Ratio and Plancherel–Rotach asymptotics for Meixner–Sobolev orthogonal polynomials

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Abstract

We study the analytic properties of the monic Meixner–Sobolev polynomials $\{Q_n\}$ orthogonal with respect to the inner product involving differences

$$(p,q)_S=\sum_{i=0}^{\infty}[p(i)q(i)+\lambda\Delta p(i)\Delta q(i)]\frac{\mu^i(\gamma)_i}{i!},\quad \gamma>0,\quad 0<\mu<1,$$

where $\lambda\geqslant 0$, Δ is the forward difference operator ($\Delta f(x)=f(x+1)-f(x)$) and $(\gamma)_n$ denotes the Pochhammer symbol. Relative asymptotics for Meixner–Sobolev polynomials with respect to Meixner polynomials is obtained. This relative asymptotics is also given for the scaled polynomials. Moreover, a zero distribution for the scaled Meixner–Sobolev polynomials and Plancherel–Rotach asymptotics for $\{Q_n\}$ are deduced.

MSC: 42C05; 33C25; 39A10

Keywords: Sobolev orthogonal polynomials; Meixner polynomials; Scaled polynomials; Asymptotics; Plancherel–Rotach asymptotics

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[1] The work of E.G. has been partially supported by Dirección General de Enseñanza Superior (DGES) of Spain under Grant PB-96-0952.

[2] The work of F.M. is partially supported by PB96-0120-C03-01 and INTAS-93-0219 Ext.

[3] The work of J.J.M.-B. is partially supported by Junta de Andalucía, G.I. FQM0229.

1. Introduction

The original motivation for considering polynomials orthogonal with respect to a Sobolev inner product

$$(p,q)_S = \sum_{i=0}^{N} (p^{(i)}, q^{(i)})_{v_i} = \sum_{i=0}^{N} \int_{\mathbb{I}} p^{(i)} q^{(i)} \, \mathrm{d}v_i, \tag{1}$$

where $v_i,\ i = 0, \ldots, N$ are positive measures supported on a bounded or unbounded interval $\mathbb{I}$, comes from the least-squares approximation problems [10]. More precisely, for a given function to find its best approximate polynomial of degree n with respect to the norm

$$\|p\|_S^2 = \sum_{i=0}^{N} \|p^{(i)}\|_{v_i},$$

where $\|\cdot\|_{v_i},\ i = 0, \ldots, N$, are the norms induced by the standard inner products [6,8]. In this way, it is interesting to study the monic polynomials Q_n orthogonal with respect to (1), which satisfy the extremal property

$$\|Q_n\|_S^2 = \inf_{P \in \mathbb{P}_n, P \text{ monic}} \|P\|_S^2,$$

where $\mathbb{P}_n$ is the space of polynomials with degree at most n, if we want to approximate simultaneously a function and its first N derivatives (see [7]).

For the last 50 years, these polynomials have been a subject of study, most of the times from an algebraic point of view. The study of asymptotic properties for polynomials orthogonal with respect to the Sobolev inner product (1) has undergone an increasing development in the recent time. For instance, in [11] a very general nondiagonal inner product has been considered, where $v_i,\ i = 1, \ldots, N$, are discrete measures and the absolutely continuous part of v_0 does not vanish. In this situation, which includes (1) as a particular case, the relative asymptotics $\{Q_n/P_n\}$ was obtained, where $\{Q_n\}$ is the monic orthogonal polynomial sequence (MOPS) with respect to (1) and $\{P_n\}$ is the MOPS associated with the measure v_0.

In the continuous case, i.e., when the measures $v_i,\ i = 0, \ldots, N$, have absolutely nonzero continuous part, the results are more recent. The most studied situation is when $N = 1$. Probably, a first paper in this direction was [16], followed by other works [15,21,14] in the bounded case (v_i measures with compact support) and [12,13] for the Laguerre weight function in the unbounded case. The case $N > 1$ when the measures v_i have compact support has been studied in [17]. It should be important to notice here that the case of unbounded support is less known (see [12,13]) and there does not exist a general theory as it occurs for the bounded case (see [14]).

On the other hand, Bavinck [2] studied the polynomials orthogonal with respect to the inner product involving differences

$$(f,g) = \int_{\mathbb{R}} f(x) g(x) \, \mathrm{d}\psi(x) + \lambda \Delta f(c) \Delta g(c), \tag{2}$$

where $c \in \mathbb{R}$, $\lambda \geqslant 0$, ψ is a distribution function with infinite spectrum such that ψ has no points of increase in the interval $(c, c+1)$ and Δ stands for the forward difference operator, $\Delta f(x) = f(x+1) - f(x)$. He obtained algebraic properties and some results on the distribution of the zeros for the sequence of polynomials orthogonal with respect to (2). Subsequent study for the general forward difference operator has been done in [3]. In some specific situations [4,5], these polynomials are eigenfunctions of difference operators of infinite order. This kind of inner products (2) appears as a discretization of (1) when $N = 1$ and v_1 is an atomic measure.

We focus our interest on the study of the inner product

$$(p,q)_S = (p,q)_{v_0} + \lambda(\Delta p, \Delta q)_{v_1}, \quad \lambda \geqslant 0, \tag{3}$$

when v_1 is a discrete measure with an infinite set as support. Note that (2) comes from (3) when the measure v_1 for the second term is atomic and supported in $\{c\}$. Moreover, (3) is a discretization of (1) for $N = 1$ when we substitute the derivative by the forward difference operator Δ and v_i, $i = 0, 1$, are discrete measures. A functional study of such kind of Sobolev spaces defined by differences was introduced in [9].

In the present contribution, we analyze (3) when $v_0 = v_1$ is the Pascal distribution from probability theory

$$v_0 = v_1 = \sum_{i=0}^{\infty} \frac{\mu^i (\gamma)_i}{i!} \delta_i, \quad 0 < \mu < 1, \quad \gamma > 0, \tag{4}$$

where $(\gamma)_0 = 1$, $(\gamma)_i = \gamma(\gamma+1)\cdots(\gamma+i-1)$, for $i \geqslant 1$ denotes the Pochhammer symbol and δ_i denotes the Dirac measure supported at i, which is a new example on the unbounded case. More concretely, we shall deal with the inner product

$$(p,q)_S = (p,q) + \lambda(\Delta p, \Delta q) = \sum_{i=0}^{\infty} [p(i)q(i) + \lambda \Delta p(i) \Delta q(i)] \frac{\mu^i (\gamma)_i}{i!}, \tag{5}$$

where $\gamma > 0$, $0 < \mu < 1$ and $\lambda \geqslant 0$. The monic polynomials $M_n^{(\gamma,\mu)}(x)$ orthogonal with respect to measure (4) are the Meixner polynomials [19]. We denote $\{Q_n\}$ the MOPS with respect to (5), which is called the Meixner–Sobolev MOPS (see [1], where algebraic properties and location of zeros of these polynomials have been studied).

The aim of this paper is to obtain the asymptotic behaviour of the sequence $\{Q_n\}$. We give a recurrence relation as well as bounds for $\tilde{k}_n = (Q_n, Q_n)_S$, which allows us to obtain the limit behaviour of the ratio $\tilde{k}_n / k_n^{(\gamma,\mu)}$, where $k_n^{(\gamma,\mu)} = (M_n^{(\gamma,\mu)}, M_n^{(\gamma,\mu)})$. Later, the relative asymptotics $\{Q_n(x)/M_n^{(\gamma,\mu)}(x)\}$, with $x \in \mathbb{C} \setminus [0, \infty)$, is given.

As a new subject of study in the theory of Sobolev orthogonal polynomials (see [13] where other results in this direction have been obtained in the Laguerre–Sobolev case), using a scaling in the variable, the relative asymptotics for the scaled Sobolev polynomials with respect to the scaled Meixner polynomials is deduced (see Theorem 6). A result about the contracted zero distribution of Meixner–Sobolev orthogonal polynomials is derived, and moreover, we deduce that the contracted zeros accumulate on $[0, (1+\sqrt{\mu})^2/(1-\mu)]$. Finally, we obtain Plancherel–Rotach asymptotics [23] for the scaled Meixner–Sobolev polynomials $\{Q_n(nx)\}$.

The structure of the paper is as follows: in Section 2 we introduce some preliminary results concerning Meixner polynomials as well as an algebraic relation between the Meixner and the Meixner–Sobolev orthogonal polynomials. In Section 3 we give the ratio asymptotics for such polynomials (Theorem 6) as well as the ratio asymptotics when we scale the variable (Theorem 7). Later, Plancherel–Rotach asymptotics for the Meixner–Sobolev polynomials is obtained. Finally, in Section 4 the proofs of such results are given.

2. Preliminary results

We present the basic results that will be used in the next sections. First, we begin with some properties of Meixner polynomials. These properties can be found in [22], with different normalization, and they are collected in the next theorem.

Theorem 1. *Let* $\{M_n^{(\gamma,\mu)}\}$ *be the Meixner's MOPS and let* $\{m_n^{(\gamma,\mu)}\}$ *be the Meixner orthonormal polynomials, where* $0 < \mu < 1$ *and* $\gamma > 0$. *Then,*

(a) *Squared norm* (*see* [22, (2.5.1), p. 40]):

$$k_n^{(\gamma,\mu)} = (M_n^{(\gamma,\mu)}, M_n^{(\gamma,\mu)}) = \frac{\mu^n(\gamma)_n n!}{(1-\mu)^{2n+\gamma}}, \tag{6}$$

where $(\gamma)_n$ *denotes the Pochhammer symbol.*

(b) *Three-term recurrence relation* (*see* [22, p. 44]):

$$xm_n^{(\gamma,\mu)}(x) = a_{n+1}m_{n+1}^{(\gamma,\mu)}(x) + b_n m_n^{(\gamma,\mu)}(x) + a_n m_{n-1}^{(\gamma,\mu)}(x), \quad n \geqslant 0, \tag{7}$$

where

$$a_n = \frac{\sqrt{\mu n(n+\gamma-1)}}{1-\mu}, \qquad b_n = \frac{(1+\mu)n+\gamma\mu}{1-\mu}, \tag{8}$$

with $m_{-1}^{(\gamma,\mu)}(x) = 0$ *and* $m_0^{(\gamma,\mu)}(x) = 1/\sqrt{(1-\mu)^\gamma}$.

About relative asymptotics, we have the following proposition which can be obtained from (7) using the Poincaré's Theorem (see [24, Section 2, pp. 213–217, Section 6, p. 237], for secondary sources, [20, Section 17.1, p. 526] or [18]).

Proposition 2. *Let* $\gamma > 0$ *and* $0 < \mu < 1$. *Then*

$$\begin{aligned} \lim_{n\to\infty} \frac{m_{n+1}^{(\gamma,\mu)}(x)}{m_n^{(\gamma,\mu)}(x)} &= -\frac{1}{\sqrt{\mu}}, \\ \lim_{n\to\infty} \frac{M_{n+1}^{(\gamma,\mu)}(x)}{nM_n^{(\gamma,\mu)}(x)} &= \frac{1}{\mu-1} \end{aligned} \tag{9}$$

hold uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

Let us denote $\tilde{k}_n := (Q_n, Q_n)_S$. The following relation between monic Meixner and monic Meixner–Sobolev polynomials can be found in [1].

Proposition 3. *For* $n \geqslant 1$,

$$M_n^{(\gamma,\mu)}(x) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(x) = Q_n(x) + n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}}Q_{n-1}(x). \tag{10}$$

Finally, a recurrence relation for $\tilde{k}_n$ appears in [1].

Proposition 4. *For* $n \geqslant 1$,

$$\tilde{k}_n = k_n^{(\gamma,\mu)} + \left(\lambda + \left(\frac{\mu}{1-\mu}\right)^2\right) n^2 k_{n-1}^{(\gamma,\mu)} - n^2 \left(\frac{\mu}{1-\mu}\right)^2 \frac{(k_{n-1}^{(\gamma,\mu)})^2}{\tilde{k}_{n-1}}, \tag{11}$$

with the initial condition $\tilde{k}_0 = k_0^{(\gamma,\mu)} = 1/(1-\mu)^\gamma$.

3. Main results

From now on we shall assume that $\lambda > 0$ (if $\lambda = 0$ then $Q_n(x) = M_n^{(\gamma,\mu)}(x)$ for $n \geqslant 0$). As a first step, using Proposition 4, bounds and the limit behaviour for $\{\tilde{k}_n / k_n^{(\gamma,\mu)}\}$ are obtained.

Proposition 5. (a) *For* $n \geqslant 1$, *we have*

$$1 + \lambda \frac{(1-\mu)^2 n}{\mu(n+\gamma-1)} \leqslant \frac{\tilde{k}_n}{k_n^{(\gamma,\mu)}} \leqslant 1 + \left(\lambda + \left(\frac{\mu}{1-\mu}\right)^2\right) \frac{(1-\mu)^2 n}{\mu(n+\gamma-1)}. \tag{12}$$

(b) *The following asymptotic behaviour holds*:

$$\eta := \lim_{n\to\infty} \frac{\tilde{k}_n}{k_n^{(\gamma,\mu)}} = \frac{1 + \lambda(1-\mu)^2/\mu + \mu + (1-\mu)\sqrt{(1+\lambda(1-\mu)/\mu)^2 + 4\lambda}}{2} > 1. \tag{13}$$

It is important to remark the following uniform bound for $k_n^{(\gamma,\mu)}/\tilde{k}_n$, with $n \geqslant 1$, that we shall use in the proof of Theorem 7 on scaled asymptotics. From the inequality on the left-hand side of (12) we have

$$\frac{k_n^{(\gamma,\mu)}}{\tilde{k}_n} \leqslant \mathscr{C} < 1 \quad \text{for all } n \geqslant 1, \tag{14}$$

where $\mathscr{C}$ can be taken as

$$\mathscr{C} = \frac{1}{1 + \lambda(1-\mu)^2/(\mu \max\{1,\gamma\})}. \tag{15}$$

The above proposition together with Proposition 3 allows us to deduce the following relative asymptotics.

Theorem 6. *Let* $0 < \mu < 1$ *and* $\gamma > 0$. *Then*,

$$\ell := \lim_{n\to\infty} \frac{Q_n(x)}{M_n^{(\gamma,\mu)}(x)} = \frac{1-\mu}{1-\mu/\eta}, \tag{16}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$, *where* η *is given in* (13). *Moreover*, $\ell \in (1-\mu, 1)$.

Next, we shall state results on the relative asymptotics for the scaled polynomials. In what follows we shall denote $\varphi(x) = x + \sqrt{x^2-1}$ and $\sqrt{x^2-1} > 0$ if $x > 1$.

Theorem 7. *Let* $0 < \mu < 1$ *and* $\gamma > 0$. *We have*

$$t := \lim_{n\to\infty} \frac{Q_n(nx)}{M_n^{(\gamma,\mu)}(nx)} = \frac{\eta[\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu}) + \sqrt{\mu}]}{\eta\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu}) + \sqrt{\mu}}, \tag{17}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, (1+\sqrt{\mu})^2/(1-\mu)]$, *where* η *is given in* (13).

From this theorem and using the asymptotics for the scaled Meixner polynomials, we get

Corollary 8. *Let* $\{x_{i,n}\}_{i=1}^n$ *be the zeros of* Q_n. *The contracted zeros* $\{x_{i,n}/n\}_{i=1}^n$ *of* Q_n *accumulate on* $[0, (1+\sqrt{\mu})^2/(1-\mu)]$.

Corollary 9 (Plancherel–Rotach asymptotics)**.** *The following limit relation*

$$\lim_{n\to\infty} \frac{Q_n(nx)}{\sqrt{k_n^{(\gamma,\mu)}}\prod_{i=1}^n \varphi\left(\frac{nx-b_i}{2a_i}\right)} = t\left(\frac{\mu(x+1)^2-(x-1)^2}{(\mu-1)x^2}\right)^{-1/4}$$
$$\times \exp\left(\frac{1+\mu}{2(1-\mu)}\int_0^1 \frac{\mathrm{d}s}{\sqrt{s^2+x^2+2sx(\mu+1)/(\mu-1)}}\right), \tag{18}$$

holds uniformly on compact subsets of $\mathbb{C} \setminus [0, (1+\sqrt{\mu})^2/(1-\mu)]$, *where* a_i, b_i *and* t *are given in* (8) *and* (17), *respectively.*

4. Proofs

Proof of Proposition 5. (a) The inequality on the right-hand side of (12) is straightforward from Proposition 4 and (6). On the other hand, using the extremal property of $k_n^{(\gamma,\mu)}$, i.e.,

$$k_n^{(\gamma,\mu)} = \inf_{p\in\mathbb{P}_n,\, p \text{ monic}} (p, p)$$

we get,

$$\tilde{k}_n = (Q_n, Q_n)_S = (Q_n, Q_n) + \lambda(\Delta Q_n, \Delta Q_n) \geqslant k_n^{(\gamma,\mu)} + \lambda n^2 k_{n-1}^{(\gamma,\mu)}.$$

It only remains to use (6) in order to obtain the other inequality.

(b) If we divide (11) by $k_n^{(\gamma,\mu)}$, from (6) we get

$$\frac{\tilde{k}_n}{k_n^{(\gamma,\mu)}} = 1 + \left(\lambda + \left(\frac{\mu}{1-\mu}\right)^2\right)\frac{n(1-\mu)^2}{\mu(n+\gamma-1)} - \frac{n\mu}{n+\gamma-1}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}}. \tag{19}$$

Let us define $s_{n+1} = \tilde{k}_n s_n / k_n^{(\gamma,\mu)}$ with the initial condition $s_0 = 1$. Therefore, (19) can be rewritten as

$$s_{n+1} - \left(1 + \left(\lambda + \left(\frac{\mu}{1-\mu}\right)^2\right)\frac{n(1-\mu)^2}{\mu(n+\gamma-1)}\right)s_n + \frac{n\mu}{n+\gamma-1}s_{n-1} = 0, \tag{20}$$

where $s_0 = 1$ and $s_1 = \tilde{k}_1/k_1^{(\gamma,\mu)}$. Thus

$$\lim_{n\to\infty}\left(1+\left(\lambda+\left(\frac{\mu}{1-\mu}\right)^2\right)\frac{n(1-\mu)^2}{\mu(n+\gamma-1)}\right)=1+\frac{\lambda(1-\mu)^2}{\mu}+\mu,$$

$$\lim_{n\to\infty}\frac{n\mu}{n+\gamma-1}=\mu.$$

On the other hand, the roots of the limit characteristic equation of (20)

$$z^2-\left(1+\frac{\lambda(1-\mu)^2}{\mu}+\mu\right)z+\mu=0$$

are

$$z_1=\frac{1+\lambda(1-\mu)^2/\mu+\mu+(1-\mu)\sqrt{(1+\lambda(1-\mu)/\mu)^2+4\lambda}}{2},$$

$$z_2=\frac{1+\lambda(1-\mu)^2/\mu+\mu-(1-\mu)\sqrt{(1+\lambda(1-\mu)/\mu)^2+4\lambda}}{2}.$$

Using Poincaré's Theorem, the sequence $\tilde{k}_n/k_n^{(\gamma,\mu)} = s_{n+1}/s_n$ converges to z_1 or z_2. Straightforward computations show that $z_1 > 1$ (and, therefore, $z_2 < 1$) and, on the other hand, $\tilde{k}_n \geqslant k_n^{(\gamma,\mu)}$, for each $n \geqslant 0$. Then, we obtain

$$\lim_{n\to\infty}\frac{\tilde{k}_n}{k_n^{(\gamma,\mu)}}=z_1. \qquad \square$$

Next we deduce the relative asymptotics $\{Q_n(x)/M_n^{(\gamma,\mu)}(x)\}$, when $x \in \mathbb{C}\setminus[0,\infty)$.

Proof of Theorem 6. Let us denote

$$C_n(x) := \frac{Q_n(x)}{M_n^{(\gamma,\mu)}(x)}.$$

From Proposition 3 we have

$$1+n\frac{\mu}{1-\mu}\frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)}=C_n(x)+n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}(x)}{\tilde{k}_{n-1}}\frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)}C_{n-1}(x),\quad n\geqslant 1. \tag{21}$$

Then, using Propositions 2 and 5 we get

$$\lim_{n\to\infty} n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}}\frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)}=-\frac{\mu}{\eta}$$

uniformly on compact subsets $\mathbb{C}\setminus[0,\infty)$. Since $\mu/\eta < 1$, for a fixed compact set $K \subset \mathbb{C}\setminus[0,\infty)$ there exist constants ε_1 with $0 < \varepsilon_1 < 1$ and $n_1 \in \mathbb{N}$ such that

$$\left|n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}}\frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)}\right|\leqslant\varepsilon_1, \tag{22}$$

when $x \in K$ and $n \geqslant n_1$. From (22) and Proposition 5, we get

$$\left| n \frac{\mu}{1-\mu} \frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)} \right| \leqslant \varepsilon_2 \tag{23}$$

for $n \geqslant n_2$ with $0 < \varepsilon_2 < 1$ and $x \in K$.

From (22) and (23) we deduce,

$$|C_n(x)| \leqslant \varepsilon_3 + \varepsilon_1 |C_{n-1}(x)|$$

for $n \geqslant n_0 = \max\{n_1, n_2\}$ and $x \in K$, where $\varepsilon_3 = 1 + \varepsilon_2$.

Thus, $\{C_n\}$ is uniformly bounded on compact subsets of $\mathbb{C} \setminus [0, \infty)$. We can rewrite (21) as

$$C_n(x) = \frac{\mu}{\eta} C_{n-1}(x) + 1 - \mu - \left(\frac{\mu}{\eta} + n \frac{\mu}{1-\mu} \frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}} \frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)} \right) C_{n-1}(x) + n \frac{\mu}{1-\mu} \frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)} + \mu.$$

If we denote

$$\delta_n(x) := -\left(\frac{\mu}{\eta} + n \frac{\mu}{1-\mu} \frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}} \frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)} \right) C_{n-1}(x) + n \frac{\mu}{1-\mu} \frac{M_{n-1}^{(\gamma,\mu)}(x)}{M_n^{(\gamma,\mu)}(x)} + \mu,$$

then

$$\lim_{n \to \infty} \delta_n(x) = 0 \tag{24}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

Since $\{C_n\}$ is a normal family in $\mathbb{C} \setminus [0, \infty)$, we can take an arbitrary subsequence $\{C_{n_j}\}_{j=1}^{\infty}$ such that

$$C_{n_{j+1}}(x) - C_{n_j}(x) = \left(\left(\frac{\mu}{\eta} \right)^{n_{j+1}-n_j} - 1 \right) \cdot \left(C_{n_j}(x) + \frac{1-\mu}{\frac{\mu}{\eta} - 1} \right) + \sum_{k=0}^{n_{j+1}-n_j-1} \left(\frac{\mu}{\eta} \right)^k \delta_{n_{j+1}-1-k}(x). \tag{25}$$

Since $\mu/\eta < 1$, using (24) we obtain that

$$\lim_j \sum_{k=0}^{n_{j+1}-n_j-1} \left(\frac{\mu}{\eta} \right)^k \delta_{n_{j+1}-1-k}(x) = 0$$

holds uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$ and

$$\left| \left(\frac{\mu}{\eta} \right)^{n_{j+1}-n_j} - 1 \right| > 1 - \left| \frac{\mu}{\eta} \right|.$$

Thus, since the left-hand side in (25) tends to zero when $n \to \infty$, we have

$$\lim_j C_{n_j} = \frac{1-\mu}{1-\mu/\eta},$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$. As the convergent subsequence is arbitrary we have proved (16).

Finally, we obtain bounds for ℓ. It is clear that $1-\mu/\eta < 1$, then the lower bound is straightforward. Since $\eta > 1$ then $\mu/\eta < \mu$. Thus, $1 - \mu/\eta > 1 - \mu$ and we deduce the upper bound. □

From Proposition 3, we can also study the relative asymptotics for the scaled polynomials.

Proof of Theorem 7. From the expressions given in (8) for the coefficients of the three-term recurrence relation satisfied by the Meixner orthonormal polynomials, we have for $j \in \mathbb{R}$ fixed

$$\lim_{n\to\infty} \frac{a_n}{n-j} = \frac{\sqrt{\mu}}{1-\mu} > 0, \qquad \lim_{n\to\infty} \frac{b_n}{n-j} = \frac{1+\mu}{1-\mu} \in \mathbb{R}. \tag{26}$$

Applying a known result due to van Assche (see [25, p. 117] or [26, p. 455]) the ratio asymptotics for two consecutive scaled Meixner orthonormal polynomials is

$$\lim_{n\to\infty} \frac{m_{n-1}^{(\gamma,\mu)}((n-j)x)}{m_n^{(\gamma,\mu)}((n-j)x)} = \frac{1}{\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}, \quad j \in \mathbb{R} \quad \text{fixed} \tag{27}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,(1+\sqrt{\mu})^2/(1-\mu)]$. Note that the contracted zeros of Meixner polynomials live in $\left[0,(1+\sqrt{\mu})^2/(1-\mu)\right]$ (see [25, p. 117, 26, p. 455]).

If we make the change of variable $x \to nx$ in (10) we get

$$M_n^{(\gamma,\mu)}(nx) + n\frac{\mu}{1-\mu}M_{n-1}^{(\gamma,\mu)}(nx) = Q_n(nx) + n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}}Q_{n-1}(nx). \tag{28}$$

Now, we can use (28) in a recursive way and after straightforward computations we get

$$Q_n(nx) = \sum_{j=0}^{n}(-1)^j b_{n-j}^{(n)}\left(M_{n-j}^{(\gamma,\mu)}(nx) + (n-j)\frac{\mu}{1-\mu}M_{n-j-1}^{(\gamma,\mu)}(nx)\right), \tag{29}$$

where $M_{-1}^{(\gamma,\mu)}(x) := 0$ and

$$\begin{aligned} b_n^{(n)} &= 1, \\ b_{n-j}^{(n)} &= \left(\frac{\mu}{1-\mu}\right)^j \prod_{i=1}^{j}(n-i+1)\frac{k_{n-i}^{(\gamma,\mu)}}{\tilde{k}_{n-i}}, \quad j=1,\ldots,n. \end{aligned} \tag{30}$$

If we divide (29) by $M_n^{(\gamma,\mu)}(nx)$ then we obtain

$$\frac{Q_n(nx)}{M_n^{(\gamma,\mu)}(nx)} = \sum_{j=0}^{n}(-1)^j b_{n-j}^{(n)}\frac{M_{n-j}^{(\gamma,\mu)}(nx) + (n-j)(\mu/(1-\mu))M_{n-j-1}^{(\gamma,\mu)}(nx)}{M_n^{(\gamma,\mu)}(nx)}. \tag{31}$$

From (6) and (27), we have for $j \in \mathbb{R}$ fixed

$$\begin{aligned} \lim_{n\to\infty} n\frac{\mu}{1-\mu}\frac{M_{n-1}^{(\gamma,\mu)}((n-j)x)}{M_n^{(\gamma,\mu)}((n-j)x)} &= \lim_{n\to\infty} n\frac{\mu}{1-\mu}\sqrt{\frac{k_{n-1}^{(\gamma,\mu)}}{k_n^{(\gamma,\mu)}}}\frac{m_{n-1}^{(\gamma,\mu)}((n-j)x)}{m_n^{(\gamma,\mu)}((n-j)x)} \\ &= \frac{\sqrt{\mu}}{\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}, \end{aligned}$$

and therefore,

$$\lim_{n\to\infty} n\frac{\mu}{1-\mu}\frac{k_{n-1}^{(\gamma,\mu)}}{\tilde{k}_{n-1}}\frac{M_{n-1}^{(\gamma,\mu)}((n-j)x)}{M_n^{(\gamma,\mu)}((n-j)x)} = \frac{\sqrt{\mu}}{\eta}\frac{1}{\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}, \tag{32}$$

both uniformly on compact subsets of $\mathbb{C}\setminus[0,(1+\sqrt{\mu})^2/(1-\mu)]$.

Taking into account that

$$\left|\varphi\left(\frac{(1-\mu)x-(1+\mu)}{2\sqrt{\mu}}\right)\right| > 1$$

when $x \in \mathbb{C} \setminus [0, (1+\sqrt{\mu})^2/(1-\mu)]$, we have

$$\left| \frac{\sqrt{\mu}}{\eta} \frac{1}{\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})} \right| < 1.$$

On the other hand, from (30), for j fixed,

$$\lim_{n\to\infty} \frac{b_{n-j}^{(n)}(\mu/(1-\mu))^{-j}}{n(n-1)\dots(n-j+1)} = \lim_{n\to\infty} \prod_{i=1}^{j} \frac{k_{n-i}^{(\gamma,\mu)}}{\tilde{k}_{n-i}} = \eta^{-j}, \tag{33}$$

and using (32), we get for j fixed

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{\mu}{1-\mu}\right)^{j} \prod_{i=1}^{j}(n-i+1) \frac{M_{n-j}^{(\gamma,\mu)}(nx)}{M_n^{(\gamma,\mu)}(nx)} \\
&\quad = \lim_{n\to\infty} \left(\frac{\mu}{1-\mu}\right)^{j} (n-j+1) \frac{M_{n-j}^{(\gamma,\mu)}(nx)}{M_{n-j+1}^{(\gamma,\mu)}(nx)} \dots n \frac{M_{n-1}^{(\gamma,\mu)}(nx)}{M_n^{(\gamma,\mu)}(nx)} \\
&\quad = \left(\frac{\sqrt{\mu}}{\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}\right)^{j}
\end{aligned} \tag{34}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, (1+\sqrt{\mu})^2/(1-\mu)]$. Note that the absolute value of the above limit is at most 1. Actually, if $j \geqslant 1$ it is strictly less than 1 and it is $\ll 1$ when j is large.

If we denote

$$f_{n,j}(nx) = \begin{cases} (-1)^j b_{n-j}^{(n)} \dfrac{M_{n-j}^{(\gamma,\mu)}(nx) + (n-j)(\mu/(1-\mu)) M_{n-j-1}^{(\gamma,\mu)}(nx)}{M_n^{(\gamma,\mu)}(nx)} & \text{if } 0 \leqslant j \leqslant n, \\ 0 & \text{if } j > n \end{cases}$$

then (31) can be rewritten as

$$\frac{Q_n(nx)}{M_n^{(\gamma,\mu)}(nx)} = \sum_{j=0}^{n} f_{n,j}(nx). \tag{35}$$

Therefore, it is clear by (33) and (34) that for j fixed

$$\begin{aligned}
&\lim_{n\to\infty} f_{n,j}(nx) \\
&\quad = \left[\left(\frac{-\sqrt{\mu}}{\eta\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}\right)^{j} + (-1)^j \eta \left(\frac{\sqrt{\mu}}{\eta\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}\right)^{j+1} \right] \\
&\quad = \left(1 + \frac{\sqrt{\mu}}{\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}\right) \left(\frac{-\sqrt{\mu}}{\eta\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}\right)^{j} := f_j(x),
\end{aligned} \tag{36}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, (1+\sqrt{\mu})^2/(1-\mu)]$. On the other hand, letting x on a compact subset of $\mathbb{C} \setminus [0, (1+\sqrt{\mu})^2/(1-\mu)]$, using (14) and (34), we have, if n large enough and $0 \leqslant j \leqslant n$,

$$|f_{n,j}(nx)| \leqslant \mathscr{K} \mathscr{C}^j, \tag{37}$$

where $\mathscr{K}$ is a constant and $\mathscr{C} < 1$ is the constant given in (15). Then, from (37) we have a dominant for (35). From (35) we can write

$$\frac{Q_n(nx)}{M_n^{(\gamma,\mu)}(nx)} = \sum_{j=0}^{n} f_{n,j}(nx) = \int f_{n,j}(nx)\,\mathrm{d}\mu(j),$$

where $\mathrm{d}\mu(j) = \sum_{j\in\mathbb{N}\cup\{0\}} \delta_j$, i.e., the discrete measure supported on the nonnegative integers with mass equal to one in each point of the support. Then, we are in a position to apply the Lebesgue's dominated convergence theorem and we get

$$\begin{aligned}\lim_{n\to\infty} \frac{Q_n(nx)}{M_n^{(\gamma,\mu)}(nx)} &= \int \lim_{n\to\infty} f_{n,j}(nx)\mathrm{d}\mu(j) = \int f_j(x)d\mu(j) = \sum_{j=0}^{\infty} f_j(x)\\ &= \left(1 + \frac{\sqrt{\mu}}{\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}\right) \frac{1}{1+\frac{\sqrt{\mu}}{\eta\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu})}}\\ &= \frac{\eta[\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu}) + \sqrt{\mu}]}{\eta\varphi(((1-\mu)x-(1+\mu))/2\sqrt{\mu}) + \sqrt{\mu}}\end{aligned}$$

which proves the theorem. □

Remark. Indeed, the technique used in the proof of the above theorem can also be applied to establish Theorem 6, but we have preferred a more standard tool in asymptotics of Sobolev polynomials for Theorem 6 which can not be applied in a straightforward way in Theorem 7.

Corollary 8 is a straightforward consequence of Theorem 7.

Proof of Corollary 9. The sequence $\{c_n = n,\ n = 1, 2, \ldots\}$ is a regularly varying sequence with index one (see [25, p. 120] or [26, p. 455]). Since (26) holds and

$$\lim_{n\to\infty}(a_{n+1}-a_n) = \frac{\sqrt{\mu}}{1-\mu} > 0, \quad \lim_{n\to\infty}(b_{n+1}-b_n) = \frac{1+\mu}{1-\mu} \in \mathbb{R},$$

we can apply a result due to van Assche and Geronimo [27] about the Plancherel–Rotach asymptotics for polynomials with unbounded recurrence coefficients (see also [25, Theorem 4.15, p. 126]) in order to obtain

$$\begin{aligned}&\lim_{n\to\infty} \frac{m_n^{(\gamma,\mu)}(nx)}{\prod_{i=1}^{n}\varphi((nx-b_i)/2a_i)}\\ &\quad = \left(\frac{(x-(1+\mu)/(1-\mu))^2 - 4\mu/(1-\mu)^2}{x^2}\right)^{-1/4}\\ &\qquad \times \exp\left(\frac{1+\mu}{2(1-\mu)}\int_0^1 \frac{\mathrm{d}s}{\sqrt{(x-((1+\mu)/(1-\mu))s)^2 - (4\mu/(1-\mu)^2)s^2}}\right), \qquad (38)\end{aligned}$$

where a_i and b_i are the corresponding coefficients in the recurrence relation (7).

Therefore, if we write (38) for monic Meixner polynomials and using Theorem 7, we obtain the Plancherel–Rotach asymptotics (18) for the scaled Meixner–Sobolev polynomials $Q_n(nx)$. □

Acknowledgements

The authors thank Prof. Andrei Martínez Finkelshtein for helpful discussions.

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