Inner products involving q-differences: the little q-Laguerre–Sobolev polynomials

I. Area[a], E. Godoy[b], F. Marcellán[c,*], J.J. Moreno-Balcázar[d,e]

[a] *Departamento de Matemática Aplicada, Escola Politécnica Superior, Universidade de Santiago de Compostela, Campus de Lugo, 27002 Lugo, Spain*

[b] *Departamento de Matemática Aplicada, E.T.S.I. Industriales y Minas, Universidad de Vigo, Campus Lagoas-Marcosende, 36200 Vigo, Spain*

[c] *Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III, C. Butarque, 15, 28911 Leganés-Madrid, Spain*

[d] *Departamento de Estadística y Matemática Aplicada, Edificio Científico Técnico III, Universidad de Almería, 04120 Almería, Spain*

[e] *Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Spain*

Abstract

In this paper, polynomials which are orthogonal with respect to the inner product

$$\langle p, r\rangle_S = \sum_{k=0}^{\infty} p(q^k) r(q^k) \frac{(aq)^k (aq;q)_\infty}{(q;q)_k} + \lambda \sum_{k=0}^{\infty} (D_q p)(q^k)(D_q r)(q^k) \frac{(aq)^k (aq;q)_\infty}{(q;q)_k},$$

where D_q is the q-difference operator, $\lambda \geqslant 0$, $0 < q < 1$ and $0 < aq < 1$ are studied. For these polynomials, algebraic properties and q-difference equations are obtained as well as their relation with the monic little q-Laguerre polynomials. Some properties about the zeros of these polynomials are also deduced. Finally, the relative asymptotics $\{Q_n(x)/p_n(x;a|q)\}_n$ on compact subsets of $\mathbb{C} \setminus [0,1]$ is given, where $Q_n(x)$ is the nth degree monic orthogonal polynomial with respect to the above inner product and $p_n(x;a|q)$ denotes the monic little q-Laguerre polynomial of degree n.

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* Corresponding author.

E-mail addresses: area@lugo.usc.es (I. Area), egodoy@dma.uvigo.es (E. Godoy), pacomarc@ing.uc3m.es (F. Marcellán), jmoreno@ualm.es (J.J. Moreno-Balcázar)

1. Introduction

The original motivation for considering Sobolev orthogonal polynomials comes from the least-squares approximation problems [15]. More precisely, for a given function the problem was to find its best polynomial approximant of degree n with respect to the norm

$$||g(x)||^2 = \int_{\mathbb{R}} [g(x)]^2 \, d\boldsymbol{v}_0(x) + \lambda \int_{\mathbb{R}} [g'(x)]^2 \, d\boldsymbol{v}_1(x)$$

over all $g \in \mathscr{C}^{(1)}$, being $\boldsymbol{v}_i(x)$, $i = 0, 1$, positive Borel measures on the real line $\mathbb{R}$ having bounded or unbounded support [4,6,19].

Also, in order to find the best polynomial approximation $p(x)$ of a function $f(x)$ where besides function values $f(x_i)$ also difference derivatives at the knots are given, the following minimization problem appears in a natural way:

$$\min \sum_{k=0}^{r} \left(\sum_{x_s=a_k}^{b_k-k-1} (\Delta^k p(x_s) - \Delta^k f(x_s))^2 \omega_k(x_s) \right), \qquad \begin{array}{l} \Delta h(x) = h(x+1) - h(x), \\ \Delta^k h(x) = \Delta^{k-1}(\Delta h(x)), \end{array}$$

where $\omega_k(x)$ are discrete weight functions on $[a_k, b_k)$, i.e., each $\omega_k(x)$ is the piecewise constant function with jumps $\omega_k(x_i)$ at the points $x = x_i$ for which $x_{i+1} = x_i + 1$ and $a_k \leqslant x_i \leqslant b_k - 1$. In this situation, it seems to be interesting the analysis of the polynomials which are orthogonal with respect to the inner product

$$(p, q)_S = \sum_{k=0}^{r} \left(\sum_{x_s=a_k}^{b_k-k-1} \Delta^k p(x_s) \Delta^k q(x_s) \omega_k(x_s) \right).$$

In recent works [1,2], we have studied algebraic and analytic properties of the polynomials orthogonal with respect to a particular case ($r = 1$, $\omega_0 \equiv \omega_1$) of the above inner product

$$\langle p, q \rangle = \sum_{s=0}^{\infty} p(s)q(s)\omega(s) + \lambda \sum_{s=0}^{\infty} \Delta p(s) \Delta q(s) \omega(s),$$

where $\lambda \geqslant 0$ and $\omega(s)$ is the Pascal distribution from probability theory, $\omega(s) = \mu^s \Gamma(\gamma+s)/\Gamma(s+1)\Gamma(\gamma)$, $\gamma > 0$, $0 < \mu < 1$. For $\lambda = 0$ the corresponding polynomials are the Meixner polynomials, introduced in [23].

Moreover, if the knots are $x_{i+1} = qx_i$ (nonequidistant mesh widths), where q is a real number $q \neq 1$, and if we want to involve in the approximation the value of the function at the knots as well as the q-differences of the function, we arrive to the following minimization problem:

$$\min \sum_{k=0}^{r} \left(\sum_{x_s=a_k}^{b_k-q^{k-1}} ((D_q^k p)(x_s) - (D_q^k f)(x_s))^2 \rho_k(x_s) \right) \quad \text{with } (D_q^k h)(x) = (D_q^{k-1}(D_q h))(x),$$

where ρ_k are q-weight functions and the q-difference operator D_q is defined by

$$(D_q h)(x) = \frac{h(qx) - h(x)}{(q-1)x}, \quad x \neq 0, \; q \neq 1, \quad (D_q h)(0) = h'(0).$$

Thus, the Fourier projector in terms of polynomials which are orthogonal with respect to the inner product

$$\langle p, r\rangle_W = \sum_{k=0}^{r} \left(\sum_{x_s=a_k}^{b_k - q^{k-1}} (D_q^k p)(x_s)(D_q^k r)(x_s)\rho_k(x_s) \right)$$

allows to give the explicit expression for such a best polynomial approximation. Since we are dealing with a nonuniform lattice [27], this q-approximation constitutes a natural tool for obtaining a more accurate estimation when the first derivative of the function to be approximated increases very quickly in a neighborhood of an end point.

The aim of this paper is the study of polynomials which are orthogonal with respect to a particular case ($r = 1,\ \rho_0 \equiv \rho_1$) of the above inner product

$$\langle p, r\rangle_S = \langle p, r\rangle + \lambda\langle D_q p, D_q r\rangle = \sum_{k=0}^{\infty} p(q^k)r(q^k)\rho(q^k) + \lambda \sum_{s=0}^{\infty} (D_q p)(q^k)(D_q r)(q^k)\rho(q^k), \quad (1.1)$$

where $\lambda \geqslant 0$ and ρ is the little q-Laguerre step function [3,5,13] whose jumps are $\rho(q^k) = (aq)^k(aq;q)_\infty/(q;q)_k$, $k = 0, 1, 2, \ldots, 0 < aq < 1, 0 < q < 1$. We call (1.1) the little q-Laguerre–Sobolev inner product, by analogy with the continuous case [20].

Koekoek [11,12] studied the q-analogues of Sobolev-type orthogonal polynomials in the Laguerre case. He defined the inner product on $\mathbb{P}$, the linear space of polynomials with real coefficients,

$$(f, g)_S = \frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha + 1)} \int_0^\infty \frac{x^\alpha}{(-(1-q)x;q)_\infty} f(x)g(x)\,\mathrm{d}x + \sum_{v=0}^{N} M_v (D_q^v f)(0)(D_q^v g)(0),$$

where $\alpha > -1$, N is a nonnegative integer, $M_v \geqslant 0$ for all $v \in \{0, 1, 2, \ldots, N\}$. Note that the mass points are zeros of the polynomial $\sigma(x)$ appearing in the Pearson-type q-difference equation for the orthogonality weight function $\rho(x)$, i.e., $(D_q(\sigma\rho))(x) = \tau(x)\rho(x)$. This location of the mass points allowed to the author to obtain a representation of the polynomials orthogonal with respect to the above inner product as a basic hypergeometric series ${}_{N+2}\varphi_{N+2}$. In the present work, we consider a different problem since we deal with an inner product involving q-differences of order 1 and nonatomic measures. Furthermore, if we consider an appropriate limit of the polynomials which are orthogonal with respect to (1.1) we recover the so-called Laguerre–Sobolev orthogonal polynomials (see e.g. [20,28]).

The structure of the paper is as follows: Section 2 contains some basic definitions and notations as well as the relations for monic little q-Laguerre polynomials $\{p_n(x; a|q)\}_n$ which will be useful within the paper. In Section 3, we introduce the monic little q-Laguerre–Sobolev orthogonal polynomials $\{Q_n(x)\}_n$. In Section 4, a linear q-difference operator $\mathscr{S}$ on $\mathbb{P}$ is defined. We prove that $\mathscr{S}$ is a symmetric operator with respect to the little q-Laguerre–Sobolev inner product and we find a nonstandard four-term recurrence relation for the polynomials $\{Q_n(x)\}_n$. In Section 5, we study the zero distribution of little q-Laguerre–Sobolev orthogonal polynomials. Finally, in Section 6, the relative asymptotics $\{Q_n(x)/p_n(x; a|q)\}_n$ and the asymptotic behavior of the ratio of two consecutive little q-Laguerre–Sobolev orthogonal polynomials are obtained.

2. Basic definitions and notations

2.1. Linear functionals

Let $\mathbb{P}$ be the linear space of real polynomials and let $\mathbb{P}'$ be its algebraic dual space. We denote by $(\boldsymbol{u}, f)$ the duality bracket for $\boldsymbol{u} \in \mathbb{P}'$ and $f \in \mathbb{P}$, and by $(\boldsymbol{u})_n = (\boldsymbol{u}, x^n)$, with $n \geqslant 0$, the canonical moments of $\boldsymbol{u}$.

Definition 1. Given a real number c, the Dirac functional $\boldsymbol{\delta}_c$ is defined by $(\boldsymbol{\delta}_c, p(x)) := p(c)$, for every $p \in \mathbb{P}$.

Definition 2. Given a linear functional $\boldsymbol{u}$, we define, for each polynomial p, the linear functional $p\boldsymbol{u}$ as follows $(p\boldsymbol{u}, r(x)) := (\boldsymbol{u}, p(x)r(x))$, for every $r \in \mathbb{P}$. For each real number c the linear functional $(x-c)^{-1}\boldsymbol{u}$ is given by

$$((x-c)^{-1}\boldsymbol{u}, r(x)) := (\boldsymbol{u}, (r(x)-r(c))/(x-c)) \quad \text{for every } r \in \mathbb{P}. \tag{2.1}$$

Note that

$$(x-c)^{-1}((x-c)\boldsymbol{u}) = \boldsymbol{u} - (\boldsymbol{u})_0 \boldsymbol{\delta}_c \quad \text{for every } \boldsymbol{u} \in \mathbb{P}', \tag{2.2}$$

while

$$(x-c)((x-c)^{-1}\boldsymbol{u}) = \boldsymbol{u}. \tag{2.3}$$

2.2. Monic little q-Laguerre orthogonal polynomials

In what follows, we shall always assume that $0 < q < 1$. The q-derivative operator D_q is defined by [7, Eq. (2.3)]

$$(D_q p)(x) = \frac{p(qx) - p(x)}{(q-1)x}, \quad x \neq 0 \tag{2.4}$$

and $(D_q p)(0) = p'(0)$ by continuity, for $p \in \mathbb{P}$. Note that $\lim_{q \uparrow 1} (D_q p)(x) = p'(x)$, for every $p \in \mathbb{P}$. This q-difference operator satisfies the following properties which will be useful in the next sections:

$$(D_q(pr))(x) = r(x)(D_q p)(x) + p(qx)(D_q r)(x) \quad \text{for every } r, p \in \mathbb{P}. \tag{2.5}$$

$$(D_q p)(x) = (D_{q^{-1}} p)(qx) \quad \text{for every } p \in \mathbb{P}. \tag{2.6}$$

Monic little q-Laguerre polynomials $p_n(x; a|q)$ are the polynomials orthogonal with respect to the inner product (see [13])

$$\langle p, r \rangle = \sum_{k=0}^{\infty} \frac{(aq)^k (aq;q)_\infty}{(q;q)_k} p(q^k) r(q^k), \quad 0 < aq < 1, \quad \text{for every } p, r \in \mathbb{P}, \tag{2.7}$$

where the q-shifted factorials are defined [5, pp. 3,6]

$$(b;q)_0 = 1, \quad (b;q)_k = \prod_{j=1}^{k}(1-bq^{j-1}), \quad k\geqslant 1 \quad \text{and} \quad (b;q)_\infty = \lim_{n\to\infty}(b;q)_n = \prod_{j=1}^{\infty}(1-bq^{j-1}). \tag{2.8}$$

Using the q-integral introduced by Thomae [30,31] and Jackson [10]

$$\int_0^b f(x)d_q(x) := (1-q)b\sum_{n=0}^{\infty} f(bq^n)q^n, \quad b>0$$

and since $(q;q)_k = (q;q)_\infty/(q^{k+1};q)_\infty$, the inner product (2.7) can be written for $a=q^\alpha$ as

$$\langle p,r\rangle = \frac{(q^{\alpha+1};q)_\infty}{(1-q)(q;q)_\infty}\int_0^1 p(x)r(x)x^\alpha(qx;q)_\infty d_q(x), \quad \alpha > -1.$$

Little q-Laguerre polynomials are a particular family of little q-Jacobi polynomials and they constitute q-analogues of Laguerre polynomials (see [14, p. 117] and [8, p. 272]). They are related with monic orthogonal Wall polynomials $\{W_n(x;b,q)\}_n$ (see [3,5, p. 198, p. 196]) as follows:

$$p_n(x;a|q) = \frac{W_n(qx;aq,q)}{q^n}, \quad n\geqslant 0.$$

They are also related with monic q-Laguerre polynomials $\{L_n^{(\alpha)}(x;q)\}_n$ (see [5,9,25, p. 194]) by

$$p_n(x;q^\alpha|q) = \left(\frac{1-q}{q}\right)^n L_n^{(\alpha)}\left(\frac{qx}{1-q};q\right), \quad n\geqslant 0,\ \alpha > -1.$$

For monic little q-Laguerre polynomials the following properties are known.

2.2.1. Three-term recurrence relation

We have [13]

$$\begin{aligned} x\,p_n(x;a|q) &= p_{n+1}(x;a|q) + B_n\,p_n(x;a|q) + C_n\,p_{n-1}(x;a|q), \quad n\geqslant 1,\\ B_n &= q^n(1+a-aq^n(q+1)), \quad C_n = aq^{2n-1}(q^n-1)(aq^n-1) \end{aligned} \tag{2.9}$$

with the initial conditions $p_0(x;a|q)=1$ and $p_1(x;a|q) = x-(1-aq)$.

2.2.2. q-difference representations

In [22] we can find

$$p_n(x;a|q) = \frac{D_q\,p_{n+1}(x;a|q)}{[n+1]_q} + aq^n(1-q)D_q\,p_n(x;a|q), \quad n\geqslant 0, \tag{2.10}$$

where

$$[0]_q = 0, \quad [n]_q = \frac{q^n-1}{q-1}, \quad n\geqslant 1. \tag{2.11}$$

Moreover, since

$$p_n(x;aq|q) = \frac{D_q\,p_{n+1}(x;a|q)}{[n+1]_q}, \quad n\geqslant 0,$$

we get

$$p_n\left(x;\frac{a}{q}\middle|q\right)=p_n(x;a|q)+aq^{n-1}(1-q^n)p_{n-1}(x;a|q),\quad n\geqslant 1. \tag{2.12}$$

2.2.3. Representation as a basic hypergeometric function

We have [5,14,22]

$$\begin{aligned}p_n(x;a|q)&=(-1)^n q^{n(n-1)/2}(aq;q)_n\,{}_2\varphi_1\left(\begin{matrix}q^{-n}, & 0\\ & aq\end{matrix}\middle|q;qx\right)\\&=(-1)^n q^{n(n-1)/2}(aq;q)_n\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k(aq;q)_k}(qx)^k.\end{aligned} \tag{2.13}$$

From the above hypergeometric representation we get

$$p_n(0;a|q)=(-1)^n q^{n(n-1)/2}(aq;q)_n. \tag{2.14}$$

2.2.4. Squared norm and moments

Let us denote for $n\geqslant 0$,

$$k_n=\langle p_n(x;a|q),p_n(x;a|q)\rangle=\sum_{k=1}^{\infty}\frac{(aq)^k(aq;q)_\infty}{(q;q)_k}(p_n(q^k;a|q))^2=(aq^n)^n(q;q)_n(aq;q)_n \tag{2.15}$$

(see [13]). From the definition of k_n we can deduce the following relations:

$$k_0=1,\quad k_n=aq^{2n-1}(q^n-1)(aq^n-1)k_{n-1},\quad n\geqslant 1. \tag{2.16}$$

If we denote the little q-Laguerre linear functional $\boldsymbol{u}^{(a)}$ as

$$(\boldsymbol{u}^{(a)},p(x))=\sum_{k=0}^{\infty}\frac{(aq)^k(aq;q)_\infty}{(q;q)_k}p(q^k)\quad\text{for every }p\in\mathbb{P} \tag{2.17}$$

then the canonical moments associated with $\boldsymbol{u}^{(a)}$ can be derived from [3, p. 198] and they are

$$(\boldsymbol{u}^{(a)},x^n)=(aq;q)_n,\quad n\geqslant 0. \tag{2.18}$$

3. Little q-Laguerre–Sobolev orthogonal polynomials

Let us consider the Sobolev inner product defined on $\mathbb{P}$ by

$$\langle p,r\rangle_S=\sum_{k=0}^{\infty}p(q^k)r(q^k)\frac{(aq)^k(aq;q)_\infty}{(q;q)_k}+\lambda\sum_{k=0}^{\infty}(D_qp)(q^k)(D_qr)(q^k)\frac{(aq)^k(aq;q)_\infty}{(q;q)_k}, \tag{3.1}$$

where D_q is the q-difference operator defined in (2.4), $0<aq<1$ and $\lambda\geqslant 0$.

We shall denote by $\{Q_n(x;a;\lambda|q)\}_n\equiv\{Q_n(x)\}_n$ the MOPS associated with the (Sobolev) inner product $\langle .,.\rangle_S$. Such a sequence is said to be the little q-Laguerre–Sobolev MOPS.

Proposition 1. *Let* $\{p_n(x;a|q)\}_n$ *be the little q-Laguerre MOPS and let* $\{Q_n(x)\}_n$ *be the little q-Laguerre–Sobolev MOPS. The following relation holds*:

$$p_n(x;a|q) + aq^{n-1}(1-q^n)p_{n-1}(x;a|q) = Q_n(x) + d_{n-1}(\lambda,a|q)Q_{n-1}(x), \quad n\geqslant 2, \tag{3.2}$$

where

$$d_{n-1}(\lambda,a|q) \equiv d_{n-1}(\lambda) = aq^{n-1}(1-q^n)\frac{k_{n-1}}{\tilde{k}_{n-1}}, \tag{3.3}$$

$$\tilde{k}_n = \langle Q_n, Q_n\rangle_S. \tag{3.4}$$

Proof. If we expand

$$p_n(x;a|q) + aq^{n-1}(1-q^n)p_{n-1}(x;a|q) = Q_n(x) + \sum_{i=0}^{n-1} f_{i,n}Q_i(x),$$

then

$$f_{i,n} = \frac{\langle p_n(x;a|q) + aq^{n-1}(1-q^n)p_{n-1}(x;a|q), Q_i(x)\rangle_S}{\langle Q_i, Q_i\rangle_S}.$$

Thus, for $0\leqslant i\leqslant n-1$ we have, by using (2.10),

$$\begin{aligned} f_{i,n} &= \frac{1}{\tilde{k}_i}\{\langle p_n(x;a|q) + aq^{n-1}(1-q^n)p_{n-1}(x;a|q), Q_i(x)\rangle_S\} \\ &= \frac{1}{\tilde{k}_i}\{\langle p_n(x;a|q) + aq^{n-1}(1-q^n)p_{n-1}(x;a|q), Q_i(x)\rangle + \lambda[n]_q\langle p_{n-1}(x;a|q), D_qQ_i(x)\rangle\} \\ &= \frac{\langle p_n(x;a|q) + aq^{n-1}(1-q^n)p_{n-1}(x;a|q), Q_i(x)\rangle}{\tilde{k}_i}. \end{aligned}$$

Then,

$$f_{i,n} = \begin{cases} 0 & \text{if } 0\leqslant i\leqslant n-2, \\ aq^{n-1}(1-q^n)\dfrac{k_{n-1}}{\tilde{k}_{n-1}} & \text{if } i=n-1. \quad \square \end{cases}$$

Corollary 1. *The little q-Laguerre–Sobolev orthogonal polynomials satisfy*

$$Q_n(x) = \sum_{j=1}^{n} e_{j,n}p_j(x;a|q), \quad n\geqslant 1, \tag{3.5}$$

where

$$e_{n,n} = 1, \tag{3.6}$$

$$e_{j,n} = (-1)^{n-j}(d_j(\lambda) + aq^j(q^{j+1}-1))\prod_{s=j+1}^{n-1} d_s(\lambda), \quad 1\leqslant j\leqslant n-1,\ n\geqslant 2, \tag{3.7}$$

with the convention $\prod_{s=m}^{m-1} d_s(\lambda) = 1$.

Proof. From (3.2) and taking into account that $Q_1(x) = p_1(x; a|q)$, (3.5) is derived. □

We can compute recursively the coefficients $d_n(\lambda)$ and the norms $\tilde{k}_n$ defined in (3.3) and (3.4), respectively, by means of

Proposition 2. *For* $n \geqslant 2$, *we have*

$$d_n(\lambda) = \frac{-a^2 q^{3n+1}(q^n - 1)(aq^n - 1)(q^{n+1} - 1)}{aq^{2n+1}(q^n - 1)(aq^n - 1) + ([n]_q)^2(\lambda q^2 + a^2(q-1)^2 q^{2n}) + aq^{n+1}(q^n - 1)d_{n-1}(\lambda)}, \tag{3.8}$$

with the initial condition

$$d_1(\lambda) = -\frac{a^2(q-1)q^2(aq-1)(q^2-1)}{\lambda + a(q-1)q(aq-1)}. \tag{3.9}$$

Moreover, we have

$$\tilde{k}_n = k_n + (\lambda + (a(1-q)q^{n-1})^2)([n]_q)^2 k_{n-1} - (a(1-q^n)q^{n-1})^2 \frac{(k_{n-1})^2}{\tilde{k}_{n-1}}, \quad n \geqslant 1 \tag{3.10}$$

with $\tilde{k}_0 = k_0 = 1$.

Proof. If we denote $p_n(x; a|q) \equiv p_n(x)$, from the definition of $\tilde{k}_n$ in (3.4), using (2.10) and (3.2), we get

$$\begin{aligned}
\tilde{k}_n &= \langle Q_n(x), Q_n(x)\rangle_S = \langle Q_n(x), p_n(x)\rangle_S = k_n + \lambda\langle (D_q Q_n)(x), (D_q p_n)(x)\rangle \\
&= k_n + \lambda\langle (D_q Q_n)(x), [n]_q p_{n-1}(x) - a(1-q)[n]_q q^{n-1}(D_q p_{n-1})(x)\rangle \\
&= k_n + \lambda([n]_q)^2 k_{n-1} - \lambda a q^{n-1}(1-q^n)\langle (D_q Q_n)(x), (D_q p_{n-1})(x)\rangle \\
&= k_n + \lambda([n]_q)^2 k_{n-1} - aq^{n-1}(1-q^n)\{\langle Q_n(x), p_{n-1}(x)\rangle_S - \langle Q_n(x), p_{n-1}(x)\rangle\} \\
&= k_n + \lambda([n]_q)^2 k_{n-1} + aq^{n-1}(1-q^n)\langle Q_n(x), p_{n-1}(x)\rangle \\
&= k_n + \lambda([n]_q)^2 k_{n-1} + aq^{n-1}(1-q^n) \\
&\quad \times \langle p_n(x) + aq^{n-1}(1-q^n)p_{n-1}(x) - d_{n-1}(\lambda)Q_{n-1}(x), p_{n-1}(x)\rangle \\
&= k_n + \lambda([n]_q)^2 k_{n-1} + (aq^{n-1}(1-q^n))^2 k_{n-1} - aq^{n-1}(1-q^n)d_{n-1}(\lambda)k_{n-1}.
\end{aligned}$$

Thus, from (2.16) and (3.3) we get (3.8) as well as the initial condition (3.9). Eq. (3.10) is a consequence of the above equality and (3.3). □

Remark. Although the coefficients $d_n(\lambda)$ appear in the previous results for $n \geqslant 1$, we can start the recurrence relation (3.8) with the initial condition $d_0(\lambda) = a(1-q)$, obtaining the same coefficients for $n \geqslant 1$.

Monic little q-Laguerre orthogonal polynomials $\{p_n(x; a|q)\}_n$ are related with monic Laguerre orthogonal polynomials $\{L_n^{(\alpha)}(x)\}_n$ by means of the following limit relation (see [13, p. 105])

$$\lim_{q \uparrow 1} \frac{p_n((1-q)x; q^\alpha|q)}{(1-q)^n} = L_n^{(\alpha)}(x), \quad n \geqslant 0, \ \alpha > -1. \tag{3.11}$$

Monic Laguerre–Sobolev polynomials $\{Q_n^{(\alpha)}(x)\}_n$ (see e.g. [20,28]) are the polynomials orthogonal with respect to the Sobolev inner product

$$(p,r)_S = \int_0^\infty p(x)r(x)x^\alpha e^{-x} + \lambda \int_0^\infty p(x)r(x)x^\alpha e^{-x}, \quad \alpha > -1. \tag{3.12}$$

Monic Laguerre–Sobolev polynomials and monic Laguerre polynomials are related by means of

$$Q_n^{(\alpha)}(x) + d_{n-1}(\lambda,\alpha)Q_{n-1}^{(\alpha)}(x) = L_n^{(\alpha)}(x) + nL_{n-1}^{(\alpha)}(x), \tag{3.13}$$

where the coefficients $d_n(\lambda,\alpha)$ satisfy the following first-order recurrence relation

$$d_n(\lambda,\alpha) = \frac{(1+n)(\alpha+n)}{\alpha+(2+\lambda)n - d_{n-1}(\lambda,\alpha)}, \quad d_1(\lambda,\alpha) = \frac{2(1+\alpha)}{1+\alpha+\lambda}. \tag{3.14}$$

A limit relation between monic little q-Laguerre–Sobolev orthogonal polynomials and monic Laguerre–Sobolev orthogonal polynomials appears in a natural way.

Proposition 3. *The following limit relation holds*:

$$\lim_{q\uparrow 1} \frac{Q_n((1-q)x;q^\alpha;\lambda(1-q)^2|q)}{(1-q)^n} = Q_n^{(\alpha)}(x), \quad n \geqslant 1, \tag{3.15}$$

where $\{Q_n^{(\alpha)}(x)\}_n$ *are the monic Laguerre–Sobolev polynomials.*

Proof. From (3.2) we have for $n \geqslant 2$,

$$\frac{Q_n((1-q)x;q^\alpha;\lambda(1-q)^2|q)}{(1-q)^n} = \frac{p_n((1-q)x;q^\alpha|q)}{(1-q)^n} + q^\alpha[n]_q q^{n-1}\frac{p_{n-1}((1-q)x;q^\alpha|q)}{(1-q)^{n-1}} - \frac{d_{n-1}(\lambda(1-q)^2,q^\alpha|q)}{(1-q)}\frac{Q_{n-1}((1-q)x;q^\alpha;\lambda(1-q)^2|q)}{(1-q)^{n-1}}. \tag{3.16}$$

Since $\lim_{q\uparrow 1}[n]_q = n$ and $d_n(\lambda(1-q)^2, q^\alpha|q)/(1-q)$ converges to the coefficients $d_n(\lambda,\alpha)$ given in (3.14) when $q \uparrow 1$, the result follows by using the limit relation (3.11) as well as the equality

$$Q_1((1-q)x;q^\alpha;\lambda(1-q)^2|q) = p_1((1-q)x;q^\alpha|q). \quad \square$$

4. The linear operator $\mathscr{S}$

Even the inner product in (3.1) no longer satisfies the basic property $\langle xp(x), r(x)\rangle_S = \langle p(x), xr(x)\rangle_S$, i.e. $\{Q_n\}_n$ does not satisfy a three-term recurrence relation, there exists an operator $\mathscr{S}$ which is symmetric with respect to the inner product (3.1).

Proposition 4. *Let* $h(x)$ *be the polynomial*

$$h(x) = a(q-1)x \tag{4.1}$$

with $0 < aq < 1$, *and let* $\mathscr{S}$ *be the q-difference operator*

$$\mathscr{S} \equiv h(x)\mathscr{I} - \lambda a D_q + \lambda(1-x)D_{q^{-1}}, \tag{4.2}$$

where $\mathscr{I}$ *is the identity operator. Then*

$$\langle h(x)p(x), r(x)\rangle_S = \langle p(x), (\mathscr{S}r)(x)\rangle \quad \textit{for every } p,r \in \mathbb{P}. \tag{4.3}$$

Proof. The weight function

$$\rho(q^k) = \frac{(aq)^k (aq;q)_\infty}{(q;q)_k}$$

satisfies

$$(1 - q^k)\rho(q^k) = aq\rho(q^{k-1}), \quad k \geqslant 1. \tag{4.4}$$

Using (2.5) we get

$$\begin{aligned}\langle h(x)p(x), r(x)\rangle_S &= \langle h(x)p(x), r(x)\rangle + \lambda\langle (D_q(hp))(x), (D_q r)(x)\rangle \\ &= \langle h(x)p(x), r(x)\rangle - \lambda a\langle p(x), (D_q r)(x)\rangle + \lambda aq\langle p(qx), (D_q r)(x)\rangle \\ &= \langle p(x), h(x)r(x) - \lambda a(D_q r)(x)\rangle + \lambda\langle (1-x)p(x), (D_{q^{-1}} r)(x)\rangle,\end{aligned}$$

where the last equality is a consequence of (4.4). Then, the result holds. □

Theorem 1. *The linear operator* $\mathscr{S}$ *defined in* (4.2) *is symmetric with respect to the Sobolev inner product* (3.1), *i.e.*,

$$\langle (\mathscr{S}p)(x), r(x)\rangle_S = \langle p(x), (\mathscr{S}r)(x)\rangle_S \quad \textit{for every } p, r \in \mathbb{P}. \tag{4.5}$$

Proof. Since $\mathscr{S}$ is a linear operator, it is sufficient to prove that

$$\langle (S\vartheta_n)(x), \vartheta_m(x)\rangle_S - \langle \vartheta_n(x), (S\vartheta_m)(x)\rangle_S = 0 \quad \text{for every } n, m \geqslant 0, \tag{4.6}$$

with $\vartheta_k(x) = x^k$.

If $n + m \leqslant 2$, it is easy to obtain the result for each particular case.

On the other hand, if $n+m > 2$, from (2.17) and (4.2) the left-hand side of (4.6) is $(\boldsymbol{u}^{(a)}, t_{n+m}(x))$, where

$$\begin{aligned}t_{n+m}(x) = {}& \lambda q\{(q^{-m}[m]_q - q^{-n}[n]_q)x^{n+m} + ((a - q^{-m})[m]_q - (a - q^{-n})[n]_q)x^{n+m-1}\} \\ & + \lambda^2 q^{1-m-n}[n]_q[m]_q\{(q^n[m]_q - q^m[n]_q)x^{n+m-2} \\ & + (q^m(1 - aq^{n-1})[n-1]_q - q^n(1 - aq^{m-1})[m-1]_q)x^{n+m-3}\}.\end{aligned} \tag{4.7}$$

From (2.18) and after some straightforward computations we deduce that $(\boldsymbol{u}^{(a)}, t_{n+m}(x)) = 0$. □

Proposition 5. *Let* $\{p_n(x;a|q)\}_n$ *be the little q-Laguerre MOPS and* $\{Q_n(x)\}_n$ *be the little q-Laguerre–Sobolev MOPS. We have*

$$h(x)p_n(x;a|q) = a(q-1)Q_{n+1}(x) + a_{n,n}Q_n(x) + a_{n-1,n}Q_{n-1}(x), \quad n \geqslant 2, \tag{4.8}$$

where

$$a_{n,n} = a(q-1)(d_n(\lambda) + q^n(1 - aq^n)), \tag{4.9}$$

$$a_{n-1,n} = aq^n(q-1)(1 - aq^n)d_{n-1}(\lambda) \tag{4.10}$$

with $h(x)$ *and* $d_n(\lambda)$ *defined in* (4.1) *and* (3.3), *respectively.*

Proof. By using the three-term recurrence relation for the little q-Laguerre polynomials (2.9) we have

$$\begin{aligned} h(x)p_n(x;a|q) &= a(q-1)xp_n(x;a|q) = a(q-1)(p_{n+1}(x;a|q) + B_n p_n(x;a|q) + C_n p_{n-1}(x;a|q)) \\ &= a(q-1)(p_{n+1}(x;a|q) + aq^n(1-q^{n+1})p_n(x;a|q) \\ &\quad + q^n(1-aq^n)p_n(x;a|q) + C_n p_{n-1}(x;a|q)) \\ &= a(q-1)(p_{n+1}(x;a|q) + aq^n(1-q^{n+1})p_n(x;a|q) \\ &\quad + q^n(1-aq^n)(p_n(x;a|q) + aq^{n-1}(1-q^n)p_{n-1}(x;a|q))). \end{aligned}$$

From (3.2) the result holds. □

Taking into account Propositions 1 and 5, we deduce:

Corollary 2. *The little q-Laguerre–Sobolev orthogonal polynomials* $\{Q_n(x)\}_n$ *satisfy the following recurrence relation*:

$$\begin{aligned} Q_{n+1}(x) &= (q^n(aq^n-1) + x - d_n(\lambda) + aq^{n-1}(q^n-1))Q_n(x) \\ &\quad - (aq^{2(n-1)}(q^n-1)(aq^{n-1}-1) + d_{n-1}(\lambda)(q^n(aq^n-1) + x + aq^{n-1}(q^n-1)))Q_{n-1}(x) \\ &\quad - aq^{2n-3}(q^n-1)(aq^n-q)d_{n-2}(\lambda)Q_{n-2}(x), \quad n \geqslant 1, \end{aligned} \tag{4.11}$$

where $d_n(\lambda)$ *are defined in* (3.3), *with the convention* $d_{-1}(\lambda) = 0$ *and* $d_0(\lambda) = a(1-q)$, *and the initial conditions* $Q_{-1}(x) = 0$, $Q_0(x) = 1$ *and* $Q_1(x) = p_1(x;a|q)$.

Proof. From (3.2) and using (4.8) we obtain the above four-term recurrence relation. □

Proposition 6. *Let* $\{p_n(x;a|q)\}_n$ *be the little q-Laguerre MOPS*, $\{Q_n(x)\}_n$ *be the little q-Laguerre–Sobolev MOPS and* $\mathscr{S}$ *be the linear q-difference operator introduced in* (4.2). *We have*

$$(\mathscr{S}Q_n)(x) = a(q-1)p_{n+1}(x;a|q) + b_{n,n}p_n(x;a|q) + b_{n-1,n}p_{n-1}(x;a|q), \quad n \geqslant 2, \tag{4.12}$$

where

$$b_{n,n} = \frac{a^2q^n(1-q)(q^{n+1}-1)(q^n - aq^{2n} + d_n(\lambda))}{d_n(\lambda)}, \tag{4.13}$$

$$b_{n-1,n} = \frac{a^3q^{3n-1}(q-1)(q^n-1)(1-aq^n)(q^{n+1}-1)}{d_n(\lambda)} \tag{4.14}$$

and the coefficients $d_n(\lambda)$ *are given in* (3.3).

Proof. If we expand $(\mathscr{S}Q_n)(x)$ in terms of the sequence $\{p_n(x;a|q)\}_n$,

$$(\mathscr{S}Q_n)(x) = a(q-1)p_{n+1}(x;a|q) + \sum_{i=0}^{n} b_{i,n}p_i(x;a|q),$$

then from (4.3) we get

$$b_{i,n} = \frac{\langle (\mathscr{S}Q_n)(x), p_i(x;a|q)\rangle}{k_i} = \frac{\langle Q_n(x), h(x)p_i(x;a|q)\rangle_S}{k_i}.$$

Hence $b_{i,n}=0$ for $i=0,1,\ldots,n-2$. On the other hand,

$$b_{n-1,n}=\frac{\langle Q_n(x),h(x)p_{n-1}(x;a|q)\rangle_S}{k_{n-1}}=a(q-1)\frac{\tilde{k}_n}{k_{n-1}}$$

and

$$b_{n,n}=\frac{\langle Q_n(x),h(x)p_n(x;a|q)\rangle_S}{k_n}$$

$$=\frac{\langle Q_n(x),a(q-1)Q_{n+1}(x)+a_{n,n}Q_n(x)+a_{n-1,n}Q_{n-1}(x)\rangle_S}{k_n}=a_{n,n}\frac{\tilde{k}_n}{k_n}$$

using (4.8). □

Proposition 7. *Let* $\{p_n(x;a|q)\}_n$ *be the little q-Laguerre MOPS,* $\{Q_n(x)\}_n$ *be the little q-Laguerre–Sobolev MOPS and* $\mathscr{S}$ *be the linear q-difference operator defined in* (4.2). *The following relation holds*:

$$(\mathscr{S}Q_n)(x)=a(q-1)Q_{n+1}(x)+c_{n,n}Q_n(x)+c_{n-1,n}Q_{n-1}(x),\quad n\geqslant 2, \tag{4.15}$$

where

$$c_{n,n}=\frac{a(q-1)(aq^{2n}(aq^n-1)(q^{n+1}-1)+d_n(\lambda)^2)}{d_n(\lambda)}, \tag{4.16}$$

$$c_{n-1,n}=\frac{a^2q^{2n}(q-1)(aq^n-1)(q^{n+1}-1)d_{n-1}(\lambda)}{d_n(\lambda)} \tag{4.17}$$

and the coefficients $d_n(\lambda)$ *are given in* (3.3).

Proof. If we expand the polynomial $(\mathscr{S}Q_n)(x)$ in terms of the polynomials $\{Q_n(x)\}_n$,

$$(\mathscr{S}Q_n)(x)=a(q-1)Q_{n+1}(x)+\sum_{i=0}^{n}c_{i,n}Q_i(x)$$

and using the symmetric character of the linear operator $\mathscr{S}$, we get

$$c_{i,n}=\frac{\langle(\mathscr{S}Q_n)(x),Q_i(x)\rangle_S}{\langle Q_i(x),Q_i(x)\rangle_S}=\frac{\langle Q_n(x),(\mathscr{S}Q_i)(x)\rangle_S}{\tilde{k}_i}.$$

Thus, $c_{i,n}=0$ for $i=0,1,\ldots,n-2$. Moreover,

$$c_{n-1,n}=\frac{\langle Q_n(x),(\mathscr{S}Q_{n-1})(x)\rangle_S}{\tilde{k}_{n-1}}=a(q-1)\frac{\tilde{k}_n}{\tilde{k}_{n-1}}.$$

Finally, from Proposition 6 we get

$$c_{n,n}=\frac{\langle Q_n(x),(\mathscr{S}Q_n)(x)\rangle_S}{\tilde{k}_n}$$

$$=\frac{\langle a(q-1)p_{n+1}(x;a|q)+b_{n,n}p_n(x;a|q)+b_{n-1,n}p_{n-1}(x;a|q),Q_n(x)\rangle_S}{\tilde{k}_n}$$

$$=a(q-1)\frac{\langle p_{n+1}(x;a|q),Q_n(x)\rangle_S}{\tilde{k}_n}+b_{n,n},$$

where $b_{n,n}$ is given in (4.13). So we must compute $\langle p_{n+1}(x;a|q), Q_n(x)\rangle_S$. Since

$$b_{n,n} = \frac{\langle Q_n(x), h(x)p_n(x;a|q)\rangle_S}{k_n} = a(q-1)\frac{\langle Q_n(x), xp_n(x;a|q)\rangle_S}{k_n}$$
$$= \frac{a(q-1)}{k_n}(\langle Q_n(x), p_{n+1}(x;a|q)\rangle_S + B_n\tilde{k}_n),$$

where $h(x) = a(q-1)x$ and B_n is given in (2.9), we can express $\langle p_{n+1}(x;a|q), Q_n(x)\rangle_S$ in terms of $b_{n,n}$ and after some straightforward computations the result holds. □

5. Zeros

The zeros of the polynomial $p_n(x;a|q)$ are all real and distinct, they live on the interval of orthogonality $(0,1)$, and they separate the zeros of $p_{n-1}(x;a|q)$. In this section we study the location of the zeros of little q-Laguerre–Sobolev orthogonal polynomials $\{Q_n(x)\}_n$. An interlacing property which relates the zeros of $Q_n(x)$ with the zeros of $p_n(x;a|q)$ is proved.

Lemma 1. *For $n \geqslant 0$ and $a \leqslant 1$ we have $(-1)^n Q_n(0) > 0$.*

Proof. We shall prove that $Q_n(0)/p_n(0;a|q) > 0$, for every $n \geqslant 0$. Thus, by using the value of $p_n(0;a|q)$ given in (2.14) the result follows.

From (3.2) we obtain the following recurrence relation for $Q_n(0)$:

$$Q_n(0) = p_n(0;a|q) + aq^{n-1}(1-q^n)p_{n-1}(0;a|q) - d_{n-1}(\lambda)Q_{n-1}(0),$$
$$Q_1(0) = p_1(0;a|q).$$

By using (2.14),

$$p_n(0;a|q) + aq^{n-1}(1-q^n)p_{n-1}(0;a|q) = p_{n-1}(0;a|q)q^{n-1}(a-1).$$

Thus,

$$Q_n(0) = p_{n-1}(0;a|q)q^{n-1}(a-1) - d_{n-1}(\lambda)Q_{n-1}(0), \quad Q_1(0) = p_1(0;a|q) = aq-1.$$

From (2.14) we can also obtain

$$p_n(0;a|q) = q^{n-1}(aq^n-1)p_{n-1}(0;a|q).$$

Taking into account the last expression,

$$\frac{Q_n(0)}{p_n(0;a|q)} = \frac{a-1}{aq^n-1} - \frac{d_{n-1}(\lambda)}{q^{n-1}(aq^n-1)}\frac{Q_{n-1}(0)}{p_{n-1}(0;a|q)}.$$

If we denote $\mathscr{A}_n = Q_n(0)/p_n(0;a|q)$, the last equality reads

$$\mathscr{A}_n = \frac{a-1}{aq^n-1} - \frac{d_{n-1}(\lambda)}{q^{n-1}(aq^n-1)}\mathscr{A}_{n-1}.$$

From (3.3) we obtain

$$\mathscr{A}_1 = \frac{Q_1(0)}{p_1(0;a|q)} = 1, \quad \mathscr{A}_n = \frac{1}{1-aq^n}\left(1-a+a(1-q^n)\frac{k_{n-1}}{\tilde{k}_{n-1}}\mathscr{A}_{n-1}\right).$$

Since $k_1/\tilde{k}_1 > 0$, the coefficient $\mathscr{A}_2$ is positive for $a \leqslant 1$ and then $\mathscr{A}_n \geqslant 0$, for every $n \geqslant 1$. Thus, $\operatorname{sgn}(Q_n(0)) = \operatorname{sgn}(p_n(0; a|q))$, for every $n \geqslant 1$, so we get $Q_{2k}(0) > 0$ and $Q_{2k+1}(0) < 0$. The case $n = 0$ follows from $Q_0(0) = p_0(0; a|q) = 1$. □

Lemma 2. *Let $p_k(x)$ be a polynomial of degree k. If $\lambda = 0$ or $a = 1$, there exists a unique polynomial $g_k(x)$ of degree k such that*

$$(\mathscr{S} g_k)(x) = h(x) p_k(x), \tag{5.1}$$

where the linear operator $\mathscr{S}$ and the polynomial $h(x)$ are defined in Proposition 4.

Proof. If $\lambda = 0$ the linear operator $\mathscr{S}$ becomes $h(x)\mathscr{I}$, where $\mathscr{I}$ stands for the identity operator. Then, it is sufficient to take $g_k(x) = p_k(x)$.

If $a = 1$ the linear operator $\mathscr{S}$ can be written

$$\mathscr{S} \equiv (q-1)x\mathscr{I} - \lambda D_q + \lambda(1-x)D_{q^{-1}}.$$

Let us expand

$$g_k(x) = \sum_{i=0}^{k} b_i x^i, \quad h(x) p_k(x) = (q-1)x p_k(x) = \sum_{i=1}^{k+1} a_i x^i.$$

The action of the linear operator $\mathscr{S}$ on $g_k(x)$ yields

$$(\mathscr{S} g_k)(x) = \sum_{i=0}^{k} b_i \left((q-1)x^{i+1} - \lambda \frac{[i]_q}{q^{i-1}} x^i + \lambda [i]_q \left(\frac{1}{q^{i-1}} - 1 \right) x^{i-1} \right).$$

From the equality $(\mathscr{S} g_k)(x) = h(x) p_k(x)$ we obtain a system of $k+1$ linear equations with $k+1$ unknowns. It has a unique solution which can be found by using the forward substitution method. □

Lemma 3. *Let $p_k(x)$ be a polynomial of degree k and assume $a = 1$. Let $g_k(x)$ be the polynomial of degree k given in Lemma 2 such that $(\mathscr{S} g_k)(x) = h(x) p_k(x)$. Then*

$$\langle r(x), g_k(x) \rangle_S = \langle r(x), p_k(x) \rangle - \frac{\lambda}{1-q} r(0)(D_{q^{-1}} g_k)(0) \quad \textit{for every } r \in \mathbb{P}. \tag{5.2}$$

Proof. For each polynomial $r(x)$ we have

$$r(x) = r(qx) - h(x)(D_q r)(x). \tag{5.3}$$

Since $(\mathscr{S} g_k)(0) = 0$, from (2.1), (2.3), (2.6), (3.1), (4.2), (5.1) and (5.3) we get

$$\begin{aligned}
\langle r(x), p_k(x) \rangle &= \left\langle r(x), \frac{(\mathscr{S} g_k)(x)}{h(x)} \right\rangle = \left(\boldsymbol{u}^{(1)}, \frac{r(x)(\mathscr{S} g_k)(x)}{h(x)} \right) \\
&= \left(\boldsymbol{u}^{(1)}, \frac{r(x)(\mathscr{S} g_k)(x) - r(0)(\mathscr{S} g_k)(0)}{h(x)} \right) = ((h(x))^{-1} \boldsymbol{u}^{(1)}, r(x)(\mathscr{S} g_k)(x)) \\
&= \langle r(x), g_k(x) \rangle_S - \lambda((h(x))^{-1} \boldsymbol{u}^{(1)}, r(qx)(D_{q^{-1}} g_k)(qx) - r(x)(D_{q^{-1}} g_k)(x)) \\
&\quad - \frac{\lambda}{q-1} (\boldsymbol{u}^{(1)}, r(x)(D_{q^{-1}} g_k)(x))
\end{aligned}$$

$$= \langle r(x), g_k(x)\rangle_S - \lambda((h(x))^{-1}\boldsymbol{u}^{(1)}, h(x)D_q[r(x)(D_{q^{-1}}g_k)(x)])$$
$$- \frac{\lambda}{q-1}(\boldsymbol{u}^{(1)}, r(x)(D_{q^{-1}}g_k)(x))$$
$$= \langle r(x), g_k(x)\rangle_S - \lambda(\boldsymbol{u}^{(1)}, D_q[r(x)(D_{q^{-1}}g_k)(x)]) - \frac{\lambda}{q-1}(\boldsymbol{u}^{(1)}, r(x)(D_{q^{-1}}g_k)(x)).$$

Consequently, if we denote $t(x) = r(x)\ (D_{q^{-1}}g_k)(x)$ we have

$$\langle r(x), p_k(x)\rangle = \langle r(x), g_k(x)\rangle_S - \lambda\left(\boldsymbol{u}^{(1)}, (D_q t)(x) + \frac{t(x)}{q-1}\right). \tag{5.4}$$

Since $t(x)$ is a polynomial, writing

$$t(x) = \sum_{k=0}^{m} t_k x^k, \tag{5.5}$$

we obtain

$$\left(\boldsymbol{u}^{(1)}, (D_q t)(x) + \frac{t(x)}{q-1}\right) = \frac{(\boldsymbol{u}^{(1)})_0 t_0}{q-1} + \sum_{k=1}^{m}\left([k]_q(\boldsymbol{u}^{(1)})_{k-1} + \frac{(\boldsymbol{u}^{(1)})_k}{q-1}\right) t_k \tag{5.6}$$

and from (2.11) and (2.18) we get

$$\left(\boldsymbol{u}^{(1)}, (D_q t)(x) + \frac{t(x)}{q-1}\right) = \frac{1}{q-1}\, r(0)(D_{q^{-1}}g_k)(0). \tag{5.7}$$

Substitution of this relation into (5.4) yields (5.2). □

Lemma 4. *Let $p_k(x)$ be a polynomial of degree $k > 1$. If $\lambda > 0$ and $a \neq 1$, there exist a unique polynomial $g_k(x)$ of degree k and a unique constant c_p (depending on the polynomial p_k) such that*

$$(\mathscr{S} g_k)(x) = h(x)(p_k(x) + xc_p). \tag{5.8}$$

Proof. Let us expand

$$g_k(x) = \sum_{i=0}^{k} b_i x^i, \quad h(x)p_k(x) = a(q-1)x p_k(x) = \sum_{i=1}^{k+1} a_i x^i.$$

The action of the linear operator $\mathscr{S}$ on $g_k(x)$ can be written as

$$(\mathscr{S} g_k)(x) = \sum_{i=0}^{k} b_i\left(a(q-1)x^{i+1} - \lambda\frac{[i]_q}{q^{i-1}}x^i + \lambda[i]_q\left(\frac{1}{q^{i-1}} - a\right)x^{i-1}\right).$$

From the equality $(\mathscr{S} g_k)(x) = h(x)(p_k(x) + xc_p)$ we get the following system of $(k+1)$ linear equations with $(k+1)$ unknowns

$$a(q-1)b_k = a_{k+1},$$

$$a(q-1)b_{k-1} - \lambda\frac{[k]_q}{q^{k-1}}\, b_k = a_k,$$

$$a(q-1)b_{i-2} - \lambda \frac{[i-1]_q}{q^{i-2}} b_{i-1} + \lambda [i]_q \left(\frac{1}{q^{i-1}} - a \right) b_i = a_{i-1}, \quad (i = 4, \ldots, k)$$

$$a(q-1)b_1 - \lambda \frac{[2]_q}{q} b_2 + \lambda [3]_q \left(\frac{1}{q^2} - a \right) b_3 = a_2 + a(q-1)c_p,$$

$$a(q-1)b_0 - \lambda b_1 + \lambda [2]_q \left(\frac{1}{q} - a \right) b_2 = a_1,$$

$$(1-a)\lambda b_1 = 0.$$

This linear system has a unique solution $(b_0, b_1, \ldots, b_k, c_p)$ which can be found by using the forward substitution method. Moreover, since $\lambda > 0$ and $a \neq 1$ then $b_1 = 0$. □

Lemma 5. *Let $p_k(x)$ be a polynomial of degree $k > 1$. Let $g_k(x)$ be the polynomial of degree k and c_p be the constant given in the previous lemma such that* (5.8) *is verified. Then,*

$$\langle r(x), g_k(x) \rangle_S = \langle r(x), p_k(x) + c_p x \rangle \quad \textit{for every } r \in \mathbb{P}. \tag{5.9}$$

Proof. By using the same technique as in Lemma 3 we get

$$\langle r(x), g_k(x) \rangle_S = \langle r(x), p_k(x) + c_p x \rangle + \frac{\lambda}{a(q-1)} (D_{q^{-1}} g_k)(0) r(0).$$

Since, as stated in the previous lemma, the coefficient $b_1 = g_k'(0) = 0$, then $(D_{q^{-1}} g_k)(0) = g_k'(0) = 0$ and the result holds. □

Lemma 6. *Let $k > 1$, let $0 < x_1 < x_2 < \cdots < x_k$ be nonnegative real numbers and let be*

$$p(x) = \prod_{i=1}^{k} (x - x_i). \tag{5.10}$$

Then, $(-1)^k c_p > 0$ where c_p is the constant obtained for $p(x)$ in Lemma 4 when $\lambda > 0$ and $0 < aq < 1$.

Proof. Using the notation of Lemma 4 and since $a_{k+1} = a(q-1)$, from the Cardano-Vieta relations between coefficients of a polynomial and their zeros, we get

$$b_k = 1,$$

$$b_{k-1} = \frac{a_k}{a(q-1)} + \lambda \frac{[k]_q}{q^{k-1}} \frac{b_k}{a(q-1)} < 0,$$

$$b_{k-2} = \frac{a_{k-1}}{a(q-1)} + \lambda \frac{[k-1]_q}{q^{k-2}} \frac{b_{k-1}}{a(q-1)} - \lambda [k]_q \left(\frac{1}{q^{k-1}} - a \right) \frac{b_k}{a(q-1)} > 0,$$

since $0 < q < 1$ and $0 < aq < 1$. Thus, it is straightforward to prove that $(-1)^{k-j} b_j > 0$, $j = 2, 3, \ldots, k$. Furthermore, since $b_1 = 0$ we obtain

$$(-1)^k c_p = -\lambda \frac{[2]_q}{q} \frac{(-1)^k b_2}{a(q-1)} + \lambda [3]_q \left(\frac{1}{q^2} - a \right) \frac{(-1)^k b_3}{a(q-1)} - \frac{(-1)^k a_2}{a(q-1)} > 0. \quad □$$

Theorem 2. *Let* $\{p_n(x;a|q)\}_n$ *be the little q-Laguerre MOPS with* $0 < aq < 1$ *and* $\{Q_n(x)\}_n$ *be the little q-Laguerre–Sobolev MOPS. For each* $\lambda > 0$ *the polynomial* $Q_n(x)$, $n \geqslant 2$, *has exactly n real and distinct zeros, where at least* $n-1$ *of them are positive. Moreover, if* $a \leqslant 1$ *then all the zeros of* $Q_n(x)$ *are positive. If we denote by* $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ *the zeros of* $p_n(x;a|q)$ *and if we denote by* $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ *the n different zeros of* $Q_n(x)$ *then*

$$y_{n,1} < x_{n,1} < y_{n,2} < x_{n,2} < \cdots < y_{n,n} < x_{n,n}. \tag{5.11}$$

Proof. Assume $n > 2$. Let $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ be the zeros of $p_n(x;a|q)$ and let us define for $1 \leqslant i \leqslant n$ the polynomial $w_{n-1}^{(i)}$ of degree $n-1$ as

$$w_{n-1}^{(i)} = \prod_{j=1, j \neq i}^{n} (x - x_{n,j}). \tag{5.12}$$

• If $a \neq 1$ by using Lemma 4 we obtain a unique polynomial $g_{n-1}^{(i)}(x)$ of degree $n-1$ as well as a unique constant c_i such that $(\mathscr{S} g_{n-1}^{(i)})(x) = h(x)(w_{n-1}^{(i)}(x) + c_i x)$. From Lemma 5 we get

$$\begin{aligned} 0 = \langle Q_n(x), g_{n-1}^{(i)}(x)\rangle_S &= \langle w_{n-1}^{(i)}(x), Q_n(x)\rangle + c_i \langle x, Q_n(x)\rangle \\ &= \sum_{k=0}^{\infty} w_{n-1}^{(i)}(q^k) Q_n(q^k)\rho(q^k) + c_i \sum_{k=0}^{\infty} q^k Q_n(q^k)\rho(q^k). \end{aligned}$$

The Gaussian-type quadrature formula based in the zeros of $p_n(x;a|q)$ yields

$$0 = \langle Q_n(x), g_{n-1}^{(i)}(x)\rangle_S = \mu_i w_{n-1}^{(i)}(x_{n,i}) Q_n(x_{n,i}) + c_i \sum_{k=0}^{\infty} q^k Q_n(q^k)\rho(q^k). \tag{5.13}$$

Let us compute the second term of the above sum by using (3.5)

$$c_i \sum_{k=0}^{\infty} q^k Q_n(q^k)\rho(q^k) = c_i \langle x, Q_n(x)\rangle = c_i \sum_{j=1}^{n} e_{j,n} \langle x, p_j(x;a|q)\rangle = c_i e_{1,n} k_1.$$

Since $k_1 > 0$, $\operatorname{sgn}(e_{1,n}) = (-1)^{n-2}$ and $\operatorname{sgn}(c_i) = (-1)^{n-1}$, we obtain that the right-hand side of the above expression is always negative.

From (5.13) we deduce

$$\mu_i w_{n-1}^{(i)}(x_{n,i}) Q_n(x_{n,i}) = -c_i \sum_{k=0}^{\infty} q^k Q_n(q^k)\rho(q^k) > 0$$

and then $Q_n(x_{n,i}) \neq 0$. Moreover $\operatorname{sgn}(Q_n(x_{n,i})) = \operatorname{sgn}(w_{n-1}^{(i)}(x_{n,i})) = (-1)^{n-i}$, so $Q_n(x)$ changes sign between two consecutive zeros of $p_n(x;a|q)$.

Finally, since $\operatorname{sgn}(Q_n(x_{n,1})) = (-1)^{n-1}$ then $Q_n(x)$ has n different real zeros, which separate those of $p_n(x;a|q)$ as stated.

• If $a = 1$, we use the orthogonality of $Q_n(x)$, Lemma 3 and the Gaussian-type quadrature formula for evaluating sums in order to obtain

$$\begin{aligned} 0 = \langle Q_n(x), g_{n-1}^{(i)}(x)\rangle_S &= \langle Q_n(x), w_{n-1}^{(i)}(x)\rangle - \frac{\lambda}{1-q} Q_n(0)(D_{q^{-1}} g_{n-1}^{(i)})(0) \\ &= \mu_i w_{n-1}^{(i)}(x_{n,i}) Q_n(x_{n,i}) - \frac{\lambda}{1-q} Q_n(0)(D_{q^{-1}} p_i)(0), \end{aligned}$$

where $w_{n-1}^{(i)}(x)$ are the polynomials defined in (5.12). Thus,

$$\mu_i w_{n-1}^{(i)}(x_{n,i}) Q_n(x_{n,i}) = \frac{\lambda}{1-q} Q_n(0)(D_{q^{-1}} g_{n-1}^{(i)})(0).$$

From Lemma 1 and repeating the arguments of Lemma 6 we get $(-1)^{n-2}(D_{q^{-1}} g_{n-1}^{(i)})(0) > 0$. Hence we deduce $\mu_i w_{n-1}^{(i)}(x_{n,i}) Q_n(x_{n,i}) > 0$, and then $Q_n(x_{n,i}) \neq 0$ as well as $\mathrm{sgn}(Q_n(x_{n,i})) = (-1)^{n-i}$. The proof follows in the same way as in the case already discussed.

Finally, by Lemma 1, if $a \leqslant 1$ we have $(-1)^n Q_n(0) > 0$, and all zeros are positive.

The result for $n = 2$ is a consequence of

$$p_2(x; a|q) - Q_2(x) = \frac{a\lambda q(-1+q^2)(-1+aq+x)}{\lambda + a(-1+q)q(-1+aq)}. \quad \square$$

6. Asymptotic properties

In this section asymptotic properties of the sequence $\{Q_n(x)\}_n$ on compact subsets of $\mathbb{C} \setminus [0,1]$ are studied (results on asymptotic properties for polynomials orthogonal with respect to different Sobolev inner products have been given in, e.g., [2,16,18,21,26]). More precisely, we shall derive the relative asymptotics $\{Q_n(x)/p_n(x; a|q)\}$ and the asymptotic behavior of the ratio of two consecutive little q-Laguerre–Sobolev orthogonal polynomials $\{Q_{n+1}(x)/Q_n(x)\}$ on compact subsets of $\mathbb{C} \setminus [0,1]$. As a consequence, we recover the relative asymptotics between monic Laguerre–Sobolev orthogonal polynomials $\{Q_n^{(\alpha)}(x)\}_n$ and monic Laguerre orthogonal polynomials $\{L_n^{(\alpha)}(x)\}_n$ given in [17].

First, using [32, Theorem 1, p. 263] we can give the ratio asymptotics for little q-Laguerre polynomials.

Proposition 8. *Let $\{p_n(x; a|q)\}_n$ be the little q-Laguerre MOPS. The following ratio asymptotics holds*:

$$\lim_{n\to\infty} \frac{p_n(x; a|q)}{p_{n+1}(x; a|q)} = \frac{1}{x}, \quad 0 < aq < 1 \tag{6.1}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,1]$.

Now, we give bounds for the squared norm $\tilde{k}_n$ defined in (3.4).

Proposition 9. *Let k_n and $\tilde{k}_n$ given in* (2.15) *and* (3.4), *respectively. For $n \geqslant 1$, we have*

$$k_n + \lambda([n]_q)^2 k_{n-1} \leqslant \tilde{k}_n \leqslant k_n + (\lambda([n]_q)^2 + (a(1-q^n)q^{n-1})^2) k_{n-1}, \tag{6.2}$$

where $[n]_q$ is given in (2.11).

Proof. The inequality on the right-hand side of (6.2) is straightforward from (3.10). Using the extremal property of k_n we have, for $n \geqslant 1$,

$$\tilde{k}_n = \langle Q_n(x), Q_n(x)\rangle_S = \langle Q_n(x), Q_n(x)\rangle + \lambda\langle (D_q Q_n)(x), (D_q Q_n)(x)\rangle \geqslant k_n + \lambda([n]_q)^2 k_{n-1}. \quad \square$$

Remark. If we divide (6.2) by k_n and we take limit when $\lambda \to \infty$ we obtain

$$\lim_{\lambda\to\infty} \frac{k_n}{\tilde{k}_n} = 0, \quad n \geqslant 1$$

and then, by using (3.3) we get

$$\lim_{\lambda\to\infty} d_n(\lambda) = 0, \quad n \geqslant 1.$$

Next, we give the asymptotic behaviour for the sequence $\{q^{2n-1}\tilde{k}_n/k_n\}_n$.

Proposition 10. *Let* k_n *and* $\tilde{k}_n$ *given in* (2.15) *and* (3.4), *respectively. Then*

$$\lim_{n\to\infty} q^{2n-1}\frac{\tilde{k}_n}{k_n} = \frac{\lambda}{a(1-q)^2} > 0, \quad 0 < aq < 1.$$

Proof. If we divide (3.10) by k_n and using (2.15) we get

$$\frac{\tilde{k}_n}{k_n} = 1 + \frac{(\lambda + (a(1-q)q^{n-1})^2)([n]_q)^2}{aq^{2n-1}(q^n-1)(aq^n-1)} - \frac{a(q^n-1)}{q(aq^n-1)}\frac{k_{n-1}}{\tilde{k}_{n-1}}. \tag{6.3}$$

Let us define

$$s_{n+1} = q^{2n-1}\frac{\tilde{k}_n}{k_n}s_n, \quad n \geqslant 0$$

with the initial condition $s_0 = 1$. Therefore, from (6.3) we get

$$s_{n+1} - q^{2n-1}\left(1 + \frac{(\lambda + (a(1-q)q^{n-1})^2)([n]_q)^2}{aq^{2n-1}(q^n-1)(aq^n-1)}\right)s_n + \frac{a(q^n-1)}{(aq^n-1)}q^{4n-5}s_{n-1} = 0, \tag{6.4}$$

where $s_0 = 1$ and $s_1 = \tilde{k}_1/k_1$. It is straightforward to deduce

$$\lim_{n\to\infty} q^{2n-1}\left(1 + \frac{(\lambda + (a(1-q)q^{n-1})^2)([n]_q)^2}{aq^{2n-1}(q^n-1)(aq^n-1)}\right) = \frac{\lambda}{a(1-q)^2},$$

$$\lim_{n\to\infty} \frac{a(q^n-1)}{(aq^n-1)}q^{4n-5} = 0.$$

Since the roots of the limit characteristic equation of (6.4) are

$$z_1 = 0, \quad z_2 = \frac{\lambda}{a(1-q)^2}$$

from the Poincaré's Theorem [29] (see also [24]), the sequence $\{s_{n+1}/s_n\}_n$ converges to z_1 or z_2. By using the inequality on the left-hand side of (6.2), we get

$$\lim_{n\to\infty} q^{2n-1}\frac{\tilde{k}_n}{k_n} = z_2. \qquad \square$$

The above results allow us to deduce the relative asymptotics $\{Q_n(x)/p_n(x;a|q)\}$.

Theorem 3. *Let* $\{Q_n(x)\}_n$ *be the little* q*-Laguerre–Sobolev MOPS and let* $\{p_n(x;a|q)\}_n$ *be the little* q*-Laguerre MOPS. The following limit relation holds*:

$$\lim_{n\to\infty} \frac{Q_n(x)}{p_n(x;a|q)} = 1, \quad 0 < aq < 1 \tag{6.5}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,1]$.

Proof. From (3.2) and (3.3) we can write for $n \geqslant 1$,

$$1 + a(1-q^n)q^{n-1}\frac{p_{n-1}(x;a|q)}{p_n(x;a|q)} = \frac{Q_n(x)}{p_n(x;a|q)} + a(1-q^n)q^{n-1}\frac{k_{n-1}}{\tilde{k}_{n-1}}\frac{p_{n-1}(x;a|q)}{p_n(x;a|q)}\frac{Q_{n-1}(x)}{p_{n-1}(x;a|q)}. \tag{6.6}$$

From (6.1) and Proposition 10,

$$\lim_{n\to\infty} a(1-q^n)q^{n-1}\frac{k_{n-1}}{\tilde{k}_{n-1}}\frac{p_{n-1}(x;a|q)}{p_n(x;a|q)} = 0 \tag{6.7}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,1]$. For a fixed compact set K in $\mathbb{C} \setminus [0,1]$, there exist $n_0 \in \mathbb{N}$, ϵ_0, $\epsilon_1 \in \mathbb{R}^+$ such that

$$\left|\frac{Q_n(x)}{p_n(x;a|q)}\right| \leqslant 1 + \epsilon_0 + \epsilon_1 \left|\frac{Q_{n-1}(x)}{p_{n-1}(x;a|q)}\right|.$$

Thus, $\{Q_n(x)/p_n(x;a|q)\}_n$ is uniformly bounded on K. Taking limits in (6.6) when $n \to \infty$, from (6.1) and (6.7) we get the result. □

From Theorems 2 and 3 we obtain

Corollary 3. *The set of zeros of* $Q_n(x)$ *accumulates on* $[0,1]$. *If* $a > 1$, *then*

$$\lim_{n\to\infty} y_{n,1} = 0,$$

where $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ *are the zeros of the nth degree monic little* q*-Laguerre–Sobolev polynomial* $Q_n(x)$.

Next, we give the asymptotic behavior of the ratio of two consecutive little q-Laguerre–Sobolev orthogonal polynomials, which is a consequence of Proposition 8 and Theorem 3.

Corollary 4. *Let* $\{Q_n(x)\}_n$ *be the little* q*-Laguerre–Sobolev MOPS. Then*

$$\lim_{n\to\infty} \frac{Q_{n+1}(x)}{Q_n(x)} = x \tag{6.8}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,1]$.

Remark. It should be finally mentioned that we can recover the relative asymptotics between monic Laguerre–Sobolev orthogonal polynomials $\{Q_n^{(\alpha)}(x)\}_n$ and monic Laguerre orthogonal polynomials

$\{L_n^{(\alpha)}(x)\}_n$ given in [17]. From (2.12) and (3.2) when $a = q^\alpha$ we get

$$\begin{aligned}&p_n((1-q)x; q^{\alpha-1}|q)\\ &\quad = Q_n((1-q)x; q^\alpha, \lambda(1-q)^2|q) + d_{n-1}(\lambda(1-q)^2, q^\alpha|q) Q_{n-1}((1-q)x; q^\alpha, \lambda(1-q)^2|q).\end{aligned}$$

If we take the limit in the above expression when $q \uparrow 1$ (see Proposition 3 and (3.11)) we obtain

$$L_n^{(\alpha-1)}(x) = Q_n^{(\alpha)}(x) + d_{n-1}(\lambda, \alpha) Q_{n-1}^{(\alpha)}(x), \tag{6.9}$$

where $\{L_n^{(\alpha)}(x)\}_n$ are the monic Laguerre polynomials, $\{Q_n^{(\alpha)}(x)\}_n$ are the monic Laguerre–Sobolev polynomials and the coefficients $d_n(\lambda, \alpha)$ are defined by means of the recurrence relation given in (3.14). From (6.9), using the same technique as in Theorem 3, it can be deduced that

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha-1)}(x)} = \frac{2}{\sqrt{\lambda^2 + 4\lambda} - \lambda} \tag{6.10}$$

holds uniformly on compact subsets of $\mathbb{C}\backslash[0, \infty)$, which is the monic form of the asymptotic behavior obtained in [17].

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References

[1] I. Area, E. Godoy, F. Marcellán, Inner products involving differences: the Meixner–Sobolev polynomials, J. Difference Equations Appl., in print.

[2] I. Area, E. Godoy, F. Marcellán, J.J. Moreno-Balcázar, Ratio and Plancherel–Rotach asymptotics for Meixner–Sobolev orthogonal polynomials, J. Comput. Appl. Math. 116 (2000) 63–75.

[3] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[4] W.D. Evans, L.L. Littlejohn, F. Marcellán, C. Markett, A. Ronveaux, On recurrence relations for Sobolev orthogonal polynomials, SIAM J. Math. Anal. 26 (2) (1995) 446–467.

[5] G. Gasper, M. Rahman, Basic hypergeometric series, Encyclopedia of Mathematics and its Applications, vol. 35, Cambridge University Press, Cambridge, 1990. Electronic version of errata available at http://www.math.nwu.edu/preprints/gasper/bhserrata.

[6] W. Gautschi, M. Zhang, Computing orthogonal polynomials in Sobolev spaces, Numer. Math. 71 (2) (1995) 159–183.

[7] W. Hahn, Über Orthogonalpolynome, die q-Differenzengleichungen genügen, Math. Nach. 2 (1949) 4–34.

[8] W. Hahn, Über Polynome, die gleichzetig zwei verschiedenen Orthogonalsystemen angehören, Math. Nach. 2 (1949) 263–278.

[9] M.E.H. Ismail, M. Rahman, The q-Laguerre polynomials and related moment problems, J. Math. Anal. Appl. 218 (1998) 155–174.

[10] F.H. Jackson, On q-definite integrals, Quart. J. Pure Appl. Math. 41 (1910) 193–203.

[11] R. Koekoek, Generalizations of the classical Laguerre polynomials and some q-analogues, Doctoral Dissertation, Delft University of Technology, 1990.

[12] R. Koekoek, Generalizations of the classical Laguerre polynomials and some q-analogues, J. Approx. Theory 69 (1992) 55–83.
[13] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Reports of the Faculty of Technical Mathematics and Informatics, 98-17, Delft University of Technology, Delft 1998.
[14] T.H. Koornwinder, Compact quantum groups and q-special functions, in V. Baldoni, M.A. Picardello (Eds.), Representations of Lie Groups and Quantum Groups, Pitman Research Notes in Mathematics Series, Vol. 311, Longman Scientific & Technical, Harlow, 1994, pp. 46–128.
[15] D.C. Lewis, Polynomial least square approximations, Amer. J. Math. 69 (1947) 273–278.
[16] G. López, F. Marcellán, W. van Assche, Relative asymptotics for polynomials with respect to a discrete Sobolev inner product, Constr. Approx. 11 (1995) 107–137.
[17] F. Marcellán, H.G. Meijer, T.E. Pérez, M.A. Piñar, An asymptotic result for Laguerre–Sobolev orthogonal polynomials, J. Comput. Appl. Math. 87 (1997) 87–94.
[18] F. Marcellán, J.J. Moreno-Balcázar, Strong and Plancherel–Rotach asymptotics of non-diagonal Laguerre–Sobolev polynomials, submitted.
[19] F. Marcellán, T.E. Pérez, M.A. Piñar, Orthogonal polynomials on weighted Sobolev spaces: the semiclassical case, Ann. Numer. Math. 2 (1995) 93–122.
[20] F. Marcellán, T.E. Pérez, M.A. Piñar, Laguerre–Sobolev orthogonal polynomials, J. Comput. Appl. Math. 71 (1996) 245–265.
[21] A. Martínez-Finkelshtein, Bernstein–Szegö's theorem for Sobolev orthogonal polynomials, Constr. Approx. 16 (2000) 73–84.
[22] J.C. Medem, Polinomios Ortogonales q-semiclásicos, Doctoral Dissertation, Universidad Politécnica de Madrid, 1996 (in Spanish).
[23] J. Meixner, Orthogonale Polynomsysteme mit einer besonderen gestalt der erzeugenden Funktionen, J. London Math. Soc. 9 (1934) 6–13.
[24] L.M. Milne-Thomson, The Calculus of Finite Differences, MacMillan, London, 1933.
[25] D.S. Moak, The q-analogue of the Laguerre polynomials, J. Math. Anal. Appl. 81 (1981) 20–47.
[26] J.J. Moreno-Balcázar, Propiedades Analíticas de los Polinomios Ortogonales de Sobolev Continuos, Doctoral Dissertation, Universidad de Granada, 1997 (in Spanish).
[27] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer, Berlin, 1991.
[28] T.E. Pérez, Polinomios Ortogonales Respecto a Productos de Sobolev: El Caso Continuo, Doctoral Dissertation, Universidad de Granada, 1994 (in Spanish).
[29] H. Poincaré, Sur les Equations Linéaires aux Différentielles ordinaires et aux Différences finies, Amer. J. Math. 7 (1885) 203–258.
[30] J. Thomae, Beiträge zur Theorie der durch die Heinesche Reihe; $1 + ((1-q^{\alpha})(1-q^{\beta})/(1-q)(1-q^{\delta}))x + \cdots$ darstellbaren funktionen, J. Reine Angew. Math. 70 (1869) 258–281.
[31] J. Thomae, Les séries Heinéennes supérieures, ou les séries de la forme .., Ann. Mat. Pura Appl. 4 (1870) 105–138.
[32] W. van Assche, Asymptotic properties of orthogonal polynomials from their recurrence formula, I, J. Approx. Theory 44 (1985) 258–276.