Orthogonal Polynomials and Cubic Polynomial Mappings II : The Positive-Definite Case

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Abstract

Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials orthogonal with respect to some distribution function σ and let $\{Q_n\}_{n\geq 0}$ be a simple set (i.e., each Q_n has degree exactly n) of polynomials such that

$$Q_{3n+m}(x) = \theta_m(x)P_n(\pi_3(x)) \quad \text{for all} \quad n = 0,1,2,\ldots$$

where π_3 is a fixed monic polynomial of degree 3 and θ_m a fixed polynomial of degree m with $0 \leq m \leq 2$. We give necessary and sufficient conditions in order that $\{Q_n\}_{n\geq 0}$ be a sequence of polynomials orthogonal with respect to some distribution function $\widetilde{\sigma}$. Under these conditions, we prove that

$$\mathrm{d}\widetilde{\sigma}(x) = \sum_{i=1}^{m} M_i \delta_{x_i}(x)\,\mathrm{d}x + \chi_{\pi_3^{-1}([\xi,\eta])}(x) \left|\frac{\theta_{2-m}(x)}{\theta_m(x)}\right| \frac{\mathrm{d}\sigma(\pi_3(x))}{\pi_3'(x)}$$

where χ_A means the characteristic function of the set A, $[\xi,\eta]$ is the support of $\mathrm{d}\sigma$, θ_{2-m} denote a polynomial of degree exactly $2-m$ and, if $m \geq 1$, M_i is a mass located at the zero x_i of $\theta_m(x) \equiv \prod_{i=1}^{m}(x-x_i)$, $\delta_{x_i}(x)$ being the Dirac functional at the point x_i.

Key words and Phrases: Orthogonal Polynomials, Polynomial mappings, Stieltjes transforms, Chain sequences.

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1 Introduction

Let $\{L_n^{(\alpha)}\}_{n\geq 0}$ be the sequence of the Laguerre monic orthogonal polynomials, characterized by the orthogonality relation

$$\int_0^{+\infty} L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)x^{\alpha}\mathrm{e}^{-x}\mathrm{d}x = k_n\delta_{nm} \quad (k_n = \text{const.} > 0)\,, \quad n,m = 0,1,2,\ldots$$

provided that $\alpha > -1$. If we make the change of variables $x \to x^3$ and choose $\alpha = -2/3$, then

$$\int_0^{+\infty} Q_{3n}(x)Q_{3m}(x)\mathrm{e}^{-x^3}\mathrm{d}x = \frac{k_n}{3}\delta_{nm}\,, \quad n,m = 0,1,2,\ldots$$

where we wrote $Q_{3n}(x) := L_n^{(-2/3)}(x^3)$. This leads to the following question: if we define $Q_{3n}(x) := L_n^{(-2/3)}(x^3)$ for all $n = 0,1,\ldots$, can we complete the system $\{Q_n\}_{n\geq 0}$ (i.e., how to

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define Q_{3n+1} and Q_{3n+2} ?) so that $\{Q_n\}_{n\geq 0}$ be a sequence of (monic) polynomials orthogonal with respect to the weight function e^{-x^3} supported on the interval $]0,+\infty[$? It can be a little unexpected, but the answer to this question is negative (one of the reasons is that $]0,+\infty[$ is not a bounded interval - see Remark after Theorem 3.1). More generally, we consider the following problem:

Problem: *Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials orthogonal with respect to some distribution function σ and let $\{Q_n\}_{n\geq 0}$ be a simple set (i.e., each Q_n has degree n) of polynomials such that*

$$Q_{3n+m}(x) = \theta_m(x) P_n(\pi_3(x)) \quad \textit{for all} \quad n = 0, 1, 2, \ldots$$

where π_3 is a fixed monic polynomial of degree 3 and θ_m is a fixed polynomial of degree m with $0 \leq m \leq 2$. To find necessary and sufficient conditions in order that $\{Q_n\}_{n\geq 0}$ be a sequence of polynomials orthogonal with respect to some distribution function $\widetilde{\sigma}$.

Notice that we must distinguish three cases, so we will refer as P1, P2 or P3 to the above problem for the three possible choices $m = 0$, $m = 1$ or $m = 2$ (resp.). In [10] we analyzed these three problems without the assumption that the given sequence $\{P_n\}_{n\geq 0}$ and the required sequence $\{Q_n\}_{n\geq 0}$ be orthogonal in the positive-definite sense (i.e., with respect to some distribution function), so that, $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ were considered orthogonal with respect to some regular (or quasi-definite) linear moment functional [4, p.16]. In this paper we will assume the results and the notations of [10]. We will consider monic orthogonal polynomial systems (MOPS), which are characterized by a three-term recurrence relation

$$\begin{aligned} &xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 1, \\ &P_0(x) = 1, \quad P_1(x) = x - \beta_0. \end{aligned} \tag{1}$$

In the general case, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ are sequences of complex numbers, with $\gamma_n \neq 0$ for all $n \geq 1$; in the positive definite case each β_n is a real number and $\gamma_n > 0$ for all n. In the next it is important to have in mind the definitions of some useful MOPS's (cf. also [10]): $\{P_n^*(c;.)\}_{n\geq 0}$ (where c is a real number such that $P_n(c) \neq 0$ for all $n \geq 0$), the well known sequence of monic kernel polynomials of K-parameter c associated with $\{P_n\}_{n\geq 0}$ [4, p.35]; $\{P_n^{(1)}\}_{n\geq 0}$, the sequence of the monic associated polynomials of the first kind of $\{P_n\}_{n\geq 0}$ [4, p.86]; $\{P_n^{\lambda}\}_{n\geq 0}$ (λ a fixed real number), which is called the co-recursive sequence of $\{P_n\}_{n\geq 0}$ for the modification λ (CHIHARA, [5]); and $\{P_n^{\lambda,c}\}_{n\geq 0}$, (where λ and c are real numbers such that $\lambda \neq 0$ and $P_n^{\lambda}(c) \neq 0$ for all $n \geq 0$), which is the MOPS corresponding to the linear functional $\mathbf{u}^{\lambda,c}$ defined by $(x-c)\mathbf{u}^{\lambda,c} = -\lambda \mathbf{u}$, or $\mathbf{u}^{\lambda,c} = u_0\delta_c - \lambda(x-c)^{-1}\mathbf{u}$, where $\mathbf{u}$ is the regular linear functional associated with $\{P_n\}_{n\geq 0}$, δ_c is the Dirac functional at the point c, $\langle \delta_c, f\rangle := f(c)$ and $(x-c)^{-1}\mathbf{u}$ is the linear functional defined by $\langle (x-c)^{-1}\mathbf{u}, f\rangle := \langle \mathbf{u}, [f(x)-f(c)]/(x-c)\rangle$, $f \in \mathbb{P}$ (MARONI, [12]). Here, $\mathbb{P}$ denotes the set of polynomials with complex coefficients. Notice that

$$P_n^{\lambda}(x) = P_n(x) - \lambda P_{n-1}^{(1)}(x), \quad P_n^{\lambda,c}(x) = P_n(x) - \frac{P_n^{\lambda}(c)}{P_{n-1}^{\lambda}(c)} P_{n-1}(x)$$

for all $n = 0, 1, 2 \ldots$. As in [10], for $x, y, \zeta \in \mathbb{R}$, we will use the notations:

$$D_n^{\lambda,\mu}(x,y) := \begin{vmatrix} P_n^{\lambda}(x) & P_{n+1}^{\lambda}(x) \\ P_n^{\mu}(y) & P_{n+1}^{\mu}(y) \end{vmatrix}, n = 0, 1, 2 \ldots; \quad P(\zeta) := \lim_{n\to+\infty} \frac{P_n(\zeta)}{P_{n-1}^{(1)}(\zeta)}.$$

2 Two Preliminary Lemmas

In [10] we have considered "formal orthogonality", in the sense that the orthogonal polynomials $\{P_n\}_{n\geq 0}$ are related to a numerical sequence $u_n := \langle \mathbf{u}, x^n\rangle$, $n = 0, 1, 2, \ldots$, independently of the fact that these numbers are moments of some weight or distribution function on some support or not. In order to study when this can occur, we must analyze when a regular linear functional $\mathbf{u}$ is positive definite, i.e., $\langle \mathbf{u}, f\rangle > 0$ for all $f \in \mathbb{P}$ such that $f(x) \geq 0$, $\forall x \in \mathbb{R}$ and $f \not\equiv 0$. By a well known representation Theorem [4, Chap.II] a linear regular functional $\mathbf{u}$ is positive definite if and only if it admits a Stieltjes integral representation $\langle \mathbf{u}, f\rangle = \int_{-\infty}^{+\infty} f(x)\mathrm{d}\sigma(x)$, for every polynomial f, where σ is a distribution function [4, p.51]. Given a sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ satisfying (1) with $\beta_n \in \mathbb{R}$ and $\gamma_n > 0$ for all n, it is possible to obtain the corresponding distribution function, σ, from the asymptotic behavior of $P_n(x)$ and $P_{n-1}^{(1)}(x)$, according to Markov's theorem (see [4, p.89], e.g.),

$$-u_0 \lim_{n\to\infty} \frac{P_{n-1}^{(1)}(z)}{P_n(z)} = F(z;\sigma) := \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(t)}{t-z}, \quad z \in \mathbb{C}\backslash \mathrm{supp}(\mathrm{d}\sigma),$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathrm{supp}(\mathrm{d}\sigma)$, provided that $\mathrm{supp}(\mathrm{d}\sigma)$ is compact (the function $F(.;\sigma)$ is called the Stieltjes function of the distribution function σ). The distribution function $\sigma(x)$ can be recovered from the above limit relation by applying Stieltjes inversion formula: at the points t_1 and t_2 where σ is a continuous function,

$$\sigma(t_2) - \sigma(t_1) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int_{t_1}^{t_2} [F(x+i\varepsilon;\sigma) - F(x-i\varepsilon;\sigma)]\,\mathrm{d}x.$$

Notice that, if λ is real and $\{P_n\}_{n\geq 0}$ is an MOPS with respect to a positive definite linear functional, then so is $\{P_n^{\lambda}\}_{n\geq 0}$, as well as $\{P_n^{(1)}\}_{n\geq 0}$. We denote by sgn the sign function.

Lemma 2.1 *Assume that $\{P_n\}_{n\geq 0}$ is an MOPS with respect to a positive definite linear functional and let $[\xi,\eta]$ be the corresponding true interval of orthogonality* [4, p.29]. *Then the true interval of orthogonality of $\{P_n^{\lambda}\}_{n\geq 0}$ is contained in $[\xi+\lambda,\eta]$ if $\lambda < 0$ and is contained in $[\xi,\eta+\lambda]$ if $\lambda > 0$. As a consequence, for a fixed pair (λ,μ) of real numbers and $n \in \mathbb{N}_0$,*

$$\begin{array}{lll}
\lambda \leq 0, & x \leq \xi+\lambda & \Rightarrow \quad \mathrm{sgn}[P_n^{\lambda}(x)] = (-1)^n \\
\mu \geq 0, & y \geq \eta+\mu & \Rightarrow \quad \mathrm{sgn}[P_n^{\mu}(x)] = +1 \\
\lambda \leq 0, & \mu \geq 0, \quad x \leq \xi+\lambda, \quad y \geq \eta+\mu & \Rightarrow \quad \mathrm{sgn}[D_n^{\lambda,\mu}(x,y)] = (-1)^n .
\end{array}$$

Furthermore, for any λ, the zeros of P_n^{λ} are in $]\xi,\eta[$ for every n if and only if $P(\xi) \leq \lambda \leq P(\eta)$, where $P(\xi)$ (resp. $P(\eta)$) must be replaced by $-\infty$ (resp. $+\infty$) if $\xi = -\infty$ (resp. $\eta = +\infty$).

Proof. By using results from [5], was shown in [9] that if $\lambda < 0$ then the true interval of orthogonality of $\{P_n^{\lambda}\}_{n\geq 0}$ is contained in $[\xi,\eta+\lambda]$. The proof is similar for $\lambda > 0$. The last statement in the Lemma is a result proved in [5].

Let $\{P_n\}_{n\geq 0}$ be orthogonal in the positive definite sense with respect to the distribution function σ and let $[\xi,\eta]$ be the corresponding true interval of orthogonality. Fix $\lambda, c \in \mathbb{R}$. According to Favard's Theorem, it follows from Lemma 2.1 and the explicit expressions for the coefficients of the three-term recurrence relation for $\{P_n^{\lambda,c}\}_{n\geq 0}$ (cf. [10]) that this MOPS is orthogonal in the positive definite sense if one of the following four sets of conditions holds:

$$\begin{array}{llll}
\text{(i)} & \lambda < 0 \text{ and } c \leq \xi+\lambda, \text{ or} & \text{(iii)} & \lambda < 0,\ P(\xi) \leq \lambda \leq P(\eta) \text{ and } c \leq \xi, \text{ or} \\
\text{(ii)} & \lambda > 0 \text{ and } c \geq \eta+\lambda, \text{ or} & \text{(iv)} & \lambda > 0,\ P(\xi) \leq \lambda \leq P(\eta) \text{ and } c \geq \eta.
\end{array} \qquad (2)$$

Moreover, if $[\xi,\eta]$ is compact and one of the conditions (2) holds, then $\mathbf{u}^{\lambda,c}$ is represented by the distribution function $\sigma^{\lambda,c}$ defined by

$$\mathrm{d}\sigma^{\lambda,c}(x) = M\delta_c(x) - \frac{\lambda}{x-c}\chi_{]\xi,\eta[}(x)\mathrm{d}\sigma(x), \quad M := u_0 + \lambda F(c;\sigma), \qquad (3)$$

χ_E being the characteristic function of a set E. Notice that any set of conditions in (2) implies

$$\int_\xi^\eta \frac{d\sigma(x)}{|x-c|} < +\infty \quad \text{and} \quad M := u_0 + \lambda F(c;\sigma) \geq 0\,.$$

Furthermore, in cases (i) and (ii) it is true that $M > 0$, and in cases (iii) and (iv) we also have $M > 0$ if $c \neq \xi$ or $c \neq \eta$, respectively. If $c = \xi$ or $c = \eta$ it can occur $M = 0$. Hence, $\sigma^{\lambda,c}$ is, in fact, a distribution function. The next proposition is motivated by the results of [7].

Lemma 2.2 *Let σ be a distribution function with* $\mathrm{supp}(d\sigma) \subset [\xi,\eta]$, $-\infty < \xi < \eta < +\infty$. *Let T be a real and monic polynomial of degree $k \geq 2$ such that the derivative T' has $k-1$ real and distinct zeros, denoted in increasing order by $y_1 < y_2 < \ldots < y_{k-1}$. Assume that either $T(y_{2i-1}) \geq \eta$ and $T(y_{2i}) \leq \xi$ (for all possible i) if k is odd, or $T(y_{2i-1}) \leq \xi$ and $T(y_{2i}) \geq \eta$ if k is even. Let A and B be two real and monic polynomials such that* $\deg(A) = k-1-m$ *and* $\deg(B) = m$ *, with $0 \leq m \leq k-1$. Assume also that the zeros of AB are real and distinct, AB and T' have the same sign in each point of the set $T^{-1}([\xi,\eta])$ and, if $m \geq 1$,*

$$\int_\xi^\eta \frac{d\sigma(y)}{|y-T(b_j)|} < +\infty \tag{4}$$

for $j = 1,\ldots,m$, where $b_1, b_2, \ldots, b_m$ denote the zeros of B. Let

$$F(z) := \frac{1}{B(z)}\left[A(z)F(T(z);\sigma) - L_{m-1}(z)\right], \quad z \in \mathbb{C}\backslash T^{-1}([\xi,\eta])\,, \tag{5}$$

where $L_{m-1}(z) := \sum_{j=1}^m M_j B(z)/(z-b_j)$, $M_j := A(b_j)F(T(b_j);\sigma)/B'(b_j)$ for $j = 1,\ldots,m$ ($L_0(z) \equiv 0$), i.e., L_{m-1} is the Lagrange interpolatory polynomial of degree $m-1$ that coincides with $A(z)F(T(z);\sigma)$ at the zeros of B. Then, F is the Stieltjes function of the distribution τ defined by

$$d\tau(x) := \left|\frac{A(x)}{B(x)}\right| \chi_{T^{-1}([\xi,\eta])}(x) \frac{d\sigma(T(x))}{T'(x)}\,. \tag{6}$$

Proof. According to the hypothesis, one can write $T^{-1}([\xi,\eta]) = \cup_{j=1}^k E_j$ where $E_1,\ldots,E_k$ are k closed intervals in the real line such that E_j and E_{j+1} have at most one common point. Consider the functions $T_j : D_j \to T(D_j)$ $(j = 1,\ldots,k)$ such that $x \in D_j \mapsto T_j(x) := T(x)$, where $D_1 :=]-\infty, y_1]$, $D_j := [y_{j-1}, y_j]$ $(j = 2,\ldots,k-1)$ and $D_k := [y_{k-1}, +\infty[$. Then, each T_j is bijective and $T_j(E_j) = [\xi,\eta]$ for $j = 1,\ldots,k$. Now, by hypothesis, AB and T' have the same sign in each interval E_j of $T^{-1}([\xi,\eta])$. This implies that the zeros of AB are located between the intervals $E_1,\ldots,E_k$, hence the zeros $a_1,\ldots,a_{k-1-m}$ of A and $b_1,\ldots,b_m$ of B, satisfy $T(a_i) \notin]\xi,\eta[$ $(i = 1,\ldots,k-1-m)$ and $T(b_j) \notin]\xi,\eta[$ $(j = 1,\ldots,m)$. We first prove that, under the hypothesis of the Lemma, τ given by (6) defines, in fact, a distribution function. For that, it is sufficient to show that

$$\int_{T^{-1}([\xi,\eta])} \left|\frac{A(x)}{B(x)}\right| \frac{d\sigma(T(x))}{T'(x)} < +\infty \tag{7}$$

(because $[\xi,\eta]$ is compact). Since the zeros of B and T' are real and simple, considering the sets of indices J_1, J_2 and J_3 defined as $J_1 := \{1,\ldots,m\} \setminus \{j : b_j = y_i \text{ for some } i \in \{1,\ldots,k-1\}\}$, $J_2 := \{1,\ldots,k-1\} \setminus \{j : y_j = b_i \text{ for some } i \in \{1,\ldots,m\}\}$ and $J_3 := \{1,\ldots,m\} \cap \{j : b_j = y_i$ for some $i \in \{1,\ldots,k-1\}\}$, one can write

$$\frac{1}{B(x)T'(x)} = \sum_{j\in J_1} \frac{\alpha_{1j}}{x-b_j} + \sum_{j\in J_2} \frac{\alpha_{2j}}{x-y_j} + \sum_{j\in J_3} \frac{\alpha_{3j}}{(x-b_j)^2}\,,$$

with α_{ij}'s real numbers. But, $T(x) = T(b_j) + T'(b_j)(x-b_j) + \frac{T''(b_j)}{2!}(x-b_j)^2 + \ldots + (x-b_j)^k$ (for a fixed j). Hence, if $j \in J_1$ then $T'(b_j) \neq 0$ and $T(x) - T(b_j) = (x-b_j)[T'(b_j) + G_{1j}(x)]$, where $G_{1j}(x)$ is a polynomial of degree $k-1$ such that $G_{1j}(b_j) = 0$. Thus, it follows that

$$\frac{|\alpha_{1j}A(x)|}{|x-b_j|} \leq \frac{K_{1j}}{|T(x) - T(b_j)|}, \quad x \in T^{-1}(]\xi,\eta[), \quad j \in J_1$$

where $K_{1j} := \sup_{x\in T^{-1}(]\xi,\eta[)} |\alpha_{1j}A(x)||T'(b_j)+G_{1j}(x)| < \infty$. Similarly, if $j \in J_3$ then $T'(b_j) = 0$ and $T''(b_j) \neq 0$ so that $T(x)-T(b_j) = (x-b_j)^2[T''(b_j)/2+G_{3j}(x)]$, where $G_{3j}(x)$ is a polynomial of degree $k-2$ such that $G_{3j}(b_j) = 0$. Then, we deduce

$$\frac{|\alpha_{3j}A(x)|}{(x-b_j)^2} \leq \frac{K_{3j}}{|T(x) - T(b_j)|}, \quad x \in T^{-1}(]\xi,\eta[), \quad j \in J_3$$

where $K_{3j} := \sup_{x\in T^{-1}(]\xi,\eta[)} |\alpha_{3j}A(x)||T''(b_j)/2 + G_{3j}(x)| < \infty$. Finally, if $j \in J_2$ then $y_j \neq b_i$ for all $i \in \{1,\ldots,m\}$ and, therefore, if $T(y_j) = \xi$ or $T(y_j) = \eta$ for some j then necessarily $y_j = a_i$ for some $i \in \{1,\ldots,k-1-m\}$, and we can conclude

$$K_{2j} := \sup_{x\in T^{-1}(]\xi,\eta[)} \frac{|\alpha_{2j}A(x)|}{|x-y_j|} < \infty, \quad j \in J_2 .$$

This is straightforward if $T(y_j) \neq \xi$ and $T(y_j) \neq \eta$ for all j. Hence, for $x \in T^{-1}(]\xi,\eta[)$,

$$\left|\frac{A(x)}{B(x)}\frac{1}{T'(x)}\right| \leq K_1 \sum_{j\in J_1} \frac{1}{|T(x)-T(b_j)|} + n_2K_2 + K_3 \sum_{j\in J_3} \frac{1}{|T(x)-T(b_j)|},$$

where $K_i := \max_{j\in J_i} K_{ij}$ $(i = 1,2,3)$ and $n_2 = \#J_2$. Thus, the integral in (7) becames

$$\begin{aligned}\int_{T^{-1}(]\xi,\eta[)} \left|\frac{A(x)}{B(x)}\frac{1}{T'(x)}\right| \operatorname{sgn} T'(x) \mathrm{d}\sigma(T(x)) &\leq K_1 \sum_{j\in J_1}\sum_{i=1}^{k} \int_{E_i} \frac{\operatorname{sgn} T'(x)}{|T(x)-T(b_j)|}\mathrm{d}\sigma(T(x)) + \\ + n_2K_2 \sum_{i=1}^{k} \int_{E_i} \operatorname{sgn} T'(x)\mathrm{d}\sigma(T(x)) &+ K_3 \sum_{j\in J_3}\sum_{i=1}^{k} \int_{E_i} \frac{\operatorname{sgn} T'(x)}{|T(x)-T(b_j)|}\mathrm{d}\sigma(T(x)) = \\ = kK_1 \sum_{j\in J_1} \int_\xi^\eta \frac{1}{|y-T(b_j)|}\mathrm{d}\sigma(y) &+ kn_2K_2u_0 + kK_3 \sum_{j\in J_3} \int_\xi^\eta \frac{1}{|y-T(b_j)|}\mathrm{d}\sigma(y)\end{aligned}$$

(by making the substitution $y = T(x)$, $x \in E_i$). This proves (7), according to the hypothesis (4), and we conclude that τ, defined by (6), is a distribution function with $\operatorname{supp}(\mathrm{d}\tau) \subset T^{-1}([\xi,\eta])$. Hence, for $z \in \mathbb{C}\backslash T^{-1}([\xi,\eta])$, making the substitution $s = T(x)$ (notice that AB and T' have the same sign in each interval E_j) we get

$$F(z;\tau) = \sum_{j=1}^{k} \int_{E_j} \frac{1}{x-z} \left|\frac{A(x)}{B(x)}\right| \frac{\mathrm{d}\sigma(T(x))}{T'(x)} = \int_\xi^\eta \frac{1}{T_j^{-1}(s)-z} \frac{A(T_j^{-1}(s))}{B(T_j^{-1}(s))} \frac{\mathrm{d}\sigma(s)}{T'(T_j^{-1}(s))} .$$

Now, since $T(b_j) \notin]\xi,\eta[$ for $j = 1,\ldots,m$, if $s \in]\xi,\eta[$ one can write

$$\frac{A(z)}{B(z)}\frac{1}{s-T(z)} = \sum_{j=1}^{k} \frac{A(T_j^{-1}(s))}{(T_j^{-1}(s)-z)B(T_j^{-1}(s))T'(T_j^{-1}(s))} + \sum_{j=1}^{m} \frac{A(b_j)}{B'(b_j)(T(b_j)-s)(b_j-z)} .$$

Thus, for $z \in \mathbb{C}\backslash T^{-1}([\xi,\eta])$, we get

$$F(z;\tau) = \int_\xi^\eta \frac{A(z)}{B(z)} \frac{1}{s-T(z)} \mathrm{d}\sigma(s) - \sum_{j=1}^{m} \frac{A(b_j)}{B'(b_j)(z-b_j)} \int_\xi^\eta \frac{1}{s-T(b_j)}\mathrm{d}\sigma(s) = F(z) .$$

3 The Positive Definite Case

We are now in position to characterize the solution for problems P1, P2 and P3 (in the positive definite case). We mention that problem P1 has been studied by P.BARRUCAND and D.DICKINSON [2] for a particular cubic transformation and for general k by J.GERONIMO and W.VAN ASSCHE [7]. In fact, these authors have proved that given a sequence $\{p_n\}_{n\geq 0}$ of polynomials orthonormal with respect to some positive measure μ_0 supported on the bounded interval $[-1,1]$ and a polynomial $T(x)$ of fixed degree $k \geq 2$ with distinct zeros and such that $|T(y_j)| \geq 1$, where y_j $(j = 1,\ldots,k-1)$ are the zeros of T', then there exists a positive measure μ and a sequence of polynomials $\{q_n\}_{n\geq 0}$ orthonormal with respect to μ such that $q_{kn}(x) = p_n(T(x))$ $(n = 0,1,\ldots)$. As we will see, the Stieltjes functions associated with the orthogonal polynomials in problems P1, P2 and P3 satisfy relations of the form

$$F(z;\widetilde{\sigma}) = \frac{A(z) + B(z)F(\pi_3(z);\sigma)}{C(z)}$$

where A, B and C are polynomials (σ and $\widetilde{\sigma}$ denote the distribution functions corresponding to $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively). This fact establishes the connection between this kind of problems with problems involving sequences of orthogonal polynomials defined by general blocks of recurrence relations studied by J.CHARRIS, M.E.H.ISMAIL and S.MONSALVE [3], in connection with sieved orthogonal polynomials. Finally, we refer that such a kind of polynomial transformations lead to orthogonal polynomials whose measure of orthogonality is supported on several intervals. These kind of sequences of orthogonal polynomials have been the subject of several investigations – among others, we refer the mentioned paper by J.GERONIMO and W.VAN ASSCHE [7], as well as the contributions of M.E.H.ISMAIL [8] and F.PEHERSTORFER [13], [14]. Our main interest is to establish necessary and sufficient conditions in order to characterize the positive definite case.

Theorem 3.1 *Let $\{P_n\}_{n\geq 0}$ be an MOPS in the positive definite sense, orthogonal with respect to some distribution function σ and let $[\xi,\eta]$ be the true interval of orthogonality of $\{P_n\}_{n\geq 0}$. Let $\{Q_n\}_{n\geq 0}$ be a simple set of monic polynomials such that*

$$Q_1(0) = -\beta\,, \quad Q_2(\beta) = -\gamma\,, \quad Q_{3n}(x) = P_n(\pi_3(x))\,, \quad n = 0,1,2,\ldots\,, \tag{8}$$

where π_3 is a real (monic) polynomial of degree 3 and β,γ are real numbers. Consider the polynomial

$$\rho(x) := \gamma + [\pi_3(x) - \pi_3(\beta)]/(x-\beta)\,, \tag{9}$$

denote by a_1 and a_2 the zeros of ρ and put $c_1 := \pi_3(a_1)$ and $c_2 := \pi_3(a_2)$.

Then, for a fixed pair (β,γ), $\{Q_n\}_{n\geq 0}$ is an MOPS in the positive definite sense if and only if a_1 and a_2 are real (we assume, without loss of generality, that $a_1 \leq a_2$) and the following conditions hold

$$\gamma > 0 \quad , \quad c_2 \leq \xi \quad , \quad \eta \leq c_1\,, \tag{10}$$

$$Q_{3n+1}(x) = \frac{1}{\rho(x)}\left[P_{n+1}(\pi_3(x)) + \gamma\frac{P_n^*(c_1;c_2)}{P_n(c_2)}\left(x - a_2 - \frac{1}{\gamma}\frac{P_{n+1}(c_2)}{P_n^*(c_1;c_2)}\right)P_n(\pi_3(x))\right]\,, \tag{11}$$

$$Q_{3n+2}(x) = \frac{1}{\rho(x)}\left[\left(x - a_1 + \frac{1}{\gamma}\frac{P_{n+1}(c_2)}{P_n^*(c_1;c_2)}\right)P_{n+1}(\pi_3(x)) - \frac{1}{\gamma}\frac{P_{n+1}(c_1)P_{n+1}(c_2)}{P_n(c_1)P_n^*(c_1;c_2)}P_n(\pi_3(x))\right] \tag{12}$$

for all $n = 0,1,2,\ldots$. Under these conditions, $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to the uniquely determined distribution function $\widetilde{\sigma}$

$$\mathrm{d}\widetilde{\sigma}(x) = |\rho(x)|\,\chi_{\pi_3^{-1}(]\xi,\eta[)}(x)\frac{\mathrm{d}\sigma(\pi_3(x))}{\pi_3'(x)}\,. \tag{13}$$

Proof. First assume that a_1 and a_2 are real ($a_1 \leq a_2$) and that conditions (10)-(12) hold. Since, for each n, the zeros of P_n are in $]\xi,\eta[$, the conditions $c_2 \leq \xi$ and $\eta \leq c_1$ yield $P_n(c_1) \neq 0$ and $P_n(c_2) \neq 0$ for all $n = 0,1,2,\ldots$. Moreover, condition $c_2 \leq \xi$ ensures that $\{P_n^*(c_2;.)\}_{n\geq 0}$ is also an MOPS in the positive definite sense and the corresponding interval of orthogonality is contained in $[\xi,\eta]$ (cf. [4, Th.7.1, p.36]). Therefore $P_n^*(c_2;c_1) \neq 0$ for all $n = 0,1,2,\ldots$ and so also $P_n^*(c_1;c_2) = P_n(c_2)P_n^*(c_2;c_1)/P_n(c_1) \neq 0$ for all $n = 0,1,2,\ldots$. It follows from Theorem 2.1 in [10] that $\{Q_n\}_{n\geq 0}$ is an MOPS. To conclude that it is an MOPS with respect to a positive measure, we only need to show that the coefficients $\{\tilde\beta_n, \tilde\gamma_{n+1}\}_{n\geq 0}$ of the three-term recurrence relation corresponding to $\{Q_n\}_{n\geq 0}$ satisfy the conditions $\tilde\beta_n$ is real and $\tilde\gamma_{n+1}$ is positive for every $n = 0,1,2,\ldots$. In fact, we have proved in [10] that

$$\tilde\beta_{3n} = \beta\,,\; \tilde\beta_{3n+1} = a_1 - \frac{1}{\gamma}\frac{P_{n+1}(c_2)}{P_n^*(c_1;c_2)}\,,\; \tilde\beta_{3n+2} = a_2 + \frac{1}{\gamma}\frac{P_{n+1}(c_2)}{P_n^*(c_1;c_2)}\,,$$

$$\tilde\gamma_{3n} = -\gamma\gamma_n\frac{P_{n-1}(c_1)P_{n-1}^*(c_1;c_2)}{P_n(c_1)P_n(c_2)}\,,\; \tilde\gamma_{3n+1} = \gamma\frac{P_n^*(c_1;c_2)}{P_n(c_2)}\,,\; \tilde\gamma_{3n+2} = -\frac{1}{\gamma^2}\frac{P_{n+1}(c_1)P_{n+1}(c_2)}{P_n^*(c_1;c_2)P_n^*(c_2;c_1)}\,.$$

Hence, it is clear that $\tilde\beta_n$ is real. Furthermore, using Lemma 2.1 (with $\lambda = 0 = \mu$), we see that $\operatorname{sgn} P_n(c_1) = \operatorname{sgn} P_n^*(c_2;c_1) = +1$, $\operatorname{sgn} P_n(c_2) = \operatorname{sgn} P_n^*(c_1;c_2) = (-1)^n$ and then we deduce $\tilde\gamma_n > 0$ for all $n = 1,2,\ldots$.

Conversely, assume that $\{Q_n\}_{n\geq 0}$ is an MOPS with respect to a positive definite linear functional. It follows from Theorem 2.1 in [10] that (11) and (12) hold. Now, from the hypothesis, $\tilde\gamma_n > 0$ for $n \geq 1$; in particular, $\gamma = \tilde\gamma_1 > 0$. To prove the remaining statements in (10), we will show [4, p.108] that (i1) $c_2 < \beta_n < c_1$ for $n = 0,1,2,\ldots$, and (i2) $\{\alpha_n(c_i)\}_{n\geq 1}$ is a chain sequence for $i = 1,2$, where $\alpha_n(x) := \gamma_n/[(\beta_{n-1} - x)(\beta_n - x)]$ for $n = 1,2,\ldots$. From [10] we know that $(x - a_1)(x - a_2) = \rho(x) = (x - \tilde\beta_{3n+1})(x - \tilde\beta_{3n+2}) - \tilde\gamma_{3n+2}$, so that $\rho(\tilde\beta_{3n+1}) < 0$ and $\rho(\tilde\beta_{3n+2}) < 0$, hence $a_1 < \tilde\beta_{3n+i} < a_2$ for all $n = 0,1,\ldots$, $i = 1,2$. Thus, (i1) follows from $\beta_n = c_i + \tilde\gamma_{3n+1}(a_i - \tilde\beta_{3n+2}) + \tilde\gamma_{3n}(a_i - \tilde\beta_{3n-2})$ ($n = 0,1,\ldots$, $i = 1,2$). To prove (i2) define

$$m_n(c_i) := 1 - \frac{P_{n+1}(c_i)}{(c_i - \beta_n)P_n(c_i)} \equiv \frac{\gamma_n P_{n-1}(c_i)}{(c_i - \beta_n)P_n(c_i)}\,, \quad n = 0,1,\ldots, i = 1,2 \quad (P_{-1} \equiv 0)\,.$$

Then $\alpha_n(c_i) = m_n(c_i)[1 - m_{n-1}(c_i)]$ for $n = 1,2,\ldots$, $i = 1,2$. Now, using $P_{n+1}(c_i)/P_n(c_i) = -\tilde\gamma_{3n+1}(a_i - \tilde\beta_{3n+2})$ and taking into account $a_1 < \tilde\beta_{3n+i} < a_2$ ($i = 1,2$) and (i1), for $n \geq 1$ we have $m_n(c_i) = 1 - P_{n+1}(c_i)/[(c_i - \beta_n)P_n(c_i)] = 1 + \tilde\gamma_{3n+1}(a_i - \tilde\beta_{3n+2})/(c_i - \beta_n) < 1$ and $m_n(c_i) = \gamma_n P_{n-1}(c_i)/[(c_i - \beta_n)P_n(c_i)] = \gamma_n/[\tilde\gamma_{3n-2}(\beta_n - c_i)(a_i - \tilde\beta_{3n-1})] > 0$. Hence $m_0(c_i) = 0$ and $0 < m_n(c_i) < 1$ for $n = 1,2,\ldots$, $i = 1,2$. It follows that $\{\alpha_n(c_i)\}_{n\geq 1}$ is a chain sequence ($i = 1,2$), $\{m_n(c_i)\}_{n\geq 0}$ being the corresponding minimal parameter sequence [4, p.110]. Thus (i2) is proved.

Now, under the above conditions, let $\tilde\sigma$ be the distribution function such that $\{Q_n\}_{n\geq 0}$ is the corresponding MOPS. Notice that conditions (10) imply that the true interval of orthogonality of $\{P_n\}_{n\geq 0}$, $[\xi,\eta]$, must be bounded. Therefore, σ is uniquely determined by the corresponding sequence of moments and then it is easy to check that $\tilde\sigma$ is also uniquely determined by the corresponding sequence of moments. Since $Q_{3n+2}^{(1)}(x) = \rho(x)P_n^{(1)}(\pi_3(x))$ (cf. [10]), by Markov's Theorem we deduce

$$F(z;\tilde\sigma) = -u_0 \lim_{n\to+\infty} \frac{\rho(z)P_{n-1}{}^{(1)}(\pi_3(z))}{P_n(\pi_3(z))} = \rho(z)F(\pi_3(z);\sigma)\,, \quad z \notin \pi_3^{-1}([\xi,\eta])\,,$$

hence (13) follows from Lemma 2.2. Notice that, since π_3 is a real monic polynomial of degree 3 and satisfies $\pi_3(a_2) \leq \xi < \eta \leq \pi_3(a_1)$ with $a_1 < a_2$, then it is easy to see (using Rolle's Theorem) that π_3' has two real and simple zeros.

Remarks. (i) Conditions (10) imply that in order to the existence of sequences $\{Q_n\}_{n\geq 0}$, orthogonal in the positive definite sense and satisfying the conditions of the Theorem, we must start with sequences $\{P_n\}_{n\geq 0}$ such that the corresponding true interval of orthogonality, $[\xi,\eta]$, is bounded, and such that the condition $c_1 \neq c_2$ holds (or, equivalently, $a_1 \neq a_2$).

(ii) In [7], the authors have imposed, a priori, the restrictions "supp($d\sigma$) compact" and "$|\pi_3(y_i)| \geq 1$ on the zeros y_i of π_3' $(i=1,2)$", which in our case corresponds to the condition $\pi_3(y_i) \notin]\xi,\eta[$. We have shown that the condition "supp($d\sigma$) compact" is necessary for the orthogonality of $\{Q_n\}_{n\geq 0}$. Furthermore, assuming (without loss of generality) that $y_1 < y_2$, since in these points $\pi_3(y)$ attains its extremum values, we have $\pi_3(y_2) \leq \pi_3(a_2) = c_2 \leq \xi < \eta \leq c_1 = \pi_3(a_1) \leq \pi_3(y_1)$, hence again $\pi_3(y_i) \notin]\xi,\eta[$ must hold necessarily for the orthogonality of $\{Q_n\}_{n\geq 0}$. We also mention that the measure (13) agrees with the results of [7].

(iii) In [11] it was pointed out that "there does not exist a pair of orthogonal polynomial sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ associated with positive definite symmetric functionals such that $Q_{3n}(x) = P_n(x^3)$". We can generalize this observation, removing the restriction about the symmetry of the functionals, since for any choice of the pair (β,γ), with $\gamma > 0$, the polynomial $\rho(x) := \gamma + (x^3-\beta^3)/(x-\beta) = x^2+\beta x+\beta^2+\gamma$ has no real zeros.

The proof of the next Theorem 3.2 is similar to the proof of Theorem 3.3, so we omit it.

Theorem 3.2 *Let $\{P_n\}_{n\geq 0}$ be an MOPS in the positive definite sense, orthogonal with respect to some distribution function σ, and let $[\xi,\eta]$ be the true interval of orthogonality of $\{P_n\}_{n\geq 0}$. Let $\{Q_n\}_{n\geq 0}$ be a simple set of monic polynomials such that*

$$Q_2(x) = (x-\beta)(x-a)-\gamma, \quad Q_{3n+1}(x) = (x-a)P_n(\pi_3(x))$$

for all $n = 0,1,2,\ldots$, where a, β and γ are fixed real numbers and π_3 a real polynomial of degree 3. Put $b := -[a+\beta+\pi_3''(0)/2]$, $c := \pi_3(b)$, $d := \pi_3(a)$.

1. *If $\{Q_n\}_{n\geq 0}$ is an MOPS in the positive definite sense, then*

$$\gamma > 0 \quad \text{and} \quad \text{either} \quad \mu > 0 \quad \text{if} \quad a < b, \quad \text{or} \quad \mu < 0 \quad \text{if} \quad a > b\,; \tag{14}$$

$$Q_{3n+2}(x) = \frac{1}{x-b}\left[P_{n+1}(\pi_3(x)) + \frac{d_n^{0,\mu}(c,d)}{P_n(c)P_n^{\mu}(d)}\left(x-b-\frac{P_{n+1}(c)P_n^{\mu}(d)}{d_n^{0,\mu}(c,d)}\right)P_n(\pi_3(x))\right], \tag{15}$$

$$Q_{3n+3}(x) = \frac{1}{x-b}\left[\left(x-a+\frac{P_{n+1}(c)P_n^{\mu}(d)}{d_n^{0,\mu}(c,d)}\right)P_{n+1}(\pi_3(x)) - \frac{P_{n+1}(c)P_{n+1}^{\mu}(d)}{d_n^{0,\mu}(c,d)}P_n(\pi_3(x))\right] \tag{16}$$

for all $n = 0,1,2,\ldots$, where $\mu := \gamma(b-a)$ and $d_n^{0,\mu}(c,d) := D_n^{0,\mu}(c,d)/(b-a)$.

2. *Conversely, if conditions (14)-(16) hold as well as one of the following: if $a < b$, either $c \leq \xi$, $d \geq \eta+\mu$, or $c \leq \xi$, $d \geq \eta$, $-\infty < P(\xi) \leq \mu \leq P(\eta) < +\infty$; and, if $a > b$, either $d \leq \xi+\mu$, $c \geq \eta$, or $d \leq \xi$, $c \geq \eta$, $-\infty < P(\xi) \leq \mu \leq P(\eta) < +\infty$. Then $\{Q_n\}_{n\geq 0}$ is an MOPS in the positive definite sense, orthogonal with respect to*

$$d\tilde{\sigma}(x) = M\delta_a(x) + \gamma\left|\frac{x-b}{x-a}\right|\chi_{\pi_3^{-1}(]\xi,\eta[)}(x)\frac{d\sigma(\pi_3(x))}{\pi_3'(x)}, \quad M := u_0 + \mu F(d;\sigma) \geq 0\,.$$

Theorem 3.3 *Let $\{P_n\}_{n\geq 0}$ be an MOPS in the positive definite sense, orthogonal with respect to some distribution function σ, and let $[\xi,\eta]$ be the true interval of orthogonality of $\{P_n\}_{n\geq 0}$. Let $\{Q_n\}_{n\geq 0}$ a simple set of monic polynomials such that*

$$Q_{3n+2}(x) = (x-a)(x-b)P_n(\pi_3(x))\,, \quad n = 0,1,2,\ldots \tag{17}$$

where a and b are fixed real numbers and π_3 is a polynomial of degree 3. Without loss of generality, suppose $a > b$. Write $Q_1(x) = x - \alpha$, $Q_3(x) = (x-\beta)Q_2(x) - \gamma Q_1(x)$ (α, γ real numbers) and denote $c := \pi_3(a)$, $d := \pi_3(b)$, $\lambda := -\gamma(a-\alpha)$ and $\mu := -\gamma(b-\alpha)$.

1. If $\{Q_n\}_{n\geq 0}$ is an MOPS in the positive definite sense, then

$$\gamma > 0 \quad , \quad \lambda < 0 < \mu \quad , \quad \beta = -[a+b+\pi_3''(0)/2] \ , \tag{18}$$

$$Q_{3n+3}(x) = P_{n+1}(\pi_3(x)) + \frac{d_n^{\lambda,\mu}(c,d)}{P_n^\lambda(c)P_n^\mu(d)}\left(x - a - \frac{P_{n+1}^\lambda(c)P_n^\mu(d)}{d_n^{\lambda,\mu}(c,d)}\right)P_n(\pi_3(x)), \tag{19}$$

$$Q_{3n+4}(x) = \left(x - b + \frac{P_{n+1}^\lambda(c)P_n^\mu(d)}{d_n^{\lambda,\mu}(c,d)}\right)P_{n+1}(\pi_3(x)) - \frac{P_{n+1}^\lambda(c)P_{n+1}^\mu(d)}{d_n^{\lambda,\mu}(c,d)}P_n(\pi_3(x)) \tag{20}$$

hold for $n = 0, 1, 2, \ldots$, where $d_n^{\lambda,\mu}(c,d) := D_n^{\lambda,\mu}(c,d)/(a-b)$.

2. Conversely, if conditions (18)-(20) hold and either (i) $c \leq \xi + \lambda$, $\eta + \mu \leq d$, or (ii) $c \leq \xi$, $d \geq \eta$, $-\infty < P(\xi) \leq \lambda$, $\mu \leq P(\eta) < +\infty$, then $\{Q_n\}_{n\geq 0}$ is an MOPS in the positive definite sense, orthogonal with respect to

$$\mathrm{d}\widetilde{\sigma}(x) = M\delta_a(x) + N\delta_b(x) - \frac{\lambda\mu/\gamma}{|(x-a)(x-b)|}\chi_{\pi_3^{-1}([\xi,\eta])}(x)\frac{\mathrm{d}\sigma(\pi_3(x))}{\pi_3'(x)}, \tag{21}$$

where $M := \frac{\mu/\gamma}{a-b}[u_0 + \lambda F(c;\sigma)] \geq 0$, $N := -\frac{\lambda/\gamma}{a-b}[u_0 + \mu F(d;\sigma)] \geq 0$.

Proof. Suppose that $\{Q_n\}_{n\geq 0}$ is an MOPS in the positive definite sense. It follows from Theorem 2.3 in [10] that (19) and (20) hold, as well as $\beta = -[a+b+\pi_3''(0)/2]$. To prove the first two conditions of (18), just notice that, by hypothesis, $\widetilde{\gamma}_n > 0$ for every n, and so, using again Theorem 2.3 in [10], $\gamma = \widetilde{\gamma}_2 > 0$ and $0 < \widetilde{\gamma}_1 = -\lambda\mu/\gamma^2 = -(\alpha-a)(\alpha-b)$, hence $b < \alpha < a$ (because $a > b$), which implies $\lambda < 0 < \mu$.

Conversely, assume that (18)-(20) hold as well as one of the conditions (i) or (ii). It is straightforward to show, using Lemma 2.1 and the expressions given in [10] for $\widetilde{\beta}_n$ and $\widetilde{\gamma}_n$ that $\widetilde{\beta}_n$ is real and $\widetilde{\gamma}_n > 0$ for every n, so that $\{Q_n\}_{n\geq 0}$ is an MOPS in the positive definite sense.

Finally, using $Q_{3n+1}^{(1)}(x) = (x-a-b+\alpha)P_n(\pi_3(x)) - (\lambda\mu/\gamma)P_{n-1}^{(1)}(\pi_3(x))$ (see [10]) and Markov's Theorem, we find

$$F(z;\widetilde{\sigma}) = \frac{k_1}{a-z} + \frac{k_2}{b-z} - \frac{\lambda\mu}{\gamma}\frac{F(\pi_3(z);\sigma)}{(z-a)(z-b)}, \quad z \notin \pi_3^{-1}([\xi,\eta]) \cup \{a,b\},$$

with $k_1 := u_0\mu/\gamma(a-b)$, $k_2 := -u_0\lambda/\gamma(a-b)$. Hence, using Lemma 2.2, one can easily check that (21) holds, which completes the proof.

As an example related with the last theorem, consider the set $\{Q_n\}_{n\geq 0}$ of monic polynomials such that $Q_1(x) = x - \alpha$, $Q_3(x) = (x-\beta)Q_2(x) - \gamma Q_1(x)$ and

$$Q_{3n+2}(x) = (x^2 - \frac{1}{4})P_n(x^3 - \frac{3}{4}x), \quad n = 0, 1, 2, \ldots,$$

with α, β and γ real parameters, where $P_n(x) := \frac{n!}{2^{3n}(\nu+1)_n}C_n^{\nu+1}(4x)$, $\{C_n^\nu\}_{n\geq 0}$ being the sequence of the Gegenbauer or ultraspherical polynomials, where it is assumed that $\nu > -1/2$, so that $\{C_n^\nu\}_{n\geq 0}$ is orthogonal in the positive definite sense on $[-1,1]$ with respect to the absolutely continuous measure $\mathrm{d}\sigma^\nu(x) := (1-x^2)^{\nu-\frac{1}{2}}\mathrm{d}x$ [6, p.175]. Then, the sequence $\{P_n\}_{n\geq 0}$

is an MOPS orthogonal on $\left[-\frac{1}{4},\frac{1}{4}\right]$ with respect to the weight $w(x) := 4(1-16x^2)^{\nu+\frac{1}{2}}$. This example has been considered in [10], where we have shown how to define Q_{3n} and Q_{3n+1} in order that $\{Q_n\}_{n\geq 0}$ be an MOPS (in the formal sense). Now, from Theorem 3.3 we have $\pi_3(x) := x^3 - \frac{3}{4}x$, $a=-b=\frac{1}{2}$ and $c=-d=-\frac{1}{4}$, and choosing $\alpha = 0$ we have $\lambda = -\mu = -\frac{\gamma}{2}$ and then, since $F(\pm\frac{1}{4};\sigma) = 4F(\pm 1;\sigma^{\nu+1}) = \mp\frac{8(\nu+1)}{2\nu+1}u_0$ $(\nu > -1/2)$, Theorem 3.3 ensures that $\{Q_n\}_{n\geq 0}$ is an MOPS in the positive definite sense if $0<\gamma\leq\frac{2\nu+1}{4(\nu+1)}$ and $\beta = 0$, with respect to

$$d\widetilde{\sigma}(x) = \frac{u_0}{2}\left(1-\frac{4(\nu+1)\gamma}{2\nu+1}\right)\left(\delta_{-\frac{1}{2}}(x)+\delta_{\frac{1}{2}}(x)\right)dx + \frac{4\gamma\left[1-(4x^3-3x)^2\right]^{\nu+\frac{1}{2}}}{|4x^2-1|}\chi_{]-1,1[}(x)dx\,.$$

Notice that for $0<\gamma<\frac{2\nu+1}{4(\nu+1)}$ there exist mass points at $x=\pm\frac{1}{2}$, interior to the interval $[-1,1]$. For $\gamma = \frac{2\nu+1}{4(\nu+1)}$, we get (up to a constant factor) $d\widetilde{\sigma}(x) = \sin\theta\,|\sin 3\theta|^{2\nu}\,d\theta$ $(x=\cos\theta)$, so that $\{Q_n\}_{n\geq 0}$ is, up to normalization, the sequence of the sieved ultraspherical polynomials of the second kind, $\{B_n^\nu(.,3)\}_{n\geq 0}$, introduced by AL-SALAM, ALLAWAY and ASKEY in [1].

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