ORTHOGONAL POLYNOMIALS AND CUBIC POLYNOMIAL MAPPINGS I

F.MARCELLÁN AND J.PETRONILHO

March 21, 2000

Abstract

We present characterization theorems for orthogonal polynomials obtained from a given system of orthogonal polynomials by a cubic polynomial transformation in the variable. Since such polynomials are the denominators of the approximants for the expansion in continued fractions of the z-transform of the moment sequences associated with the linear functionals with respect to which such polynomials are orthogonal, we state the explicit relation for the corresponding formal Stieltjes series. As an application, we study the eigenvalues of a tridiagonal 3-Toeplitz matrix. Finally, we deduce the second order linear differential equation satisfied by the new family of orthogonal polynomials, when the initial family satisfies such a kind of differential equation.

Key words and Phrases: Orthogonal polynomials, Polynomial mappings, Recurrence coefficients, Stieltjes formal series, Toeplitz matrices, Sieved orthogonal polynomials.

AMS (MSC 1991) Subject Classification: Primary 42C05.

1 Introduction and preliminaries

In this paper we analyze some problems related to cubic transformations in the variable of a sequence of monic polynomials orthogonal with respect to some moment linear functional. The first problem is the following:

P1. *Let* $\{P_n\}_{n\geq0}$ *be a fixed sequence of monic orthogonal polynomials,* π_3 *a fixed cubic monic polynomial and* $\{Q_n\}_{n\geq0}$ *a simple set (i.e.,* $\deg Q_n = n$ *for all* $n = 0,1,2,\ldots$*) of monic polynomials such that*

$$Q_{3n}(x) = P_n(\pi_3(x)) \qquad (1)$$

for all $n = 0,1,2,\ldots$. *To find necessary and sufficient conditions in order to* $\{Q_n\}_{n\geq0}$ *be a sequence of orthogonal polynomials. In such conditions, what is the relation between the moment linear functionals associated with the sequences* $\{P_n\}_{n\geq0}$ *and* $\{Q_n\}_{n\geq0}$*?*

This problem has been partially solved in [2] and [15] . In fact, in [2], P.BARRUCAND and D.DICKINSON have considered P1 under the assumption that $\{P_n\}_{n\geq0}$ is a symmetric orthogonal polynomial system and π_3 a general odd polynomial of degree 3, and requiring that $\{Q_n\}_{n\geq0}$ be also a symmetric orthogonal polynomial system; on the other hand, in [15], F.MARCELLÁN and G.SANSIGRE studied the case $\pi_3(x) \equiv x^3$ without any restriction on $\{P_n\}_{n\geq0}$ or $\{Q_n\}_{n\geq0}$. Nevertheless, in these contributions, only sufficient conditions were found and the relations between the corresponding functionals were not studied.

From another point of view, this problem was studied by J.GERONIMO and W.VAN ASSCHE [9] (for a general polynomial of degree k, instead of $k = 3$) and also by J.CHARRIS, M.E.H.ISMAIL and S.MONSALVE [4] (in the framework of families of orthogonal polynomials defined by general blocks of recurrence relations).

In connection with problem P1 two other similar problems arise in a natural way, eventually with more useful applications. Thus, a second problem to be considered is the following:

P2. *The same assumptions and questions as in* P1, *but with* (1) *replaced by*

$$Q_{3n+1}(x) = (x-a)P_n(\pi_3(x))$$

for all $n = 0, 1, 2, \ldots$, *where* a *is a fixed complex number.*

The third problem can be stated as follows:

P3. *The same assumptions and questions as in* P1, *but with* (1) *replaced by*

$$Q_{3n+2}(x) = (x-a)(x-b)P_n(\pi_3(x))$$

for all $n = 0, 1, 2, \ldots$, *where* a *and* b *are fixed complex numbers.*

In the remainder of the paper we will make use of some well known concepts and results in the theory of orthogonal polynomials. We mention the most important for our purposes. First, we recall the definition of an orthogonal polynomial system (OPS). If $\mathbf{u}$ is a linear functional on $\mathbb{P}$ (the linear space of all polynomials with complex coefficients) and $\{P_n\}_{n\geq 0}$ a sequence of polynomials, $\{P_n\}_{n\geq 0}$ is said to be orthogonal with respect to $\mathbf{u}$ if $\{P_n\}_{n\geq 0}$ is a simple set and

$$\langle \mathbf{u}, P_n P_m \rangle = k_n \delta_{nm}$$

for all $n = 0, 1, 2, \ldots$, where $\{k_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers ($\langle .,. \rangle$ means the duality bracket). The functional $\mathbf{u}$ is said to be regular or quasi-definite when such a sequence of orthogonal polynomials exists (cf. [5]). In this paper we will consider monic orthogonal polynomial systems (MOPS). All MOPS are characterized by a three-term recurrence relation

$$\begin{array}{l} xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)\,, \quad n \geq 1, \\ P_0(x) = 1\,, \quad P_1(x) = x - \beta_0 \end{array} \tag{2}$$

where $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ are sequences of complex numbers, with $\gamma_n \neq 0$ for all $n \geq 1$. If c is a complex number such that $P_n(c) \neq 0$ for all $n \geq 0$, then the sequence of monic kernel polynomials of K-parameter c associated with $\{P_n\}_{n\geq 0}$, which will be denoted by $\{P_n^*(c;.)\}_{n\geq 0}$, is

$$P_n^*(c;x) := \frac{1}{x-c}\left[P_{n+1}(x) - \frac{P_{n+1}(c)}{P_n(c)}P_n(x)\right]\,, \quad n \geq 0\,. \tag{3}$$

(cf. [5, p.35]). If $\{P_n\}_{n\geq 0}$ is an MOPS associated with the moment linear functional $\mathbf{u}$, then $\{P_n^*(c;.)\}_{n\geq 0}$ is an MOPS associated with

$$\mathbf{u}^*(c) := (x-c)\mathbf{u}\,.$$

Furthermore, the coefficients $\beta_n^* \equiv \beta_n^*(c)$ and $\gamma_n^* \equiv \gamma_n^*(c)$ for the three-term recurrence relation corresponding to $\{P_n^*(c;.)\}_{n\geq 0}$ are given explicitly by

$$\beta_n^* = \beta_{n+1} + \frac{P_{n+2}(c)}{P_{n+1}(c)} - \frac{P_{n+1}(c)}{P_n(c)}\,, \quad \gamma_{n+1}^* = \gamma_{n+1}\frac{P_{n+2}(c)P_n(c)}{P_{n+1}^2(c)}\,, \quad n \geq 0\,. \tag{4}$$

Given an MOPS $\{P_n\}_{n\geq 0}$ which satisfies (2), the polynomials $\{P_n^{(1)}\}_{n\geq 0}$ defined by the shifted recurrence relation

$$\begin{array}{l} xP_n^{(1)}(x) = P_{n+1}^{(1)}(x) + \beta_{n+1}P_n^{(1)}(x) + \gamma_{n+1}P_{n-1}^{(1)}(x) \quad , \quad n \geq 1 \\ P_0^{(1)}(x) = 1 \quad , \quad P_1^{(1)}(x) = x - \beta_1 \end{array} \tag{5}$$

are called the associated polynomials of the first kind of $\{P_n\}_{n\geq 0}$. These polynomials are also given by

$$P_n^{(1)}(x) = \frac{1}{u_0}\langle \mathrm{u}_y, \frac{P_{n+1}(x) - P_{n+1}(y)}{x-y}\rangle , \quad n \geq 0$$

where $u_0 := \langle \mathrm{u}, 1\rangle$. For a fixed complex number λ , the sequence $\{P_n^\lambda\}_{n\geq 0}$ defined by the same recurrence relation (2) as $\{P_n\}_{n\geq 0}$, but with a different initial condition, namely

$$\begin{array}{l} xP_n^\lambda(x) = P_{n+1}^\lambda(x) + \beta_n P_n^\lambda(x) + \gamma_n P_{n-1}^\lambda(x) , \quad n \geq 1 \\ P_0^\lambda(x) = 1 \quad , \quad P_1^\lambda(x) = x - \beta_0 - \lambda , \end{array} \tag{6}$$

is the co-recursive sequence of $\{P_n\}_{n\geq 0}$ for the modification λ. These polynomials were introduced and studied by T.S.CHIHARA in [6]. Notice that

$$P_n^\lambda(x) = P_n(x) - \lambda P_{n-1}^{(1)}(x) . \tag{7}$$

The following notation will be used:

$$D_n^{\lambda,\mu}(x,y) := \left| \begin{array}{cc} P_n^\lambda(x) & P_{n+1}^\lambda(x) \\ P_n^\mu(y) & P_{n+1}^\mu(y) \end{array} \right| \tag{8}$$

for $x, y \in \mathbf{C}$ and $n \geq 0$. Notice that if $P_n(c) \neq 0$ then $D_n^{0,0}(c,x) = (x-c)P_n(c)P_n^*(c;x)$. Moreover, using the well known relation

$$P_n^{(1)}(x)P_n(x) - P_{n+1}(x)P_{n-1}^{(1)}(x) = \prod_{i=1}^n \gamma_i , \quad n \geq 0 ,$$

(with the convention that the product over an empty set equals unity) we can show that $D_n^{0,\mu}(x,x)$ is also independent of the choice of x:

$$D_n^{0,\mu}(x,x) = -\mu \prod_{i=1}^n \gamma_i , \quad n \geq 0 . \tag{9}$$

The co-recursive polynomials are important in order to establish the regularity conditions for a linear functional associated with an inverse polynomial modification of a regular functional. In fact, if u is regular, for fixed $\lambda, c \in \mathbf{C}$ and with $\mathrm{u}^{\lambda,c}$ defined by the distributional equation $(x-c)\mathrm{u}^{\lambda,c} = -\lambda \mathrm{u}$ (recall that given a linear functional v and a polynomial ϕ, then $\phi\mathrm{v}$ is the linear functional defined by $\langle \phi\mathrm{v}, f\rangle := \langle \mathrm{v}, \phi f\rangle$ for every polynomial f), i.e.,

$$\mathrm{u}^{\lambda,c} = u_0\delta_c - \lambda(x-c)^{-1}\mathrm{u} , \tag{10}$$

where δ_c means the Dirac functional at the point c, $\langle \delta_c, f\rangle := f(c)$ $(f \in \mathbb{P})$, and $(x-c)^{-1}\mathrm{u}$ is the linear functional defined by

$$\langle (x-c)^{-1}\mathrm{u}, f\rangle := \langle \mathrm{u}, \frac{f(x) - f(c)}{x-c}\rangle , \quad f \in \mathbb{P} ,$$

then $\mathbf{u}^{\lambda,c}$ is regular if and only if $\lambda \neq 0$ and $P_n^\lambda(c) \neq 0$ for all $n = 0, 1, 2, \dots$ (MARONI [16]). In such conditions the corresponding MOPS, $\{P_n^{\lambda,c}\}_{n\geq 0}$, is given by

$$P_n^{\lambda,c}(x) := P_n(x) - \frac{P_n^\lambda(c)}{P_{n-1}^\lambda(c)} P_{n-1}(x) \quad , \quad n \geq 0. \tag{11}$$

The coefficients $\{\beta_n^{\lambda,c}, \gamma_{n+1}^{\lambda,c}\}_{n\geq 0}$ for the three-term recurrence relation corresponding to $\{P_n^{\lambda,c}\}_{n\geq 0}$ are given explicitly by

$$\beta_0^{\lambda,c} = \beta_0 + P_1^\lambda(c), \quad \beta_n^{\lambda,c} = \beta_n + \frac{P_{n+1}^\lambda(c)}{P_n^\lambda(c)} - \frac{P_n^\lambda(c)}{P_{n-1}^\lambda(c)}, \tag{12}$$

$$\gamma_1^{\lambda,c} = \lambda P_1^\lambda(c), \quad \gamma_{n+1}^{\lambda,c} = \gamma_n \frac{P_{n+1}^\lambda(c) P_{n-1}^\lambda(c)}{[P_n^\lambda(c)]^2} \tag{13}$$

for all $n = 1, 2, \dots$. Notice also that the moments of $\mathbf{u}^{\lambda,c}$ can be obtained from the moments $u_n := \langle \mathbf{u}, x^n \rangle$ of $\mathbf{u}$ according to the formula

$$\langle \mathbf{u}^{\lambda,c}, x^n \rangle = u_0 c^n - \lambda \sum_{i=0}^{n-1} c^{n-1} u_{n-1-i} \quad , \quad n \geq 0$$

(with the convention that summation over an empty set of indices is zero).

Finally, we recall the Stieltjes formal series, $S_{\mathbf{u}}(z)$, corresponding to a given moment linear functional $\mathbf{u}$:

$$S_{\mathbf{u}}(z) := -\sum_{n\geq 0} \frac{u_n}{z^{n+1}} \equiv \langle \mathbf{u}_x, \frac{1}{x-z} \rangle, \quad u_n := \langle \mathbf{u}, x^n \rangle$$

where the subscript x in $\mathbf{u}_x$ means that $\mathbf{u}$ acts on functions of the variable x. In fact, (formally) we have

$$S_{\mathbf{u}}(z) = -\sum_{n\geq 0} \frac{u_n}{z^{n+1}} = -\sum_{n\geq 0} \frac{\langle \mathbf{u}_x, x^n \rangle}{z^{n+1}} = -\frac{1}{z} \langle \mathbf{u}_x, \sum_{n\geq 0} \left(\frac{x}{z}\right)^n \rangle = \langle \mathbf{u}_x, \frac{1}{x-z} \rangle.$$

The structure of the paper is the following. In Section 2 we give the characterizations for the solutions of problems P1, P2 and P3 and in Section 3 we prove the statements on Section 2. In such a way we obtain the relation between the coefficients of the expansion in continued fractions of two Stieltjes functions related by

$$\widetilde{S}(z) = \frac{A(z) S(\pi_3(z)) + B(z)}{C(z)}$$

where A, B and C are polynomials of degrees less than or equal to 2. As an application of our results, in Section 4 we solve the eigenvalue problem for a tridiagonal 3-Toeplitz matrix. In Section 5 we analyze the semiclassical case and we give the structure relation for the new sequence $\{Q_n\}_{n\geq 0}$ in problem P3, from which the second order linear differential equation can be deduced. Finally, we make the connection of this kind of problems with the sieved orthogonal polynomials and we recover the structure relation and the second order differential equation for a special case of the sieved ultraspherical polynomials of the second kind.

We notice that this paper deals with algebraic (or formal) orthogonal polynomials, so that the important tool is the three-term recurrence relation that each sequence of orthogonal polynomials satisfies. In a future paper [14] we will analyze the positive-definite case and we will give the corresponding measures of orthogonality.

2 The Main Results

The solution of problem P1 can be characterized as follows.

Theorem 2.1 *Let $\{P_n\}_{n\geq 0}$ be a system of monic orthogonal polynomials and $\{Q_n\}_{n\geq 0}$ a simple set of monic polynomials such that*

$$Q_1(0) = -\beta\,,\quad Q_2(\beta) = -\gamma\,,\quad Q_{3n}(x) = P_n(\pi_3(x))\,,\quad n = 0,1,2,\ldots\,, \tag{14}$$

where π_3 is a (monic) polynomial of degree 3 and $\beta,\gamma \in \mathbb{C}$. Consider the polynomial

$$\rho(x) := \gamma + \frac{\pi_3(x) - \pi_3(\beta)}{x - \beta}\,. \tag{15}$$

Let a_1 and a_2 be the zeros of ρ and put

$$c_1 := \pi_3(a_1)\,,\quad c_2 := \pi_3(a_2)\,.$$

Then, for a fixed pair (β,γ), $\{Q_n\}_{n\geq 0}$ is an MOPS if and only if

$$\gamma \neq 0\,,\quad P_n(c_1) \neq 0\,,\quad P_n(c_2) \neq 0\,,\quad P_n^*(c_1;c_2) \neq 0\,, \tag{16}$$

$$Q_{3n+1}(x) = \frac{1}{\rho(x)}\left[P_{n+1}(\pi_3(x)) + \gamma\frac{P_n^*(c_1;c_2)}{P_n(c_2)}\left(x - a_2 - \frac{1}{\gamma}\frac{P_{n+1}(c_2)}{P_n^*(c_1;c_2)}\right)P_n(\pi_3(x))\right]\,, \tag{17}$$

$$Q_{3n+2}(x) = \frac{1}{\rho(x)}\left[\left(x - a_1 + \frac{1}{\gamma}\frac{P_{n+1}(c_2)}{P_n^*(c_1;c_2)}\right)P_{n+1}(\pi_3(x)) - \frac{1}{\gamma}\frac{P_{n+1}(c_1)P_{n+1}(c_2)}{P_n(c_1)P_n^*(c_1;c_2)}P_n(\pi_3(x))\right] \tag{18}$$

hold for all $n = 0,1,2,\ldots$.

In such conditions, if $\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (2), then the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ for the corresponding three-term recurrence relation satisfied by $\{Q_n\}_{n\geq 0}$ are given by

$$\tilde{\beta}_{3n} = \beta\,,\quad \tilde{\beta}_{3n+1} = a_1 - \frac{1}{\gamma}\frac{P_{n+1}(c_2)}{P_n^*(c_1;c_2)}\,,\quad \tilde{\beta}_{3n+2} = a_2 + \frac{1}{\gamma}\frac{P_{n+1}(c_2)}{P_n^*(c_1;c_2)}\,, \tag{19}$$

$$\tilde{\gamma}_{3n} = -\gamma\gamma_n\frac{P_{n-1}(c_1)P_{n-1}^*(c_1;c_2)}{P_n(c_1)P_n(c_2)}\,, \tag{20}$$

$$\tilde{\gamma}_{3n+1} = \gamma\frac{P_n^*(c_1;c_2)}{P_n(c_2)}\,,\quad \tilde{\gamma}_{3n+2} = -\frac{1}{\gamma^2}\frac{P_{n+1}(c_1)P_{n+1}(c_2)}{P_n^*(c_1;c_2)P_n^*(c_2;c_1)}\,. \tag{21}$$

Finally, denote by $\mathbf{u}$ the moment linear functional such that $\{P_n\}_{n\geq 0}$ is the corresponding MOPS. Let $\{A_n\}_{n\geq 0}$ be the basis of $\mathbb{P}$ defined by

$$A_{3n}(x) := \pi_3^n(x)\,,\quad A_{3n+1}(x) := (x-\beta)\pi_3^n(x)\,,\quad A_{3n+2}(x) := (x^2 - \beta x - \gamma)\pi_3^n(x) \tag{22}$$

for $n = 0,1,2,\ldots$. Then, under the above conditions, $\{Q_n\}_{n\geq 0}$ is an MOPS associated with the moment linear functional $\mathbf{v}$ defined on the basis $\{A_n\}_{n\geq 0}$ by

$$\langle \mathbf{v}, A_{3n}(x)\rangle := \langle \mathbf{u}, x^n\rangle\,,\quad \langle \mathbf{v}, A_{3n+1}(x)\rangle := 0\,,\quad \langle \mathbf{v}, A_{3n+2}(x)\rangle := 0 \tag{23}$$

for all $n = 0,1,2,\ldots$.

Corollary 2.1 *Under the hypotheses of Theorem 2.1,*

$$Q^{(1)}_{3n+2}(x) = \rho(x)P_n{}^{(1)}(\pi_3(x)) \tag{24}$$

holds for all $n = 0, 1, 2, \ldots$. Furthermore, the formal Stieltjes series corresponding to the moment linear functionals $\mathbf{u}$ *and* $\mathbf{v}$ *as in Theorem 2.1 are related by*

$$S_{\mathbf{v}}(z) = \rho(z)S_{\mathbf{u}}(\pi_3(z))\,. \tag{25}$$

The solution for problem P2 is

Theorem 2.2 *Let $\{P_n\}_{n\geq 0}$ be an MOPS and $\{Q_n\}_{n\geq 0}$ a simple set of monic polynomials such that*

$$Q_2(x) = (x-\beta)(x-a) - \gamma\,, \quad Q_{3n+1}(x) = (x-a)P_n(\pi_3(x)) \tag{26}$$

for all $n = 0, 1, 2, \ldots$, where a, β and γ are fixed complex numbers and π_3 a monic polynomial of degree 3. Put

$$b := -[a+\beta+\pi_3''(0)/2]\,, \quad c := \pi_3(b)\,, \quad d := \pi_3(a)\,. \tag{27}$$

Then:

(i) *if $b \neq a$, or $b = a$ and $\pi_3'(a) = 0$, for fixed (β, γ), $\{Q_n\}_{n\geq 0}$ is an MOPS if and only if*

$$\gamma \neq 0\,, \quad P_n(c) \neq 0\,, \quad P_n^{\mu}(d) \neq 0\,, \quad d_n^{0,\mu}(c,d) \neq 0\,, \tag{28}$$

$$Q_{3n+2}(x) = \frac{1}{x-b}\left[P_{n+1}(\pi_3(x)) + \frac{d_n^{0,\mu}(c,d)}{P_n(c)P_n^{\mu}(d)}\left(x - b - \frac{P_{n+1}(c)P_n^{\mu}(d)}{d_n^{0,\mu}(c,d)}\right)P_n(\pi_3(x))\right],$$

$$Q_{3n+3}(x) = \frac{1}{x-b}\left[\left(x - a + \frac{P_{n+1}(c)P_n^{\mu}(d)}{d_n^{0,\mu}(c,d)}\right)P_{n+1}(\pi_3(x)) - \frac{P_{n+1}(c)P_{n+1}^{\mu}(d)}{d_n^{0,\mu}(c,d)}P_n(\pi_3(x))\right]$$

for all $n = 0, 1, 2, \ldots$, where

$$\mu := \gamma(b-a)\,, \quad d_n^{0,\mu}(c,d) := \begin{cases} D_n^{0,\mu}(c,d)/(b-a) & \text{if} \quad b \neq a \\ D_n^{0,\gamma}(c,c) & \text{if} \quad b = a \text{ and } \pi_3'(a) = 0\,. \end{cases}$$

In such conditions, if we assume that $\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (2), then the coefficients $\widetilde{\beta}_n$ and $\widetilde{\gamma}_n$ for the corresponding three-term recurrence relation satisfied by $\{Q_n\}_{n\geq 0}$ are given by

$$\widetilde{\beta}_0 = a\,, \quad \widetilde{\beta}_{3n+1} = \beta\,, \quad n \geq 0$$

$$\widetilde{\beta}_{3n+2} = a - \frac{P_{n+1}(c)P_n^{\mu}(d)}{d_n^{0,\mu}(c,d)}\,, \quad \widetilde{\beta}_{3n+3} = b + \frac{P_{n+1}(c)P_n^{\mu}(d)}{d_n^{0,\mu}(c,d)}\,, \quad n \geq 0$$

$$\widetilde{\gamma}_1 = \gamma\,,$$

$$\widetilde{\gamma}_{3n+4} = -\gamma_{n+1}\frac{d_n^{0,\mu}(c,d)}{P_{n+1}(c)P_{n+1}^{\mu}(d)}\,, \quad \widetilde{\gamma}_{3n+2} = \frac{d_n^{0,\mu}(c,d)}{P_n(c)P_n^{\mu}(d)}\,, \quad n \geq 0$$

$$\widetilde{\gamma}_{3n+3} = -\frac{P_n(c)P_n^{\mu}(d)P_{n+1}(c)P_{n+1}^{\mu}(d)}{\left(d_n^{0,\mu}(c,d)\right)^2}\,, \quad n \geq 0\,.$$

(ii) *If $b = a$ and $\pi_3'(a) \neq 0$, for fixed (β, γ), $\{Q_n\}_{n\geq 0}$ is a MOPS if and only if*

$$\gamma \neq 0\,, \quad \delta \neq 0\,, \quad P_n(c) \neq 0\,, \quad R_n^{\nu}(c) \neq 0\,,$$

$$Q_{3n+2}(x) = \frac{1}{x-a}\left[P_{n+1}(\pi_3(x)) - \delta\frac{R_n^\nu(c)}{P_n(c)}\left(x-a+\frac{1}{\delta}\frac{P_{n+1}(c)}{R_n^\nu(c)}P_n(\pi_3(x))\right)\right],$$

$$Q_{3n+3}(x) = \frac{1}{x-a}\left[\left(x-a-\frac{1}{\delta}\frac{P_{n+1}(c)}{R_n^\nu(c)}\right)P_{n+1}(\pi_3(x)) + \frac{1}{\delta}\frac{P_{n+1}^2(c)}{P_n(c)R_n^\nu(c)}P_n(\pi_3(x))\right]$$

for all $n = 0,1,2,\ldots$, *where*

$$R_n(x) := P_n^*(c;x), \quad \delta := \gamma + \pi_3'(a), \quad \nu := \gamma P_1(c)/\delta.$$

Under such conditions,

$$\widetilde{\beta}_0 = a, \quad \widetilde{\beta}_{3n+1} = \beta, \quad n \geq 0$$

$$\widetilde{\beta}_{3n+2} = a + \frac{1}{\delta}\frac{P_{n+1}(c)}{R_n^\nu(c)}, \quad \widetilde{\beta}_{3n+3} = a - \frac{1}{\delta}\frac{P_{n+1}(c)}{R_n^\nu(c)}, \quad n \geq 0$$

$$\widetilde{\gamma}_1 = \gamma, \quad \widetilde{\gamma}_{3n+4} = \delta\gamma_{n+1}\frac{P_n(c)R_n^\nu(c)}{P_{n+1}^2(c)}, \quad n \geq 0$$

$$\widetilde{\gamma}_{3n+2} = -\delta\frac{R_n^\nu(c)}{P_n(c)}, \quad \widetilde{\gamma}_{3n+3} = -\left(\frac{1}{\delta}\frac{P_{n+1}(c)}{R_n^\nu(c)}\right)^2, \quad n \geq 0.$$

Furthermore, let u *be the moment linear functional such that* $\{P_n\}_{n\geq 0}$ *is the corresponding MOPS, and consider the basis* $\{B_n\}_{n\geq 0}$ *of* $\mathbb{P}$ *such that*

$$B_0(x) := 1, \quad B_{3n+1}(x) := (x-a)\pi_3^n(x),$$
$$B_{3n+2}(x) := (x-a)^2\pi_3^n(x), \quad B_{3n+3}(x) := (x-a)^2(x-\beta)\pi_3^n(x)$$

for $n = 0,1,2,\ldots$. *Then, in any of the situations* (i) *or* (ii), $\{Q_n\}_{n\geq 0}$ *is an MOPS associated to the moment linear functional* v *defined by*

$$\langle \mathrm{v}, B_0(x)\rangle := u_0, \quad \langle \mathrm{v}, B_{3n+1}(x)\rangle := 0,$$
$$\langle \mathrm{v}, B_{3n+2}(x)\rangle := \gamma\langle \mathrm{u}, x^n\rangle, \quad \langle \mathrm{v}, B_{3n+3}(x)\rangle := 0 \tag{29}$$

for $n = 0,1,2,\ldots$.

Remark 2.1 *In the case* $a = b$ *and* $\pi_3'(a) = 0$, *we have* $P_n^\mu(d) = P_n(c)$ *and since, according to* (9), $D_n^{0,\gamma}(c,c) = -\gamma\prod_{i=1}^n \gamma_i$, *the conditions* (28) *reduce to*

$$\gamma \neq 0, \quad P_n(c) \neq 0.$$

Corollary 2.2 *Under the hypotheses of Theorem 2.2,*

$$Q_{3n}^{(1)}(x) = P_n(\pi_3(x)) + \gamma(x-b)P_{n-1}^{(1)}(\pi_3(x)) \tag{30}$$

holds for all $n = 0,1,2,\ldots$. *Moreover,*

$$S_{\mathrm{v}}(z) = -\frac{u_0 - \gamma(z-b)S_{\mathrm{u}}(\pi_3(z))}{z-a}. \tag{31}$$

Finally, we give the solution for problem P3:

Theorem 2.3 *Let $\{P_n\}_{n\geq 0}$ be an MOPS and $\{Q_n\}_{n\geq 0}$ a simple set of monic polynomials such that*

$$Q_{3n+2}(x) = (x-a)(x-b)P_n(\pi_3(x)), \quad n = 0,1,2,\ldots \tag{32}$$

where a and b are fixed complex numbers and π_3 is a polynomial of degree 3. Without loss of generality, write

$$Q_1(x) = x-\alpha, \quad Q_3(x) = (x-\beta)Q_2(x) - \gamma Q_1(x)$$

and denote

$$c := \pi_3(a), \quad d := \pi_3(b), \quad \lambda := -\gamma(a-\alpha), \quad \mu := -\gamma(b-\alpha).$$

Then,

(i) *If $b \neq a$, or $b = a$ and $\pi_3'(a) = 0$, for fixed (α,γ), $\{Q_n\}_{n\geq 0}$ is an MOPS if and only if*

$$\lambda\mu \neq 0, \; \beta = -[a+b+\pi_3''(0)/2], \quad P_n^{\lambda}(c) \neq 0, \; P_n^{\mu}(d) \neq 0, \; d_n^{\lambda,\mu}(c,d) \neq 0, \tag{33}$$

$$Q_{3n+3}(x) = P_{n+1}(\pi_3(x)) + \frac{d_n^{\lambda,\mu}(c,d)}{P_n^{\lambda}(c)P_n^{\mu}(d)}\left(x - a - \frac{P_{n+1}^{\lambda}(c)P_n^{\mu}(d)}{d_n^{\lambda,\mu}(c,d)}\right)P_n(\pi_3(x)),$$

$$Q_{3n+4}(x) = \left(x - b + \frac{P_{n+1}^{\lambda}(c)P_n^{\mu}(d)}{d_n^{\lambda,\mu}(c,d)}\right)P_{n+1}(\pi_3(x)) - \frac{P_{n+1}^{\lambda}(c)P_{n+1}^{\mu}(d)}{d_n^{\lambda,\mu}(c,d)}P_n(\pi_3(x))$$

for $n = 0,1,2,\ldots$, where

$$d_n^{\lambda,\mu}(c,d) := \begin{cases} D_n^{\lambda,\mu}(c,d)/(a-b) & \text{if} \quad b \neq a \\ D_n^{0,\gamma}(c,c) & \text{if} \quad b = a \text{ and } \pi_3'(a) = 0. \end{cases}$$

Under such conditions, if $\{P_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (2), then the coefficients $\widetilde{\beta}_n$ and $\widetilde{\gamma}_n$ for the corresponding three-term recurrence relation satisfied by $\{Q_n\}_{n\geq 0}$ are given by

$$\widetilde{\beta}_0 = \alpha, \quad \widetilde{\beta}_1 = a+b-\alpha, \quad \widetilde{\beta}_{3n+2} = \beta, \quad n \geq 0$$

$$\widetilde{\beta}_{3n+3} = b - \frac{P_{n+1}^{\lambda}(c)P_n^{\mu}(d)}{d_n^{\lambda,\mu}(c,d)}, \quad \widetilde{\beta}_{3n+4} = a + \frac{P_{n+1}^{\lambda}(c)P_n^{\mu}(d)}{d_n^{\lambda,\mu}(c,d)}, \quad n \geq 0$$

$$\widetilde{\gamma}_1 = -\lambda\mu/\gamma^2, \quad \widetilde{\gamma}_2 = \gamma$$

$$\widetilde{\gamma}_{3n} = \frac{d_{n-1}^{\lambda,\mu}(c,d)}{P_{n-1}^{\lambda}(c)P_{n-1}^{\mu}(d)}, \quad \widetilde{\gamma}_{3n+2} = -\gamma_n\frac{d_{n-1}^{\lambda,\mu}(c,d)}{P_n^{\lambda}(c)P_n^{\mu}(d)}, \quad n \geq 1$$

$$\widetilde{\gamma}_{3n+1} = -\frac{P_{n-1}^{\lambda}(c)P_{n-1}^{\mu}(d)P_n^{\lambda}(c)P_n^{\mu}(d)}{\left(d_{n-1}^{\lambda,\mu}(c,d)\right)^2}, \quad n \geq 1.$$

(ii) *If $b = a$ and $\pi_3'(a) \neq 0$, for fixed (α,γ), $\{Q_n\}_{n\geq 0}$ is an MOPS if and only if*

$$\lambda \neq 0, \quad \delta \neq 0, \quad \beta = -[2a + \pi_3''(0)/2], \quad P_n^{\lambda}(c) \neq 0, \quad R_n^{\nu}(c) \neq 0,$$

$$Q_{3n+3}(x) = P_{n+1}(\pi_3(x)) - \delta\frac{R_n^{\nu}(c)}{P_n^{\lambda}(c)}\left(x - a + \frac{1}{\delta}\frac{P_{n+1}^{\lambda}(c)}{R_n^{\nu}(c)}P_n(\pi_3(x))\right),$$

$$Q_{3n+4}(x) = \left(x - a - \frac{1}{\delta}\frac{P_{n+1}^{\lambda}(c)}{R_n^{\nu}(c)}\right)P_{n+1}(\pi_3(x)) + \frac{1}{\delta}\frac{(P_{n+1}^{\lambda}(c))^2}{P_n^{\lambda}(c)R_n^{\nu}(c)}P_n(\pi_3(x))$$

for $n = 0, 1, 2, \ldots$, where

$$R_n(x) := [P_n^\lambda]^*(c;x)\,, \quad \delta := \gamma + \pi_3'(a)\,, \quad \nu := \gamma P_1^\lambda(c)/\delta\,.$$

Under such conditions,

$$\widetilde{\beta}_0 = \alpha\,, \quad \widetilde{\beta}_1 = 2a - \alpha\,, \quad \widetilde{\beta}_{3n+2} = \beta\,, \quad n \geq 0$$

$$\widetilde{\beta}_{3n+3} = a + \frac{1}{\delta}\frac{P_{n+1}^\lambda(c)}{R_n^\nu(c)}\,, \quad \widetilde{\beta}_{3n+4} = a - \frac{1}{\delta}\frac{P_{n+1}^\lambda(c)}{R_n^\nu(c)}\,, \quad n \geq 0$$

$$\widetilde{\gamma}_1 = -\lambda^2/\gamma^2\,, \quad \widetilde{\gamma}_2 = \gamma\,, \quad \widetilde{\gamma}_{3n+3} = -\delta\frac{R_n^\nu(c)}{P_n^\lambda(c)}\,, \quad n \geq 0$$

$$\widetilde{\gamma}_{3n+4} = -\left(\frac{1}{\delta}\frac{P_{n+1}^\lambda(c)}{R_n^\nu(c)}\right)^2\,, \quad \widetilde{\gamma}_{3n+5} = \delta\gamma_{n+1}\frac{P_n^\lambda(c)R_n^\nu(c)}{\left(P_{n+1}^\lambda(c)\right)^2}\,, \quad n \geq 0\,.$$

Furthermore, let u *be the moment linear functional such that $\{P_n\}_{n\geq 0}$ is the corresponding MOPS and consider the basis $\{C_n\}_{n\geq 0}$ of* $\mathbb{P}$ *such that*

$$\begin{array}{l} C_0(x) := 1\,, \quad C_{3n+1}(x) := (x-b)\pi_3^n(x)\,, \\ C_{3n+2}(x) := (x-a)(x-b)\pi_3^n(x)\,, \quad C_{3n+3}(x) := (x-a)(x-b)^2\pi_3^n(x) \end{array}$$

for $n = 0, 1, 2, \ldots$. Then, in any of the situations (i) *or* (ii), *$\{Q_n\}_{n\geq 0}$ is an MOPS with respect to the moment linear functional* v *defined by*

$$\begin{array}{l} \langle \mathrm{v}, C_0(x)\rangle := u_0\,, \quad \langle \mathrm{v}, C_{3n+1}(x)\rangle := \frac{\mu}{\gamma}\langle \mathrm{u}^{\lambda,c}, x^n\rangle\,, \\ \langle \mathrm{v}, C_{3n+2}(x)\rangle := 0\,, \quad \langle \mathrm{v}, C_{3n+3}(x)\rangle := 0 \end{array} \tag{34}$$

for $n = 0, 1, 2, \ldots$.

Remark 2.2 *In the case $a = b$ and $\pi_3'(a) = 0$, conditions* (33) *reduce to*

$$\lambda \neq 0\,, \quad \beta = -[2a + \pi_3''(0)/2]\,, \quad P_n^\lambda(c) \neq 0\,.$$

Corollary 2.3 *Under the hypotheses of Theorem 2.3,*

$$Q_{3n+1}^{(1)}(x) = (x - a - b + \alpha)\,P_n(\pi_3(x)) - \frac{\lambda\mu}{\gamma}\,P_{n-1}^{(1)}(\pi_3(x)) \tag{35}$$

holds for all $n = 0, 1, 2, \ldots$, and

$$S_{\mathrm{v}}(z) = -\frac{u_0(z - a - b + \alpha) + (\lambda\mu/\gamma)S_{\mathrm{u}}(\pi_3(z))}{(z-a)(z-b)}\,. \tag{36}$$

As a consequence of the previous results, we can state that if $\{P_n\}_{n\geq 0}$ is a semiclassical MOPS then $\{Q_n\}_{n\geq 0}$ is a semiclassical MOPS too (for the definition of "semiclassical MOPS" see Maroni [17], e.g.). In fact,

Corollary 2.4 *If S_{u} satisfies*

$$\phi(z)S_{\mathrm{u}}'(z) = C(z)S_{\mathrm{u}}(z) + D(z)$$

where ϕ, C and D are polynomials, then $S_{\rm v}$ satisfies

$$\widetilde{\phi}(z)S'_{\rm v}(z)=\widetilde{C}(z)S_{\rm v}(z)+\widetilde{D}(z)$$

where

$$\text{(i)}\quad\begin{cases}\widetilde{\phi}(z)=\rho(z)\phi(\pi_3(z))\\ \widetilde{C}(z)=\rho'(z)\phi(\pi_3(z))+\rho(z)\pi_3'(z)C(\pi_3(z))\\ \widetilde{D}(z)=\rho^2(z)\pi_3'(z)D(\pi_3(z))\end{cases}$$

for problem P1;

$$\text{(ii)}\quad\begin{cases}\widetilde{\phi}(z)=(z-a)(z-b)\phi(\pi_3(z))\\ \widetilde{C}(z)=(b-a)\phi(\pi_3(z))+(z-a)(z-b)\pi_3'(z)C(\pi_3(z))\\ \widetilde{D}(z)=u_0\phi(\pi_3(z))+(z-b)\pi_3'(z)[u_0C(\pi_3(z))+\gamma(z-b)D(\pi_3(z))]\end{cases}$$

for problem P2; finally, for problem P3,

$$\text{(iii)}\quad\begin{cases}\widetilde{\phi}(z)=(z-a)(z-b)\phi(\pi_3(z))\\ \widetilde{C}(z)=-(2z-a-b)\phi(\pi_3(z))+(z-a)(z-b)\pi_3'(z)C(\pi_3(z))\\ \widetilde{D}(z)=-u_0+\pi_3'(z)[u_0(z-a-b+\alpha)C(\pi_3(z))-(\lambda\mu/\gamma)D(\pi_3(z))]\,.\end{cases}$$

Moreover, if ϕ, C and D have no common zeros, so that $s:=\max\{\deg C-1,\deg D\}$ is the class of u, *then the class of* v *is at most $\widetilde{s}:=\max\{\deg\widetilde{C}-1,\deg\widetilde{D}\}\leq 3s+6$ (for any of the above three problems).*

3 Proofs

We will only prove Theorems 2.1 and 2.3 and Corollaries 2.1 and 2.3. The proof of Theorem 2.2 follows using the same ideas as in the proofs of Theorems 2.1 and 2.3. In the same way, we can prove Corollary 2.2 by using the same technique as in Corollaries 2.1 and 2.3. Corollary 2.4 is straightforward consequence of Corollaries 2.1, 2.2 and 2.3.

3.1 Proof of Theorem 2.1

Assume that $\{Q_n\}_{n\geq 0}$ is an MOPS. Thus, it satisfies a three-term recurrence relation

$$\begin{aligned}&xQ_n(x)=Q_{n+1}(x)+\tilde{\beta}_nQ_n(x)+\tilde{\gamma}_nQ_{n-1}(x)\,,\quad n\geq 1\\ &Q_0(x)=1\,,\quad Q_1(x)=x-\tilde{\beta}_0\end{aligned}\tag{37}$$

with $\tilde{\gamma}_n\neq 0$ for $n\geq 1$. It is clear that $\tilde{\beta}_0=\beta$, and then also $\tilde{\gamma}_1=\gamma$. If

$$\pi_3(x)=x^3+px^2+qx+r$$

notice that, with this notation, the polynomial ρ can be written

$$\rho(x)=(x+\beta)(x+p)+\beta^2+\gamma+q\,.$$

In the three-term recurrence relation (2) for $\{P_n\}_{n\geq 0}$ replace x by x^3+px^2+qx+r, so that

$$(x^3+px^2+qx+r)Q_{3n}(x)=Q_{3n+3}(x)+\beta_nQ_{3n}(x)+\gamma_nQ_{3n-3}(x)\,,\quad n\geq 1\,.\tag{38}$$

Now, use successively (37) to expand $xQ_{3n}(x)$, $x^2Q_{3n}(x)$ and $x^3Q_{3n}(x)$ as

$$xQ_{3n}(x) = Q_{3n+1}(x) + \tilde{\beta}_{3n}Q_{3n}(x) + \tilde{\gamma}_{3n}Q_{3n-1}(x),$$

$$\begin{aligned}x^2Q_{3n}(x) = Q_{3n+2}(x) + (\tilde{\beta}_{3n+1} + \tilde{\beta}_{3n})Q_{3n+1}(x) + (\tilde{\gamma}_{3n+1} + \tilde{\beta}^2_{3n} + \tilde{\gamma}_{3n})Q_{3n}(x) + \\ + \tilde{\gamma}_{3n}(\tilde{\beta}_{3n} + \tilde{\beta}_{3n-1})Q_{3n-1}(x) + \tilde{\gamma}_{3n}\tilde{\gamma}_{3n-1}Q_{3n-2}(x)\end{aligned}$$

and

$$\begin{aligned}x^3Q_{3n}(x) &= Q_{3n+3}(x) + (\tilde{\beta}_{3n+2} + \tilde{\beta}_{3n+1} + \tilde{\beta}_{3n})Q_{3n+2}(x) + \\ &+ [\tilde{\gamma}_{3n+2} + (\tilde{\beta}_{3n+1} + \tilde{\beta}_{3n})\tilde{\beta}_{3n+1} + \tilde{\gamma}_{3n+1} + \tilde{\beta}^2_{3n} + \tilde{\gamma}_{3n}]\, Q_{3n+1}(x) + \\ &+ [\tilde{\gamma}_{3n+1}(\tilde{\beta}_{3n+1} + \tilde{\beta}_{3n}) + \tilde{\beta}_{3n}(\tilde{\gamma}_{3n+1} + \tilde{\beta}^2_{3n} + \tilde{\gamma}_{3n}) + \tilde{\gamma}_{3n}(\tilde{\beta}_{3n} + \tilde{\beta}_{3n-1})]\, Q_{3n}(x) + \\ &+ \tilde{\gamma}_{3n}[\tilde{\gamma}_{3n+1} + \tilde{\beta}^2_{3n} + \tilde{\gamma}_{3n} + \tilde{\beta}_{3n-1}(\tilde{\beta}_{3n} + \tilde{\beta}_{3n-1}) + \tilde{\gamma}_{3n-1}]\, Q_{3n-1}(x) + \\ &+ \tilde{\gamma}_{3n-1}\tilde{\gamma}_{3n}(\tilde{\beta}_{3n} + \tilde{\beta}_{3n-1} + \tilde{\beta}_{3n-2})Q_{3n-2}(x) + \\ &+ \tilde{\gamma}_{3n-2}\tilde{\gamma}_{3n-1}\tilde{\gamma}_{3n}Q_{3n-3}(x).\end{aligned}$$

Substitution of these expressions in the left-hand side of (38) yields a linear combination of a finite number of terms of the sequence $\{Q_n\}_{n\geq 0}$ which vanish identically. Therefore, since this sequence is a basis of $\mathbb{P}$, we find the following relations:

$$\tilde{\beta}_{3n+2} + \tilde{\beta}_{3n+1} + \tilde{\beta}_{3n} + p = 0 \tag{39}$$

$$\tilde{\gamma}_{3n+2} + \tilde{\gamma}_{3n+1} + \tilde{\gamma}_{3n} + \tilde{\beta}^2_{3n+1} + \tilde{\beta}_{3n}\tilde{\beta}_{3n+1} + \tilde{\beta}^2_{3n} + p(\tilde{\beta}_{3n} + \tilde{\beta}_{3n+1}) + q = 0 \tag{40}$$

$$\begin{aligned}\tilde{\gamma}_{3n+1}(\tilde{\beta}_{3n+1} + \tilde{\beta}_{3n}) + \tilde{\beta}_{3n}(\tilde{\gamma}_{3n+1} + \tilde{\beta}^2_{3n} + \tilde{\gamma}_{3n}) + \tilde{\gamma}_{3n}(\tilde{\beta}_{3n} + \tilde{\beta}_{3n-1}) + \\ + p(\tilde{\gamma}_{3n+1} + \tilde{\beta}^2_{3n} + \tilde{\gamma}_{3n}) + q\tilde{\beta}_{3n} + r = \beta_n\end{aligned} \tag{41}$$

$$\tilde{\gamma}_{3n+1} + \tilde{\beta}^2_{3n} + \tilde{\gamma}_{3n} + \tilde{\beta}_{3n-1}(\tilde{\beta}_{3n} + \tilde{\beta}_{3n-1}) + \tilde{\gamma}_{3n-1} + p(\tilde{\beta}_{3n} + \tilde{\beta}_{3n-1}) + q = 0 \tag{42}$$

$$\tilde{\beta}_{3n} + \tilde{\beta}_{3n-1} + \tilde{\beta}_{3n-2} + p = 0 \tag{43}$$

$$\tilde{\gamma}_{3n}\tilde{\gamma}_{3n-1}\tilde{\gamma}_{3n-2} = \gamma_n. \tag{44}$$

Combining equation (43) (after the change of indices $n \to n+1$) with (39) leads to $\tilde{\beta}_{3n+3} = \tilde{\beta}_{3n}$ for $n \geq 0$, so that

$$\tilde{\beta}_{3n} = \tilde{\beta}_0 = \beta \quad , \quad n \geq 0 \quad . \tag{45}$$

Consequently, equation (40) can be rewritten as

$$\tilde{\gamma}_{3n+2} + \tilde{\gamma}_{3n+1} + \tilde{\gamma}_{3n} + \beta^2 + (\tilde{\beta}_{3n+1} + \beta)(\tilde{\beta}_{3n+1} + p) + q = 0 \quad ,$$

or, according to (39) and (45),

$$\tilde{\gamma}_{3n+2} + \tilde{\gamma}_{3n+1} + \tilde{\gamma}_{3n} + \beta^2 + (\tilde{\beta}_{3n+2} + \beta)(\tilde{\beta}_{3n+2} + p) + q = 0 \quad . \tag{46}$$

Notice that (46) holds for $n = 0, 1, 2, \ldots$, with the convention $\tilde{\gamma}_0 = 0$. Now, from (42), we get

$$\tilde{\gamma}_{3n+1} + \tilde{\gamma}_{3n} + \tilde{\gamma}_{3n-1} + \beta^2 + (\tilde{\beta}_{3n-1} + \beta)(\tilde{\beta}_{3n-1} + p) + q = 0 \quad . \tag{47}$$

If we change the indices $n \to n+1$ in (47) and then compare the resulting equation with (46), it follows that $\tilde{\gamma}_{3n+4} + \tilde{\gamma}_{3n+3} = \tilde{\gamma}_{3n+1} + \tilde{\gamma}_{3n}$. Hence

$$\tilde{\gamma}_{3n+1} + \tilde{\gamma}_{3n} = \tilde{\gamma}_1 = \gamma \quad , \quad n \geq 0 \quad . \tag{48}$$

Now, from (42), (45) and (48) we have

$$\gamma + \beta^2 + \tilde{\beta}_{3n-1}(\beta + \tilde{\beta}_{3n-1}) + \tilde{\gamma}_{3n-1} + p(\beta + \tilde{\beta}_{3n-1}) + q = 0 \quad ,$$

so that, after the change of indices $n \to n+1$,

$$\tilde{\gamma}_{3n+2} + (\tilde{\beta}_{3n+2} + \beta)(\tilde{\beta}_{3n+2} + p) = -(\beta^2 + \gamma + q) \quad , \quad n \geq 0 \quad . \tag{49}$$

Next, we will see how to define Q_{3n+1} and Q_{3n+2}. We have

$$(x - \tilde{\beta}_{3n+1})Q_{3n+1}(x) = Q_{3n+2}(x) + \tilde{\gamma}_{3n+1}Q_{3n}(x) \tag{50}$$

and

$$Q_{3n+2}(x) = \frac{Q_{3n+3}(x) + \tilde{\gamma}_{3n+2}Q_{3n+1}(x)}{x - \tilde{\beta}_{3n+2}} . \tag{51}$$

Substitution of (51) in (50) yields

$$Q_{3n+1}(x) = \frac{P_{n+1}(\pi_3(x)) + (x - \tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1}P_n(\pi_3(x))}{(x - \tilde{\beta}_{3n+1})(x - \tilde{\beta}_{3n+2}) - \tilde{\gamma}_{3n+2}} . \tag{52}$$

But, using (39) and (49), we have

$$\begin{aligned}(x - \tilde{\beta}_{3n+1})(x - \tilde{\beta}_{3n+2}) - \tilde{\gamma}_{3n+2} = \\ &= x^2 - (\tilde{\beta}_{3n+1} + \tilde{\beta}_{3n+2})x + \tilde{\beta}_{3n+1}\tilde{\beta}_{3n+2} - \tilde{\gamma}_{3n+2} \\ &= x^2 + (p + \beta)x + p\beta + \beta^2 + \gamma + q \\ &= \rho(x)\end{aligned} \tag{53}$$

and, therefore, (52) reduces to

$$Q_{3n+1}(x) = \frac{1}{\rho(x)} \{ P_{n+1}(\pi_3(x)) + (x - \tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1}P_n(\pi_3(x)) \} \quad . \tag{54}$$

Moreover, substitution of (54) in (51) leads to

$$Q_{3n+2}(x) = \frac{1}{\rho(x)} \{ (x - \tilde{\beta}_{3n+1})P_{n+1}(\pi_3(x)) + \tilde{\gamma}_{3n+1}\tilde{\gamma}_{3n+2}P_n(\pi_3(x)) \} \quad . \tag{55}$$

Now, we will prove that conditions (16) must hold. We need to distinguish two cases:

CASE 1: $a_1 \neq a_2$. According to (54), for each n the polynomial $P_{n+1}(\pi_3(x)) + (x - \tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1}P_n(\pi_3(x))$ vanishes at the zeros of ρ, i.e.,

$$P_{n+1}(c_i) + (a_i - \tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1}P_n(c_i) = 0 \quad , \quad i = 1, 2 . \tag{56}$$

From (53) it follows that

$$\rho(\tilde{\beta}_{3n+2}) = -\tilde{\gamma}_{3n+2} \quad , \tag{57}$$

and then $\tilde{\beta}_{3n+2}$ is not a zero of ρ. Hence, we have $a_i - \tilde{\beta}_{3n+2} \neq 0$ for all n and $i = 1, 2$. Since P_n and P_{n+1} are coprime, we deduce from (56) that

$$P_n(c_i) \neq 0 \quad , \quad i = 1, 2 \tag{58}$$

for all $n = 0, 1, 2, \ldots$. Now, write (56) for $i = 1, 2$:

$$(a_1 - \tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1} = -\frac{P_{n+1}(c_1)}{P_n(c_1)} \tag{59}$$

$$(a_2 - \tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1} = -\frac{P_{n+1}(c_2)}{P_n(c_2)} . \tag{60}$$

Thus

$$a_1 - \tilde{\beta}_{3n+2} = \frac{P_{n+1}(c_1)P_n(c_2)}{P_{n+1}(c_2)P_n(c_1)}(a_2 - \tilde{\beta}_{3n+2})\,. \tag{61}$$

Therefore, since $a_1 \neq a_2$, it follows that $P_{n+1}(c_1)P_n(c_2) \neq P_{n+1}(c_2)P_n(c_1)$, which is equivalent to

$$P_n^*(c_1; c_2) \neq 0\,. \tag{62}$$

Notice that the conditions $a_1 \neq a_2$ and $\gamma \neq 0$ yield $c_1 \neq c_2$. In order to verify this assertion, use the relation $\pi_3(x) = (x-\beta)[\rho(x) - \gamma] + \pi_3(\beta)$, put $x = a_1$ and $x = a_2$ and then subtract the above relations in order to find

$$c_2 - c_1 = -\gamma(a_2 - a_1)\,. \tag{63}$$

Hence, conditions (16) follow from (58) and (62).

Now, we will prove relations (19)–(21). In fact, formula (19) for $\tilde{\beta}_{3n+2}$ is an easy consequence of (61) and (63). The expression for $\tilde{\beta}_{3n+1}$ follows from $\tilde{\beta}_{3n+1} + \tilde{\beta}_{3n+2} = a_1 + a_2$ (this last relation can be easily obtained taking derivatives in (53)). We get the first formula of (21) subtracting (59) from (60). Now, from (59) and (60) we deduce

$$\rho(\tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1}^2 = \frac{P_{n+1}(c_1)P_{n+1}(c_2)}{P_n(c_1)P_n(c_2)}\,.$$

So, taking into account (57) and the first formula of (21), the second formula of (21) is obtained. Finally, (20) follows from (21) and (44).

Case 2: $a_1 = a_2$. We will see that this is a limit situation of case 1 when $a_2 \to a_1$. We write $a_1 = a_2 = a$ and $c_1 = c_2 = c$. From (54) and (55) we can see that a is a double zero of both polynomials $P_{n+1}(\pi_3(x)) + (x - \tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1}P_n(\pi_3(x))$ and $(x - \tilde{\beta}_{3n+1})P_{n+1}(\pi_3(x)) + \tilde{\gamma}_{3n+1}\tilde{\gamma}_{3n+2}P_n(\pi_3(x))$. Then the relations

$$P_{n+1}(c) + (a - \tilde{\beta}_{3n+2})\tilde{\gamma}_{3n+1}P_n(c) = 0 \tag{64}$$

$$\pi_3'(a)P_{n+1}'(c) + \tilde{\gamma}_{3n+1}[P_n(c) + (a - \tilde{\beta}_{3n+2})\pi_3'(a)P_n'(c)] = 0 \tag{65}$$

$$(a - \tilde{\beta}_{3n+1})P_{n+1}(c) + \tilde{\gamma}_{3n+1}\tilde{\gamma}_{3n+2}P_n(c) = 0 \tag{66}$$

$$P_{n+1}(c) + \pi_3'(a)[(a - \tilde{\beta}_{3n+1})P_{n+1}'(c) + \tilde{\gamma}_{3n+1}\tilde{\gamma}_{3n+2}P_n'(c)] = 0 \tag{67}$$

hold for $n = 0, 1, 2, \ldots$.

As in case 1, it follows that $\rho(\tilde{\beta}_{3n+2}) = -\tilde{\gamma}_{3n+2}$, so that $\tilde{\beta}_{3n+2} \neq a$ and then from (64) $P_n(c) \neq 0$ for $n = 0, 1, 2, \ldots$ follows. Since $\pi_3'(a) = -\gamma$ (take derivatives in the expression $\pi_3(x) = (x-\beta)[\rho(x) - \gamma] + \pi_3(\beta)$, then put $x = a$ and use $\rho(a) = \rho'(a) = 0$), from (64) and (65), and taking into account that

$$P_n^*(c; c) = P_{n+1}'(c) - \frac{P_{n+1}(c)}{P_n(c)}P_n'(c) \quad ,$$

we find $\tilde{\gamma}_{3n+1}P_n(c) = \gamma P_n^*(c; c)$. This proves $P_n^*(c; c) \neq 0$ for $n = 0, 1, 2, \ldots$ and also the first formula of (21) when $a_1 = a_2$. The expression for $\tilde{\beta}_{3n+2}$ follows from (64) and the first formula of (21) when $a_1 = a_2$. Now, by $\tilde{\gamma}_{3n+2} = -\rho(\tilde{\beta}_{3n+2}) = -(a - \tilde{\beta}_{3n+2})^2$, we find the second formula of (21). Finally, the expression for $\tilde{\beta}_{3n+1}$ can be deduced from $\tilde{\beta}_{3n+1} + \tilde{\beta}_{3n+2} = 2a$, as well as the expression for $\tilde{\gamma}_{3n}$ from (44).

In any case, the representations (17) and (18), can be obtained by substitution of formulas (19) and (21) in (54) and (55).

To complete the proof, it remains to show that $\{Q_n\}_{n\geq 0}$ is an MOPS with respect to the linear functional v defined by (23). First, notice that relations (23) yield

$$\begin{aligned}\langle \mathrm{v}, f(\pi_3(x))\rangle &= \langle \mathrm{u}, f(x)\rangle \ , &(68)\\ \langle \mathrm{v}, (x-\beta) f(\pi_3(x))\rangle &= 0 \ , &(69)\\ \langle \mathrm{v}, (x^2-\beta x-\gamma) f(\pi_3(x))\rangle &= 0 &(70)\end{aligned}$$

for every polynomial f . To prove that $\{Q_n\}_{n\geq 0}$ is an MOPS with respect to v, we only need to show that

$$\langle \mathrm{v}, 1\rangle \neq 0 \quad , \quad \langle \mathrm{v}, Q_n(x)\rangle = 0 \, , n \geq 1$$

(because we already know that $\{Q_n\}_{n\geq 0}$ is an MOPS). From (23), for $n=0$ we get

$$\langle \mathrm{v}, 1\rangle = \langle \mathrm{v}, A_0\rangle = u_0 \neq 0 \, , \quad \langle \mathrm{v}, Q_1(x)\rangle = \langle \mathrm{v}, x-\beta\rangle = \langle \mathrm{v}, A_1(x)\rangle = 0 \, .$$

Using (68),

$$\langle \mathrm{v}, Q_{3n}(x)\rangle = \langle \mathrm{v}, P_n(\pi_3(x))\rangle = \langle \mathrm{u}, P_n(x)\rangle = 0 \, , n \geq 1 \, . \qquad (71)$$

Moreover, from $Q_{3n+1}(x)+\tilde{\gamma}_{3n}Q_{3n-1}(x) = (x-\beta)Q_{3n}(x) = (x-\beta)P_n(\pi_3(x))$ and (69), we get

$$\langle \mathrm{v}, Q_{3n+1}(x)\rangle = -\tilde{\gamma}_{3n}\langle \mathrm{v}, Q_{3n-1}(x)\rangle \, , n \geq 1 \, . \qquad (72)$$

Now, since $Q_2(x) = (x-\tilde{\beta}_1)(x-\beta)-\gamma = (x^2-\beta x-\gamma)-\tilde{\beta}_1(x-\beta)$, from (23) we get

$$\langle \mathrm{v}, Q_2(x)\rangle = \langle \mathrm{v}, x^2-\beta x-\gamma\rangle - \tilde{\beta}_1\langle \mathrm{v}, x-\beta\rangle = 0 \, .$$

Furthermore, from the three-term recurrence relation for $\{Q_n\}_{n\geq 0}$ and applying (72) and (71),

$$\begin{aligned}\langle \mathrm{v}, Q_{3n+2}\rangle &= \langle \mathrm{v}, xQ_{3n+1}\rangle - \tilde{\beta}_{3n+1}\langle \mathrm{v}, Q_{3n+1}\rangle - \tilde{\gamma}_{3n+1}\langle \mathrm{v}, Q_{3n}\rangle \\ &= \langle \mathrm{v}, x[(x-\beta)Q_{3n} - \tilde{\gamma}_{3n}Q_{3n-1}]\rangle + \tilde{\beta}_{3n+1}\tilde{\gamma}_{3n}\langle \mathrm{v}, Q_{3n-1}\rangle \\ &= \langle \mathrm{v}, x(x-\beta)Q_{3n}\rangle - \tilde{\gamma}_{3n}\langle \mathrm{v}, Q_{3n} + \tilde{\beta}_{3n-1}Q_{3n-1} + \tilde{\gamma}_{3n-1}Q_{3n-2}\rangle + \\ &\quad + \tilde{\beta}_{3n+1}\tilde{\gamma}_{3n}\langle \mathrm{v}, Q_{3n-1}\rangle \\ &= \langle \mathrm{v}, (x^2-\beta x-\gamma)Q_{3n}\rangle + (\gamma-\tilde{\gamma}_{3n})\langle \mathrm{v}, Q_{3n}\rangle - \tilde{\gamma}_{3n}\tilde{\beta}_{3n-1})\langle \mathrm{v}, Q_{3n-1}\rangle - \\ &\quad - \tilde{\gamma}_{3n}\tilde{\gamma}_{3n-1}\langle \mathrm{v}, Q_{3n-2}\rangle \\ &= -\tilde{\gamma}_{3n}\tilde{\beta}_{3n-1}\langle \mathrm{v}, Q_{3n-1}\rangle - \tilde{\gamma}_{3n}\tilde{\gamma}_{3n-1}\langle \mathrm{v}, Q_{3n-2}\rangle \, , \quad n \geq 1 \, .\end{aligned}$$

Therefore, for $n=1$ it follows that $\langle \mathrm{v}, Q_5\rangle = -\tilde{\gamma}_3\tilde{\beta}_2\langle \mathrm{v}, Q_2\rangle - \tilde{\gamma}_3\tilde{\gamma}_2\langle \mathrm{v}, Q_1\rangle = 0$; and, for $n \geq 2$, since, by (72), $\langle \mathrm{v}, Q_{3n-2}\rangle = -\tilde{\gamma}_{3n-3}\langle \mathrm{v}, Q_{3n-4}\rangle$, we deduce

$$\langle \mathrm{v}, Q_{3n+2}\rangle = -\tilde{\gamma}_{3n}\tilde{\beta}_{3n-1}\langle \mathrm{v}, Q_{3n-1}\rangle + \tilde{\gamma}_{3n}\tilde{\gamma}_{3n-1}\tilde{\gamma}_{3n-3}\langle \mathrm{v}, Q_{3n-4}\rangle \, .$$

Hence, since $\langle \mathrm{v}, Q_2\rangle = \langle \mathrm{v}, Q_5\rangle = 0$, it follows by iteration that

$$\langle \mathrm{v}, Q_{3n+2}\rangle = 0 \quad , \quad n \geq 0 \, .$$

This relation, together with (72) leads to

$$\langle \mathrm{v}, Q_{3n+1}\rangle = 0 \quad , \quad n \geq 0 \, .$$

This completes the proof of the necessity of the conditions. Now, it can be easily verified that the conditions are also sufficient, since them imply that $\{Q_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (37), with the sequences $\{\tilde{\beta}_n\}_{n\geq 0}$ and $\{\tilde{\gamma}_n\}_{n\geq 1}$ defined by formulas (19)–(21).

3.2 Proof of Corollary 2.1

Put

$$P_n(x) \equiv \sum_{i=0}^{n} a_i^{(n)} x^i \quad , \tag{73}$$

so that

$$P_n(x) - P_n(y) = (x-y) \sum_{i=0}^{n-1} \sum_{j=0}^{i} a_{i+1}^{(n)} x^{i-j} y^j \tag{74}$$

and then also

$$P_n(\pi_3(x)) - P_n(\pi_3(y)) = [\pi_3(x) - \pi_3(y)] \sum_{i=0}^{n-1} \sum_{j=0}^{i} a_{i+1}^{(n)} \pi_3^{i-j}(x) \pi_3^j(y) \quad . \tag{75}$$

One can easily check that

$$\pi_3(x) - \pi_3(y) = (x-y)[\rho(x) + (x - a_1 - a_2)(y - \beta) + (y^2 - \beta y - \gamma)] \quad . \tag{76}$$

Therefore, using (75) and (76), for $n \geq 1$ it follows that

$$\begin{aligned}
Q_{3n-1}^{(1)}(x) &= \frac{1}{v_0} \langle \mathrm{v}_y, \frac{Q_{3n}(x) - Q_{3n}(y)}{x-y} \rangle = \frac{1}{u_0} \langle \mathrm{v}_y, \frac{P_n(\pi_3(x)) - P_n(\pi_3(y))}{x-y} \rangle \\
&= \frac{1}{u_0} \langle \mathrm{v}_y, [\rho(x) + (x - a_1 - a_2)(y - \beta) + (y^2 - \beta y - \gamma)] \times \\
&\qquad \times \sum_{i=0}^{n-1} \sum_{j=0}^{i} a_{i+1}^{(n)} \pi_3^{i-j}(x) \pi_3^j(y) \rangle \\
&= \frac{1}{u_0} \sum_{i=0}^{n-1} \sum_{j=0}^{i} a_{i+1}^{(n)} \pi_3^{i-j}(x) \, \{\rho(x) \langle \mathrm{v}_y, A_{3j}(y) \rangle + \\
&\qquad + (x - a_1 - a_2) \langle \mathrm{v}_y, A_{3j+1}(y) \rangle + \langle \mathrm{v}_y, A_{3j+2}(y) \rangle \} \\
&= \frac{1}{u_0} \sum_{i=0}^{n-1} \sum_{j=0}^{i} a_{i+1}^{(n)} \pi_3^{i-j}(x) \rho(x) \langle \mathrm{u}_y, y^j \rangle \\
&= \rho(x) \frac{1}{u_0} \langle \mathrm{u}_y, \frac{P_n(\pi_3(x)) - P_n(y)}{\pi_3(x) - y} \rangle \quad , \quad \text{by (74)} \\
&= \rho(x) P_{n-1}^{(1)}(\pi_3(x)) \quad ,
\end{aligned}$$

which proves (24). Now, consider the representation

$$S_{\mathrm{v}}(z) = \langle \mathrm{v}_x, \frac{1}{x-z} \rangle \quad ,$$

and taking into account (from (76))

$$\frac{1}{x-z} = \frac{\rho(z)}{\pi_3(x) - \pi_3(z)} + \frac{(z - a_1 - a_2)(x - \beta)}{\pi_3(x) - \pi_3(z)} + \frac{x^2 - \beta x - \gamma}{\pi_3(x) - \pi_3(z)} \quad ,$$

we obtain

$$S_{\mathrm{v}}(z) = \rho(z) \langle \mathrm{v}_x, \frac{1}{\pi_3(x) - \pi_3(z)} \rangle + (z - a_1 - a_2) \langle \mathrm{v}_x, \frac{x - \beta}{\pi_3(x) - \pi_3(z)} \rangle + \langle \mathrm{v}_x, \frac{x^2 - \beta x - \gamma}{\pi_3(x) - \pi_3(z)} \rangle .$$

Since (formally) $1/[\pi_3(x) - \pi_3(z)] = -\sum_{n\geq 0} \pi_3{}^n(x)/\pi_3^{n+1}(z)$, according to the relations (23) we have

$$S_u(\pi_3(z)) = -\sum_{n\geq 0} \frac{u_n}{\pi_3^{n+1}(z)} = -\sum_{n\geq 0} \frac{\langle v_x, \pi_3{}^n(x)\rangle}{\pi_3^{n+1}(z)} = \langle v_x, \frac{1}{\pi_3(x) - \pi_3(z)}\rangle \quad ,$$

$$\langle v_x, \frac{x-\beta}{\pi_3(x) - \pi_3(z)}\rangle = -\sum_{n\geq 0} \frac{\langle v_x, (x-\beta)\pi_3{}^n(x)\rangle}{\pi_3^{n+1}(z)} = 0 \quad ,$$

$$\langle v_x, \frac{x^2 - \beta x - \gamma}{\pi_3(x) - \pi_3(z)}\rangle = -\sum_{n\geq 0} \frac{\langle v_x, (x^2 - \beta x - \gamma)\pi_3{}^n(x)\rangle}{\pi_3^{n+1}(z)} = 0 \quad .$$

Hence,

$$S_v(z) = \rho(z) S_u(\pi_3(z)) \quad .$$

3.3 Proof of Theorem 2.3

The first part of the proof is similar to the proof of Theorem 2.1 and thus we will be rather sketchy. Hence, let

$$\pi_3(x) = x^3 + px^2 + qx + r$$

and assume that $\{Q_n\}_{n\geq 0}$ is an MOPS, so that

$$\begin{aligned} &xQ_n(x) = Q_{n+1}(x) + \tilde{\beta}_n Q_n(x) + \tilde{\gamma}_n Q_{n-1}(x) \quad , \quad n \geq 1 \\ &Q_0(x) = 1 \quad , \quad Q_1(x) = x - \tilde{\beta}_0 \end{aligned} \tag{77}$$

with $\tilde{\gamma}_n \neq 0$ for $n \geq 1$. Then, $\tilde{\beta}_0 = \alpha$, $\tilde{\beta}_2 = \beta$ and $\tilde{\gamma}_2 = \gamma \neq 0$. As in the proof of Theorem 2.1, in the three-term recurrence relation (2) replace x by $x^3 + px^2 + qx + r$ and then multiply by $(x-a)(x-b)$ to find

$$(x^3 + px^2 + qx + r)Q_{3n+2}(x) = Q_{3n+5}(x) + \beta_n Q_{3n+2}(x) + \gamma_n Q_{3n-1}(x) \quad , \quad n \geq 1 . \tag{78}$$

Then use successively (77) to expand $xQ_{3n+2}(x)$, $x^2Q_{3n+2}(x)$ and $x^3Q_{3n+2}(x)$ as linear combinations of polynomials Q_i. This leads to the following system:

$$\tilde{\beta}_{3n+4} + \tilde{\beta}_{3n+3} + \tilde{\beta}_{3n+2} + p = 0 \tag{79}$$

$$\tilde{\gamma}_{3n+4} + \tilde{\gamma}_{3n+3} + \tilde{\gamma}_{3n+2} + \tilde{\beta}^2_{3n+3} + \tilde{\beta}_{3n+2}\tilde{\beta}_{3n+3} + \tilde{\beta}^2_{3n+2} + p(\tilde{\beta}_{3n+2} + \tilde{\beta}_{3n+3}) + q = 0 \tag{80}$$

$$\begin{aligned} &\tilde{\gamma}_{3n+3}(\tilde{\beta}_{3n+3} + \tilde{\beta}_{3n+2}) + \tilde{\beta}_{3n+2}(\tilde{\gamma}_{3n+3} + \tilde{\beta}^2_{3n+2} + \tilde{\gamma}_{3n+2}) + \tilde{\gamma}_{3n+2}(\tilde{\beta}_{3n+2} + \tilde{\beta}_{3n+1}) + \\ &\quad + p(\tilde{\gamma}_{3n+3} + \tilde{\beta}^2_{3n+2} + \tilde{\gamma}_{3n+2}) + q\tilde{\beta}_{3n+2} + r = \beta_n \end{aligned} \tag{81}$$

$$\tilde{\gamma}_{3n+3} + \tilde{\beta}^2_{3n+2} + \tilde{\gamma}_{3n+2} + \tilde{\beta}_{3n+1}(\tilde{\beta}_{3n+2} + \tilde{\beta}_{3n+1}) + \tilde{\gamma}_{3n+1} + p(\tilde{\beta}_{3n+2} + \tilde{\beta}_{3n+1}) + q = 0 \tag{82}$$

$$\tilde{\beta}_{3n+2} + \tilde{\beta}_{3n+1} + \tilde{\beta}_{3n} + p = 0 \tag{83}$$

$$\tilde{\gamma}_{3n+2}\tilde{\gamma}_{3n+1}\tilde{\gamma}_{3n} = \gamma_n . \tag{84}$$

Again, notice that this system is very similar to the system (39)-(44) and then, using the same technique as in the proof of Theorem 2.1, we easily get the following relations:

$$\tilde{\beta}_{3n+2} = \tilde{\beta}_2 = \beta \tag{85}$$

$$\tilde{\beta}_{3n+1} + \tilde{\beta}_{3n} = a + b \tag{86}$$

$$\tilde{\gamma}_{3n+3} + \tilde{\gamma}_{3n+2} = \tilde{\gamma}_3 + \tilde{\gamma}_2 \tag{87}$$

$$\tilde{\gamma}_{3n+1} + (\tilde{\beta}_{3n+1} + \beta)(\tilde{\beta}_{3n+1} + p) = -(\beta^2 + \tilde{\gamma}_3 + \tilde{\gamma}_2 + q) \tag{88}$$

$$\beta_n = \pi_3(\beta) + (\tilde{\gamma}_3 + \tilde{\gamma}_2)(\beta + p) + (\beta + \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n+2} - (p + \tilde{\beta}_{3n+4})\tilde{\gamma}_{3n+3} \tag{89}$$

for $n \geq 0$. Notice that from $(x - \tilde{\beta}_1)(x - \tilde{\beta}_0) - \tilde{\gamma}_1 = Q_2(x) = (x-a)(x-b)$, by identification of coefficients, we get

$$\tilde{\beta}_1 = a + b - \tilde{\beta}_0 = a + b - \alpha \quad , \quad \tilde{\gamma}_1 = \tilde{\beta}_0\tilde{\beta}_1 - ab = -(\alpha - a)(\alpha - b) = -\lambda\mu/\gamma^2$$

and then, since (by setting $n = 0$ in (83)) $\tilde{\beta}_2 + \tilde{\beta}_1 + \tilde{\beta}_0 + p = 0$, we find

$$\beta = \tilde{\beta}_2 = -(a + b + p) = -\left[a + b + \pi_3''(0)/2\right] \quad .$$

Also, from (88) for $n = 0$ one sees that

$$\tilde{\gamma}_3 + \tilde{\gamma}_2 = -[\beta^2 + (a + \beta)(a + p) + q] \quad . \tag{90}$$

Therefore, in this problem, we can consider as free parameters only $\tilde{\beta}_0$ and $\tilde{\gamma}_2$, i.e., α and γ. As in the proof of Theorem 2.1, for the representations of Q_{3n} and Q_{3n+1}, we can find

$$Q_{3n}(x) = \frac{(x-a)(x-b)}{(x - \tilde{\beta}_{3n})(x - \tilde{\beta}_{3n+1}) - \tilde{\gamma}_{3n+1}} \left\{ P_n(\pi_3(x)) + (x - \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n} P_{n-1}(\pi_3(x)) \right\},$$

$$Q_{3n+1}(x) = \frac{Q_{3n+2}(x) + \tilde{\gamma}_{3n+1} Q_{3n}(x)}{x - \tilde{\beta}_{3n+1}} \quad ,$$

and since (as in the deduction of (53))

$$(x - \tilde{\beta}_{3n})(x - \tilde{\beta}_{3n+1}) - \tilde{\gamma}_{3n+1} = (x-a)(x-b) \tag{91}$$

the above relations reduce to

$$Q_{3n}(x) = P_n(\pi_3(x)) + (x - \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n} P_{n-1}(\pi_3(x)) \, , \tag{92}$$

$$Q_{3n+1}(x) = (x - \tilde{\beta}_{3n}) P_n(\pi_3(x)) + \tilde{\gamma}_{3n}\tilde{\gamma}_{3n+1} P_{n-1}(\pi_3(x)) \, . \tag{93}$$

If we compare with the proof of Theorem 2.1, we see that there is a difference between the cofresponding situations. In fact, in this case the right-hand sides of (92) and (93) do not appear divided by a polynomial and then we can't deduce any regularity condition immediately from (92) or (93). So, we will work directly with the difference equations. First, notice that if we replace in $Q_{3n+3} = (x - \beta)Q_{3n+2} - \tilde{\gamma}_{3n+2}Q_{3n+1}$ the expressions for $Q_{3n+2} = (x-a)(x-b)P_n(\pi_3(x))$, Q_{3n+3} and Q_{3n+1} given by (92) and (93), we deduce

$$P_{n+1}(\pi_3(x)) = \left[(x - \beta)(x-a)(x-b) - (x - \tilde{\beta}_{3n+4})\tilde{\gamma}_{3n+3} - (x - \tilde{\beta}_{3n})\tilde{\gamma}_{3n+2}\right] P_n(\pi_3(x)) \\ -\gamma_n P_{n-1}(\pi_3(x))$$

from which, by comparison with the three-term recurrence relation (2), one obtains

$$\pi_3(x) - \beta_n = (x - \beta)(x-a)(x-b) - (x - \tilde{\beta}_{3n+4})\tilde{\gamma}_{3n+3} - (x - \tilde{\beta}_{3n})\tilde{\gamma}_{3n+2} \quad , \quad n \geq 0 \quad . \tag{94}$$

Setting $x = a$ and $x = b$ in this relation and according to (86), we find

$$\beta_n = c + (a - \tilde{\beta}_{3n+4})\tilde{\gamma}_{3n+3} - (b - \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n+2} \quad , \tag{95}$$

$$\beta_n = d + (b - \tilde{\beta}_{3n+4})\tilde{\gamma}_{3n+3} - (a - \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n+2} \quad . \tag{96}$$

From (91) we see that $a \neq \tilde{\beta}_{3n}$, $a \neq \tilde{\beta}_{3n+1}$, $b \neq \tilde{\beta}_{3n}$ and $b \neq \tilde{\beta}_{3n+1}$ for all n. Now, from (91) for $x = a$ and using again (86), we deduce

$$b - \tilde{\beta}_{3n+1} = -\frac{\tilde{\gamma}_{3n+1}}{a - \tilde{\beta}_{3n+1}} \quad , \tag{97}$$

and substituting in (95), we obtain

$$\beta_n = c + (a - \tilde{\beta}_{3n+4})\tilde{\gamma}_{3n+3} + \frac{\tilde{\gamma}_{3n+1}\tilde{\gamma}_{3n+2}}{a - \tilde{\beta}_{3n+1}}, \quad n \geq 0 . \tag{98}$$

So, from (84),

$$\beta_n = c + (a - \tilde{\beta}_{3n+4})\tilde{\gamma}_{3n+3} + \frac{\gamma_n}{(a - \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n}} \quad , \quad n \geq 1 \quad . \tag{99}$$

Now, introducing a sequence of auxiliary parameters, $\{\theta_n\}_{n\geq 1}$,

$$\theta_n := -(a - \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n} \quad , \quad n \geq 1 \tag{100}$$

it follows that $\theta_n \neq 0$ for all $n = 1, 2, \ldots$ and (99) can be rewritten as

$$\beta_n + \theta_{n+1} + \frac{\gamma_n}{\theta_n} = c \quad , \quad n \geq 1 \quad . \tag{101}$$

To get the solution of this nonlinear difference equation we introduce the sequence of parameters $\{y_n\}_{n\geq 0}$

$$y_0 := 1 \quad , \quad y_{n+1} := \theta_{n+1} y_n \quad , \quad n \geq 0 \tag{102}$$

so that

$$y_n \neq 0 \quad , \quad n \geq 0 \quad . \tag{103}$$

Therefore, from (101) and (102) we get

$$y_{n+1} = (c - \beta_n) y_n - \gamma_n y_{n-1} \quad , \quad n \geq 1 \quad . \tag{104}$$

For y_1, we deduce

$$y_1 = \theta_1 = -(a - \tilde{\beta}_4)\tilde{\gamma}_3 = c - \beta_0 + \tilde{\gamma}_2(a - \tilde{\beta}_0)$$

where the last equality can be justified by using (98) and (97) for $n = 0$ and (86), so that

$$y_1 = c - (\beta_0 + \lambda) \quad , \quad \lambda := -\tilde{\gamma}_2(a - \tilde{\beta}_0) = -\gamma(a - \alpha) \quad . \tag{105}$$

We notice that $\lambda \neq 0$. It follows from (104) and (105) that

$$y_n = P_n^\lambda(c) \equiv P_n(c) - \lambda P_{n-1}^{(1)}(c) \quad , \quad n \geq 0 \quad . \tag{106}$$

Hence, the condition $P_n^\lambda(c) \neq 0$ holds for all $n \geq 0$. Moreover, since $\theta_n = y_n / y_{n-1} = P_n^\lambda(c) / P_{n-1}^\lambda(c)$ for $n \geq 1$, we find from (100) the relation

$$(a - \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n} = -\frac{P_n^\lambda(c)}{P_{n-1}^\lambda(c)} \quad , \quad n \geq 1 \quad . \tag{107}$$

Notice that if we start again with (91), but now taking $x = b$, and change the roles of a and b in the previous reasoning, we also derive $P_n^{\mu}(d) \neq 0$ for all $n \geq 0$ and

$$(b - \tilde{\beta}_{3n+1})\tilde{\gamma}_{3n} = -\frac{P_n^{\mu}(d)}{P_{n-1}^{\mu}(d)} \quad , \quad n \geq 1 \quad . \tag{108}$$

Subtracting (107) from (108), we get

$$(a-b)\tilde{\gamma}_{3n} = \frac{P_{n-1}^{\lambda}(c)P_n^{\mu}(d) - P_n^{\lambda}(c)P_{n-1}^{\mu}(d)}{P_{n-1}^{\lambda}(c)P_{n-1}^{\mu}(d)} = \frac{D_{n-1}^{\lambda,\mu}(c,d)}{P_{n-1}^{\lambda}(c)P_{n-1}^{\mu}(d)} \quad , \quad n \geq 1 \, . \tag{109}$$

Now, notice that, taking into account (94) and (87), the equality

$$\pi_3(x) - \pi_3(y) = (x-y)[(x-a)(x-b) + (x+y-a-b)(y-\beta) - (\tilde{\gamma}_3 + \tilde{\gamma}_2)] \tag{110}$$

holds, from which it follows that

$$\zeta := \tilde{\gamma}_3 + \tilde{\gamma}_2 = \begin{cases} -(d-c)/(b-a) & \text{if} \quad b \neq a \\ -\pi_3'(a) & \text{if} \quad b = a \end{cases} \, . \tag{111}$$

Hence, we distinguish two cases:

CASE 1: $b \neq a$. We conclude from (109) that the conditions $D_n^{\lambda,\mu}(c,d) \neq 0$ hold for $n \geq 0$, i.e., $d_n^{\lambda,\mu}(c,d) \neq 0$ for $n \geq 0$. Then (109) also gives the expression for $\tilde{\gamma}_{3n}$, and following the same steps as in the proof of Theorem 2.1, it is easy to show that Q_{3n+3} and Q_{3n+4}, as well as the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$, are given by the expressions indicated in the Theorem.

CASE 2: $b = a$. Then (109) reduces to a trivial equality. From (107), we have

$$P_{n+1}^{\lambda}(c) + (a - \tilde{\beta}_{3n+4})\tilde{\gamma}_{3n+3}P_n^{\lambda}(c) = 0 \quad , \quad n \geq 0 \, . \tag{112}$$

Thus, multiplying by $a - \tilde{\beta}_{3n+3}$ $(= \tilde{\gamma}_{3n+4}/(a - \tilde{\beta}_{3n+4})$, according to (91)),

$$(a - \tilde{\beta}_{3n+3})P_{n+1}^{\lambda}(c) + \tilde{\gamma}_{3n+3}\tilde{\gamma}_{3n+4}P_n^{\lambda}(c) = 0 \quad , \quad n \geq 0 \, . \tag{113}$$

Therefore, by (113), (86) and (112), we have

$$\frac{P_n^{\lambda}(c)}{P_{n+1}^{\lambda}(c)}\tilde{\gamma}_{3n+3}\tilde{\gamma}_{3n+4} = -(a - \tilde{\beta}_{3n+3}) = a - \tilde{\beta}_{3n+4} = -\frac{P_{n+1}^{\lambda}(c)}{P_n^{\lambda}(c)}\frac{1}{\tilde{\gamma}_{3n+3}}$$

so that, taking into account (84) and (87),

$$\frac{P_n^{\lambda}(c)}{P_{n+1}^{\lambda}(c)}\frac{\gamma_{n+1}}{\zeta - \tilde{\gamma}_{3n+6}} = -\frac{P_{n+1}^{\lambda}(c)}{P_n^{\lambda}(c)}\frac{1}{\tilde{\gamma}_{3n+3}} \quad , \quad n \geq 0 \tag{114}$$

where $\zeta \equiv \tilde{\gamma}_3 + \tilde{\gamma}_2 = -\pi_3'(a)$, according to (111). Now, we analyze the following two sub-cases:

SUB-CASE 2.1: $\zeta = 0$. It follows from (114) that $\tilde{\gamma}_{3n+6} = \gamma_{n+1}[P_n^{\lambda}(c)/P_{n+1}^{\lambda}(c)]^2\tilde{\gamma}_{3n+3}$ and by iteration of this equality we get $\tilde{\gamma}_{3n+6} = \{\gamma_1\gamma_2 \cdots \gamma_{n+1}/[P_{n+1}^{\lambda}(c)]^2\}\tilde{\gamma}_3$, so that, since in this sub-case $\tilde{\gamma}_3 = -\tilde{\gamma}_2 = -\gamma$ and taking into account (9),

$$\tilde{\gamma}_{3n+3} = -\frac{\gamma}{[P_n^{\lambda}(c)]^2}\prod_{i=1}^{n}\gamma_i = \frac{D_n^{0,\gamma}(c,c)}{[P_n^{\lambda}(c)]^2} \quad , \quad n \geq 0 \, .$$

From this we can easily conclude the proof of the Theorem in this sub-case 2.1, i.e., $b = a$ and $\pi_3'(a) = 0$. This also finish the proof of part (i) of the Theorem.

SUB-CASE 2.2: $\zeta \neq 0$. Introduce a sequence of auxiliary parameters, $\{\xi_n\}_{n\geq 0}$

$$\xi_n := -\frac{1}{\gamma_{n+1}} \frac{P_{n+1}^{\lambda}(c)}{P_n^{\lambda}(c)} \frac{1}{\tilde{\gamma}_{3n+3}}, \quad n \geq 0. \tag{115}$$

Then we have $\xi_n \neq 0$ for all $n \geq 0$ and (114) can be rewritten as

$$-\frac{1}{\gamma_{n+2}} \frac{P_{n+2}^{\lambda}(c)}{P_{n+1}^{\lambda}(c)} \frac{\xi_n}{\xi_{n+1}} = \zeta\xi_n - \frac{P_n^{\lambda}(c)}{P_{n+1}^{\lambda}(c)}, \quad n \geq 0. \tag{116}$$

Introduce another sequence of auxiliary parameters, $\{\eta_n\}_{n\geq 0}$, by

$$\eta_n := -\gamma_{n+1} \left[\zeta\xi_n - \frac{P_n^{\lambda}(c)}{P_{n+1}^{\lambda}(c)}\right], \quad n \geq 0. \tag{117}$$

Then, by (116) also $\eta_n \neq 0$ for all $n \geq 0$, and substituting in (116) the expressions of ξ_n and ξ_{n+1} given by (117) – notice that $\zeta \neq 0$ – we find, after multiplying by ζ,

$$-\frac{\gamma_{n+2} P_{n+1}^{\lambda}(c)}{P_{n+2}^{\lambda}(c)} + \eta_{n+1} = -\gamma_{n+1} \frac{P_{n+2}^{\lambda}(c) P_n^{\lambda}(c)}{[P_{n+1}^{\lambda}(c)]^2} \frac{1}{\eta_n} + \frac{P_{n+2}^{\lambda}(c)}{P_{n+1}^{\lambda}(c)}.$$

Thus, taking into account that $\gamma_{n+2} P_{n+1}^{\lambda}(c) = -P_{n+3}^{\lambda}(c) + (c - \beta_{n+2}) P_{n+2}^{\lambda}(c)$, as well as (4) with $P_i(c)$ replaced by $P_i^{\lambda}(c)$,

$$\eta_{n+1} + \beta_{n+1}^{\lambda*} + \frac{\gamma_{n+1}^{\lambda*}}{\eta_n} = c, \quad n \geq 0, \tag{118}$$

where $\{\beta_n^{\lambda*}\}_{n\geq 0}$ and $\{\gamma_{n+1}^{\lambda*}\}_{n\geq 0}$ denote the sequences of the coefficients which appear in the three-term recurrence relation for the MOPS of the kernel polynomials of K-parameter c, $\{[P_n^{\lambda}]^*(c;.)\}_{n\geq 0}$, corresponding to the sequence $\{P_n^{\lambda}\}_{n\geq 0}$. As indicated in the Theorem, we will denote $R_n(x) := [P_n^{\lambda}]^*(c;x)$ for $n \geq 0$. Notice that $\{R_n\}_{n\geq 0}$ is in fact an MOPS, since we know that $P_n^{\lambda}(c) \neq 0$ holds for all $n \geq 0$. In order to find the solution of the difference equation (118) we consider the sequence $\{z_n\}_{n\geq 0}$

$$z_0 := 1 \quad , \quad z_{n+1} := \eta_n z_n \quad , \quad n \geq 0.$$

Then $z_n \neq 0$ for all $n \geq 0$ and, by (118),

$$z_{n+1} = (c - \beta_n^{\lambda*}) z_n - \gamma_n^{\lambda*} z_{n-1} \quad , \quad n \geq 1.$$

For z_1, we have:

$$\begin{aligned} z_1 &= \eta_0 = -\gamma_1 \left[\zeta\xi_0 - \frac{P_0^{\lambda}(c)}{P_1^{\lambda}(c)}\right] = -\gamma_1 \left[-\frac{\zeta}{\gamma_1} \frac{P_1^{\lambda}(c)}{P_0^{\lambda}(c)} \frac{1}{\tilde{\gamma}_3} - \frac{1}{P_1^{\lambda}(c)}\right] \\ &= \frac{\zeta P_1^{\lambda}(c)}{\tilde{\gamma}_3} + \frac{\gamma_1}{P_1^{\lambda}(c)} = \frac{\zeta P_1^{\lambda}(c)}{\tilde{\gamma}_3} + \beta_0^{\lambda} - \beta_0^{\lambda*} = c - \beta_0^{\lambda*} + \frac{\zeta P_1^{\lambda}(c)}{\tilde{\gamma}_3} - P_1^{\lambda}(c) \\ &= c - \beta_0^{\lambda*} + \frac{\zeta - \tilde{\gamma}_3}{\tilde{\gamma}_3} P_1^{\lambda}(c) = c - (\beta_0^{\lambda*} + \nu), \end{aligned}$$

with $\nu := -\gamma P_1^{\lambda}(c)/(\zeta - \gamma) = \gamma P_1^{\lambda}(c)/(\pi_3'(a) + \gamma)$. Therefore, we conclude that

$$z_n = R_n^{\nu}(c) \quad , \quad n \geq 0,$$

and then $R_n^\nu(c) \neq 0$ for all $n \geq 0$. Consequently, going to back, we deduce successively

$$\frac{R_{n+1}^\nu(c)}{R_n^\nu(c)} = \eta_n = -\gamma_{n+1}\left[\zeta\xi_n - \frac{P_n^\lambda(c)}{P_{n+1}^\lambda(c)}\right] = \frac{\gamma_{n+1}}{\gamma_{n+2}}\frac{P_{n+2}^\lambda(c)}{P_{n+1}^\lambda(c)}\frac{\xi_n}{\xi_{n+1}} = \frac{P_{n+1}^\lambda(c)}{P_n^\lambda(c)}\frac{\tilde{\gamma}_{3n+6}}{\tilde{\gamma}_{3n+3}},$$

hence, $P_{n+1}^\lambda(c)\tilde{\gamma}_{3n+6}/R_{n+1}^\nu(c) = P_n^\lambda(c)\tilde{\gamma}_{3n+3}/R_n^\nu(c)$ $(n \geq 0)$, which leads to

$$\tilde{\gamma}_{3n+3} = (\zeta - \gamma)\frac{R_n^\nu(c)}{P_n^\lambda(c)} = -\delta\frac{R_n^\nu(c)}{P_n^\lambda(c)}, \quad n \geq 0,$$

where $\delta := \pi_3'(a) + \gamma$. From this we easily conclude the proof of part (ii) of the Theorem.

Finally, we show that $\{Q_n\}_{n\geq 0}$ is an MOPS with respect to the linear functional $\mathbf{v}$ defined by (34). First, notice that (34) implies

$$\begin{aligned}
\langle \mathbf{v}, (x-b)f(\pi_3(x))\rangle &= \frac{\mu}{\gamma}\langle \mathbf{u}^{\lambda,c}, f(x)\rangle, \\
\langle \mathbf{v}, (x-a)(x-b)f(\pi_3(x))\rangle &= 0, \\
\langle \mathbf{v}, (x-a)(x-b)^2 f(\pi_3(x))\rangle &= 0
\end{aligned}$$

for every polynomial f. Since $\langle \mathbf{v}, 1\rangle = u_0 \neq 0$, to prove that $\{Q_n\}_{n\geq 0}$ is an MOPS with respect to $\mathbf{v}$, we only need to show that

$$\langle \mathbf{v}, Q_n(x)\rangle = 0 \quad , \quad n = 1, 2, \ldots.$$

In fact, it is obvious that

$$\langle \mathbf{v}, Q_{3n+2}(x)\rangle = \langle \mathbf{v}, (x-a)(x-b)P_n(\pi_3(x))\rangle = 0 \quad , \quad n \geq 0.$$

Since $Q_{3n+3}(x) = (x-\beta)Q_{3n+2}(x) - \tilde{\gamma}_{3n+2}Q_{3n+1}(x) = (x-a)(x-b)^2 P_n(\pi_3(x)) + (b-\beta)(x-a)(x-b)P_n(\pi_3(x)) - \tilde{\gamma}_{3n+2}Q_{3n+1}(x)$, it follows that

$$\langle \mathbf{v}, Q_{3n+3}(x)\rangle = -\tilde{\gamma}_{3n+2}\langle \mathbf{v}, Q_{3n+1}(x)\rangle \quad , \quad n \geq 0. \tag{119}$$

Now, notice that, from the expressions for $Q_{3n+3}(x)$ and $\tilde{\gamma}_{3n+3}$ (in the Theorem), and taking into account (11), we can write

$$Q_{3n+3}(x) = P_{n+1}^{\lambda,c}(\pi_3(x)) + \tilde{\gamma}_{3n+3}(x-a)P_n(\pi_3(x)) \quad , \quad n \geq 0$$

(in both cases (i) and (ii)). Hence, for $n \geq 1$,

$$\begin{aligned}
Q_{3n+1}(x) &= (x-b)Q_{3n}(x) + (b-\tilde{\beta}_{3n})Q_{3n}(x) - \tilde{\gamma}_{3n}Q_{3n-1}(x) \\
&= (x-b)P_n^{\lambda,c}(\pi_3(x)) + (b-\tilde{\beta}_{3n})Q_{3n}(x)
\end{aligned}$$

and thus, since $\langle \mathbf{v}, (x-b)P_n^{\lambda,c}(\pi_3(x))\rangle = \frac{\mu}{\gamma}\langle \mathbf{u}^{\lambda,c}, P_n^{\lambda,c}(x)\rangle = 0$ for $n \geq 1$, we find $\langle \mathbf{v}, Q_{3n+1}\rangle = (b-\tilde{\beta}_{3n})\langle \mathbf{v}, Q_{3n}\rangle$, or, according to (119),

$$\langle \mathbf{v}, Q_{3n+1}\rangle = -\tilde{\gamma}_{3n-1}(b-\tilde{\beta}_{3n})\langle \mathbf{v}, Q_{3n-2}\rangle \quad , \quad n \geq 1.$$

Therefore, since

$$\begin{aligned}
\langle \mathbf{v}, Q_1\rangle &= \langle \mathbf{v}, x-\alpha\rangle = \langle \mathbf{v}, C_1(x)\rangle + (b-\alpha)v_0 = \frac{\mu}{\gamma}\langle \mathbf{u}^{\lambda,c}, 1\rangle + (b-\alpha)u_0 \\
&= (\frac{\mu}{\gamma} + b - \alpha)u_0 = 0,
\end{aligned}$$

it follows recurrently that

$$\langle \mathrm{v}, Q_{3n+1}\rangle = 0 \quad , \quad n \geq 0 .$$

This relation, together with (119) leads to

$$\langle \mathrm{v}, Q_{3n+3}\rangle = 0 \quad , \quad n \geq 0 ,$$

which completes the proof of Theorem 2.3.

3.4 Proof of Corollary 2.3

We have

$$Q^{(1)}_{3n+1}(x) = \frac{1}{v_0}\langle \mathrm{v}_y, \frac{(x-a)(x-b)P_n(\pi_3(x)) - (y-a)(y-b)P_n(\pi_3(y))}{x-y}\rangle . \qquad (120)$$

Put $P_n(x) \equiv \sum_{i=0}^{n} a_i^{(n)} x^i$, so that (74) and (75) hold. So, taking into account (110) and (75), we deduce

$$\begin{aligned} &P_n(\pi_3(x)) - P_n(\pi_3(y)) = \\ &= (x-y)[(x-a)(x-b) + (x+y-a-b)(y-\beta) - (\tilde{\gamma}_3+\tilde{\gamma}_2)] \sum_{i=0}^{n-1}\sum_{j=0}^{i} a_{i+1}^{(n)} \pi_3^{i-j}(x)\pi_3^j(y) . \end{aligned}$$

Since

$$\begin{aligned} &(x-a)(x-b)P_n(\pi_3(x)) - (y-a)(y-b)P_n(\pi_3(y)) = \\ &\quad = (x-y)(x+y-a-b)P_n(\pi_3(x)) + (y-a)(y-b)\,[P_n(\pi_3(x)) - P_n(\pi_3(y))] \quad , \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{(x-a)(x-b)P_n(\pi_3(x)) - (y-a)(y-b)P_n(\pi_3(y))}{x-y} = \\ &= (x+y-a-b)P_n(\pi_3(x)) + (y-a)(y-b)\times \\ &\quad \times[(x-a)(x-b) + (x+y-a-b)(y-\beta) - (\tilde{\gamma}_3+\tilde{\gamma}_2)] \sum_{i=0}^{n-1}\sum_{j=0}^{i} a_{i+1}^{(n)} \pi_3^{i-j}(x)\pi_3^j(y) . \end{aligned}$$

Therefore, from (120) and $\langle \mathrm{v}_y, (y-a)(y-b)\pi_3^j(y)\rangle = 0$, we find

$$\begin{aligned} Q^{(1)}_{3n+1}(x) &= \frac{1}{v_0}\langle \mathrm{v}_y, x+y-a-b)P_n(\pi_3(x)) + \\ &\quad + \frac{1}{v_0}\sum_{i=0}^{n-1}\sum_{j=0}^{i} a_{i+1}^{(n)}\pi_3^{i-j}(x)\langle \mathrm{v}_y, (x+y-a-b)(y-\beta)(y-a)(y-b)\pi_3^j(y)\rangle \\ &= (x-a-b+\alpha)P_n(\pi_3(x)) + \\ &\quad + \frac{1}{v_0}\sum_{i=0}^{n-1}\sum_{j=0}^{i} a_{i+1}^{(n)}\pi_3^{i-j}(x)\langle \mathrm{v}, (y-b)(y-\beta)(y-a)(y-b)\pi_3^j(y)\rangle . \end{aligned}$$

By (110) for $x = a$, we get $(y-\beta)(y-a)(y-b) = \pi_3(y) - c + (\tilde{\gamma}_3+\tilde{\gamma}_2)(y-a)$, so that

$$\begin{aligned} &\langle \mathrm{v}, (y-b)(y-\beta)(y-a)(y-b)\pi_3^j(y)\rangle = \\ &= \langle \mathrm{v}, (y-b)[\pi_3(y) - c]\pi_3^j(y)\rangle = \tfrac{\mu}{\gamma}\langle \mathrm{u}^{\lambda,c}, (y-c)y^j\rangle = -\tfrac{\lambda\mu}{\gamma}\langle \mathrm{u}, y^j\rangle . \end{aligned}$$

Consequently,

$$
\begin{aligned}
Q^{(1)}_{3n+1}(x) &= (x-a-b+\alpha)P_n(\pi_3(x)) - \frac{\lambda\mu}{\gamma}\frac{1}{u_0}\langle \mathrm{u}_y, \sum_{i=0}^{n-1}\sum_{j=0}^{i} a^{(n)}_{i+1}\pi_3^{i-j}(x)y^j\rangle \\
&= (x-a-b+\alpha)P_n(\pi_3(x)) - \frac{\lambda\mu}{\gamma}P^{(1)}_{n-1}(\pi_3(x)).
\end{aligned}
$$

Using (110), it is straightforward to verify that

$$
\begin{aligned}
\tfrac{(z-a)(z-b)}{x-z} &= (z-a)(z-b)\tfrac{(x-a)(x-b)}{\pi_3(x)-\pi_3(z)} - (z-a-b+\alpha) - (x-\alpha)+ \\
&+ \tfrac{[\pi_3(x)-c](x-b)}{\pi_3(x)-\pi_3(z)} + (z-a)\tfrac{(x-a)(x-b)^2+(b-\beta)(x-a)(x-b)}{\pi_3(x)-\pi_3(z)}.
\end{aligned}
$$

Therefore, we deduce

$$
\begin{aligned}
(z-a)(z-b)S_{\mathrm{v}}(z) &= \langle \mathrm{v}_x, \frac{(z-a)(z-b)}{x-z}\rangle \\
&= -(z-a)(z-b)\sum_{n\geq 0}\frac{\langle \mathrm{v}, C_{3n+2}\rangle}{\pi_3^{n+1}(z)} - u_0(z-a-b+\alpha) - \\
&\quad -\langle \mathrm{v}, x-\alpha\rangle - \sum_{n\geq 0}\frac{\langle \mathrm{v}, (x-b)[\pi_3(x)-c]\pi_3{}^n(x)\rangle}{\pi_3^{n+1}(z)} - \\
&\quad -(z-a)\sum_{n\geq 0}\frac{\langle \mathrm{v}, C_{3n+3}\rangle + (b-\beta)\langle \mathrm{v}, C_{3n+2}\rangle}{\pi_3^{n+1}(z)} \\
&= -u_0(z-a-b+\alpha) + \frac{\lambda\mu}{\gamma}\sum_{n\geq 0}\frac{\langle \mathrm{u}, x^n\rangle}{\pi_3^{n+1}(z)} \\
&= -u_0(z-a-b+\alpha) - \frac{\lambda\mu}{\gamma}S_{\mathrm{u}}(\pi_3(z)) \quad ,
\end{aligned}
$$

which proves (36).

4 The eigenvalues of a tridiagonal 3-Toeplitz matrix

Theorem 2.3 can be applied to the determination of the eigenvalues of a finite tridiagonal 3-Toeplitz matrix, which as the general form

$$
B_n = \begin{bmatrix}
a_1 & b_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
c_1 & a_2 & b_2 & 0 & 0 & 0 & 0 & \cdots \\
0 & c_2 & a_3 & b_3 & 0 & 0 & 0 & \cdots \\
0 & 0 & c_3 & a_1 & b_1 & 0 & 0 & \cdots \\
0 & 0 & 0 & c_1 & a_2 & b_2 & 0 & \cdots \\
0 & 0 & 0 & 0 & c_2 & a_3 & b_3 & \cdots \\
0 & 0 & 0 & 0 & 0 & c_3 & a_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \in \mathbb{R}^{(n,n)}. \qquad (121)
$$

It is assumed that B_n is irreducible, i.e., $b_i c_i \neq 0$ for $i = 1, 2, 3$. We mention that the corresponding problem for a finite tridiagonal 2-Toeplitz matrix was solved by M.J.C.GOVER [10] (see also [12] as well as [7] where a more general situation for band matrices is considered). Given B_n, define recurrently a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ by (2), with

$$\beta_n = b_1c_1 + b_2c_2 + b_3c_3 \quad , \quad \gamma_{n+1} = b_1b_2b_3c_1c_2c_3 \quad , \quad n \geq 0 .$$

According to the Favard theorem (see [5, p.21]), $\{P_n\}_{n\geq 0}$ is an MOPS. Furthermore, since the coefficients β_n and γ_n are constant, then $P_n \equiv P_n^{(1)}$, so that $P_n^\lambda(x) = P_n(x) - \lambda P_{n-1}(x)$. Furthermore, one easily see that

$$P_n(x) = \left(2\sqrt{b_1b_2b_3c_1c_2c_3}\right)^n \widehat{U}_n\left(\frac{x - b_1c_1 - b_2c_2 - b_3c_3}{2\sqrt{b_1b_2b_3c_1c_2c_3}}\right) \quad , \quad n \geq 0$$

where $\{\widehat{U}_n\}_{n\geq 0}$ are the monic Tchebyshev polynomials of second kind, $\widehat{U}_n(x) = 2^{-n}U_n(x)$, and

$$U_n(x) := \frac{\sin(n+1)\theta}{\sin\theta} \, , \; x = \cos\theta \, .$$

The polynomials $\{\widehat{U}_n\}_{n\geq 0}$ satisfy the three-term recurrence relation $x\widehat{U}_n(x) = \widehat{U}_{n+1}(x) + \frac{1}{4}\widehat{U}_{n-1}(x)$ $(n \geq 0)$, $\widehat{U}_{-1}(x) = 0$, $\widehat{U}_0(x) = 1$. Remark that the zeros of $\widehat{U}_n$ are $x_{n,k} = \cos\frac{k\pi}{n+1}$, $k = 1, 2, \ldots, n$. Motivated by (91) and (94), let a and b be the zeros of the quadratic polynomial

$$(x - a_1)(x - a_2) - b_1c_1 \tag{122}$$

and define

$$\pi_3(x) := \begin{vmatrix} x - a_1 & 1 & 1 \\ b_1c_1 & x - a_2 & 1 \\ b_3c_3 & b_2c_2 & x - a_3 \end{vmatrix} . \tag{123}$$

Choose $\alpha := a_1$ and $\gamma := b_2c_2$. Now, according to Theorem 2.3, let $c := \pi_3(a)$, $d := \pi_3(b)$, $\lambda := -b_2c_2(a - a_1)$ and $\mu := -b_2c_2(b - a_1)$. With these notations, using induction and the three-term recurrence relation (6), it is straightforward to prove that

$$P_n^\lambda(c) = [b_3c_3(a_2 - a)]^n \neq 0 , \quad P_n^\mu(d) = [b_3c_3(a_2 - b)]^n \neq 0 ,$$
$$D_n^{\lambda,\mu}(c, d) = (a - b)(b_3c_3)^{2n+1}(-b_1c_1)^n \neq 0 \quad \text{if} \quad b \neq a ,$$
$$D_n^{0,\gamma}(c, c) = -b_2c_2(b_1c_1b_2c_2b_3c_3)^n$$

for all $n = 0, 1, 2, \ldots$. (We can infer the first two formulas from (107) and (108)). Furthermore, if $a = b \equiv (a_1 + a_2)/2$ then $b_1c_1 = -(a_1 - a_2)^2/4$ and $\pi'(a) = -(b_2c_2 + b_3c_3)$. Hence if $a = b$ and $\pi'(a) \neq 0$, we have $\delta := \gamma + \pi'(a) = -b_3c_3$, $\nu := \gamma P_1^\lambda(c)/\delta = b_2c_2(a_1 - a_2)/2 \equiv \lambda$. The polynomial R_n defined in (ii) on Theorem 2.3 coincides with P_n, so that

$$R_n^\nu(c) = P_n^\lambda(c) = [b_3c_3(a_2 - a_1)/2]^n \neq 0$$

for all $n = 0, 1, 2, \ldots$. Therefore, if we define a sequence of polynomials $\{Q_n\}_{n\geq 0}$ such that

$$\begin{aligned} Q_{3n}(x) &:= P_n(\pi_3(x)) + b_3c_3(x - a_2)P_{n-1}(\pi_3(x)) \\ Q_{3n+1}(x) &:= (x - a_1)P_n(\pi_3(x)) + b_1c_1b_3c_3P_{n-1}(\pi_3(x)) \\ Q_{3n+2}(x) &:= (x - a)(x - b)P_n(\pi_3(x)) , \end{aligned} \tag{124}$$

then Theorem 2.3 ensures that $\{Q_n\}_{n\geq 0}$ is an MOPS and the parameters for the corresponding three-term recurrence relation are

$$\tilde{\beta}_{3n} = a_1 \, , \; \tilde{\beta}_{3n+1} = a_2 \, , \; \tilde{\beta}_{3n+2} = a_3 \, , \; \tilde{\gamma}_{3n+1} = b_1c_1 \, , \; \tilde{\gamma}_{3n+2} = b_2c_2 \, , \; \tilde{\gamma}_{3n+3} = b_3c_3 \, .$$

It follows from a well known result of the theory of tridiagonal matrices that the eigenvalues of B_n coincide with the eigenvalues of the triadiagonal Jacobi matrix of order n, $[\alpha_{ij}]_{i,j=1}^n$, corresponding to the MOPS $\{Q_n\}_{n\geq 0}$ – which, by definition, has entries $\alpha_{ii} := \tilde{\beta}_{i-1}$, $\alpha_{i,i+1} := 1$, $\alpha_{i+1,i} := \tilde{\gamma}_i$. Hence, the eigenvalues of B_n are the zeros of Q_n, and the following proposition follows:

Theorem 4.1 *Let B_n be an irreducible tridiagonal 3-Toeplitz matrix, given by* (121). *Define a polynomial π_3 as* (123) *and denote a and b as in* (122). *For a fixed $n \geq 0$,*
(i) *the eigenvalues of B_n are the zeros of Q_n defined according to relations* (124)*;*
(ii) *in particular, the eigenvalues of B_{3n+2} are a, b and the roots of the cubic equations*

$$\pi_3(x) = b_1c_1 + b_2c_2 + b_3c_3 + 2\sqrt{b_1b_2b_3c_1c_2c_3}\cos\frac{k\pi}{n+1} \quad , \quad k = 1, \ldots, n.$$

5 Second order linear differential equation

According to Corollary 2.4, we know that if $\{P_n\}_{n\geq 0}$ is a semiclassical MOPS, then so is $\{Q_n\}_{n\geq 0}$, in any of the above three problems. In particular, this means that if the polynomials of the sequence $\{P_n\}_{n\geq 0}$ satisfy a second order differential equation, say

$$J_n(x)y'' + K_n(x)y' + L_n(x)y = 0 \quad (y = P_n(x)) \tag{125}$$

($n = 0, 1, 2, \ldots$) where J_n, K_n and L_n are polynomials in the variable x, whose coefficients depend on n but whose degrees are uniformly bounded by a positive integer number independent of n, then the polynomials of the sequence $\{Q_n\}_{n\geq 0}$ also satisfy a second order differential equation of the same type, say

$$\tilde{J}_n(x)y'' + \tilde{K}_n(x)y' + \tilde{L}_n(x)y = 0 \quad (y = Q_n(x)) \tag{126}$$

($n = 0, 1, 2, \ldots$) with $\tilde{J}_n$, $\tilde{K}_n$ and $\tilde{L}_n$ polynomials whose coefficients can be dependent on n but whose degrees are also uniformly bounded by a number independent of n.

In fact, if $\{P_n\}_{n\geq 0}$ is semiclassical of class s, so that the polynomials ϕ, C and D which appear in the first order linear differential equation satisfied by the Stieltjes function associated to $\mathbf{u}$ (see Corollary 2.4) are co-prime and $s = \max\{\deg C - 1\,,\, \deg D\}$, then

$$\deg J_n \leq 2s + 2\,, \quad \deg K_n \leq 2s + 1\,, \quad \deg L_n \leq 2s\,,$$

and analogous upper bounds can be given for the degrees of the polynomials $\tilde{J}_n$, $\tilde{K}_n$ and $\tilde{L}_n$ which appear on equation (126) in terms of the class $\tilde{s}$ of $\{Q_n\}_{n\geq 0}$.

We want to determine the $\tilde{J}_n$'s, $\tilde{K}_n$'s and $\tilde{L}_n$'s in terms of the J_n's, K_n's and L_n's. We will do that only for problem P3 (for the others it is similar).

In order to obtain the differential equation (125) for a given semiclassical MOPS $\{P_n\}_{n\geq 0}$, an important tool is the so called structure relation

$$\phi(x)P_n'(x) = M_n(x)P_{n+1}(x) + N_n(x)P_n(x) \tag{127}$$

($n = 0, 1, 2, \ldots$), where M_n and N_n are also polynomials in the variable x, whose coefficients can be dependent on n but whose degrees are uniformly bounded by a number independent of n. In fact,

$$\deg M_n \leq s\,, \quad \deg N_n \leq s + 1\,.$$

The structure relation (127) is also a characteristic property for a semiclassical OPS, and if we know the above polynomials ϕ, C and D, then the polynomials M_n and N_n in (127) can be successively deduced from the mixed recurrence relations

$$N_n = -C - N_{n-1} - (x-\beta_n)M_n \tag{128}$$

$$\gamma_{n+1}M_{n+1} = -\phi + \gamma_n M_{n-1} + (x-\beta_n)(N_{n-1} - N_n) \tag{129}$$

$(n = 0, 1, 2, \ldots)$, with initial conditions

$$N_{-1} = C\,, \quad M_{-1} = 0\,, \quad M_0 = u_0^{-1}D \tag{130}$$

$(u_0 := \langle u, 1\rangle)$. Now, the polynomials J_n, K_n and L_n in (125) can be computed by means of the relations

$$J_n = \phi M_n\,, \quad K_n = W(M_n, \phi) + CM_n\,, \quad L_n = W(N_n, M_n) - M_n \sum_{i=0}^{n} M_i \tag{131}$$

$(n = 0, 1, 2, \ldots)$, where $W(f, g) := fg' - f'g$ (the Wronskian). By combination of (128) and (129) we can give the following alternative expression for L_n:

$$L_n = W(N_n, M_n) - (\gamma_{n+1}M_nM_{n+1} - N_n^2 - CN_n)M_n/\phi\,.$$

We remark that the previous results follow from the theory of semiclassical orthogonal polynomials presented by P.Maroni in [17].

The next result gives the structure relation for the polynomials of the sequence $\{Q_n\}_{n\geq 0}$ corresponding to problem P3.

Theorem 5.1 *Under the conditions of Theorem 2.3, if the polynomials $\{P_n\}_{n\geq 0}$ satisfy the structure relation (127), then the polynomials $\{Q_n\}_{n\geq 0}$ satisfy*

$$\tilde{\phi}(x)Q_n'(x) = \tilde{M}_n(x)Q_{n+1}(x) + \tilde{N}_n(x)Q_n(x) \tag{132}$$

$(n = 0, 1, 2, \ldots)$, where $\tilde{\phi}$, $\tilde{M}_n$ and $\tilde{N}_n$ are explicitly given by

$$\begin{aligned}
\tilde{\phi}(x) &= (x-a)(x-b)\phi(\pi_3(x)) \\
\tilde{M}_{3n+1}(x) &= \phi(\pi_3(x)) \;+\; \pi_3'(x)\{\tilde{\gamma}_{3n}\tilde{\gamma}_{3n+1}M_{n-1}(\pi_3(x)) - (x-\tilde{\beta}_{3n})\times \\
&\qquad \times [C(\pi_3(x)) + 2N_{n-1}(\pi_3(x)) - \tilde{\gamma}_{3n+2}(x-\tilde{\beta}_{3n})M_n(\pi_3(x))]\} \\
\tilde{M}_{3n+2}(x) &= (x-a)^2(x-b)^2\pi_3'(x)M_n(\pi_3(x)) \\
\tilde{M}_{3n+3}(x) &= -\phi(\pi_3(x)) \;+\; \pi_3'(x)\{\tilde{\gamma}_{3n+4}\tilde{\gamma}_{3n+5}M_{n+1}(\pi_3(x)) - (x-\tilde{\beta}_{3n+4})\times \\
&\qquad \times [C(\pi_3(x)) + 2N_n(\pi_3(x)) - \tilde{\gamma}_{3n+3}(x-\tilde{\beta}_{3n+4})M_n(\pi_3(x))]\} \\
\tilde{N}_{3n+1}(x) &= (x-a)(x-b)\pi_3'(x)[N_{n-1}(\pi_3(x)) - \tilde{\gamma}_{3n+2}(x-\tilde{\beta}_{3n})M_n(\pi_3(x))] \\
\tilde{N}_{3n+2}(x) &= (2x-a-b)\phi(\pi_3(x)) + \\
&\qquad + (x-a)(x-b)\pi_3'(x)[N_n(\pi_3(x)) - \tilde{\gamma}_{3n+3}(x-\tilde{\beta}_{3n+4})M_n(\pi_3(x))] \\
\tilde{N}_{3n+3}(x) &= (x-\tilde{\beta}_{3n+3})\phi(\pi_3(x)) + \\
&\qquad + \pi_3'(x)\{\,(x-a)(x-b)[N_n(\pi_3(x)) - \tilde{\gamma}_{3n+4}(x-\tilde{\beta}_{3n+2})M_n(\pi_3(x))] + \\
&\qquad\qquad + \tilde{\gamma}_{3n+4}[N_n(\pi_3(x)) - \tilde{\gamma}_{3n+5}(x-\tilde{\beta}_{3n+3})M_{n+1}(\pi_3(x))] - \\
&\qquad\qquad - \tilde{\gamma}_{3n+4}[N_{n-1}(\pi_3(x)) - \tilde{\gamma}_{3n+2}(x-\tilde{\beta}_{3n})M_n(\pi_3(x))]\,\}\,.
\end{aligned} \tag{133}$$

Proof. Making the substitution $x \to \pi_3(x)$ in (127), then multiplying the resulting equation by $(x-a)^2(x-b)^2\pi_3'(x)$ and using the relations $Q_{3n+2}(x) = (x-a)(x-b)P_n(\pi_3(x))$ and $Q_{3n+2}'(x) = (2x-a-b)P_n(\pi_3(x)) + (x-a)(x-b)\pi_3'(x)P_n'(\pi_3(x))$, we get

$$\tilde{\phi}(x)Q_{3n+2}'(x) = (x-a)(x-b)\pi_3'(x)M_n(\pi_3(x))Q_{3n+5}(x)+ \\ +\left[(x-a)(x-b)\pi_3'(x)N_n(\pi_3(x)) + (2x-a-b)\phi(\pi_3(x))\right]Q_{3n+2}(x).$$

But, using the three-term recurrence relation for $\{Q_n\}_{n\geq 0}$ we can write, according to (91),

$$Q_{3n+5}(x) = (x-a)(x-b)Q_{3n+3}(x) - \tilde{\gamma}_{3n+3}(x-\tilde{\beta}_{3n+4})Q_{3n+2}(x)$$

and so the above equation reduces to

$$\tilde{\phi}(x)Q_{3n+2}'(x) = \tilde{M}_{3n+2}(x)Q_{3n+3}(x) + \tilde{N}_{3n+2}(x)Q_{3n+2}(x),$$

with $\tilde{M}_{3n+2}$ and $\tilde{N}_{3n+2}$ defined as in (133). Now, since we have determined $\tilde{M}_{3n+2}$ and $\tilde{N}_{3n+2}$ for all $n = 0, 1, 2, \ldots$, we can determine all the others $\tilde{M}_i$'s and $\tilde{N}_i$'s, by using the relations corresponding to (128) and (129) for the $\tilde{M}_i$'s and $\tilde{N}_i$'s. In fact, from (128) one immediately find

$$\tilde{N}_{3n+1}(x) = -[\tilde{C} + \tilde{N}_{3n+2}(x) + (x-\tilde{\beta}_{3n+2})\tilde{M}_{3n+2}(x)],$$

so that, using the expressions for $\tilde{M}_{3n+2}$ and $\tilde{N}_{3n+2}$ already determined as well as the expression for $\tilde{C}$ as in Corollary 2.4, one obtain the expression for $\tilde{N}_{3n+1}$ given in (133). Now, combining (128) and (129) one easily see that

$$\tilde{N}_{3n+3} = \frac{\tilde{\gamma}_{3n+4}(\tilde{C} + \tilde{N}_{3n+4}) + (x-\tilde{\beta}_{3n+4})[-\tilde{\phi} + \tilde{\gamma}_{3n+3}\tilde{M}_{3n+2} + (x-\tilde{\beta}_{3n+3})\tilde{N}_{3n+2}]}{(x-\tilde{\beta}_{3n+3})(x-\tilde{\beta}_{3n+4}) - \tilde{\gamma}_{3n+4}},$$

and notice again that $(x-\tilde{\beta}_{3n+3})(x-\tilde{\beta}_{3n+4}) - \tilde{\gamma}_{3n+4} = (x-a)(x-b)$. Hence, using the expressions for $\tilde{N}_{3n+4}$, $\tilde{N}_{3n+2}$ and $\tilde{M}_{3n+2}$ already calculated, after some simplifications we get the expression for $\tilde{N}_{3n+3}$ as in (133). Finally, since we know all the $\tilde{N}_i$'s, from (128) we determine the remainder $\tilde{M}_i$'s.

6 Connection with sieved orthogonal polynomials

Consider the sequence $\{C_n^\nu\}_{n\geq 0}$ of the Gegenbauer or ultraspherical polynomials, which is usually defined by the three-term recurrence relation

$$2(n+\nu)xC_n^\nu(x) = (n+1)C_{n+1}^\nu(x) + (n+2\nu-1)C_{n-1}^\nu(x), \quad n \geq 1,$$

with initial conditions $C_0^\nu(x) = 1$, $C_1^\nu(x) = 2\nu x$ (see [8, p.175]). Assuming that $\nu \neq -n/2$ ($n = 0, 1, \ldots$) one see that $\{C_n^\nu\}_{n\geq 0}$ is an orthogonal polynomial sequence. The leading coefficient of $C_n^\nu(x)$ is $2^n(\nu)_n/n!$, so one can define an MOPS $\{P_n\}_{n\geq 0}$ by

$$P_n(x) := \frac{n!}{2^{3n}(\nu+1)_n}C_n^{\nu+1}(4x), \quad n = 0, 1, 2, \ldots. \tag{134}$$

Then the coefficients of the three-term recurrence relation corresponding to $\{P_n\}_{n\geq 0}$ are given by

$$\beta_n = 0, \quad \gamma_{n+1} = \frac{(n+1)(n+2\nu+2)}{64(n+\nu+2)(n+\nu+1)}, \quad n = 0, 1, 2, \ldots.$$

(see [8, p.175], with an appropriate normalization). Hence, from formulas (133) we find, up to the factor $\frac{1}{3}U_2(x)\pi_3'(x)$,

$$\tilde{\phi}(x) = (1-x^2)U_2(x)\,, \quad \tilde{C}(x) = -x[(6\nu+1)U_2(x) - 12\nu]\,,$$
$$\tilde{M}_{3n}(x) = -[(6n+6\nu+2)U_2(x) - 6\nu]\,, \quad \tilde{N}_{3n}(x) = x[(3n+6\nu+2)U_2(x) - 6\nu]\,,$$
$$\tilde{M}_{3n+1}(x) = -[(6n+6\nu+4)U_2(x) + 6\nu]\,, \quad \tilde{N}_{3n+1}(x) = (3n+6\nu+3)xU_2(x)\,,$$
$$\tilde{M}_{3n+2}(x) = -(6n+6\nu+6)U_2(x)\,, \quad \tilde{N}_{3n+2}(x) = x[(3n+6\nu+4)U_2(x) - 12\nu]\,,$$

where $U_2(x) = 4x^2 - 1$. Now, we get the second order linear differential equation that each Q_n satisfies from the formulas for the $\tilde{J}_i$'s, $\tilde{K}_i$'s and $\tilde{L}_i$'s corresponding to (131). Notice that in [3] the authors considered the general case (not only $k=3$), but there is a mistake in the computation of the coefficients of the differential equation, as one of the authors confirmed to us in a private communication. For $k=3$, if we put

$$p_n(x) = B_n^\nu(x;3)\,,$$

we deduce (by the indicated way) that $\{p_n\}_{n\geq 0}$ satisfies

$$a_n(x)p_n''(x) + b_n(x)p_n'(x) + c_n(x)p_n(x) = 0\,, \quad n = 0,1,2\ldots\,,$$

where

$$\begin{array}{rcl}
a_{3m}(x) & = & (1-x^2)U_2(x)[(3m+3\nu+1)U_2(x) - 3\nu] \\
b_{3m}(x) & = & 3x\{-(2\nu+1)(3m+3\nu+1)U_2^2(x) + 3\nu(4m+6\nu+3)U_2(x) - 6\nu(2\nu+1)\} \\
c_{3m}(x) & = & 3m\{(3m+3\nu+1)(3m+6\nu+2)U_2^2(x) - 3\nu(3m+6\nu+4)U_2(x) - 6\nu\} \\
a_{3m+1}(x) & = & (1-x^2)U_2(x)[(3m+3\nu+2)U_2(x) + 3\nu] \\
b_{3m+1}(x) & = & 3x\{-(2\nu+1)(3m+3\nu+2)U_2^2(x) + 3\nu(4m+2\nu+1)U_2(x) + 6\nu(2\nu+1)\} \\
c_{3m+1}(x) & = & 3(m+2\nu+1)\{(3m+1)(3m+3\nu+2)U_2^2(x) + 3\nu(3m-1)U_2(x) - 6\nu\} \\
a_{3m+2}(x) & = & (1-x^2)U_2^2(x) \\
b_{3m+2}(x) & = & -3xU_2(x)[(2\nu+1)U_2 - 4\nu] \\
c_{3m+2}(x) & = & (3m+2)(3m+6\nu+4)U_2^2(x) - 12\nu U_2(x) - 24\nu\,.
\end{array}$$

Acknowledgments. The work of the first author was supported by *Dirección General de Enseñanza Superior* (DGES) of Spain under grant PB96-0120-C03-01 and INTAS program, INTAS-93-0219 Ext. The work of the second author was supported by *Junta Nacional de Investigação Científica e Tecnológica* (JNICT) under grant BD976 and *Centro de Matemática da Universidade de Coimbra* (CMUC) of Portugal.

References

[1] W.AL-SALAM, W.R.ALLAWAY, R.ASKEY: Sieved Ultraspherical Polynomials, *Trans. Amer. Math. Soc.*, 284, 1984, 39-55.

[2] P.BARRUCAND, D.DICKINSON: On Cubic Transformations of Orthogonal Polynomials, *Proc. Amer. Math. Soc.* 17, 1966, 810-814.

[3] J.BUSTOZ, M.E.H.ISMAIL, J.WIMP: On Sieved Orthogonal Polynomials, VI: Differential Equations, *Diff. and Int. Eq.*, 3, 1990, 757-766.

[4] J.CHARRIS, M.E.H.ISMAIL, S.MONSALVE: On Sieved Orthogonal Polynomials, X: General Blocks of Recurrence Relations, *Pacific J. of Math.*, 163, 1994, 237-267.

[5] T.S.CHIHARA: *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978 .

[6] T.S.CHIHARA: On Co-recursive Orthogonal Polynomials , *Proc. Amer. Math. Soc.*, 8, 1957, 899-905.

[7] L.ELSNER, R.M.REDHEFFER: Remark on Band Matrices , *Numer. Math.* , 10, 1967, 153-161.

[8] A.ERDÉLYI, W.MAGNUS, F.OBERHETTINGER, F.G.TRICOMI: *Higher Transcendental Functions*, vol. 2, Krieger, Malabar, Florida, 1981.

[9] J.GERONIMO, W.VAN ASSCHE: Orthogonal Polynomials on Several Intervals via a Polynomial Mapping, *Trans. Amer. Math. Soc.*, 308, 1988, 559-581.

[10] M.J.C.GOVER: The Eigenproblem of a Tridiagonal 2-Toeplitz Matrix. *Linear Alg. and Its Applic.*, 197-198, 1994, 63-78.

[11] M.E.H.ISMAIL: On Sieved Orthogonal Polynomials, III: Orthogonality on Several Intervals, *Trans. Amer. Math. Soc.*, 249, 1986, 89-111.

[12] F.MARCELLÁN, J.PETRONILHO: Eigenproblems for Tridiagonal 2-Toeplitz Matrices and Quadratic Polynomial Mappings. *Linear Alg. and Its Applic.*, 260, 1997, 169-208.

[13] F.MARCELLÁN, J.PETRONILHO: Orthogonal Polynomials and Quadratic Transformations. *Port. Math.* 56, 1999, 81-113.

[14] F.MARCELLÁN, J.PETRONILHO: Orthogonal Polynomials and Cubic Polynomial Mappings II: The Positive-Definite Case. Submitted.

[15] F.MARCELLÁN, G.SANSIGRE: Orthogonal Polynomials and Cubic Transformations, *J. Comp. App. Math.* 49, 1993, 161-168.

[16] P.MARONI: Sur la Suite de Polynômes Orthogonaux Associée à la Forme $u = \delta_c + \lambda(x - c)^{-1}\mathcal{L}$, *Periodica Mathematica Hungarica*, 21, 1990, 223-248.

[17] P.MARONI: Une Théorie Algébrique des Polynômes Orthogonaux. Application aux Polynômes Orthogonaux Semi-Classiques. In *Orthogonal Polynomials and Their Applications*, C.BREZINSKI, L.GORI, A.RONVEAUX Eds., Proc. Erice, 1990. IMACS. *Ann. Comp. Appl. Math.*, 9, 1991, 95-130.

FRANCISCO MARCELLÁN:
Address: Depto. de Matemáticas,
Escuela Politécnica Superior,
Univ. Carlos III de Madrid,
Butarque 15, 28911 Leganés,
Madrid, SPAIN.
E-Mail: pacomarc@ing.uc3m.es

JOSÉ PETRONILHO:
Address: Depto. de Matemática,
Faculdade de Ciências e Tecnologia,
Univ. Coimbra,
Apartado 3008,
3000 Coimbra, PORTUGAL.
E-Mail: josep@mat.uc.pt