Inner Products Involving Differences: The Meixner–Sobolev Polynomials

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In this paper, polynomials which are orthogonal with respect to the inner product

$$\langle p,q\rangle_S = \sum_{s=0}^{\infty} p(s)q(s)\frac{\mu^s\Gamma(\gamma+s)}{\Gamma(s+1)\Gamma(\gamma)} + \lambda\sum_{s=0}^{\infty}\Delta p(s)\Delta q(s)\frac{\mu^s\Gamma(\gamma+s)}{\Gamma(s+1)\Gamma(\gamma)},$$

where $0<\mu<1$, $\gamma>0$ and $\lambda\geq 0$ are studied. For these polynomials, algebraic properties and difference equations are obtained as well as their relation with the Meixner polynomials. Moreover, some properties about the zeros of these polynomials are deduced.

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1 INTRODUCTION

Let $\mathbb{P}$ be the linear space of polynomials with real coefficients. If we define an inner product on $\mathbb{P}$

$$(f, g) = \int_{\mathbb{R}} f(x)g(x)\,d\mu(x), \tag{1}$$

where $d\mu(x)$ is a signed measure on the real line; it is known (see [5, pp. 21–22]) that there exists a sequence of polynomials $\{P_n(x)\}_n$ such that

$$\deg P_n = n,$$
$$(P_n, P_m) = k_n\delta_{nm}, \quad k_n \neq 0.$$

We assume that the sequence $\{P_n(x)\}_n$ is monic, i.e. the leading coefficient of $P_n(x)$ is one. In these conditions, the sequence $\{P_n(x)\}_n$ is called a Monic Orthogonal Polynomial Sequence (MOPS) with respect to the inner product (1). Such a MOPS $\{P_n(x)\}_n$ satisfies a three-term recurrence relation

$$P_{n+1}(x) = (x - B_n)P_n(x) - C_nP_{n-1}(x), \quad n \geq 0,$$
$$P_0(x) = 1, \quad P_{-1}(x) = 0, \quad C_n \neq 0, \quad n = 1, 2, \ldots$$

The original motivation for considering Sobolev orthogonal polynomials comes from the least squares approximation problems [10,15]. A given function f and its derivative f' are to be approximated simultaneously by a polynomial p of degree n minimizing

$$\|p(x) - f(x)\|^2 = \int_{\mathbb{R}} [p(x) - f(x)]^2\,d\mu_0(x) + \lambda\int_{\mathbb{R}} [p'(x) - f'(x)]^2\,d\mu_1(x) \tag{2}$$

over all $p \in \mathbb{P}_n$, $d\mu_i(x)$, $i = 0, 1$, being positive Borel measures on the real line $\mathbb{R}$ having bounded or unbounded support [8,19]. Expanding p in terms of the Sobolev orthogonal polynomials we obtain the usual Fourier approximation $p(x)$ of $f(x)$ and $f'(x)$. This problem was considered in [15], but nothing was said there about the sequence of polynomials $\{Q_n(x)\}_n$ orthogonal with respect to the inner product

$$(f,g)_S = \int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}\mu_0(x) + \lambda \int_{\mathbb{R}} f'(x)g'(x)\,\mathrm{d}\mu_1(x), \quad \lambda \geq 0. \quad (3)$$

Study of polynomials orthogonal with respect to (3), called *non-discrete* or *continuous case*, can be found in [6,7,19,23]. Other kind of modifications of (1) are studied in [1,16,20]. A new attempt to the study of the non-discrete case was made in 1991 by Iserles *et al.* [11]. There the authors proved that if the Borel measures $\mathrm{d}\mu_0$ and $\mathrm{d}\mu_1$ obey a specific condition (coherent pair) then the sequence of orthogonal polynomials $\{p_n^{(\lambda)}(x)\}_n$ with respect to (3) can be expanded in terms of the polynomials orthogonal with respect to $\mathrm{d}\mu_0$ in such a way that, taking into account an adequate normalization, the expansion coefficients, up to the leading one, are independent of n and are themselves orthogonal polynomials in λ. They also explored several examples and showed how their theory can be used for an efficient evaluation of Sobolev–Fourier coefficients.

The concept of coherent pair, initially defined for measures in [11] can be characterized in terms of the MOPSs $\{P_n(x)\}_n$ and $\{T_n(x)\}_n$ associated with $\mathrm{d}\mu_0$ and $\mathrm{d}\mu_1$, respectively [11, Theorem 3] in the following way: there exists a sequence of non-zero complex numbers $\{\sigma_n\}_n$ such that

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}.$$

This concept has been extensively studied by several authors [17,18,22] and it has been recently adapted to the case of orthogonal polynomials of a discrete variable in [2], characterizing the MOPS $\{P_n(x)\}_n$ and $\{T_n(x)\}_n$ such that

$$T_n(x) = \frac{\Delta P_{n+1}(x)}{n+1} - \sigma_n \frac{\Delta P_n(x)}{n},$$

where $\{\sigma_n\}_n$ is a sequence of complex numbers and Δ stands for the forward difference operator ($\Delta h(x) = h(x+1) - h(x)$).

In order to find the best polynomial approximation $p(x)$ of a function $f(x)$ where besides function values $f(x_i)$, also difference derivatives at the knots are given, the following minimization problem appears in a natural way:

$$\min \sum_{k=0}^{r} \left(\sum_{x_s=a_k}^{b_k-k-1} (\Delta^k p(x_s) - \Delta^k f(x_s))^2 \rho_k(x_s) \right), \quad \Delta^k h(x) = \Delta^{k-1}(\Delta h(x)),$$

where $\rho_k(x)$ are discrete weight functions on $[a_k, b_k)$, i.e., each $\rho_k(x)$ is piecewise constant function with jumps $\rho_k(x_i)$ at the points $x = x_i$ for which $x_{i+1} = x_i + 1$ and $a_k \leq x_i \leq b_k - 1$.

Thus, it seems to be interesting the analysis of the polynomials which are orthogonal with respect to the inner product

$$\langle p, q \rangle_W = \sum_{k=0}^{r} \left(\sum_{x_s=a_k}^{b_k-k-1} \Delta^k p(x_s) \Delta^k q(x_s) \rho_k(x_s) \right).$$

The aim of this paper is the study of polynomials which are orthogonal with respect to a particular case ($r = 1$, $\rho_0 \equiv \rho_1$) of the above inner product:

$$\langle p, q \rangle_S = \sum_{s=0}^{\infty} p(s) q(s) \rho(s) + \lambda \sum_{s=0}^{\infty} \Delta p(s) \Delta q(s) \rho(s); \tag{4}$$

$\lambda \geq 0$ and $\rho(s)$ is the Meixner weight function [24]. We call (4) the Meixner–Sobolev inner product, by analogy with the continuous case [21].

The structure of the paper is as follows: Section 2 contains the basic relations for monic Meixner orthogonal polynomials $\{M_n^{(\gamma,\mu)}(x)\}_n$. In Section 3, we introduce the Meixner–Sobolev inner product, the monic Meixner–Sobolev orthogonal polynomials $\{Q_n(x)\}_n$ and the limit polynomials $\{R_n(x)\}_n$ obtained from $\{Q_n(x)\}_n$ when λ tends to infinity. We also give some relations among these three families of polynomials and a limit relation between Meixner–Sobolev and Laguerre–Sobolev polynomials. In Section 4, a linear difference operator $\mathcal{S}$ on $\mathbb{P}$ is defined. We prove it is a symmetric operator with respect to the Meixner–Sobolev inner product and we find a non-standard four-term recurrence relation for the $\{Q_n(x)\}_n$ polynomials. Finally, in Section 5, we study the properties of the zeros of Meixner–Sobolev orthogonal polynomials.

2 MONIC MEIXNER ORTHOGONAL POLYNOMIALS

The forward difference operator Δ and the backward difference operator ∇ are defined by

$$\Delta f(x) = f(x+1) - f(x), \qquad \nabla f(x) = f(x) - f(x-1). \tag{5}$$

These difference operators satisfy the following properties which will be useful in the next sections:

$$\begin{gathered} \Delta = \nabla + \Delta\nabla, \quad \Delta p(x) = \nabla p(x+1), \\ \Delta(p(x)q(x)) = q(x)\Delta p(x) + p(x+1)\Delta q(x). \end{gathered} \tag{6}$$

Monic Meixner orthogonal polynomials $\{M_n^{(\gamma,\mu)}(x)\}_n$ are the polynomial solution of a second order linear difference equation of hypergeometric type [9,25]

$$\begin{gathered} \sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0, \\ \sigma(x) = x, \quad \tau(x) = \gamma\mu - x(1-\mu), \quad \lambda_n = n(1-\mu). \end{gathered} \tag{7}$$

These polynomials $\{M_n^{(\gamma,\mu)}(x)\}_n$ are orthogonal with respect to the inner product

$$\begin{gathered} \langle p(x), q(x)\rangle = \sum_{s=0}^{\infty} p(s)q(s)\rho(s), \quad \rho(s) = \frac{\mu^s\Gamma(\gamma+s)}{\Gamma(s+1)\Gamma(\gamma)}, \\ s \in [0, +\infty), \quad 0 < \mu < 1, \quad \gamma > 0. \end{gathered} \tag{8}$$

For monic Meixner orthogonal polynomials the following properties are known [3,9,25].

2.1 Three-term Recurrence Relation

We have

$$xM_n^{(\gamma,\mu)}(x) = M_{n+1}^{(\gamma,\mu)}(x) + B_n M_n^{(\gamma,\mu)}(x) + C_n M_{n-1}^{(\gamma,\mu)}(x), \quad n \geq 1, \tag{9}$$

$$B_n = \frac{\gamma\mu + n(1+\mu)}{1-\mu}, \qquad C_n = \frac{\mu n(\gamma+n-1)}{(1-\mu)^2}, \tag{10}$$

with the initial conditions $M_0^{(\gamma,\mu)}(x) = 1$, $M_1^{(\gamma,\mu)}(x) = x - B_0$.

2.2 Difference Representation

We have

$$M_n^{(\gamma,\mu)}(x) = \frac{\Delta M_{n+1}^{(\gamma,\mu)}(x)}{n+1} + \frac{\mu}{1-\mu}\Delta M_n^{(\gamma,\mu)}(x), \quad n \geq 0. \tag{11}$$

2.3 Representation as Hypergeometric Function

We have

$$M_n^{(\gamma,\mu)}(x) = \left(\frac{\mu}{\mu-1}\right)^n (\gamma)_n \, {}_2F_1\left(-n, -x; \gamma; 1-\frac{1}{\mu}\right),$$

where $(a)_s$ denotes the Pochhammer symbol, $(a)_0 = 1$, $(a)_s = a(a+1)\cdots(a+s-1)$. From the above hypergeometric representation of monic Meixner polynomials we get

$$M_n^{(\gamma,\mu)}(0) = \left(\frac{\mu}{\mu-1}\right)^n (\gamma)_n, \quad n \geq 0. \tag{12}$$

2.4 Squared Norm

Let us denote

$$\begin{aligned} k_n &= \langle M_n^{(\gamma,\mu)}(x), M_n^{(\gamma,\mu)}(x)\rangle = \sum_{s=0}^{\infty}\left(M_n^{(\gamma,\mu)}(s)\right)^2 \frac{\mu^s\Gamma(\gamma+s)}{s!\Gamma(\gamma)} \\ &= \frac{n!(\gamma)_n\mu^n}{(1-\mu)^{2n+\gamma}}, \quad n \geq 0. \end{aligned} \tag{13}$$

The following relations can be easily derived from the definition of k_n:

$$k_0 = \frac{1}{(1-\mu)^\gamma}, \qquad k_n = \frac{(\gamma+n-1)\mu n}{(1-\mu)^2} k_{n-1}, \quad n \geq 1. \tag{14}$$

3 MEIXNER–SOBOLEV ORTHOGONAL POLYNOMIALS

Let us consider the Sobolev inner product defined on $\mathbb{P}$ by

$$\begin{aligned}\langle p(x), q(x)\rangle_S &= \langle p(x), q(x)\rangle + \lambda\langle \Delta p(x), \Delta q(x)\rangle \\ &= \sum_{s=0}^{\infty} p(s)q(s)\frac{\mu^s\Gamma(\gamma+s)}{s!\Gamma(\gamma)} + \lambda\sum_{s=0}^{\infty}\Delta p(s)\Delta q(s)\frac{\mu^s\Gamma(\gamma+s)}{s!\Gamma(\gamma)},\end{aligned} \tag{15}$$

where $0<\mu<1$, $\gamma>0$ and $\lambda\geq 0$.

We shall denote by $\{Q_n^{(\gamma,\mu)}(x;\lambda)\}_n \equiv \{Q_n(x)\}_n$ the MOPS associated with the inner product $\langle .,. \rangle_S$. Such a sequence is said to be the Meixner–Sobolev MOPS.

Let us denote the moments associated with the inner product (8) for the basis $\{x^{[n]}\}_n$ as

$$u_{i,j} = \langle x^{[i]}, x^{[j]}\rangle = \sum_{s=0}^{\infty} s^{[i]}s^{[j]}\rho(s) = \sum_{s=\max\{i,j\}}^{\infty} s^{[i]}s^{[j]}\rho(s), \tag{16}$$

with $\rho(x)$ defined in Eq. (8), $x^{[n]}=x(x-1)\cdots(x-n+1)$, $x^{[0]}=1$, and let us denote the moments associated with the inner product (15) for the basis $\{x^{[n]}\}_n$ as $c_{i,j}=\langle x^{[i]}, x^{[j]}\rangle_S$. Since $\Delta x^{[n]} = nx^{[n-1]}$, we get

$$\begin{aligned} c_{i,0} &= c_i = \langle x^{[i]}, 1\rangle_S = u_{i,0} = u_{0,i} \equiv u_i, \\ c_{i,j} &= \langle x^{[i]}, x^{[j]}\rangle_S = u_{i,j} + \lambda ij u_{i-1,j-1}, \quad i,j\geq 1. \end{aligned} \tag{17}$$

From the definition,

$$Q_0(x) = M_0^{(\gamma,\mu)}(x) = 1, \qquad Q_1(x) = M_1^{(\gamma,\mu)}(x) = x - \frac{\gamma\mu}{1-\mu},$$

but if $\lambda>0$ the elements of these sequences are different for degrees greater than or equal to 2.

We can write the Meixner–Sobolev polynomials in the following determinantal form:

$$Q_n(x) = \frac{\begin{vmatrix} u_0 & u_1 & \dots & u_n \\ u_1 & u_{1,1}+\lambda u_0 & \dots & u_{1,n}+\lambda n u_{n-1} \\ \vdots & \vdots & & \vdots \\ u_{n-1} & u_{1,n-1}+\lambda(n-1)u_{n-2} & \dots & u_{n,n-1}+\lambda n(n-1)u_{n-1,n-2} \\ 1 & x & \dots & x^{[n]} \end{vmatrix}}{\begin{vmatrix} u_0 & u_1 & \dots & u_{n-1} \\ u_1 & u_{1,1}+\lambda u_0 & \dots & u_{1,n-1}+\lambda(n-1)u_{n-2} \\ \vdots & \vdots & & \vdots \\ u_{n-1} & u_{1,n-1}+\lambda(n-1)u_{n-2} & \dots & u_{n-1,n-1}+\lambda(n-1)^2 u_{n-2,n-2} \end{vmatrix}}, \tag{18}$$

where each coefficient of $Q_n(x)$ in terms of $x^{[j]}$ is a rational function in λ, the numerator and the denominator being of degree $n-1$. Then, we can define a new sequence of monic polynomials $\{R_n^{(\gamma,\mu)}(x)\}_n \equiv \{R_n(x)\}_n$,

$$R_0(x) = Q_0(x) = M_0^{(\gamma,\mu)}(x) = 1, \qquad R_1(x) = Q_1(x) = M_1^{(\gamma,\mu)}(x), \tag{19}$$

$$R_n(x) = \lim_{\lambda\to\infty} Q_n(x) = \frac{\begin{vmatrix} u_0 & u_1 & \dots & u_n \\ 0 & u_0 & \dots & n u_{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & (n-1)u_{n-2} & \dots & n(n-1)u_{n-1,n-2} \\ 1 & x & \dots & x^{[n]} \end{vmatrix}}{\begin{vmatrix} u_0 & u_1 & \dots & u_{n-1} \\ 0 & u_0 & \dots & (n-1)u_{n-2} \\ \vdots & \vdots & & \vdots \\ 0 & (n-1)u_{n-2} & \dots & (n-1)^2 u_{n-2,n-2} \end{vmatrix}}. \tag{20}$$

PROPOSITION 1 *(a) For each* $n \geq 1$ *we have* $\sum_{s=0}^{\infty} R_n(s)\rho(s) = 0$, *where the weight* $\rho(s)$ *is given in* (8) *and the polynomials* $R_n(x)$ *are defined in* (19) and (20).

(b) If $n \geq 2$ *and* $0 \leq m \leq n-2$ *then* $\sum_{s=0}^{\infty} s^{[m]} \Delta R_n(s)\rho(s) = 0$, *where the polynomials* $R_n(x)$ *are defined in Eqs.* (19) *and* (20), *and* $\rho(s)$ *is given in* (8).

Proof

(a) $\sum_{s=0}^{\infty} R_n(s)\rho(s) = \sum_{s=0}^{\infty} \lim_{\lambda\to\infty} Q_n(s)\rho(s) = \lim_{\lambda\to\infty}\langle Q_n(x), 1\rangle_S = 0$.
(b) Apply the Δ operator in the definition of $R_n(x)$.

COROLLARY 1 *The following two equalities hold:*

$$\Delta R_n(x) = n M_{n-1}^{(\gamma,\mu)}(x), \quad n \geq 1; \tag{21}$$

$$\begin{aligned} R_n(x) &= M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu} M_{n-1}^{(\gamma,\mu)}(x) \\ &= M_n^{(\gamma,\mu)}(x) + \frac{\mu}{1-\mu}\Delta R_n(x), \quad n \geq 2. \end{aligned} \tag{22}$$

Remark 1 If $\gamma > 1$, then $\{R_n(x)\}_n \equiv \{M_n^{(\gamma-1,\mu)}(x)\}_n$, i.e., $R_n(x)$ is the monic Meixner polynomial of degree n associated with the weight function

$$\rho^{(\gamma-1,\mu)}(x) = \frac{\mu^x \Gamma(x+\gamma-1)}{\Gamma(x+1)\Gamma(\gamma-1)}.$$

If $0 < \gamma \leq 1$, $R_n(x)$ is a quasi-orthogonal polynomial [5, p. 64] of order one with respect to the MOPS $\{M_n^{(\gamma,\mu)}(x)\}_n$.

PROPOSITION 2 *The following relation holds:*

$$R_n(x) = Q_n(x) + d_{n-1}(\lambda) Q_{n-1}(x), \quad n \geq 2, \tag{23}$$

where

$$d_{n-1}(\lambda) = n\frac{\mu}{1-\mu}\frac{k_{n-1}}{\tilde{k}_{n-1}}, \quad n \geq 2, \tag{24}$$

$$\tilde{k}_n = \langle Q_n(x), Q_n(x)\rangle_S. \tag{25}$$

Proof If $n \geq 2$ we can expand the polynomial $R_n(x)$ in terms of Meixner–Sobolev polynomials in the following way:

$$R_n(x) = Q_n(x) + \sum_{i=0}^{n-1} f_{i,n}(\lambda) Q_i(x). \tag{26}$$

By using (21) and (22) the coefficients $f_{i,n}(\lambda)$ can be computed as

$$\begin{aligned} f_{i,n}(\lambda) &= \frac{\langle R_n, Q_i \rangle_S}{\langle Q_i, Q_i \rangle_S} \\ &= \frac{1}{\tilde{k}_i} \left\{ \sum_{s=0}^{\infty} R_n(s) Q_i(s) \rho(s) + \lambda \sum_{s=0}^{\infty} \Delta R_n(s) \Delta Q_i(s) \rho(s) \right\} \\ &= \frac{1}{\tilde{k}_i} \left\{ \sum_{s=0}^{\infty} R_n(s) Q_i(s) \rho(s) + \lambda \sum_{s=0}^{\infty} n M_{n-1}^{(\gamma,\mu)}(s) \Delta Q_i(s) \rho(s) \right\} \\ &= \frac{1}{\tilde{k}_i} \sum_{s=0}^{\infty} \left\{ M_n^{(\gamma,\mu)}(s) + n \frac{\mu}{1-\mu} M_{n-1}^{(\gamma,\mu)}(s) \right\} Q_i(s) \rho(s) \\ &= \frac{1}{\tilde{k}_i} \langle M_n^{(\gamma,\mu)}(x), Q_i(x) \rangle + \frac{n\mu}{(1-\mu)\tilde{k}_i} \langle M_{n-1}^{(\gamma,\mu)}(x), Q_i(x) \rangle, \\ & 0 \leq i \leq n-1. \end{aligned}$$

Thus,

$$f_{i,n}(\lambda) = \begin{cases} 0 & \text{if } 0 \leq i \leq n-2, \\ \dfrac{n\mu}{1-\mu} \dfrac{k_i}{\tilde{k}_i} & \text{if } i = n-1. \end{cases}$$

Then (26) becomes

$$R_n(x) = Q_n(x) + \frac{n\mu}{1-\mu} \frac{k_{n-1}}{\tilde{k}_{n-1}} Q_{n-1}(x), \quad n \geq 2.$$

COROLLARY 2 *The Meixner–Sobolev orthogonal polynomials defined in* (18) *satisfy*

$$Q_n(x) = \sum_{j=1}^{n} e_{j,n} M_j^{(\gamma,\mu)}(x), \quad n \geq 2,$$

where

$$e_{n,n} = 1,$$

$$e_{j,n} = (-1)^{n-j-1}\left[\frac{\mu}{1-\mu}(j+1) - d_j(\lambda)\right]\prod_{s=j+1}^{n-1} d_s(\lambda), \quad 1 \le j \le n-1. \tag{27}$$

Proof From Eqs. (23) and (22), we obtain

$$Q_n(x) = M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x) - d_{n-1}(\lambda)Q_{n-1}(x), \quad n \ge 2, \tag{28}$$

where repeating this substitution the result follows taking into account that $Q_1(x) = M_1^{(\gamma,\mu)}(x)$.

We can compute recursively the coefficients $d_n(\lambda)$ defined in (24) by means of

PROPOSITION 3 *The coefficients $d_n(\lambda)$ satisfy the following two-term recurrence relation:*

$$d_n(\lambda) = \frac{\mu^2(n+1)(\gamma+n-1)}{(1-\mu)\{\mu(\gamma+n-1) + n(\mu^2+\lambda(1-\mu)^2) - \mu(1-\mu)d_{n-1}(\lambda)\}}, \tag{29}$$

valid for $n \ge 2$, with the initial condition

$$d_1(\lambda) = \frac{2\gamma\mu^2}{(1-\mu)(\gamma\mu+\lambda(\mu-1)^2)}.$$

Proof From (25) and using (11) we get

$$
\begin{aligned}
\tilde{k}_n &= \langle Q_n(x), M_n^{(\gamma,\mu)}(x)\rangle_S = k_n + \lambda\langle \Delta Q_n(x), \Delta M_n^{(\gamma,\mu)}(x)\rangle \\
&= k_n + \lambda\langle \Delta Q_n(x), nM_{n-1}^{(\gamma,\mu)}(x) - \frac{n\mu}{1-\mu}\Delta M_{n-1}^{(\gamma,\mu)}(x)\rangle \\
&= k_n + \lambda n^2 k_{n-1} - \lambda\frac{n\mu}{1-\mu}\langle \Delta Q_n(x), \Delta M_{n-1}^{(\gamma,\mu)}(x)\rangle \\
&= k_n + \lambda n^2 k_{n-1} - \frac{n\mu}{1-\mu}\left[\langle Q_n(x), M_{n-1}^{(\gamma,\mu)}(x)\rangle_S - \langle Q_n(x), M_{n-1}^{(\gamma,\mu)}(x)\rangle\right] \\
&= k_n + \lambda n^2 k_{n-1} + \frac{n\mu}{1-\mu}\langle Q_n(x), M_{n-1}^{(\gamma,\mu)}(x)\rangle \\
&= k_n + \lambda n^2 k_{n-1} + \frac{n\mu}{1-\mu} \\
&\quad \times \langle M_n^{(\gamma,\mu)}(x) + \frac{n\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x) - d_{n-1}(\lambda)Q_{n-1}(x), M_{n-1}^{(\gamma,\mu)}(x)\rangle \\
&= k_n + \lambda n^2 k_{n-1} + \left(\frac{n\mu}{1-\mu}\right)^2 k_{n-1} - d_{n-1}(\lambda)\frac{n\mu}{1-\mu}k_{n-1}.
\end{aligned}
$$

Thus, from (24)

$$
d_n(\lambda) = \frac{((n+1)\mu/(1-\mu))k_n}{k_n + k_{n-1}\left(\lambda n^2 + (n\mu/(1-\mu))^2 - (n\mu/(1-\mu))d_{n-1}(\lambda)\right)}.
$$

Finally from (14) we obtain (29), and from (24) we get the initial condition.

Remark 2 Although the coefficients $d_n(\lambda)$ appear in the previous results for $n \geq 1$, we can start the recurrence relation (29) with the initial condition $d_{-1}(\lambda) = 0$, obtaining the same coefficients for $n \geq 1$. Moreover, for each fixed $n \geq 1$ the coefficient $d_n(\lambda)$ is a rational function in λ of degree $n-1$ in the numerator and of degree n in the denominator. Thus $\lim_{\lambda\to\infty} d_n(\lambda) = 0$ for all $n \geq 1$.

Remark 3 We can write the coefficients $d_n(\lambda)$ given by (29) as

$$
\begin{aligned}
d_n(\lambda) &= \frac{N_{n-1}(\lambda)}{D_n(\lambda)} = \frac{\vartheta_n}{(\lambda\varpi_n + \nu_n) - \mu(1-\mu)^2 d_{n-1}(\lambda)} \\
&= \frac{\vartheta_n}{(\lambda\varpi_n + \nu_n) - \mu(1-\mu)^2 N_{n-2}(\lambda)/D_{n-1}(\lambda)},
\end{aligned}
$$

where

$$\varpi_n = (1-\mu)^3 n, \quad \nu_n = (1-\mu)\mu(\gamma + n(\mu+1) - 1),$$
$$\vartheta_n = \mu^2(n+1)(\gamma+n-1),$$

i.e.,

$$\frac{N_{n-1}(\lambda)}{D_n(\lambda)} = \frac{\vartheta_n D_{n-1}(\lambda)}{(\lambda\varpi_n + \nu_n)D_{n-1}(\lambda) - \mu(1-\mu)^2 N_{n-2}(\lambda)}.$$

Thus, the denominators $D_n(\lambda)$ satisfy the following three-term recurrence relation:

$$D_n(\lambda) = (\lambda\varpi_n + \nu_n)D_{n-1}(\lambda) - \mu(1-\mu)^2\vartheta_{n-1}D_{n-2}(\lambda), \quad n \geq 2,$$

which in the monic case, $E_n(\lambda) = D_n(\lambda)/(n!(\mu-1)^{2n+1})$, can be written

$$E_n(\lambda) = (\lambda - \beta_n)E_{n-1}(\lambda) - \kappa_n E_{n-2}(\lambda), \quad n \geq 2,$$
$$E_0(\lambda) = 1, \qquad E_1(\lambda) = \lambda + \frac{\gamma\mu}{(1-\mu)^2},$$

where

$$\beta_n = \frac{\mu}{(1-\mu)^2}\left(\frac{1-\gamma}{n} - (\mu+1)\right), \qquad \kappa_n = \frac{\mu^3(\gamma+n-2)}{(\mu-1)^4(n-1)}.$$

If $\{P_n(x)\}_n$ is a MOPS satisfying the three-term recurrence relation

$$P_n(x) = (x - \phi_n)P_{n-1}(x) - \psi_n P_{n-2}(x), \quad n \geq 2,$$

then the polynomials $S_n(x) = \alpha^{-n}P_n(\alpha x + \omega)$, with $\alpha \neq 0$, satisfy [5, p. 25]

$$S_n(x) = \left(x - \frac{\phi_n - \omega}{\alpha}\right)S_{n-1}(x) - \frac{\psi_n}{\alpha^2}S_{n-2}(x), \quad n \geq 2.$$

In [5, p. 187, Eq. (5.18)] we find the Pollaczek polynomials $r_n(x; a, b, c) \equiv r_n(x)$ satisfy the following three-term recurrence relation:

$$
\begin{aligned}
r_n(x) = {} & \left(x - \frac{\xi a - (n+a-1)b}{2(n+a)(n+a-1)} \right) r_{n-1}(x) \\
& - \frac{(n-1)(n+c-1)}{4(n+a-1)^2} r_{n-2}(x), \quad n \geq 2, \\
& r_0(x) = 1, \quad r_1(x) = x + \frac{b-\xi}{2(1+a)},
\end{aligned}
$$

where ξ is either a root of $ax^2 + bx + a - c = 0$. If we choose

$$
\begin{aligned}
\alpha = \frac{(1-\mu)^2}{2\mu^{3/2}}, \quad \omega = \frac{\mu+1}{2\sqrt{\mu}}, \quad a = 0, \\
b = \frac{\gamma - 1}{\sqrt{\mu}}, \quad c = \gamma - 1, \quad \xi = \sqrt{\mu},
\end{aligned}
\tag{30}
$$

then

$$
E_n(\lambda) = \alpha^{-n} r_n \left(\alpha\lambda + \omega; 0, \frac{\gamma-1}{\sqrt{\mu}}, \gamma - 1 \right), \quad n \geq 0,
$$

which are orthogonal with respect to the weight function given in [5, p. 187, Eq. (5.20)]. Moreover, if $\gamma = 1$ the coefficients in the three-term recurrence relation for $E_n(\lambda)$ are constant, and $E_n(\lambda)$ are co-recursive of monic second kind Chebyshev polynomials $\{U_n(x)\}_n$ [5, p. 5]. Therefore $E_n(\lambda)$ can be computed explicitly by means of

$$
E_n(\lambda) = \alpha^{-n} \left(U_n(\alpha\lambda + \omega) - \frac{\mu^{1/2}}{2} U_{n-1}(\alpha\lambda + \omega) \right), \quad n \geq 1, \tag{31}
$$

where α and ω are defined in (30).

Monic Meixner orthogonal polynomials are related with monic Laguerre orthogonal polynomials $\{L_n^{(\alpha)}(x)\}_n$ by means of the following limit relation (see [5, p. 177, Eq. (3.8)]):

$$
\lim_{\mu \to 1} (1-\mu)^n M_n^{(\alpha+1,\mu)} \left(\frac{x}{1-\mu} \right) = L_n^{(\alpha)}(x). \tag{32}
$$

A limit relation between monic Meixner–Sobolev orthogonal polynomials $\{Q_n^{(\gamma,\mu)}(x;\lambda)\}_n$ and monic Laguerre–Sobolev orthogonal polynomials [21] appears in a natural way.

PROPOSITION 4 *The following limit relation holds*:

$$\lim_{\mu\to 1}(1-\mu)^n Q_n^{(\alpha+1,\mu)}\left(\frac{x}{1-\mu};\frac{\lambda}{(1-\mu)^2}\right)=Q_n^{(\alpha)}(x),\quad n\geq 1, \tag{33}$$

where $\{Q_n^{(\alpha)}(x)\}_n$ *are the Laguerre–Sobolev polynomials* [21].

Proof From Eq. (28) we have for $n\geq 2$

$$\begin{aligned}(1-\mu)^n Q_n^{(\alpha+1,\mu)}&\left(\frac{x}{1-\mu};\frac{\lambda}{(1-\mu)^2}\right)\\ &=(1-\mu)^n M_n^{(\alpha+1,\mu)}\left(\frac{x}{1-\mu}\right)+n\mu(1-\mu)^{n-1}M_{n-1}^{(\alpha+1,\mu)}\left(\frac{x}{1-\mu}\right)\\ &\quad-(1-\mu)d_{n-1}\left(\frac{\lambda}{(1-\mu)^2}\right)(1-\mu)^{n-1}Q_{n-1}^{(\alpha+1,\mu)}\left(\frac{x}{1-\mu};\frac{\lambda}{(1-\mu)^2}\right).\end{aligned}$$

Since $(1-\mu)d_n(\lambda/(1-\mu)^2)$ converges to the coefficients given in [21, Proposition 3.3] when $\mu\to 1$, the result follows by using the limit relation (32) as well as the equality

$$Q_1^{(\alpha+1,\mu)}\left(\frac{x}{1-\mu};\frac{\lambda}{(1-\mu)^2}\right)=M_1^{(\alpha+1,\mu)}\left(\frac{x}{1-\mu}\right).$$

4 THE LINEAR OPERATOR $\mathcal{S}$

Even the inner product in (15) no longer satisfies the basic property $\langle xp(x),q(x)\rangle_S=\langle p(x),xq(x)\rangle_S$, i.e., $\{Q_n(x)\}_n$ does not satisfy a three-term recurrence relation, this inner product is symmetric with respect to the new operator $\mathcal{S}$.

PROPOSITION 5 *If we define the polynomial*

$$h(x)=\mu(x+\gamma-1), \tag{34}$$

with $0<\mu<1$ *and* $\gamma>0$, *and the linear difference operator* $\mathcal{S}$ *by*

$$\begin{aligned}\mathcal{S} &\equiv h(x)\mathcal{I} + \lambda(x - h(x))\Delta - \lambda x \Delta\nabla \\ &\equiv h(x)\mathcal{I} - \lambda(x\Delta\nabla + (x(\mu-1) + \mu(\gamma-1))\Delta),\end{aligned} \tag{35}$$

where $\mathcal{I}$ *is the identity operator, then*

$$\langle h(x)p(x), q(x)\rangle_{\mathrm{S}} = \langle p(x), \mathcal{S}q(x)\rangle, \tag{36}$$

for every polynomial p and q.

Proof It is well known (see e.g. [25, p. 21, Eq. (2.1.17)]), that $\rho(s)$ defined in (8) satisfies

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s)+\tau(s)}{\sigma(s+1)},$$

with σ and τ given in (7). Since $\sigma(s)+\tau(s)=\mu(\gamma+s)$, we obtain

$$s\rho(s) = \mu(\gamma+s-1)\rho(s-1). \tag{37}$$

Now we can compute, by using (6)

$$\begin{aligned}&\langle h(x)p(x), q(x)\rangle_{\mathrm{S}} \\ &= \sum_{s=0}^{\infty} h(s)p(s)q(s)\rho(s) + \lambda\sum_{s=0}^{\infty}\Delta(h(s)p(s))\Delta q(s)\rho(s) \\ &= \sum_{s=0}^{\infty} p(s)(h(s)q(s) - \lambda h(s)\Delta q(s))\rho(s) \\ &\quad + \lambda\sum_{s=0}^{\infty} h(s+1)p(s+1)\Delta q(s)\rho(s) \\ &= \sum_{s=0}^{\infty} p(s)(h(s)q(s) - \lambda h(s)\Delta q(s))\rho(s) \\ &\quad + \lambda\sum_{s=1}^{\infty} h(s)p(s)\nabla q(s)\rho(s-1).\end{aligned}$$

By using Eq. (37) we can write the above expression as

$$\langle h(x)p(x), q(x)\rangle_S = \sum_{s=0}^{\infty} p(s)(h(s)q(s) - \lambda h(s)\Delta q(s) + \lambda s\nabla q(s))\rho(s)$$

and from (6) we obtain (36).

Remark 4 Notice that the linear operator $\mathcal{S}$ maps polynomials of exact degree n in polynomials of exact degree $n+1$.

THEOREM 1 *The linear operator* $\mathcal{S}$ *defined in Eq.* (35) *is symmetric with respect to the Sobolev inner product* (15), *i.e.*,

$$\langle \mathcal{S}p(x), q(x)\rangle_S = \langle p(x), \mathcal{S}q(x)\rangle_S. \tag{38}$$

Proof

$$\begin{aligned}
\langle \mathcal{S}p(x), q(x)\rangle_S &= \sum_{s=0}^{\infty} \mathcal{S}p(s)q(s)\rho(s) + \lambda\sum_{s=0}^{\infty} \Delta(\mathcal{S}p(s))\Delta q(s)\rho(s)\\
&= \sum_{s=0}^{\infty} \mathcal{S}p(s)(q(s) - \lambda\Delta q(s))\rho(s)\\
&\quad + \lambda\sum_{s=0}^{\infty} \mathcal{S}p(s+1)\Delta q(s)\rho(s)\\
&= \sum_{s=0}^{\infty} \mathcal{S}p(s)(q(s) - \lambda\Delta q(s))\rho(s)\\
&\quad + \lambda\sum_{s=1}^{\infty} \mathcal{S}p(s)\nabla q(s)\rho(s-1)\\
&= \sum_{s=0}^{\infty} \mathcal{S}p(s)(q(s) - \lambda\Delta q(s))\rho(s)\\
&\quad + \lambda\sum_{s=1}^{\infty} \mathcal{S}p(s)\nabla q(s)\rho(s-1)\frac{h(s)}{h(s)},
\end{aligned}$$

since $h(s) \neq 0$ if $s \geq 1$. At this point we must distinguish two situations: If $\gamma \neq 1$ we can write the above expression by using Eq. (37) as

$$\begin{aligned}
\langle \mathcal{S}p(x), q(x)\rangle_S &= \sum_{s=0}^{\infty} \mathcal{S}p(s)\left(q(s) - \lambda\Delta q(s) + \frac{\lambda s\nabla q(s)}{h(s)}\right)\rho(s)\\
&= \sum_{s=0}^{\infty} \frac{\mathcal{S}p(s)\mathcal{S}q(s)}{h(s)}\rho(s),
\end{aligned}$$

and this leads to the result.

Moreover, if $\gamma = 1$ then $\rho(s) = \mu^s$, $\mathcal{S} \equiv s(\mu\mathcal{I} + \lambda(1-\mu)\Delta - \lambda\Delta\nabla)$, $\rho(s-1) = \rho(s)/\mu$. We can write

$$\begin{aligned}(\mathcal{S}p(x), q(x))_S &= \sum_{s=1}^{\infty} \mathcal{S}p(s)\left(q(s) - \lambda\Delta q(s) + \frac{\lambda\nabla q(s)}{\mu}\right)\rho(s)\\ &= \sum_{s=1}^{\infty} \frac{\mathcal{S}p(s)\mathcal{S}q(s)}{h(s)}\rho(s),\end{aligned}$$

and the result holds.

Remark 5 If $\gamma = 1$ the difference operator $\mathcal{S}$ reduces to $\mathcal{S} \equiv s(\mu\mathcal{I} + \lambda(1-\mu)\,\Delta - \lambda\Delta\nabla)$ and we arrive to the analogous situation of Laguerre–Sobolev polynomials with parameter $\alpha = 0$ studied by Brenner [4].

PROPOSITION 6 *We have*

$$h(x)M_n^{(\gamma,\mu)}(x) = \mu Q_{n+1}(x) + a_{n,n}Q_n(x) + a_{n-1,n}Q_{n-1}(x), \quad n \geq 2, \tag{39}$$

where

$$a_{n,n} = \mu\left(\frac{\gamma + n - 1}{1-\mu} + d_n(\lambda)\right), \tag{40}$$

$$a_{n-1,n} = \frac{\mu}{1-\mu}(\gamma + n - 1)d_{n-1}(\lambda), \tag{41}$$

with $h(x)$ *and* $d_n(\lambda)$ *introduced in* (34) *and* (24), *respectively.*

Proof By using the three-term recurrence relation satisfied by Meixner polynomials (9) we have

$$\begin{aligned}h(x)M_n^{(\gamma,\mu)}(x) &= \mu(\gamma + x - 1)M_n^{(\gamma,\mu)}(x)\\ &= \mu M_{n+1}^{(\gamma,\mu)}(x) + \frac{\mu}{1-\mu}(\gamma + n - 1 + \mu(n+1))M_n^{(\gamma,\mu)}(x)\\ &\quad + \frac{\mu^2 n}{(1-\mu)^2}(\gamma + n - 1)M_{n-1}^{(\gamma,\mu)}(x)\\ &= \mu\left(M_{n+1}^{(\gamma,\mu)}(x) + \frac{\mu}{1-\mu}(n+1)M_n^{(\gamma,\mu)}(x)\right)\\ &\quad + \frac{\mu}{1-\mu}(\gamma + n - 1)\left(M_n^{(\gamma,\mu)}(x) + \frac{n\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x)\right).\end{aligned}$$

Using (21)–(23), the result holds.

From Propositions 2 and 6, we obtain a non-standard four-term recurrence relation for Meixner–Sobolev orthogonal polynomials.

COROLLARY 3 *The Meixner–Sobolev orthogonal polynomials* $\{Q_n(x)\}_n$ *defined in* (18) *satisfy the following recurrence relation*:

$$
\begin{aligned}
&xQ_n(x)\\
&= Q_{n+1}(x) + \left(\frac{n+\mu(\gamma+n-1)}{1-\mu} + d_n(\lambda)\right)Q_n(x)\\
&+ \left(\frac{\mu n(\gamma+n-2) - (\mu-1)(n+\mu(\gamma+n-1))d_{n-1}(\lambda)}{(\mu-1)^2} - xd_{n-1}(\lambda)\right)\\
&\times Q_{n-1}(x) + \left(\frac{\mu n(\gamma+n-2)d_{n-2}(\lambda)}{(\mu-1)^2}\right)Q_{n-2}(x), \quad n \geq 1, \qquad (42)
\end{aligned}
$$

where $d_n(\lambda)$ *are defined in* (24), *with the conventions* $d_{-1}(\lambda)=0$ *and* $d_0(\lambda)=\mu/(1-\mu)$ *and the initial conditions* $Q_{-1}(x)=0$, $Q_0(x)=1$ *and* $Q_1(x)=M_1^{(\gamma,\mu)}(x)$.

Proof Multiplying Eq. (23) by $h(x)$ and using Eq. (39) we obtain the four-term recurrence relation.

PROPOSITION 7 *We have*

$$SQ_n(x) = \mu M_{n+1}^{(\gamma,\mu)}(x) + b_{n,n}M_n^{(\gamma,\mu)}(x) + b_{n-1,n}M_{n-1}^{(\gamma,\mu)}(x), \quad n \geq 2, \qquad (43)$$

where

$$b_{n,n} = \frac{\mu^2(n+1)}{(1-\mu)}\left(1 + \frac{(\gamma+n-1)}{d_n(\lambda)(1-\mu)}\right), \qquad (44)$$

$$b_{n-1,n} = n(n+1)\left(\frac{\mu}{1-\mu}\right)^3\frac{\gamma+n-1}{d_n(\lambda)}. \qquad (45)$$

Proof If we expand $SQ_n(x)$ in terms of $\{M_n^{(\gamma,\mu)}(x)\}_n$ we can write

$$SQ_n(x) = \mu M_{n+1}^{(\gamma,\mu)}(x) + \sum_{i=0}^{n} b_{i,n}M_i^{(\gamma,\mu)}(x),$$

where

$$b_{i,n} = \frac{\langle SQ_n(x), M_i^{(\gamma,\mu)}(x)\rangle}{k_i} = \frac{\langle Q_n(x), h(x)M_i^{(\gamma,\mu)}(x)\rangle_S}{k_i}.$$

Hence $b_{i,n} = 0$ for $i = 0, 1, \ldots, n-2$, and

$$b_{n-1,n} = \frac{\langle Q_n(x), h(x)M_{n-1}^{(\gamma,\mu)}(x)\rangle_S}{k_{n-1}} = \mu\frac{\tilde{k}_n}{k_{n-1}} = \mu^2\frac{(n+1)k_n}{(1-\mu)d_n(\lambda)k_{n-1}}.$$

Finally we can compute

$$\begin{aligned} b_{n,n} &= \frac{\langle Q_n(x), h(x)M_n^{(\gamma,\mu)}(x)\rangle_S}{k_n} \\ &= \frac{\langle Q_n(x), \mu Q_{n+1}(x) + a_{n,n}Q_n(x) + a_{n-1,n}Q_{n-1}(x)\rangle_S}{k_n} \\ &= a_{n,n}\frac{\tilde{k}_n}{k_n}. \end{aligned}$$

PROPOSITION 8 *We have*

$$SQ_n(x) = \mu Q_{n+1}(x) + c_{n,n}Q_n(x) + c_{n-1,n}Q_{n-1}(x), \quad n \geq 2, \tag{46}$$

where

$$c_{n,n} = \frac{\mu^2(n+1)(\gamma+n-1)}{(1-\mu)^2 d_n(\lambda)} + \mu d_n(\lambda), \tag{47}$$

$$c_{n-1,n} = \mu\frac{\tilde{k}_n}{\tilde{k}_{n-1}} = \left(\frac{\mu}{1-\mu}\right)^2(n+1)(\gamma+n-1)\frac{d_{n-1}(\lambda)}{d_n(\lambda)}. \tag{48}$$

Proof If we expand the polynomial $SQ_n(x)$ in terms of polynomials $\{Q_n(x)\}_n$,

$$SQ_n(x) = \mu Q_{n+1}(x) + \sum_{i=0}^{n} c_{i,n}Q_i(x),$$

by using the symmetric character of the linear operator $\mathcal{S}$ we get

$$c_{i,n}=\frac{\langle \mathcal{S}Q_n(x),Q_i(x)\rangle_S}{\langle Q_i(x),Q_i(x)\rangle_S}=\frac{\langle Q_n(x),\mathcal{S}Q_i(x)\rangle_S}{\tilde{k}_i}.$$

Thus $c_{i,n}=0$ for $i=0,1,\dots,n-2$, and

$$c_{n-1,n}=\frac{\langle Q_n(x),\mathcal{S}Q_{n-1}(x)\rangle_S}{\tilde{k}_{n-1}}=\mu\frac{\tilde{k}_n}{\tilde{k}_{n-1}}.$$

Finally,

$$\begin{aligned} c_{n,n}&=\frac{\langle Q_n(x),\mathcal{S}Q_n(x)\rangle_S}{\tilde{k}_n}\\ &=\frac{\langle \mu M_{n+1}(x)+b_{n,n}M_n(x)+b_{n-1,n}M_{n-1}(x),Q_n(x)\rangle_S}{\tilde{k}_n}\\ &=\mu\frac{\langle M_{n+1}(x),Q_n(x)\rangle_S}{\tilde{k}_n}+b_{n,n}. \end{aligned}$$

So we must compute $\langle M_{n+1}(x),\ Q_n(x)\rangle_S$. Since $h(x)=\mu(x+\gamma-1)$ from (9) we get

$$\begin{aligned} b_{n,n}&=\frac{\langle Q_n(x),h(x)M_n(x)\rangle_S}{k_n}\\ &=\mu\frac{\langle Q_n(x),xM_n(x)\rangle_S}{k_n}+\mu(\gamma-1)\frac{\langle Q_n(x),M_n(x)\rangle_S}{k_n}\\ &=\mu\frac{\langle Q_n(x),M_{n+1}(x)\rangle_S}{k_n}+\mu\frac{\tilde{k}_n}{k_n}(B_n+\gamma-1), \end{aligned}$$

where B_n is given in (10). So we obtain

$$\begin{aligned} \mu\frac{\langle Q_n(x),M_{n+1}(x)\rangle_S}{\tilde{k}_n}&=\left(b_{n,n}-\mu\frac{\tilde{k}_n}{k_n}(B_n+\gamma-1)\right)\frac{k_n}{\tilde{k}_n}\\ &=a_{n,n}-\mu(B_n+\gamma-1), \end{aligned}$$

and then the result holds.

5 ZEROS

It is well known that the zeros of Meixner polynomials $M_n^{(\gamma,\mu)}(x)$ are real and distinct. They also lie on the interval of orthogonality $[0,+\infty)$ and they separate the zeros of $M_{n-1}^{(\gamma,\mu)}(x)$ (see [26] and the notes of Féjèr at the end of [12]). In this section we study the location of the zeros of Meixner–Sobolev orthogonal polynomials $\{Q_n(x)\}_n$ and an interlacing property which relates the zeros of $Q_n(x)$ to the zeros of $M_n^{(\gamma,\mu)}(x)$.

LEMMA 1 *If $n \geq 0$ and if $\gamma \geq 1$ we have $(-1)^n Q_n(0) > 0$.*

Proof We shall prove that $Q_n(0)$ and $M_n^{(\gamma,\mu)}(0)$ have the same sign or, equivalently, $Q_n(0)/M_n^{(\gamma,\mu)}(0) > 0$ for all $n \geq 0$. Then, by using the value of $M_n^{(\gamma,\mu)}(0)$ given in Eq. (12) the result follows since $0 < \mu < 1$.

If we write Eq. (28) for $x=0$ we obtain a recurrence relation for $Q_n(0)$:

$$\begin{aligned} Q_n(0) &= M_n^{(\gamma,\mu)}(0) + \frac{n\mu}{1-\mu} M_{n-1}^{(\gamma,\mu)}(0) - d_{n-1}(\lambda) Q_{n-1}(0), \quad n \geq 2, \\ Q_1(0) &= M_1^{(\gamma,\mu)}(0). \end{aligned} \tag{49}$$

By using Eq. (12) we get

$$\begin{aligned} & M_n^{(\gamma,\mu)}(0) + \frac{n\mu}{1-\mu} M_{n-1}^{(\gamma,\mu)}(0) \\ &= \left(\frac{\mu}{\mu-1}\right)^n (\gamma)_n + \frac{n\mu}{1-\mu}\left(\frac{\mu}{\mu-1}\right)^{n-1} (\gamma)_{n-1} \\ &= (\gamma-1)\frac{\mu}{\mu-1} M_{n-1}^{(\gamma,\mu)}(0). \end{aligned}$$

Thus, from the above equation we can write Eq. (49) as

$$Q_n(0) = (\gamma-1)\frac{\mu}{\mu-1} M_{n-1}^{(\gamma,\mu)}(0) - d_{n-1}(\lambda) Q_{n-1}(0).$$

From Eq. (12) we can also deduce

$$M_n^{(\gamma,\mu)}(0) = \frac{\mu}{\mu-1}(\gamma+n-1) M_{n-1}^{(\gamma,\mu)}(0).$$

So, taking into account the last expression we get

$$\frac{Q_n(0)}{M_n^{(\gamma,\mu)}(0)} = (\gamma-1)\frac{\mu}{\mu-1}\frac{M_{n-1}^{(\gamma,\mu)}(0)}{M_n^{(\gamma,\mu)}(0)} - d_{n-1}(\lambda)\frac{Q_{n-1}(0)}{M_n^{(\gamma,\mu)}(0)}$$
$$= \frac{\gamma-1}{\gamma+n-1} - d_{n-1}(\lambda)\frac{(\mu-1)}{\mu(\gamma+n-1)}\frac{Q_{n-1}(0)}{M_{n-1}^{(\gamma,\mu)}(0)}.$$

If we denote $\mathcal{A}_n = Q_n(0)/M_n^{(\gamma,\mu)}(0)$ the above expression can be written

$$\mathcal{A}_n = \frac{\gamma-1}{\gamma+n-1} - d_{n-1}(\lambda)\frac{(\mu-1)}{\mu(\gamma+n-1)}\mathcal{A}_{n-1}.$$

Finally, from Eq. (24) we obtain

$$\mathcal{A}_1 = \frac{Q_1(0)}{M_1^{(\gamma,\mu)}(0)} = 1,$$
$$\mathcal{A}_n = \frac{1}{\gamma+n-1}\left((\gamma-1) + n\frac{k_{n-1}}{\tilde{k}_{n-1}}\mathcal{A}_{n-1}\right), \quad n \geq 2.$$

Since $k_1/\tilde{k}_1 \geq 0$, $\mathcal{A}_2$ is positive for $\gamma \geq 1$, and then $\mathcal{A}_n > 0$ for all $n \geq 1$. Thus $\operatorname{sgn}(Q_n(0)) = \operatorname{sgn}(M_n^{(\gamma,\mu)}(0)) = \operatorname{sgn}((\mu-1)^n)$, for all $n \geq 1$, so we get $Q_{2k}(0) > 0$ and $Q_{2k+1}(0) < 0$. The case $n = 0$ follows from $Q_0(0) = M_0^{(\gamma,\mu)}(0) = 1$.

LEMMA 2 *Let $p(x)$ be a polynomial of degree k. If $\lambda = 0$ or $\gamma = 1$ there exists a unique polynomial $p_1(x)$ of degree k such that $\mathcal{S}p_1(x) = h(x)p(x)$, where $\mathcal{S}$ and $h(x)$ are defined in Proposition* 5.

Proof If $\lambda = 0$ the linear operator $\mathcal{S}$ becomes $\mathcal{S} \equiv h(x)\mathcal{I}$, where $\mathcal{I}$ stands for the identity operator. Then it is sufficient to take $p_1(x) = p(x)$.

If $\gamma = 1$, the linear operator $\mathcal{S}$ can be written

$$\mathcal{S} \equiv \mu x\mathcal{I} + \lambda(1-\mu)x\Delta - \lambda x\Delta\nabla.$$

Let us expand

$$p_1(x) = \sum_{i=0}^{k} b_i(x+1)^{[i]}, \qquad h(x)p(x) = \mu xp(x) = \sum_{i=0}^{k+1} a_i x^{[i]}.$$

The following basic properties will be useful in the proof:

$$xx^{[m]} = x^{[m+1]} + mx^{[m]}, \quad \Delta x^{[m]} = mx^{[m-1]}, \quad \nabla x^{[m]} = m(x-1)^{[m-1]},$$
$$h(x)(x+1)^{[m]} = \mu\big(x^{[m+1]} + 2mx^{[m]} + m(m-1)x^{[m-1]}\big),$$
$$\lambda h(x)\Delta(x+1)^{[m]} = \lambda m\mu\big(x^{[m]} + 2(m-1)x^{[m-1]}$$
$$+ (m-1)(m-2)x^{[m-2]}\big),$$
$$\lambda x\Delta x^{[m]} = \lambda m\big(x^{[m]} + (m-1)x^{[m-1]}\big).$$

Hence, the action of the operator S on $p_1(x)$ yields

$$Sp_1(x) = \sum_{i=0}^{n} b_i\Big(\mu x^{[i+1]} + i(\lambda + \mu(2-\lambda))x^{[i]}$$
$$+ i(i-1)(\lambda + \mu(1-2\lambda))x^{[i-1]} - i(i-1)(i-2)\lambda\mu x^{[i-2]}\Big).$$

From the equality $Sp_1(x) = h(x)p(x)$ we obtain a system of $(k+1)$ linear equations with $(k+1)$ unknowns. It has a unique solution which can be obtained using the forward substitution method.

COROLLARY 4 *Let $p(x)$ be a polynomial of degree k and assume $\gamma = 1$. Let $p_1(x)$ be the polynomial of degree k such that $Sp_1(x) = h(x)p(x)$. Then*

$$\langle q(x), p_1(x)\rangle_S = \langle q(x), p(x)\rangle - \lambda\frac{q(0)\nabla p_1(0)}{\mu}, \quad \forall q(x) \in \mathbb{P}. \qquad (50)$$

Proof We have

$$\langle q(x), p_1(x)\rangle_S = \sum_{s=0}^{\infty} q(s)(p_1(s) - \lambda\Delta p_1(s))\rho(s) + \lambda\sum_{s=0}^{\infty} q(s+1)\Delta p_1(s)\rho(s)$$
$$= \sum_{s=0}^{\infty} q(s)(p_1(s) - \lambda\Delta p_1(s))\rho(s) + \lambda\sum_{s=1}^{\infty} q(s)\nabla p_1(s)\rho(s-1)$$
$$= \sum_{s=0}^{\infty} q(s)(p_1(s) - \lambda\Delta p_1(s))\rho(s) + \lambda\sum_{s=1}^{\infty}\frac{q(s)\nabla p_1(s)\rho(s)}{\mu}$$
$$= \sum_{s=0}^{\infty} q(s)\left(p_1(s) - \lambda\Delta p_1(s) + \lambda\frac{\nabla p_1(s)}{\mu}\right)\rho(s) - \lambda\frac{q(0)\nabla p_1(0)}{\mu}$$
$$= \sum_{s=0}^{\infty} q(s)\frac{Sp_1(s)}{\mu s}\rho(s) - \lambda\frac{q(0)\nabla p_1(0)}{\mu}$$
$$= \sum_{s=0}^{\infty} q(s)p(s)\rho(s) - \lambda\frac{q(0)\nabla p_1(0)}{\mu}$$
$$= \langle q(x), p(x)\rangle - \lambda\frac{q(0)\nabla p_1(0)}{\mu}.$$

LEMMA 3 *Let $p(x)$ be a polynomial of degree k. If $\lambda \neq 0$ and if $\gamma \neq 1$ there exists a unique polynomial $p_1(x)$ of degree k as well as a unique constant c_p (depending on p) such that*

$$\mathcal{S}p_1(x) = h(x)(p(x) + (x + \gamma - 2)c_p). \tag{51}$$

Proof Let us write the polynomial $p_1(x)$ in terms of the basis $\{(x+\gamma)^{[i]}\}_i$ as

$$p_1(x) = \sum_{i=0}^{k} b_i (x + \gamma)^{[i]}.$$

For such a basis we have

$$\Delta(x + \gamma)^{[i]} = i(x + \gamma)^{[i-1]}, \qquad \nabla(x + \gamma)^{[i]} = i(x + \gamma - 1)^{[i-1]}.$$

Hence, if we apply $\mathcal{S}$ to any element of the above basis we obtain

$$\mathcal{S}(x + \gamma)^{[i]} \equiv h(x)(x + \gamma)^{[i]} - \lambda i h(x)(x + \gamma)^{[i-1]} + \lambda i(x + \gamma)^{[i]},$$

or, equivalently,

$$\begin{aligned}\mathcal{S}(x + \gamma)^{[i]} =& \mu(x + \gamma - 1)^{[i+1]} + (x + \gamma - 1)^{[i]}(i(\lambda + \mu(2 - \lambda))) \\ &+ (x + \gamma - 1)^{[i-1]}(\mu i(i - 1) - 2\lambda\mu i(i - 1) + \lambda i(i - \gamma)) \\ &- (x + \gamma - 1)^{[i-2]}\lambda\mu i(i - 1)(i - 2).\end{aligned}$$

Let us expand the polynomial $h(x)p(x)$ in the basis $\{(x+\gamma-1)^{[i]}\}_i$:

$$h(x)p(x) = \sum_{i=0}^{k+1} a_i (x + \gamma - 1)^{[i]}.$$

From $\mathcal{S}p_1(x) = h(x)(p(x) + (x + \gamma - 2)c_p)$ we obtain the following system of $(k + 2)$ linear equations with $(k + 2)$ unknowns:

$$\begin{aligned}\mu b_k &= a_{k+1}, \\ \mu b_{k-1} + k(\lambda + \mu(2 - \lambda))b_k &= a_k,\end{aligned}$$

$$\mu b_{k-2} + (k-1)(\lambda + \mu(2-\lambda))b_{k-1} + k(\mu(k-1)(1-2\lambda) + \lambda(k-\gamma))b_k = a_{k-1},$$

$$\mu b_{i-3} + (i-2)(\lambda + \mu(2-\lambda))b_{i-2} + (i-1)(\mu(i-2)(1-2\lambda) + \lambda(i-\gamma-1))b_{i-1} - \lambda\mu i(i-1)(i-2)b_i = a_{i-2}, \quad i = k, \dots, 5,$$

$$\mu b_1 + 2(\lambda + \mu(2-\lambda))b_2 + 3(\mu(2)(1-2\lambda) + \lambda(4-\gamma-1))b_3 - 24\lambda\mu b_4 = a_2 + c_p,$$

$$\mu b_0 + (\lambda + \mu(2-\lambda))b_1 + 2(\mu(1-2\lambda) + \lambda(3-\gamma-1))b_2 - 6\lambda\mu b_3 = a_1,$$

$$\lambda(1-\gamma)b_1 = a_0.$$

This linear system has a unique solution $(b_0, b_1, \dots, b_k, c_p)$ which can be found using the forward substitution method. Moreover, if $\gamma \neq 1$ and $\lambda \neq 0$ then $b_1 = 0$ since $a_0 = 0$.

COROLLARY 5 *Let $p(x)$ be a polynomial of degree k and let $p_1(x)$ be the polynomial of degree k and c_p be the constant given in the previous lemma such that Eq.* (51) *is verified. Then*

$$\langle q(x), p_1(x)\rangle_S = \langle q(x), p(x)\rangle + c_p\langle x+\gamma-2, q(x)\rangle, \quad \forall q \in \mathbb{P}. \qquad (52)$$

Proof We have

$$\begin{aligned}
\langle q(x), p_1(x)\rangle_S
&= \sum_{s=0}^{\infty} q(s)p_1(s)\rho(s) + \lambda \sum_{s=0}^{\infty} \Delta q(s)\Delta p_1(s)\rho(s) \\
&= \sum_{s=0}^{\infty} q(s)(p_1(s) - \lambda\Delta p_1(s))\rho(s) + \lambda \sum_{s=0}^{\infty} q(s+1)\Delta p_1(s)\rho(s) \\
&= \sum_{s=0}^{\infty} q(s)(p_1(s) - \lambda\Delta p_1(s))\rho(s) + \lambda \sum_{s=1}^{\infty} q(s)\nabla p_1(s)\rho(s-1) \\
&= \sum_{s=0}^{\infty} q(s)(p_1(s) - \lambda\Delta p_1(s))\rho(s) + \lambda \sum_{s=1}^{\infty} q(s)\nabla p_1(s)\frac{h(s)}{h(s)}\rho(s-1) \\
&= \sum_{s=0}^{\infty} q(s)(p_1(s) - \lambda\Delta p_1(s))\rho(s) + \lambda \sum_{s=1}^{\infty} q(s)\nabla p_1(s)\frac{s}{h(s)}\rho(s),
\end{aligned}$$

where the last equality is a consequence of (37). Thus

$$\begin{aligned}
&\langle q(x), p_1(x)\rangle_S \\
&= \sum_{s=0}^{\infty} q(s)\left(p_1(s) - \lambda\Delta p_1(s) + \frac{\lambda s}{h(s)}\nabla p_1(s)\right)\rho(s) \\
&= \sum_{s=0}^{\infty} q(s)\frac{Sp_1(s)}{h(s)}\rho(s) = \sum_{s=0}^{\infty} q(s)\frac{h(s)(p(s) + (s+\gamma-2)c_p)}{h(s)}\rho(s) \\
&= \langle q(x), p(x)\rangle + c_p\langle x+\gamma-2, q(x)\rangle.
\end{aligned}$$

LEMMA 4 *Let p be a polynomial of degree n. If $\lambda \neq 0$ and $\gamma \neq 1$ then we have $(-1)^n c_p > 0$, where c_p is the constant obtained for $p(x)$ by using Lemma* 3.

Proof If we write Eq. (52) for $q(x) = Q_n(x)$ it follows

$$\langle Q_n(x), p_1(x)\rangle_S = \langle Q_n(x), p(x)\rangle + c_p\langle x+\gamma-2, Q_n(x)\rangle.$$

From Eq. (27) the above equation reads

$$\langle Q_n(x), p_1(x)\rangle_S = \langle Q_n(x), p(x)\rangle + c_p e_{1,n} k_1 = k_n + c_p e_{1,n} k_1.$$

If p is a monic polynomial of degree n, then p_1 is also a monic polynomial of degree n. Thus $\tilde{k}_n = k_n + c_p e_{1,n} k_1$, or, equivalently,

$$c_p = \frac{\tilde{k}_n - k_n}{e_{1,n} k_1}.$$

The coefficient $e_{1,n}$ can be computed from (27) and (24):

$$e_{1,n} = (-1)^{n-2}\frac{2\lambda\mu(1-\mu)}{\lambda(1-\mu)^2 + \gamma\mu}\prod_{s=2}^{n-1}\frac{(s+1)\mu}{1-\mu}\frac{\tilde{k}_s}{k_s},$$

so $\operatorname{sgn}(e_{1,n}) = (-1)^{n-2}$.

Finally, from $\tilde{k}_n > k_n$ it follows that $\operatorname{sgn}(c_p) = (-1)^{n-2}$ and then $(-1)^n c_p > 0$.

THEOREM 2 *For each $\lambda > 0$ the polynomial $Q_n(x)$, $n \geq 2$, has exactly n real and distinct zeros, where at least $n-1$ of them are positive.*

Moreover, if $\gamma \geq 1$ *then all the zeros and positive. If we denote by* $\tilde{x}_{n,1} < x_{n,2} < \cdots < x_{n,n}$ *the zeros of* $M_n^{(\gamma,\mu)}(x)$ *and if we denote by* $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ *the n different zeros of* $Q_n(x)$ *then*

$$y_{n,1} < x_{n,1} < y_{n,2} < x_{n,2} < \cdots < y_{n,n} < x_{n,n}. \tag{53}$$

Proof Let $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ be the zeros of $M_n^{(\gamma,\mu)}(x)$ and let us define

$$w_i(x) = \prod_{j=1, j\neq i}^{n} (x - x_{n,j}). \tag{54}$$

- If $\gamma \neq 1$ by using Lemma 3 we obtain a unique polynomial $p_i(x)$ of degree $n-1$ as well as a constant c_i such that $\mathcal{S}p_i(x) = h(x)(w_i(x) + (x+\gamma-2)c_i)$. From Corollary 5 we get

$$\begin{aligned} 0 &= \langle Q_n(x), p_i(x)\rangle_S = \langle w_i(x), Q_n(x)\rangle + c_i\langle x+\gamma-2, Q_n(x)\rangle \\ &= \sum_{s=0}^{\infty} w_i(s)Q_n(s)\rho(s) + c_i\sum_{s=0}^{\infty}(s+\gamma-2)Q_n(s)\rho(s). \end{aligned}$$

The Gaussian type quadrature formula based in the zeros of $M_n^{(\gamma,\mu)}(x)$ [25] leads to

$$\begin{aligned} 0 &= \langle Q_n(x), p_i(x)\rangle_S \\ &= \lambda_i w_i(x_{n,i})Q_i(x_{n,i}) + c_i\sum_{s=0}^{\infty}(s+\gamma-2)Q_n(s)\rho(s). \end{aligned} \tag{55}$$

Let us compute the second term of the above sum by using (27),

$$\begin{aligned} c_i\sum_{s=0}^{\infty}(s+\gamma-2)Q_n(s)\rho(s) &= c_i\langle x+\gamma-2, Q_n(x)\rangle \\ &= c_i\sum_{j=1}^{n} e_{j,n}\langle x+\gamma-2, M_j^{(\gamma,\mu)}(x)\rangle = c_i e_{1,n}k_1. \end{aligned}$$

Hence the sign of this second term is always negative since $k_1 > 0$, $\operatorname{sgn}(e_{1,n}) = (-1)^{n-2}$ and $\operatorname{sgn}(c_i) = (-1)^{n-1}$ (from Lemma 4 because $\deg w_i(x) = n-1$).

Thus from Eq. (55) we deduce

$$\lambda_i w_i(x_{n,i}) Q_n(x_{n,i}) = -c_i \sum_{s=0}^{\infty} (s+\gamma-2) Q_n(s) \rho(s) > 0$$

and then $Q_n(x_{n,i}) \neq 0$. Moreover $\mathrm{sgn}(Q_n(x_{n,i})) = \mathrm{sgn}(w_i(x_{n,i})) = (-1)^{n-i}$, so $Q_n(x)$ changes sign between two consecutive zeros of $M_n^{(\gamma,\mu)}(x)$.

Finally, since $\mathrm{sgn}(Q_n(x_{n,1})) = (-1)^{n-1}$ then $Q_n(x)$ has n different real zeros, which separate those of $M_n^{(\gamma,\mu)}(x)$ as stated.

- If $\gamma = 1$, we use the orthogonality of $Q_n(x)$, Corollary 5 and the Gaussian type quadrature formula for evaluating sums in order to obtain

$$\begin{aligned} 0 = \langle Q_n(x), p_i(x) \rangle_S &= \langle Q_n(x), w_i(x) \rangle - \lambda \frac{Q_n(0) \nabla p_i(0)}{\mu} \\ &= \lambda_i w_i(x_{n,i}) Q_n(x_{n,i}) - \lambda \frac{Q_n(0) \nabla p_i(0)}{\mu}, \end{aligned}$$

where $w_i(x)$ are the polynomials defined in (54).

In this case we have $\lambda_i w_i(x_{n,i}) Q_n(x_{n,i}) = (\lambda/\mu) Q_n(0) \nabla p_i(0)$. From Lemma 1 we have $(-1)^n Q_n(0) > 0$, and repeating the arguments in Lemma 4 we can obtain $(-1)^{n-2} \nabla p_i(0) > 0$. Hence we obtain $\lambda_i w_i(x_{n,i}) Q_n(x_{n,i}) > 0$, and then $Q_n(x_{n,i}) \neq 0$ as well as $\mathrm{sgn}(Q_n(x_{n,i})) = (-1)^{n-i}$. The proof follows in the same way as in the case already discussed.

Finally, if $\gamma \geq 1$ we have $(-1)^n Q_n(0) > 0$, using Lemma 1, so we can deduce that all zeros are positive.

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