Orthogonality of linear combinations of two orthogonal polynomial sequences

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Abstract

We find necessary and sufficient conditions for some linear combinations of two sequences of orthogonal polynomials to be again orthogonal.

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1. Introduction

Let $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$ be two sequences of monic orthogonal polynomials with respect to quasi-definite moment functionals u and v, respectively.

It is well known [4] that such sequences of monic polynomials satisfy three-term recurrence relations

$$Q_{n+1}(x) = (x - \beta_n)Q_n(x) - \gamma_n Q_{n-1}(x), \quad n \geqslant 0, \tag{1.1}$$

where $Q_{-1}(x) = 0$, $Q_0(x) = 1$, $\gamma_0 = \langle u, 1 \rangle = 1$, $\gamma_n \neq 0$ for $n \geqslant 1$, and

$$R_{n+1}(x) = (x - \delta_n)R_n(x) - \varepsilon_n R_{n-1}(x), \quad n \geqslant 0, \tag{1.2}$$

where $R_{-1}(x) = 0$, $R_0(x) = 1$, $\varepsilon_0 = \langle v, 1 \rangle = 1$, $\delta_n \neq 0$ for $n \geqslant 1$.

In Section 2, we consider a linear perturbation problem: Introduce a sequence $\{P_n(x)\}_{n=0}^{\infty}$ of monic polynomials given by

$$P_n(x) = Q_n(x) - \alpha_n R_{n-t}(x), \quad n \geqslant 0 \ (R_n(x) = 0, \ n < 0), \tag{1.3}$$

where t is a positive integer and $\alpha_n \in \mathbb{C}$. Notice that the above conditions yield $P_n(x) = Q_n(x)$, $0 \leqslant n \leqslant t-1$.

The situation $R_n(x) \equiv Q_n(x)$ is considered in [6] where necessary and sufficient conditions for the orthogonality of the sequence $\{P_n(x)\}_{n=0}^{\infty}$ are obtained.

If $t = 1$ and $R_n(x) = Q_n^{(1)}(x)$, i.e., the sequence of monic associated polynomials of the first kind, then for $\alpha_n = \alpha$=constant we obtain the so-called sequence of co-recursive monic orthogonal polynomials (see [3,4,10,11]).

In Section 3, we consider a convex combination of $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$:

$$P_n(x) = \alpha Q_n(x) + (1-\alpha) R_n(x), \quad n \geqslant 0, \tag{1.4}$$

where $\alpha \in \mathbb{C}$. A similar problem for a convex combination of orthogonal polynomials on the unit circle was analyzed in [2].

In the present contribution, our aim is to give necessary and sufficient conditions about the sequences $\{R_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{\alpha_n\}_{n=0}^{\infty}$ or α in order for $\{P_n(x)\}_{n=0}^{\infty}$ as in (1.3) or (1.4) to be a sequence of monic orthogonal polynomials with respect to a moment functional w, i.e., $\{P_n(x)\}_{n=0}^{\infty}$ must satisfy a three-term recurrence relation,

$$P_{n+1}(x) = (x - b_n) P_n(x) - c_n P_{n-1}(x), \quad n \geqslant 0 \tag{1.5}$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$, $c_n \neq 0$ for $n \geqslant 1$.

2. Linear perturbation

In this section, we let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of monic polynomials given by the perturbation (1.3) of $\{Q_n(x)\}_{n=0}^{\infty}$ by $\{R_n(x)\}_{n=0}^{\infty}$.

Assuming that $\{P_n(x)\}_{n=0}^{\infty}$ satisfies (1.5) and substituting (1.3) into (1.5), we obtain for $n \geqslant 0$

$$\begin{aligned}
&Q_{n+1}(x) - \alpha_{n+1} R_{n-t+1}(x) \\
&\quad = (x - b_n)[Q_n(x) - \alpha_n R_{n-t}(x)] - c_n [Q_{n-1}(x) - \alpha_{n-1} R_{n-t-1}(x)] \\
&\quad = Q_{n+1}(x) + (\beta_n - b_n) Q_n(x) + (\gamma_n - c_n) Q_{n-1}(x) - \alpha_n R_{n-t+1}(x) \\
&\qquad + \alpha_n (b_n - \delta_{n-t}) R_{n-t}(x) + (\alpha_{n-1} c_n - \alpha_n \varepsilon_{n-t}) R_{n-t-1}(x)
\end{aligned}$$

or equivalently

$$\begin{aligned}
&(\beta_n - b_n) Q_n(x) + (\gamma_n - c_n) Q_{n-1}(x) \\
&\quad = \begin{cases} (\alpha_n - \alpha_{n+1}) R_{n-t+1}(x) - \alpha_n (b_n - \delta_{n-t}) R_{n-t}(x) \\ \quad -(\alpha_{n-1} c_n - \alpha_n \varepsilon_{n-t}) R_{n-t-1}(x), \quad n \geqslant t, \\ -\alpha_t, \quad n = t-1, \\ 0, \ 0 \leqslant n \leqslant t-2, \end{cases}
\end{aligned} \tag{2.1}$$

where $Q_n(x) = R_n(x) = 0$ for $n < 0$.

Taking into account the different choices for t, three different situations appear.

Theorem 2.1. *Assume $t \geqslant 3$. Then $\{P_n(x)\}_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials (SMOP) if and only if $\alpha_n = 0$, $n \geqslant 0$, which means $P_n(x) = Q_n(x)$, $n \geqslant 0$.*

Proof. $\Leftarrow$) Trivial.
$\Rightarrow$) By (2.1), we have $\alpha_n = \alpha_{n+1}$ for $n \geqslant t$ and $\alpha_t = 0$ so that $\alpha_n = 0$ for $n \geqslant 0$. $\square$

Now we will assume $t = 2$. Then (2.1) is equivalent to

$$b_n = \beta_n, \quad n \geqslant 0, \tag{2.2}$$

$$c_1 = \gamma_1 + \alpha_2 \tag{2.3}$$

and

$$(\gamma_n - c_n)Q_{n-1}(x) = (\alpha_n - \alpha_{n+1})R_{n-1}(x) \\ -\alpha_n(b_n - \delta_{n-2})R_{n-2}(x) - (\alpha_{n-1}c_n - \alpha_n\varepsilon_{n-2})R_{n-3}(x), \quad n \geqslant 2. \tag{2.4}$$

If we identify the leading coefficients on both sides of (2.4), then

$$c_n = \gamma_n - \alpha_n + \alpha_{n+1}, \quad n \geqslant 2. \tag{2.5}$$

Thus, we obtain:

Theorem 2.2. *Assume $t = 2$. Then $\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP if and only if*

$$\alpha_n - \alpha_{n+1} \neq \gamma_n, \quad n \geqslant 1$$

and

$$(\alpha_n - \alpha_{n+1})Q_{n-1}(x) = (\alpha_n - \alpha_{n+1})R_{n-1}(x) - \alpha_n(\beta_n - \delta_{n-2})R_{n-2}(x) \\ + [\alpha_n\varepsilon_{n-2} - \alpha_{n-1}\gamma_n + \alpha_{n-1}(\alpha_n - \alpha_{n+1})]R_{n-3}(x), \quad n \geqslant 2. \tag{2.6}$$

Then, we have

$$b_n = \beta_n \text{ for } n \geqslant 0 \quad \text{and} \quad c_n = \gamma_n - \alpha_n + \alpha_{n+1} \text{ for } n \geqslant 1.$$

We will analyze some particular cases.

Theorem 2.3. *Assume $t = 2$ and $\alpha_n = \alpha$ for every $n \geqslant 2$. Then $\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP if and only if either $\alpha = 0$ so that $P_n(x) = Q_n(x)$, $n \geqslant 0$ or $\alpha \neq 0, -\gamma_1$ and $\delta_{n-2} - \beta_n = \varepsilon_{n-1} - \gamma_{n+1} = 0$ for $n \geqslant 2$ so that $P_n(x) = Q_n(x) - \alpha Q_{n-2}^{(2)}(x)$, $n \geqslant 0$.*

Proof. Since $\alpha_n = \alpha$ for $n \geqslant 2$, (2.6) becomes

$$\alpha(\beta_n - \delta_{n-2})R_{n-2}(x) - \alpha(\varepsilon_{n-2} - \gamma_n)R_{n-3}(x) = 0, \quad n \geqslant 2,$$

which is equivalent to

$$\alpha(\beta_n - \delta_{n-2}) = \alpha(\varepsilon_{n-1} - \gamma_{n+1}) = 0, \quad n \geqslant 2.$$

Hence, by Theorem 2.2, $\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP if and only if

$$\gamma_1 + \alpha \neq 0 \quad \text{and} \quad \alpha(\beta_n - \delta_{n-2}) = \alpha(\varepsilon_{n-1} - \gamma_{n+1}) = 0, \; n \geqslant 2. \tag{2.7}$$

If $\alpha = 0$, then condition (2.7) holds trivially and $P_n(x) = Q_n(x)$, $n \geqslant 0$. If $\alpha \neq 0$, then condition (2.7) is equivalent to

$$\alpha \neq -\gamma_1, \quad \delta_n = \beta_{n+2} \text{ for } n \geqslant 0 \quad \text{and} \quad \varepsilon_n = \gamma_{n+2} \text{ for } n \geqslant 1,$$

so that $\{R_n(x)\}_{n=0}^{\infty} = \{Q_n^{(2)}(x)\}_{n=0}^{\infty}$ is the associated SMOP of the second kind for $\{Q_n(x)\}_{n=0}^{\infty}$. $\square$

In case $\alpha \neq 0, -\gamma_1$, $\{P_n(x)\}_{n=0}^{\infty} = \{Q_n(x) - \alpha Q_{n-2}^{(2)}(x)\}_{n=0}^{\infty}$ satisfies the three-term recurrence relation (1.5), where

$$b_n = \beta_n \text{ for } n \geqslant 0, \quad c_1 = \gamma_1 + \alpha \quad \text{and} \quad c_n = \gamma_n \text{ for } n \geqslant 2.$$

Hence, $\{P_n(x)\}_{n=0}^{\infty}$ is the co-dilated SMOP of $\{Q_n(x)\}_{n=0}^{\infty}$ at level 1(see [5]).

Remark 2.4. If $\alpha_n \neq \alpha_{n+1}$ for every $n \geqslant 2$, then (2.6) becomes

$$Q_{n-1}(x) = R_{n-1}(x) + s_n R_{n-2}(x) + t_n R_{n-3}(x), \quad n \geqslant 2,$$

where

$$s_n = \frac{\alpha_n}{\alpha_{n+1} - \alpha_n}(\beta_n - \delta_{n-2})$$

$$t_n = \alpha_{n-1} + \frac{\alpha_n \varepsilon_{n-2} - \alpha_{n-1}\gamma_n}{\alpha_n - \alpha_{n+1}}.$$

Hence, $\{Q_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal of order $\leqslant 2$ relative to v.

Since both $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$ are SMOPs with respect to u and v, respectively,

$$\begin{aligned} \langle v, Q_n \rangle &= 0, \quad n \geqslant 3, \\ \langle v, Q_2 \rangle &= t_3, \\ \langle v, Q_1 \rangle &= s_2. \end{aligned} \tag{2.8}$$

Then

$$v = \left(t_3 \frac{Q_2(x)}{\langle u, Q_2^2(x) \rangle} + s_2 \frac{Q_1(x)}{\langle u, Q_1^2(x) \rangle} + 1 \right) u,$$

so that

$$v = (ax^2 + bx + c)u,$$

where a, b, c can be deduced taking into account (2.8)

$$t_3 = \langle u, (ax^2 + bx + c)Q_2(x) \rangle,$$

$$s_2 = \langle u, (ax^2 + bx + c)Q_1(x) \rangle,$$

$$1 = \langle u, ax^2 + bx + c \rangle$$

i.e.,

$$t_3 = a\langle u, x^2 Q_2(x)\rangle = a\gamma_2\gamma_1,$$

$$s_2 = \gamma_1[b + a(\beta_0 + \beta_1)],$$

$$c = \frac{t_3 Q_2(0)}{\gamma_2\gamma_1} + \frac{s_2 Q_1(0)}{\gamma_1} + 1 = \frac{t_3(\beta_0\beta_1 - \gamma_1)}{\gamma_2\gamma_1} - \frac{s_2\beta_0}{\gamma_1} + 1.$$

Thus,

$$a = \frac{1}{\gamma_1\gamma_2}\left[\alpha_2 + \frac{\alpha_3\varepsilon_1 - \gamma_3\alpha_2}{\alpha_3 - \alpha_4}\right],$$

$$b = \frac{\alpha_2}{\gamma_1(\alpha_3 - \alpha_2)}(\beta_2 - \delta_0) - a(\beta_0 + \beta_1),$$

$$c = a(\beta_0\beta_1 - \gamma_1) - \beta_0[b + a(\beta_0 + \beta_1)] + 1.$$

In this case

$$P_n(x) = R_n(x) + s_{n+1}R_{n-1}(x) + (t_{n+1} - \alpha_n)R_{n-2}(x), \quad n \geqslant 0$$

so that $\{P_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal of order $\leqslant 2$ relative to v.

Since $\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP, from (1.5) we can easily obtain

$$\begin{aligned}
&b_n = \delta_n + s_{n+1} - s_{n+2}, \quad n \geqslant 0,\\
&c_n = \varepsilon_n - b_n s_{n+1} + s_{n+1}\delta_{n-1} + t_{n+1} - \alpha_n - t_{n+2} + \alpha_{n+1}, \quad n \geqslant 1,\\
&s_n c_n = -b_n(t_{n+1} - \alpha_n) + s_{n+1}\varepsilon_{n-1} + (t_{n+1} - \alpha_n)\delta_{n-2}, \quad n \geqslant 2,\\
&(t_n - \alpha_{n-1})c_n = (t_{n+1} - \alpha_n)\varepsilon_{n-2}, \quad n \geqslant 3.
\end{aligned} \tag{2.9}$$

Hence from the last equation in (2.9), either $t_n - \alpha_{n-1} = 0$ for all $n \geqslant 3$ or $t_n - \alpha_{n-1} \neq 0$ for all $n \geqslant 3$. In case $t_3 - \alpha_2 = 0$ (so that $t_n - \alpha_{n-1} = 0$ for all $n \geqslant 3$), P_n is quasi-orthogonal of order 1 relative to v. Notice that necessary and sufficient conditions for the quasi orthogonality of order 1 of a sequence of monic polynomials yield orthogonality can be found in [9, Theorem 2; 6 Theorem 4.2]. In case $t_3 - \alpha_2 \neq 0$ (so that $t_n - \alpha_{n-1} \neq 0$ for all $n \geqslant 3$) (i.e., $\{P_n(x)\}_{n=0}^{\infty}$ is strictly quasi-orthogonal of order 2 relative to v). Notice that necessary and sufficient conditions for the quasi-orthogonality of order 2 of a sequence of monic polynomials yield orthogonality that can be found in [1, Theorem 9].

In these cases, $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal relative to $w = p(x)v$, where

$$\deg(p) = \begin{cases} 0 \text{ if } t_3 - \alpha_2 = 0 \text{ and } s_2 = 0,\\ 1 \text{ if } t_3 - \alpha_2 = 0 \text{ and } s_2 \neq 0,\\ 2 \text{ if } t_3 - \alpha_2 \neq 0. \end{cases}$$

Remark 2.5. Notice that $s_n = 0$ for all $n \geqslant 2$ and $t_3 \neq 0$ yields, according to Theorem 4.7 in [6],

$$\delta_n = \beta_n, \quad n \geqslant 0,$$

$$\delta_n = \delta_{n-2}, \quad \varepsilon_n = \gamma_n, \ \ n \geqslant 2,$$

$$\varepsilon_1 = \gamma_1 + t_3.$$

Hence,

$$R_n(x) = A(x)Q_n(x) + B(x)Q_{n-1}^{(1)}(x).$$

Because of the initial conditions

$$R_1(x) = Q_1(x),$$

$$R_2(x) = Q_2(x) - t_3,$$

$$Q_1(x) = A(x)Q_1(x) + B(x),$$

$$Q_2(x) - t_3 = A(x)Q_2(x) + B(x)Q_1^{(1)}(x),$$

we have

$$A(x) = 1 + \frac{t_3}{\gamma_1},$$

$$B(x) = \frac{-t_3}{\gamma_1} Q_1(x).$$

Thus,

$$R_n(x) = \left(1 + \frac{t_3}{\gamma_1}\right) Q_n(x) - \frac{t_3}{\gamma_1} Q_1(x) Q_{n-1}^{(1)}(x) = Q_n(x) - t_3 Q_{n-2}^{(2)}(x).$$

Finally, we deduce that

$$P_n(x) = Q_n(x) - \alpha_n [Q_{n-2}(x) - t_3 Q_{n-4}^{(2)}]$$

as well as (see [6, Theorem 4.7])

$$\varepsilon_n - t_{n+2} + t_{n+1} = \frac{t_{n+1}}{t_n} \varepsilon_{n-2}, \quad n \geqslant 3,$$

i.e.,

$$\gamma_n - t_{n+2} + t_{n+1} = \frac{t_{n+1}}{t_n} \gamma_{n-2}, \quad n \geqslant 4,$$

$$\gamma_3 - t_5 + t_4 = \frac{t_3}{t_2}(\gamma_1 + t_3),$$

together with

$$t_n = \alpha_{n-1} + \frac{\alpha_n \gamma_{n-2} - \gamma_n \alpha_{n-1}}{\alpha_n - \alpha_{n+1}}, \quad n \geqslant 4$$

which gives the compatible values for $\{\alpha_n\}_{n=0}^{\infty}$.

Remark 2.6. If $Q_n(x)=R_n(x)$ for every $n\geqslant 0$, then we get from (2.6)

$$\alpha_n(\beta_n-\beta_{n-2})=0,\quad n\geqslant 2,$$

$$\alpha_n\gamma_{n-2}=\alpha_{n-1}(\gamma_n-\alpha_n+\alpha_{n+1}),\quad n\geqslant 3.$$

If there exists $s\geqslant 2$ such that $\alpha_s=0$, then from the second identity we deduce $\alpha_{s+1}=0$ and so $\alpha_n=0$, $n\geqslant s$. On the other hand, $\alpha_{s-1}(\gamma_s+\alpha_{s+1})=0$, i.e., $\alpha_{s-1}=0$. If we continue the process, then we deduce that $\alpha_2=0$. Thus $\alpha_n=0$ for every $n\geqslant 2$ so that $P_n(x)=Q_n(x)$ for all $n\geqslant 0$. If $\alpha_n\neq 0$ for every $n\geqslant 2$, we have $\beta_n=\beta_{n-2}$, $n\geqslant 2$ as well as

$$\gamma_n+\alpha_{n+1}=\alpha_n+\frac{\alpha_n}{\alpha_{n-1}}\gamma_{n-2},\quad n\geqslant 3.$$

In this case, according to Theorem 2.2, the sequence $\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP and the parameters in the three-term recurrence relation (1.5) are

$$b_n=\beta_n,\quad n\geqslant 0,$$

$$c_n=\frac{\alpha_n}{\alpha_{n-1}}\gamma_{n-2},\quad n\geqslant 3,$$

$$c_2=\gamma_2+(\alpha_3-\alpha_2),$$

$$c_1=\gamma_1+\alpha_2,$$

where we assume

$$\alpha_1\neq-\gamma_1\quad\text{and}\quad\alpha_3\neq\alpha_2-\gamma_2.$$

If $R_n=Q_n$ and $\alpha_n=\alpha(\neq 0,-\gamma_1)$, $n\geqslant 2$, then by Theorem 2.3, $R_n=Q_n^{(2)}$, $n\geqslant 0$, i.e.,

$$\delta_n=\beta_n=\delta_{n+2}=\beta_{n+2},\quad n\geqslant 0,$$

$$\varepsilon_n=\gamma_n=\varepsilon_{n+2}=\gamma_{n+2},\quad n\geqslant 0.$$

Hence, $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$ have 2-periodic coefficients in the three-term recurrence relation.

Finally, we will assume $t=1$. Then (2.1) becomes

$$(\beta_n-b_n)Q_n(x)+(\gamma_n-c_n)Q_{n-1}(x)$$
$$=\begin{cases}(\alpha_n-\alpha_{n+1})R_n(x)-\alpha_n(b_n-\delta_{n-1})R_{n-1}(x)-(\alpha_{n-1}c_n-\alpha_n\varepsilon_{n-1})R_{n-2}(x), & n\geqslant 1,\\ -\alpha_1, & n=0,\end{cases}\tag{2.10}$$

and taking into account the coefficients of x^n and the x^{n-1} on both sides we get

$$b_n=\beta_n+\alpha_{n+1}-\alpha_n,\quad n\geqslant 0,$$

$$c_n=\gamma_n+\alpha_n(\beta_n-\delta_{n-1})+(\alpha_{n+1}-\alpha_n)(\delta(n)-\beta(n)+\alpha_n),\quad n\geqslant 1,$$

where $\beta(n):=-\sum_{i=0}^{n-1}\beta_i$ and $\delta(n):=-\sum_{i=0}^{n-1}\delta_i$ are the coefficients of x^{n-1} in $Q_n(x)$ and $R_n(x)$, respectively.

We now analyze some particular cases.

Theorem 2.7. *Assume* $t=1$, $\alpha_n=\alpha\neq 0$ *and* $\beta_n=\delta_{n-1}$ *for every* $n\geqslant 1$. *Then* $\{P_n(x)\}_{n=0}^{\infty}$ *is an SMOP if and only if* $\gamma_n=\varepsilon_{n-1}$, $n\geqslant 2$. *In this case*, $P_n(x)=Q_n(x)-\alpha Q_{n-1}^{(1)}(x)$, $n\geqslant 0$ *and*

$$b_0=\beta_0+\alpha,\ b_n=\beta_n \text{ for } n\geqslant 1 \quad \text{and} \quad c_n=\gamma_n \text{ for } n\geqslant 1. \tag{2.11}$$

Proof. Assume $\alpha_n=\alpha\neq 0$ and $\beta_n=\delta_{n-1}$, $n\geqslant 1$. Then (2.10) becomes

$$(\beta_n-b_n)Q_n(x)+(\gamma_n-c_n)Q_{n-1}(x)$$
$$=\begin{cases}-\alpha(b_n-\beta_n)R_{n-1}(x)-\alpha(c_n-\varepsilon_{n-1})R_{n-2}(x), & n\geqslant 1,\\ -\alpha, \quad n=0.\end{cases}$$

Hence, if $\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP, then

$$\beta_0-b_0=-\alpha,\ \beta_n-b_n=0 \quad \text{for } n\geqslant 1,$$
$$\gamma_n-c_n=-\alpha(b_n-\beta_n)=0 \quad \text{for } n\geqslant 1,$$

and

$$c_n-\varepsilon_{n-1}=0 \quad \text{for } n\geqslant 2,$$

so that $\gamma_n=\varepsilon_{n-1}$, $n\geqslant 2$ and (2.11) follows.

Since $\delta_n=\beta_{n+1}$, $n\geqslant 0$ and $\varepsilon_n=\gamma_{n+1}$, $n\geqslant 1$, $\{R_n(x)\}_{n=0}^{\infty}=\{Q_n^{(1)}(x)\}_{n=0}^{\infty}$ is the associated SMOP of the first kind for $\{Q_n(x)\}_{n=0}^{\infty}$. Conversely, assume $\gamma_n=\varepsilon_{n-1}$, $n\geqslant 2$. Then $R_n(x)=Q_n^{(1)}(x)$, $n\geqslant 0$ so that $P_n(x)=Q_n(x)-\alpha Q_{n-1}^{(1)}(x)=Q_n(\alpha;x)$, $n\geqslant 0$, are the co-recursive SMOP of $\{Q_n(x)\}_{n=0}^{\infty}$ [3]. □

Remark 2.8. If $\alpha_n=\alpha$ for every $n\geqslant 1$ and $\gamma_n\neq c_n$ for every $n\geqslant 1$, then (2.10) becomes

$$Q_{n-1}(x)=R_{n-1}(x)+\alpha\frac{\varepsilon_{n-1}-c_n}{\gamma_n-c_n}R_{n-2}(x), \quad n\geqslant 1.$$

Since both $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$ are SMOPs, according to Theorem 4.2 in [6] if we assume $\varepsilon_1\neq c_2$, then $(\varepsilon_{n-1}-c_n)/(\gamma_n-c_n)\neq 0$ for $n\geqslant 2$ and

$$c=\lambda+\alpha\frac{\varepsilon_1-c_2}{\gamma_2-c_2}-\delta_0\neq 0,$$
$$\lambda=-\varepsilon_n\frac{\gamma_{n+1}-c_{n+1}}{\varepsilon_n-c_{n+1}}\frac{1}{\alpha}-\alpha\frac{\varepsilon_{n+1}-c_{n+2}}{\gamma_{n+2}-c_{n+2}}+\delta_n, \quad n\geqslant 1.$$

In such a case, $\{Q_n(x)\}_{n=0}^{\infty}$ is the SMOP with respect to the moment functional

$$u=(x-\lambda)^{-1}v-\frac{1}{c}\delta(x-\lambda),$$

i.e.,

$$(x-\lambda)u=v$$

and $\{R_n(x)\}_{n=0}^{\infty}$ are the monic kernel polynomials for $\{Q_n(x)\}_{n=0}^{\infty}$ with K-parameter λ ([4]), that is,

$$R_n(x)=Q_n^*(\lambda,x)=\frac{1}{Q_n(\lambda)}\frac{Q_{n+1}(x)Q_n(\lambda)-Q_{n+1}(\lambda)Q_n(x)}{x-\lambda}, \quad n\geqslant 0.$$

Remark 2.9. If $R_n(x)=Q_n(x)$ for every $n\geqslant 0$ so that $P_n(x)=Q_n(x)-\alpha_n Q_{n-1}(x)$, $n\geqslant 0$, then $\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP if and only if either $\alpha_n=0$, $n\geqslant 1$ or $\alpha_n\neq 0$, $n\geqslant 1$, $(\gamma_n/\alpha_n)+\alpha_{n+1}+\beta_n=\lambda$, $n\geqslant 1$, and $c:=\lambda-\alpha_1-\beta_0\neq 0$. In the latter case, $\{P_n(x)\}_{n=0}^{\infty}$ is the SMOP with respect to the moment functional

$$(x-\lambda)^{-1}u-\frac{1}{c}\delta(x-\lambda).$$

See [6,10].

Remark 2.10. Assume $\langle u,R_n\rangle=0$ for $n\geqslant 2$ and $\langle u,R_1\rangle\neq 0$. Then $u=\varphi(x)v$, where

$$\varphi(x)=\frac{\beta_0-\delta_0}{\varepsilon_1}x+\frac{\varepsilon_1-(\beta_0-\delta_0)\delta_0}{\varepsilon_1},\quad \beta_0\neq\delta_0.$$

Hence,

$$Q_n(x)=R_n^*(\lambda,x),\quad n\geqslant 0$$

and

$$R_n(x)=Q_n(x)-a_nQ_{n-1}(x),\quad n\geqslant 0,$$

where $\lambda=((\beta_0-\delta_0)\delta_0-\varepsilon_1)/(\beta_0-\delta_0)$ is the zero of $\varphi(x)$ and

$$a_n=\frac{R_{n-1}(\lambda)}{R_n(\lambda)}\varepsilon_n\neq 0,\quad n\geqslant 1.$$

Therefore,

$$P_n(x)=Q_n(x)-\alpha_nQ_{n-1}(x)+\alpha_na_{n-1}Q_{n-2}(x),\quad n\geqslant 0.$$

Then, $\{P_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal of order $\leqslant 2$ relative to u. As in Remark 2.4, if $\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP, then either $\alpha_2=0$ or $\alpha_2\neq 0$. In case $\alpha_2=0$, $\alpha_n=0$ for all $n\geqslant 1$ so that $P_n(x)=Q_n(x)$, $n\geqslant 0$. In case $\alpha_2\neq 0$, then $\alpha_n\neq 0$ for all $n\geqslant 2$ (but α_1 may or may not be 0) and $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal relative to $w=p(x)u$ for some polynomial of degree 2 (see [1, Theorem 9]).

Remark 2.11. If $\alpha_n\neq\alpha_{n+1}$ for every $n\geqslant 0$, then (2.10) becomes

$$Q_n(x)+\frac{\gamma_n-c_n}{\alpha_n-\alpha_{n+1}}Q_{n-1}(x)$$
$$=R_n(x)-\alpha_n\frac{b_n-\delta_{n-1}}{\alpha_n-\alpha_{n+1}}R_{n-1}(x)-\frac{\alpha_{n-1}c_n-\alpha_n\varepsilon_{n-1}}{\alpha_n-\alpha_{n+1}}R_{n-2}(x),\quad n\geqslant 1.$$

This leads to the following problem. Given an SMOP $\{R_n(x)\}_{n=0}^{\infty}$, find necessary and sufficient conditions for the existence of an SMOP $\{Q_n(x)\}_{n=0}^{\infty}$ such that

$$Q_n(x)+s_nQ_{n-1}(x)=R_n(x)+t_nR_{n-1}(x)+u_nR_{n-2}(x). \tag{2.12}$$

The next problem to solve is to find the relation between the corresponding linear functionals. An analog of this problem for orthogonal polynomials on the unit circle has been analyzed in [8] in a more general framework.

3. Convex combination

In this section, we analyze an analogous problem of the one solved in [2] for orthogonal polynomials on the unit circle: When is a convex linear combination of $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$ again orthogonal?

We now let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of monic polynomials given by (1.4). To avoid the trivial situation, we always assume $\alpha \neq 0, 1$. We expand $xP_n(x)$ via (1.1) and (1.2) as

$$\begin{aligned} xP_n(x) &= \alpha x Q_n(x) + (1-\alpha) x R_n(x) \\ &= \alpha[Q_{n+1}(x) + \beta_n Q_n(x) + \gamma_n Q_{n-1}(x)] + (1-\alpha)[R_{n+1}(x) + \delta_n R_n(x) + \varepsilon_n R_{n-1}(x)] \\ &= P_{n+1}(x) + \alpha(\beta_n - \delta_n) Q_n(x) + \delta_n P_n(x) + \alpha(\gamma_n - \varepsilon_n) Q_{n-1}(x) + \varepsilon_n P_{n-1}(x), \quad n \geqslant 0. \end{aligned}$$

Now, if

$$Q_n(x) = P_n(x) + \sum_{i=0}^{n-1} a_{n,i} P_i(x), \quad n \geqslant 1, \tag{3.1}$$

then

$$\begin{aligned} xP_n(x) = {} & P_{n+1}(x) + [\alpha(\beta_n - \delta_n) + \delta_n] P_n(x) + [\alpha(\beta_n - \delta_n) a_{n,n-1} + \alpha(\gamma_n - \varepsilon_n) + \varepsilon_n] P_{n-1}(x) \\ & + \sum_{i=0}^{n-2} [\alpha(\beta_n - \delta_n) a_{n,i} + \alpha(\gamma_n - \varepsilon_n) a_{n-1,i}] P_i(x), \quad n \geqslant 2, \end{aligned}$$

$$xP_1(x) = P_2(x) + [\alpha(\beta_1 - \delta_1) + \delta_1] P_1(x) + [\alpha(\beta_1 - \delta_1) a_{1,0} + \alpha(\gamma_1 - \varepsilon_1) + \varepsilon_1] P_0(x).$$

Thus we deduce the following:

Theorem 3.1. *$\{P_n(x)\}_{n=0}^{\infty}$ is an SMOP if and only if*

(i) $(\beta_n - \delta_n) a_{n,i} + (\gamma_n - \varepsilon_n) a_{n-1,i} = 0$ *for* $i = 0, 1, \ldots, n-2, \quad n \geqslant 2.$
(ii) $\alpha(\beta_n - \delta_n) a_{n,n-1} + \alpha(\gamma_n - \varepsilon_n) + \varepsilon_n \neq 0$ *for* $n \geqslant 1.$

Remark 3.2. Notice that (i) in Theorem 3.1 means that

$$\begin{aligned} & (\beta_n - \delta_n) Q_n(x) + (\gamma_n - \varepsilon_n) Q_{n-1}(x) \\ & \quad = (\beta_n - \delta_n) P_n(x) + [(\beta_n - \delta_n) a_{n,n-1} + (\gamma_n - \varepsilon_n)] P_{n-1}(x). \end{aligned} \tag{3.2}$$

Remark 3.3. Considering the coefficients of x^{n-1} on both sides of (3.1)

$$\beta(n) = b(n) + a_{n,n-1},$$

where $b(n)$ and $\beta(n)$ are the coefficients of x^{n-1} in $P_n(x)$ and $Q_n(x)$, respectively. But from (1.4)

$$b(n) = \alpha\beta(n) + (1-\alpha)\delta(n),$$

where $\delta(n)$ is the coefficient of x^{n-1} in $R_n(x)$. Thus,

$$a_{n,n-1} = (1-\alpha)(\beta(n) - \delta(n))$$

and (ii) in Theorem 3.1 becomes

$$\varepsilon_n + \alpha(\gamma_n - \varepsilon_n) + \alpha(\beta_n - \delta_n)(1-\alpha)(\beta(n) - \delta(n)) \neq 0.$$

Remark 3.4. If $\beta_s = \delta_s$ for some s, then from (i) in Theorem 3.1, $\gamma_s = \varepsilon_s$ or $a_{s-1,i} = 0$ for $i = 0, 1, \ldots, s-2$, i.e., $Q_{s-1}(x) = P_{s-1}(x)$.

Remark 3.5. From (3.2) and (1.4), we have

$$(\beta_n - \delta_n)Q_n(x) + (\gamma_n - \varepsilon_n)Q_{n-1}(x) = (\beta_n - \delta_n)(\alpha Q_n(x) + (1-\alpha)R_n(x)) \\ + [(\beta_n - \delta_n)a_{n,n-1} + (\gamma_n - \varepsilon_n)](\alpha Q_{n-1}(x) + (1-\alpha)R_{n-1}(x)),$$

in other words

$$(\beta_n - \delta_n)(1-\alpha)Q_n(x) + [(\gamma_n - \varepsilon_n)(1-\alpha) - \alpha a_{n,n-1}(\beta_n - \delta_n)]Q_{n-1}(x) \\ = (\beta_n - \delta_n)(1-\alpha)R_n(x) + (1-\alpha)[(\beta_n - \delta_n)a_{n,n-1} + (\gamma_n - \varepsilon_n)]R_{n-1}(x)$$

or, equivalently

$$(\beta_n - \delta_n)Q_n(x) + \left[(\gamma_n - \varepsilon_n) - \frac{\alpha}{1-\alpha}a_{n,n-1}(\beta_n - \delta_n)\right]Q_{n-1}(x) \\ = (\beta_n - \delta_n)R_n(x) + [(\gamma_n - \varepsilon_n) + (\beta_n - \delta_n)a_{n,n-1}]R_{n-1}(x). \tag{3.3}$$

Thus,

Theorem 3.6. $\{P_n(x)\}_{n=0}^{\infty}$ *is an SMOP if and only if* $\{Q_n(x)\}_{n=0}^{\infty}$ *and* $\{R_n(x)\}_{n=0}^{\infty}$ *satisfy*

(i) *Eq.* (3.3);
(ii) $\alpha(\beta_n - \delta_n)a_{n,n-1} + \alpha(\gamma_n - \varepsilon_n) + \varepsilon_n \neq 0$, *where* $a_{n,n-1} = (1-\alpha)\sum_{j=0}^{n-1}(\delta_j - \beta_j)$.

If $\beta_n = \delta_n$ for every $n \geqslant 0$, then from the above theorem

$$(\gamma_n - \varepsilon_n)Q_{n-1}(x) = (\gamma_n - \varepsilon_n)R_{n-1}(x) \tag{3.4}$$

together with $\alpha\gamma_n + (1-\alpha)\varepsilon_n \neq 0$.

Two cases can be considered:

(1) $\gamma_n = \varepsilon_n$ for every $n \geqslant 1$. Then $Q_n(x) = R_n(x)$ for every $n \geqslant 0$ and thus $P_n(x) = Q_n(x)$ for every $n \geqslant 0$.
(2) There exists at least one positive integer m such that $\gamma_m \neq \varepsilon_m$. Denote by s the minimum of such numbers. Then $Q_n(x) = R_n(x)$ for $n = 0, 1, \ldots, s-1, s$. Furthermore,

$$Q_{s+1}(x) = (x - \beta_s)Q_s(x) - \gamma_s Q_{s-1}(x) \\ = (x - \delta_s)R_s(x) - \varepsilon_s R_{s-1}(x) - (\gamma_s - \varepsilon_s)R_{s-1}(x) \\ = R_{s+1}(x) - (\gamma_s - \varepsilon_s)R_{s-1}(x).$$

Thus, $Q_{s+1}(x) \neq R_{s+1}(x)$ and from (3.4) $\gamma_{s+2} = \varepsilon_{s+2}$. We will distinguish two situations:

(i) If $\gamma_{s+1} = \varepsilon_{s+1}$, then

$$\begin{aligned} Q_{s+2}(x) &= (x - \beta_{s+1})Q_{s+1}(x) - \gamma_{s+1}Q_s(x) \\ &= (x - \delta_{s+1})[R_{s+1}(x) - (\gamma_s - \varepsilon_s)R_{s-1}(x)] - \gamma_{s+1}R_s(x) \\ &= R_{s+2}(x) - (\gamma_s - \varepsilon_s)(x - \delta_{s+1})R_{s-1}(x). \end{aligned}$$

Thus $R_{s+2}(x) \neq Q_{s+2}(x)$ and from (3.4) $\gamma_{s+3} = \varepsilon_{s+3}$. Now

$$\begin{aligned} Q_{s+3}(x) &= (x - \beta_{s+2})Q_{s+2}(x) - \gamma_{s+2}Q_{s+1}(x) \\ &= (x - \delta_{s+2})[R_{s+2}(x) - (\gamma_s - \varepsilon_s)(x - \delta_{s+1})R_{s-1}(x)] \\ &\quad -\gamma_{s+2}[R_{s+1}(x) - (\gamma_s - \varepsilon_s)R_{s-1}(x)] \\ &= R_{s+3}(x) - (\gamma_s - \varepsilon_s)(x - \delta_{s+2})(x - \delta_{s+1})R_{s-1}(x) \\ &\quad +\gamma_{s+2}(\gamma_s - \varepsilon_s)R_{s-1}(x). \end{aligned}$$

Thus, $R_{s+3}(x) \neq Q_{s+3}(x)$ and from (3.4), $\gamma_{s+4} = \varepsilon_{s+4}$. Then, an induction argument yields $\gamma_n = \varepsilon_n$ for $n \geqslant s + 1$ together with $\gamma_n = \varepsilon_n$ for $n < s$.

(ii) If $\gamma_{s+1} \neq \varepsilon_{s+1}$, then

$$\begin{aligned} Q_{s+2}(x) &= (x - \beta_{s+1})Q_{s+1}(x) - \gamma_{s+1}Q_s(x) \\ &= (x - \delta_{s+1})[R_{s+1}(x) - (\gamma_s - \varepsilon_s)R_{s-1}(x)] - \gamma_{s+1}R_s(x) \\ &= R_{s+2}(x) + (\varepsilon_{s+1} - \gamma_{s+1} + \varepsilon_s - \gamma_s)R_s(x) \\ &\quad + (\delta_{s-1} - \delta_{s+1})(\varepsilon_s - \gamma_s)R_{s-1}(x) - (\gamma_s - \varepsilon_s)\varepsilon_{s-1}R_{s-2}(x). \end{aligned}$$

Notice that in this case, $Q_{s+2}(x) \neq R_{s+2}(x)$ and thus $\gamma_{s+3} = \varepsilon_{s+3}$. Now since $\gamma_{s+2} = \varepsilon_{s+2}$,

$$\begin{aligned} Q_{s+3}(x) &= (x - \beta_{s+2})Q_{s+2}(x) - \gamma_{s+2}Q_{s+1}(x) \\ &= (x - \delta_{s+2})[R_{s+2}(x) + (\varepsilon_{s+1} - \gamma_{s+1} + \varepsilon_s - \gamma_s)R_s(x) \\ &\quad + (\delta_{s-1} - \delta_{s+1})(\varepsilon_s - \gamma_s)R_{s-1}(x) - (\gamma_s - \varepsilon_s)\varepsilon_{s-1}R_{s-2}(x)] \\ &\quad -\gamma_{s+2}[R_{s+1}(x) - (\gamma_s - \varepsilon_s)R_{s-1}(x)] \\ &= R_{s+3}(x) + \cdots + \varepsilon_{s-1}\varepsilon_{s-2}(\varepsilon_s - \gamma_s)R_{s-3}(x). \end{aligned}$$

Thus $Q_{s+3}(x) \neq R_{s+3}(x)$ and then from (3.4), $\gamma_{s+4} = \varepsilon_{s+4}$. An induction argument yields $\gamma_n = \varepsilon_n$ for $n \geqslant s + 2$ and $\gamma_n = \varepsilon_n$ for $n < s$.

As a conclusion:

Theorem 3.7. *If* $\beta_n = \delta_n$, $n \geqslant 0$, *then the sequence* $\{P_n(x)\}_{n=0}^{\infty}$ *of monic polynomials such that* $P_n(x) = \alpha Q_n(x) + (1 - \alpha)R_n(x)$ *is also an SMOP if and only if any one of the following holds*:

(1) $Q_n(x) = R_n(x)$ *for all* $n \geqslant 0$, *i.e.*, $\gamma_n = \varepsilon_n$ *for all* $n \geqslant 1$.
(2) *There exists a positive integer* s *such that* $\gamma_s \neq \varepsilon_s$ *and* $\gamma_n = \varepsilon_n$ *otherwise.*
(3) *There exists a positive integer* s *such that* $\gamma_s \neq \varepsilon_s$, $\gamma_{s+1} \neq \varepsilon_{s+1}$ *and* $\gamma_n = \varepsilon_n$ *otherwise.*

This statement can be given in an alternative way: at most two consecutive elements of sequence $\{\varepsilon_n - \gamma_n\}_{n=1}^{\infty}$ are not zero.

Using standard arguments about the set of solutions of the difference equation $xy_n = y_{n+1} + c_n y_n + d_n y_{n-1}$, it is very easy to deduce the following:

Corollary 3.8. *If* $\gamma_s \neq \varepsilon_s$ *and* $\gamma_n = \varepsilon_n$ *otherwise, then*

$$Q_n(x) = \frac{\gamma_s}{\varepsilon_s} R_n(x) + \left(1 - \frac{\gamma_s}{\varepsilon_s}\right) R_{n-s}^{(s)}(x) R_s(x), \quad n \geqslant s.$$

If $\gamma_s \neq \varepsilon_s$, $\gamma_{s+1} \neq \varepsilon_{s+1}$ *and* $\gamma_n = \varepsilon_n$ *otherwise, then*

$$Q_n(x) = \frac{\gamma_{s+1}}{\varepsilon_{s+1}} R_n(x) + R_{n-s-1}^{(s+1)}(x) \left[\left(1 - \frac{\gamma_{s+1}}{\varepsilon_{s+1}}\right) R_{s+1}(x) - (\gamma_s - \varepsilon_s) R_{s-1}(x)\right], \quad n \geqslant s+1.$$

Here $\{R_n^{(s)}(x)\}_{n=0}^{\infty}$ *is the associated SMOP of the sth kind for* $\{R_n(x)\}_{n=0}^{\infty}$.

This kind of perturbation has been introduced in [7].

Remark 3.9. Notice that if $\beta_n \neq \delta_n$ for every $n \geqslant$, (3.3) is a particular case of (2.12) for $u_n = 0$.

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