frac{1}{T^{2d}}\sum(y_{t-k}-\bar{y})(y_t-\bar{y})' \\
&= P(\frac{1}{T^{2d}}\sum(f_{t-k}-\bar{f})(f_t-\bar{f})')P' + P(\frac{1}{T^{2d}}\sum(f_{t-k}-\bar{f})n_t') \\
&+ P(\frac{1}{T^{2d}}\sum(f_{t-k}-\bar{f})\epsilon_t') + (\frac{1}{T^{2d}}\sum n_{t-k}(f_t-\bar{f})')P' \\
&+ \frac{1}{T^{2d}}\sum n_{t-k}n_t' + \frac{1}{T^{2d}}\sum n_{t-k}\epsilon_t' + (\frac{1}{T^{2d}}\sum \epsilon_{t-k}(f_t-\bar{f})')P' \\
&+ \frac{1}{T^{2d}}\sum \epsilon_{t-k}n_t'\frac{1}{T^{2d}}\sum \epsilon_{t-k}\epsilon_t' \\
&= P(\frac{1}{T^{2d}}\sum(f_{t-k}-\bar{f})(f_t-\bar{f})')P' + o_p(1).
\end{aligned}
$$

where vector $\bar{f}' = (\bar{f}_1', 0_{r_2}')$ and $\bar{f}_1 = 1/T\sum f_{1,i}$. It is shown in Appendix 1 that all the terms but the ones associated with the $I(d)$ factors are $o_p(1)$.

From (12), and following the notation in Tanaka (1996), the $I(d)$ factors, $f_{1,t}$ can be expressed as $f_{1,t} = f_{1,t}^{(d)} = f_{1,t}^{(d-1)} + f_{1,t-1}^{(d)} = \sum_{j=1}^{t} f_{1,j}^{(d-1)}$,where $\left\{f_{1,t}^{(d-1)}\right\}_{t=1}^{T}$ is an $I(d-1)$ process that can be defined recursively in a similar way with $f_{1,t}^{(0)} = u_{1,t}$. For example, for $I(2)$ factors $f_{1,t}^{(2)} = \sum_{j=1}^{t} f_{1,j}^{(1)}$ and $\left\{f_{1,t}^{(1)}\right\}_{t=1}^{T}$ is the $I(1)$ process $f_{1,t}^{(1)} = u_t + f_{1,t-1}^{(1)}$. With this notation, $f_{1,t-k} = f_{1,t} - \sum_{i=0}^{k-1} f_{1,t-i}^{(d-1)}$, so

$$
\begin{aligned}
\frac{1}{T^{2d}}\sum_{t=k+1}^{T}(f_{1,t-k}-\bar{f}_1)(f_{1,t}-\bar{f}_1)' &= \frac{1}{T^{2d}}\sum_{t=k+1}^{T}(f_{1,t}-\bar{f}_1)(f_{1,t}-\bar{f}_1)' - \\
&\quad \frac{1}{T^{2d}}\sum_{t=k+1}^{T}(\sum_{i=0}^{k-1} f_{1,t-i}^{(d-1)})(f_{1,t}-\bar{f}_1)'.
\end{aligned}
$$

From Chan and Wei (1988) and Tanaka (1996) $\sum f_{1,t-i}^{(d-1)} f_{1,t}'$ is $O_p(T^{2d-1})$ for finite i and i small relative to T; also

$$
\begin{aligned}
\frac{1}{T^{d+1/2}}\sum_{t=1}^{T} f_{1,t} &\Rightarrow \Psi(1)\Sigma_1^{1/2}\int_0^1 F_{d-1}(\tau)d\tau \\
\frac{1}{T^{2d}}\sum_{t=1}^{T} f_{1,t}f_{1,t}' &\Rightarrow \Psi(1)\Sigma_1^{1/2}\int_0^1 F_{d-1}(\tau)F_{d-1}(\tau)'d\tau(\Sigma_1^{1/2})'\Psi(1)'
\end{aligned}
$$

where $F_d(\tau)$ is the *d-fold integrated Brownian motion* and can be defined recursively by $F_d(\tau) = \int_0^\tau F_{d-1}(s)ds$, for $d = 1, 2, ...$ and $F_0(\tau) = W(\tau)$ where $W(\tau)$ is the r_1-dimensional

Brownian motion. Then, by the continuous mapping theorem (Billingsley, 1968)

$$\frac{1}{T^{2d}}\sum_{t=1}^{T}(f_{1,t}-\bar{f}_1)(f_{1,t}-\bar{f}_1)' \Rightarrow \Psi(1)\Sigma_1^{1/2}\int_0^1 V_{d-1}(\tau)V_{d-1}(\tau)'d\tau(\Sigma_1^{1/2})'\Psi(1)' \qquad (12)$$

where $V_d(\tau) = F_d(\tau) - \int_0^1 F_d(\tau)d\tau$. Partitioning P as $[P_1 P_2]$, where P_1 (P_2) is the $m \times r_1$, ($m \times r_2$) submatrix of the factor loading matrix associated to the nonstationay (stationary) factors and using again the continuous mapping theorem (Billingsley, 1968)

$$\begin{aligned}
\Gamma_y = \lim\ C_y(k) &= \lim\ \frac{1}{T^2}\sum(y_{t-k}-\bar{y})(y_t-\bar{y})' \\
&= \lim\ P(\frac{1}{T^{2d}}\sum(f_{t-k}-\bar{f})(f_t-\bar{f})'P' \\
&\Rightarrow [P_1 P_2]\begin{bmatrix} \Psi(1)\Sigma_1^{1/2}\int_0^1 V_{d-1}(\tau)V_{d-1}(\tau)'d\tau(\Sigma_1^{1/2})'\Psi(1)' & 0_{r_1\times r_2} \\ 0_{r_2\times r_1} & 0_{r_2\times r_2}\end{bmatrix}\begin{bmatrix} P_1' \\ P_2'\end{bmatrix} \\
&= P_1\Psi(1)\Sigma_1^{1/2}(\int_0^1 V_{d-1}(\tau)V_{d-1}(\tau)'d\tau)(\Sigma_1^{1/2})'\Psi(1)'P_1' \qquad (13)
\end{aligned}$$

Note that all generalized covariance matrices (for lag 0, as well as, for lag k, finite) have the same limiting distribution.

Let $S = \Sigma_1^{1/2}\int_0^1 V_{d-1}(\tau)V_{d-1}(\tau)'d\tau(\Sigma_1^{1/2})'$, then its spectral decomposition for each realization, leads to $S = B\Lambda B'$ so $\Gamma_y = A\Lambda A'$ where $A = P_1\Psi(1)B$ and Λ has its r_1 eigenvalues different form zero. Therefore, the number of zero eigenvalues of Γ_y is $m - r_1$. Empirically, the number of common nonstationary factors can be found as the number of nonzero eigenvalues of $C_y(k)$, since $C_y(k) \Rightarrow \Gamma_y$ and the ordered eigenvalues are continuous functions of the coefficient matrix (Lemma 2 of Anderson et al, 1983), applying the continuous mapping theorem, the ordered eigenvalues of $C_y(k)$ converge weakly to those of Γ_y.

(ii) Define the variability in $\{y_t\}_{t=1}^{T}$ as $trace(C_y(0))$. In the limit, when $T \to \infty$, the amount of variability is random (as in the finite sample case), but the subspace spanned by the eigenvectors corresponding to eigenvalues associated with nonstationary factors is constant. For matrices $C_y(k)$, we just found their limiting distribution given by (13) and by the last paragraph of (i)

$$trace(C_y(0)) \Rightarrow \sum_{i=1}^{r_1}\lambda_i$$

where $\lambda_i, i = 1, \cdots, r_1$ are the diagonal elements of Λ, random quantities. The subspace spanned by the columns of P_1 is the rank of P_1, of dimension r_1, and it can be called the subspace of nonstationarity since it is associated to the $I(d)$ factors. The null space of P_1

is orthogonal to the one spanned by P_1. It can be called the subspace of stationarity or cointegration.

The next result establish that the limit random matrix has r_1 eigenvalues strictly greater than zero almost sure.

Theorem 2. For the model of sections 2 and 3 and assumption 1, Γ_y has r_1 eigenvalues greater than zero almost sure and $m - r_1$ equals zero.

Proof For the factor model of sections 2 and 3, under assumption 1, it was proved in theorem 1, equation (12) that

$$S_T = \frac{1}{T^{2d}} \sum (f_{1,t} - \bar{f}_1)(f_{1,t} - \bar{f}_1)' \Longrightarrow \Psi(1)\Sigma_1^{1/2} \int_0^1 V_{d-1}(\tau)V_{d-1}(\tau)'d\tau(\Sigma_1^{1/2})'\Psi(1)'.$$

The eigenvalues of the limiting sequence S_T are all greater than zero, since this is always a positive definite symmetric matrix. This is easily seen if we apply the next equality given in Bellman (1960, p. 49) , for $s = 1, 2, \cdots, r_1$, which proves that all the principal minors have determinant greater than zero and therefore S_T is postive definite. Let x^i, $i = 1, 2, \cdots s$ be a set of T-dimensional vectors, $T \geq s$, given by $x^i = f_1^i - \bar{f}_1^i \mathbf{1}$, where $f_1^i = (f_{1,1}^i, f_{1,2}^i, \cdots, f_{1,T}^i)'$ is the $T \times 1$ vector of sample values of the $i-$th nonstationary factor, $\bar{f}_1^i = 1/T \sum_{j=1}^T f_{1,j}^i$ and $\mathbf{1}' = (1, \cdots, 1)$ is a $T \times 1$ vector of ones. Then

$$|(x^i, x^j)|_{i,j=1,2,\cdots,s} = \frac{1}{s!} \sum_{\{i_s\}} \begin{vmatrix} x_{i_1}^1 & x_{i_2}^1 & \cdots & x_{i_s}^1 \\ x_{i_1}^2 & x_{i_2}^2 & \cdots & x_{i_s}^2 \\ \vdots & \vdots & & \vdots \\ x_{i_1}^s & x_{i_2}^s & \cdots & x_{i_s}^s \end{vmatrix}^2$$

where $(x^i, x^j) = \sum_{k=1}^T x_k^i x_k^j$ is element i, j of the matrix in the left hand side whose determinant we are calculating and the sum in the equality is over all sets of integers $\{i_s\}$, with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq T$.

Not only the limiting sequence is positive definite, but also in the limit it cannot be zero. First, it will be shown that $M = \int_0^1 V_{d-1}(\tau)V_{d-1}(\tau)'d\tau$ is nonsingular almost sure. Denote by $V_g^j(w, \tau)$ the $j-$th component of the process $V_g(\tau)$, for $g = 0, 1, ..., d-1$. If M were singular, then $\exists\ \mathbf{c} = (c_1, ...c_{r_1})' \neq \mathbf{0}$ such that $\mathbf{c}'M\mathbf{c} = 0$. Therefore $\sum_{j=1}^{r_1} c_j V_{d-1}^j(\tau) = 0$, for $0 \leq \tau \leq 1$. Since $\mathbf{c} \neq \mathbf{0}$, $\exists\ c_i \neq 0$, such that $V_{d-1}^i(\tau) = \frac{1}{c_i} \sum_{j=1, j\neq i}^{r_1} c_j V_{d-1}^j(\tau)$. But for each realization of the proccess $V_{d-1}(\tau) = F_{d-1}(\tau) - \int_0^1 F_{d-1}(\tau)d\tau$ this means that

$$F_{d-1}^i(\tau) - \int_0^1 F_{d-1}^i(\tau)d\tau = \frac{1}{c_i} \sum_{j=1, j\neq i}^{r_1} c_j \left(F_{d-1}^j(\tau) - \int_0^1 F_{d-1}^j(\tau)d\tau \right)$$

or

$$F_{d-1}^i(\tau) = \frac{1}{c_i} \sum_{j=1,j\neq i}^{r_1} c_j F_{d-1}^j(\tau) - K,$$

where $K = 1/c_i \sum_{j=1,j\neq i}^{r_1} c_j \int_0^1 F_{d-1}^j(\tau)d\tau - \int_0^1 F_{d-1}^i(\tau)d\tau$. But with probability 1 $F_{d-1}^i(\tau)$ cannot lie in the span of F_{d-1}^1,..., F_{d-1}^{i-1}, F_{d-1}^{i+1},..., $F_{d-1}^{r_1}$, since var $(F_{d-1}(\tau)) = \tau^{2d-1}/((2d-1)((d-1)!)^2) \times I_{r_1}$ is diagonal. It can also be checked that var $(V_{d-1}(\tau))$ is a full rank diagonal matrix, then with probability 1 $V_{d-1}^i(\tau)$ cannot be a linear combination of the remaining components $V_{d-1}^j(\tau)$, $j \neq i$, $j = 1, ..., r_1$. Therefore M is nonsingular almost sure and if $\Sigma_1 > 0$, $|\Sigma_1^{1/2} M(\Sigma_1^{1/2})'| \neq 0$ almost sure. Since $\Psi(1)$ and P_1 have rank r_1, $\Gamma_y = P_1\Psi(1)\Sigma_1^{1/2} \int_0^1 V_{d-1}(\tau)V_{d-1}(\tau)'d\tau(\Sigma_1^{1/2})'\Psi(1)'P_1'$ is nonsingular almost sure. Then, for any $m \times 1$ vector $\lambda \neq 0$ and by the Portmanteau theorem (see theorem 2.1 in Billingsley, 1968),

$$1 = P_T(\lambda'(P_1 S_T P_1')\lambda > 0) \rightarrow P(\lambda'\Gamma_y\lambda > 0).$$

which constitutes the desired result: with probabillity 1, matrix Γ_y is positive definite.

Similar results are found if we use generalized sample second moments matrices, $A_y(k) = \frac{1}{T^{2d}} \sum y_t y_{t-k}'$, instead of generalized covariance matrices. In this case

$$A_y(k) \Rightarrow \gamma_y = P_1\Psi(1)\Sigma_1^{1/2} \int F_{d-1}(\tau)F_{d-1}(\tau)'d\tau(\Sigma_1^{1/2})'\Psi(1)'P_1' \qquad (14)$$

where P_1 is an $m \times r_1$ matrix. A similar result is found for the eigenvalues of γ_y.

Lemma 1. For the model of sections 2 and 3 and assumption 1, γ_y has r_1 eigenvalues greater than zero almost sure and $m - r_1$ equals zero.

Proof is given in Appendix 2.

3.1 Nonstationary I(1) factors

From a practical point of view and due to its broad applicability, special attention is paid to the $I(1)$ case. Suppose now that the vector of observed time series is $I(1)$. Then, the nonstationary factors are also $I(1)$. In particular, they can be common trends in the sense of Stock and Watson (1988).

Assumption 2. The equation for the common nonstationary factors is

$$\begin{aligned} f_{1,t} &= f_{1,t-1} + u_t \qquad (15) \\ u_t &= \Psi(B)a_{1,t} \end{aligned}$$

with E $(a_{1,t}) = 0$ and var $(a_{1,t}) = \Sigma_1 = diag(\sigma_1^2, \cdots, \sigma_{r_1}^2) > 0$, and $\sum i||\Psi_i|| < \infty$, $||\Psi_i|| = [tr(\Psi_i'\Psi)]^{1/2}$.

Lemma 2. The model given in sections 2 and 3 with nonstationary factors as in assumption 2 (model M2) is equivalent (in the sense that it gives equal first and second moments of the observed series and of the auxilliary process that defines the short run dynamics) to a model (M1) with the same number of common trends, r_1, and r_1 more stationary common factors.

Proof is given in Appendix 3.

Theorm 1 applied to the $I(1)$ case tells us that the generalized covariance matrices are now divided by T^2 and converge to

$$\Gamma_y = \lim\ C_y(k) = P_1\Psi(1)\Sigma_1^{1/2}(\int_0^1 V(\tau)V(\tau)'d\tau)(\Sigma_1^{1/2})'(\Psi(1))'P_1'$$

where $V(\tau) = W(\tau) - \int_0^1 W(\tau)d\tau$ is the *demeaned Brownian motion*, and $W(\tau)$ is the r_1-dimensional standard Brownian motion.

Note that matrix $S = \Sigma_1^{1/2}(\int_0^1 V(\tau)V(\tau)'d\tau)(\Sigma_1^{1/2})'$ is a nondiagonal matrix and that all generalized covariance matrices (for lag 0, as well as, for lag k, finite) have the same limiting distribution: they all tend to a symmetric random matrix of rank r_1.

Remarks:

(1) Similar results are found if we use generalized sample second moments matrices, $A_y(k) = \frac{1}{T^{2d}}\sum y_t y_{t-k}'$, instead of generalized covariance matrices. In this case, also with $d = 1$

$$A_y(k) \Rightarrow P_1\Psi(1)\Sigma_1^{1/2}\int W(\tau)W(\tau)'d\tau(\Sigma_1^{1/2})'\Psi(1)'P_1'. \qquad (16)$$

(2) These convergence results can also be found in Phillips and Durlauf (1986) and Chan and Wei (1988) and apply to processes that satisfy more general assumptions of the innovations, that what is needed here. In particular, these results can be generalized to the case where the innovations present heterogeneity. Also normallity is no needed.

(3) The expected value of $\Sigma_1^{1/2}\int W(\tau)W(\tau)'d\tau(\Sigma_1^{1/2})'$ is

$$\text{E}\ (\Sigma^{1/2}\int W(\tau)W(\tau)'d\tau(\Sigma^{1/2})') = \Sigma^{1/2}\int \text{E}\ (W(\tau)W(\tau)')d\tau(\Sigma^{1/2})' = \frac{1}{2}diag(\frac{1}{\sigma_1^2}, \cdots, \frac{1}{\sigma_{r_1}^2})$$

since E $(\int w_i w_j d\tau) = 1/2$ if $i = j$ and 0 otherwise.

Lemma 1 and theorem 2 also apply to the case of $I(1)$ factors. Therefore Γ_y and γ_y have almost sure r_1 eigenvalues strictly grater than zero and $m - r_1$ equals zero.

4 Conclusions

Several authors (Engle and Granger, 1987, Phillips and Durlauf, 1986, Stock, 1987) have proposed estimating a cointegration vector by using the fact that if $z_t = b'y_t$ is stationary and ergodic then the sample variance

$$T^{-1} \sum z_t^2 \to E(z_t^2),$$

whereas if z_t is nonstationary $I(1)$ then

$$T^{-2} \sum z_t^2 \to c^2 \int_0^1 W(\tau)^2 d\tau,$$

where the constant c depends on covariances of the differenced stationary process. Therefore, if z_t is nonstationary its sample variance will go to ∞. This leads to finding cointegrating vectors by minimizing the sample variance of z_t. The usual procedure is to assume a normalization of b such that the coefficient of the first component of y_t is unity, which implies finding b by regressing the first component on all the others.

If instead of looking at the cointegration relationships we look at the orthogonal factor space it is clear that a reasonable procedure for finding a vector a such that $z_t = a'y_t$ is nonstationay is by maximizing the variance of z_t which leads to a principal component analysis of the covariance matrix of the series. This approach was initially followed by Stock and Watson (1988) who proposed to base their cointegration test on the linear combinations generated by the principal components of the covariance matrix of the series, although afterwards these authors abandoned this approach in favor of a regression procedure (Stock and Watson, 1993). Their approach differs from ours in the following aspects.

First, principal components are introduced in an intuitive way, whereas in our model the formal justification is the stability of the factor space in the eigenvalues of all lag covariance matrices.

Second, our approach is general and can be applied to factors with different orders of integration. For example, suppose that there are r_1 factors, $f_{1,t}$, that are $I(d_1)$ and r_2, $f_{2,t}$, that are $I(d_2)$, with $d_1 > d_2$, plus stationary factors. We can apply the method defining generalized covariance matrices as in (10), divided by T^{2d_1}. After we find the r_1 $I(d_1)$ nonstationary factors, we define the auxilliary process $z_t = y_t - P_1 f_{1,t}$. We can now apply the same procedure to z_t defining generalized covariance matrices for z_t and normalizing them by T^{2d_2} to obtain the r_2 common $I(d_2)$ factors.

Third, the method can be generalized to nonstationary fractional factors. If instead of defining the d-fold integrated Brownian motion recursively, we use the definition valid for real d such as $d > -1/2$,

$$F_d(t) = \int_0^t \frac{(t-s)^d}{g!} dW(s)$$

convergence results could also be found.

Appendix 1

In this appendix, it is shown that for n_t, f_t and $\bar{f}$ defined as in sections 2 and 3,
(a) $\frac{1}{T^{2d}}\sum n_{t-k}n_t' \xrightarrow{p} 0_{m\times m}$, $\frac{1}{T^{2d}}\sum \epsilon_{t-k}\epsilon_t' \xrightarrow{p} 0_{m\times m}$ and $\frac{1}{T^{2d}}\sum \epsilon_{t-k}n_t' \xrightarrow{p} 0_{m\times m}$,
(b) $\frac{1}{T^{2d}}\sum (f_{t-k}-\bar{f})n_t' \xrightarrow{p} 0_{r\times m}$ and $\frac{1}{T^{2d}}\sum (f_{t-k}-\bar{f})\epsilon_t' \xrightarrow{p} 0_{r\times m}$ and
(c) $\frac{1}{T^{2d}}\sum (f_{1,t-k}-\bar{f}_1)f_{2,t}' \xrightarrow{p} 0_{r_1\times r_2}$.
In each case and in what follows 0 is a matrix of appropriate dimensions or an scalar.

(a) Let n_t be an $m\times 1$ vector of specific components. By the stationary assumption, $\frac{1}{T}\sum n_{t-k}n_t' \xrightarrow{p} E(n_{t-k}n_t')$. Since $E(n_{t-k}n_t')$ exists and is finite, $\frac{1}{T^{2d}}\sum n_{t-k}n_t' \xrightarrow{p} 0$. Also $\frac{1}{T^{2d}}\sum \epsilon_{t-k}\epsilon_t' \xrightarrow{p}$ E $(\epsilon_t\epsilon_{t-k}) = 0_{m\times m}$ and $\frac{1}{T^{2d}}\sum \epsilon_{t-k}n_t' \xrightarrow{p}$ E $(\epsilon_{t-k}n_t) = 0_{m\times m}$.

(b) Let f_t be an $r\times 1$ vector of common factors with r_1 common nonstationary factors and r_2 common, zero mean, stationary factors, such that $r = r_1 + r_2$, then

$$\frac{1}{T^{2d}}\sum (f_{t-k}-\bar{f})n_t' = \frac{1}{T^{2d}}\sum \begin{bmatrix} (f_{1,t-k}-\bar{f}_1)n_t' \\ f_{2,t-k}n_t' \end{bmatrix}$$

(b1) First, it will be shown that for the stationary factors $\frac{1}{T^{2d}}\sum f_{2,t-k}n_t' \xrightarrow{p} 0$ which is easily seen since both processes n_t and f_{2t} are stationary, then $\frac{1}{T}\sum f_{2,t-k}n_t' \xrightarrow{p} E(f_{2,t-k}n_t')$, finite, therefore $\frac{1}{T^{2d}}\sum f_{2,t-k}n_t' \xrightarrow{p} 0$, for $d \geq 1$.

(b2) Now, for the term associated with the nonstationary common factors, $\frac{1}{T^{2d}}\sum (f_{1,t-k}-\bar{f}_1)n_t' \xrightarrow{p} 0$. Denote by $f_{1,t}^i$, (n_t^i) the i-th component of vector $f_{1,t}$, (n_t). Element (i,j) of the previous matrix is defined as $\sigma_{i,j} = \frac{1}{T^{2d}}\sum (f_{1,t-k}^i - \bar{f}_1^i)n_t^j$, for $i = 1, ...r_1$, $j = 1, ..., m$. It will be shown that $\sigma_{i,j} \xrightarrow{p} 0$, for all $i = 1, ...r_1$ and $j = 1, ..., m$.

$$\begin{aligned} \frac{1}{T^{2d}}\sum (f_{1,t-k}^i - \bar{f}_1^i)n_t^j &\leq \frac{1}{T^{2d}}\sum (f_{1,t-k}^i - \bar{f}_1^i) \max_{1\leq t\leq T} |n_t^j| \\ &= \frac{1}{T^{d-1/2}} \max_{1\leq t\leq T} |n_t^j| \frac{1}{T^{d+1/2}} \sum_{t=k+1}^{T} f_{1,t-k}^i \end{aligned}$$

From Tanaka (1996) we know that $\frac{1}{T^{d+1/2}}\sum_{t=k+1}^T f_{1,t-k}^i$ is $O_p(1)$, and since n_t^j is a stationary process and $d \geq 1$, $\frac{1}{T^{d-1/2}} \max_{1\leq t\leq T} |n_t^j| \xrightarrow{p} 0$. Therefore $\sigma_{i,j} \xrightarrow{p} 0$. So, $\frac{1}{T^2}\sum (f_{1,t-k}-\bar{f}_1)n_t' \xrightarrow{p} 0$. And from (b1) and (b2), $\frac{1}{T^{2d}}\sum (f_{t-k}-\bar{f})n_t' \xrightarrow{p} 0$.

The proof for $\frac{1}{T^{2d}}\sum (f_{t-k}-\bar{f})\epsilon_t' \xrightarrow{p} 0_{r\times m}$ goes exactly like the one before since ϵ_t is also a stationary process.

(c) Now, for the term involving stationary and nonstationary common factors,

$$\frac{1}{T^{2d}}\sum (f_{1,t-k}-\bar{f}_1)f_{2,t}' \xrightarrow{p} 0$$

The proof goes like (b2), with $f_{2,t}$ instead of n_t, since it is the limit of the sum of the product of a lagged k, $k = 0, 1, ...K$, nonstationary factor by an stationary one.

Appendix 2

PROOF OF LEMMA 1. The proof goes like in Theorem 2, but now

$$A_y(k) \Rightarrow \gamma_y = P_1\Psi(1)\Sigma_1^{1/2}\int F_{d-1}(\tau)F_{d-1}(\tau)'d\tau(\Sigma_1^{1/2})'\Psi(1)'P_1'.$$

So all we have to prove is that $\int F_{d-1}(\tau)F_{d-1}(\tau)'d\tau$ is nonsingular. Denote by $F_g^j(\tau)$ the $j-$th component of the process $F_g(\tau)$, for $g = 0, 1, ...$ We will prove by induction that that $P(|\int F_{d-1}(\tau)F_{d-1}(\tau)'d\tau| = 0) = 0$, since otherwise $\exists F_{d-1}^i|F_{d-1}^i$ lies in the span of $F_{d-1}^1, ..., F_{d-1}^{i-1}, F_{d-1}^{i+1}, ..., F_{d-1}^{r_1}$ and this is not possible. For $d = 1$, $F_{d-1}(s) = W(s)$, where $W(s)$ is the r_1-dimensional standard Brownian motion, with all its components independent among them. Therefore, $P(|\int W(\tau)W(\tau)'d\tau| = 0) = 0$, since otherwise $\exists W^i|W^i$ lies in the span of $W^1, ..., W^{i-1}, W^{i+1}, ..., W^{r_1}$ and this is not possible since all the componets of $W(s)$ are independent among each other. Now suppose that it is true for $d-1$, this means that $P(|\int F_{d-1}(\tau)F_{d-1}(\tau)'d\tau| = 0) = 0$, or equivalently with probability zero F_{d-1}^i lies in the span of $F_{d-1}^1, ..., F_{d-1}^{i-1}, F_{d-1}^{i+1}, ..., F_{d-1}^{r_1}$. We will see that $P(|\int F_d(\tau)F_d(\tau)'d\tau| = 0) = 0$, because if not $\exists F_d^i|F_d^i$ lies in the span of $F_d^1, ..., F_d^{i-1}, F_d^{i+1}, ..., F_d^{r_1}$, that is F_d^i can be expressed as a linear combination of $F_d^1, ..., F_d^{i-1}, F_d^{i+1}, ..., F_d^{r_1}$ or it exists $\alpha_1, ..., \alpha_{r_1-1}$ not all of them simultaneouslly zero such that

$$F_d^i = \sum_{j=1}^{r_1-1}\alpha_j F_d^j.$$

Differentiating the above equation,

$$F_{d-1}^i = \sum_{j=1}^{r_1-1}\alpha_j F_{d-1}^j$$

which occurs with probability zero. Therefore γ_y is nonsingular almost sure.

Appendix 3

PROOF OF LEMMA 2. Since the specific components and the common stationary factors not derived by the dynamic structure of the nonstationary ones are not involved in this proof, let us suppose, just for ease of exposition, that they do not exist. To distinguish both models, the factors and system matrices will be denoted by $\tilde{}$ in M2 and without it in model M1. So, let us suppose model M2 with r_1 common $I(1)$ factors with dynamic structure expressed as in assumption 1 and a model with $r_1$common trends plus r_1 common stationary factors. We will see when they give the same first and second moments of the observed series.

First, let us show that they give the same limiting distribution of the generalized covariance matrices. For model M2 and by theorem 1

$$\Gamma_y = \lim C_y(k) = \tilde{P}_1\tilde{\Psi}(1)\tilde{\Sigma}_1^{1/2}(\int_0^1 V(\tau)V(\tau)'d\tau)(\tilde{\Sigma}_1^{1/2})'(\tilde{\Psi}(1))'\tilde{P}_1'. \tag{A3.1}$$

For model M1 and by theorm 1

$$\Gamma_y = P_1\Sigma_1^{1/2}(\int_0^1 V(\tau)V(\tau)'d\tau)(\Sigma_1^{1/2})'P_1'. \tag{A3.2}$$

These distributions are the same if and only if $\tilde{P}_1\tilde{\Psi}(1)\tilde{\Sigma}_1^{1/2} = P_1\Sigma_1^{1/2}$. If $\tilde{\Sigma}_1 = \Sigma_1$, then $P_1 = \tilde{P}_1\tilde{\Psi}(1)$, but notice that many other possibilities are still open.

Now for the short run, let us show that the dynamics generated by the structure of the $I(1)$ factors can be expressed as r_1 common stationary factors, $f_{2,t}$ in the equivalent model M1. Define the auxilliary process in model M2, $x_t = y_t - \tilde{P}_1\tilde{\Psi}(1)\tilde{f}_{1,t} = \tilde{P}_1\bar{\Psi}(L)a_{1,t} + \epsilon_t$, where $\bar{\Psi}(L) = (1-L)^{-1}(\tilde{\Psi}(L) - \tilde{\Psi}(1))$, $\bar{\Psi}_j = -\sum_{i=j+1}^{\infty}\tilde{\Psi}_i$. The mean of the auxilliary process is 0 and second moments are given by

$$\begin{aligned} \mathrm{V}\ (x_t) &= \sum_{j=0}^{\infty}\tilde{P}_1(\sum_{i=j+1}^{\infty}\tilde{\Psi}_i)\tilde{\Sigma}_1(\sum_{i=j+1}^{\infty}\tilde{\Psi}_i)'\tilde{P}_1' + \Sigma_\epsilon \\ \mathrm{E}\ (x_t x_{t-k}') &= \sum_{j=0}^{\infty}\tilde{P}_1(\sum_{i=j+1+k}^{\infty}\tilde{\Psi}_i)\tilde{\Sigma}_1(\sum_{i=j+1}^{\infty}\tilde{\Psi}_i)'\tilde{P}_1' \end{aligned}$$

Define an auxilliary process, z_t, related to model M1 as $z_t = y_t - P_1 f_{1,t} = P_2 f_{2,t} + \epsilon_t$, with $f_{2,t} = \Phi(B)a_{2,t}$, a set of generic r_2 stationary common factors. Let us see who are the $f_{2,t}$ factors. E(z_t)=0 and second moments are given by

$$\begin{aligned} \mathrm{V}\ (z_t) &= \sum_{i=0}^{\infty} P_2\Phi_i\Sigma_2\Phi_i'P_2' + \Sigma_\epsilon \\ \mathrm{E}\ (z_t z_{t-k}') &= \sum_{i=k}^{\infty} P_2\Phi_i\Sigma_2\Phi_{i-k}'P_2'. \end{aligned}$$

Both auxilliary process are the same, if we take r_1 common stationary factors in M1 and

$$\sum_{i=k}^{\infty} P_2\Phi_i\Sigma_2\Phi_{i-k}'P_2' = \sum_{j=0}^{\infty}\tilde{P}1(\sum_{i=j+1+k}^{\infty}\tilde{\Psi}_i)\tilde{\Sigma}_1(\sum_{i=j+1}^{\infty}\tilde{\Psi}_i)'\tilde{P}_1'$$

which is satisfied, for example, for $P_2 = \tilde{P}_1$ and $\Phi_i = \bar{\Psi}_i = -\sum_{i=j+1}^{\infty}\tilde{\Psi}_i$.

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