Perturbations in the Nevai matrix class of orthogonal matrix polynomials

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Abstract

In this paper we study a Jacobi block matrix and the behavior of the limit of its entries when a perturbation of its spectral matrix measure by the addition of a Dirac delta matrix measure is introduced.

Keywords Matrix measures; Orthogonal matrix polynomials; Jacobi block matrices; Asymptotics

1. Introduction

In the study of analytic properties of polynomials orthogonal with respect to a real measure supported on the real line, the existence of mass points plays an important role. In [12] the asymptotic behavior of a sequence of polynomials orthogonal with respect to a perturbation $d\alpha_t$ of a measure $d\alpha$ of the Nevai class, i.e., a measure such that the coefficient of the three-term recurrence relation which such a family satisfies converge, adding a mass concentrated at a point t is analyzed. In fact, this asymptotic behavior depends on the location of the mass point with respect to the support of the initial measure α. The explicit expression for the new coeffi cients of the three-term recurrence relation can be given. More recently, a relation for the corresponding Jacobi matrices has been found. Furthermore, if we factorize a semi-infinit tridiagonal matrix

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$$L = \begin{pmatrix} b_1 & 1 & & & \\ a_1 & b_2 & 1 & & \\ & a_2 & b_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with two factors A and B in the form

$$A = \begin{pmatrix} \alpha_1 & 1 & & & \\ 0 & \alpha_2 & 1 & & \\ & 0 & \alpha_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & & & \\ \beta_1 & 1 & 0 & & \\ & \beta_2 & 1 & 0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

it is easy to show that the factorization is not unique and depends on a free parameter. If we then form the product in the reversed order

$$\tilde{L} = BA,$$

$\tilde{L}$ is a new tridiagonal matrix which is called the Darboux transform of L. In a recent paper [8] a connection of the Darboux transform of a Jacobi matrix and the presence of a mass point has been established.

In such a way, we are interested in the study of some analog problems for matrix measures and the corresponding sequences of matrix orthogonal polynomials. However, the analysis of asymptotic properties of matrix orthogonal polynomials is very recent [2,15] and this is one of the reasons for the interest in the extension of the "scalar" case to the "matrix" case.

Matrix orthogonal polynomials appear in a natural way when one considers families of scalar sequences of polynomials satisfying higher-order recurrence relations [4] as well as when one analyzes scalar polynomials orthogonal with respect to measures supported on some kind of algebraic curves [10,11]. The connection between the discrete Sturm–Liouville operator and the scattering problem using asymptotic formulas for matrix orthogonal polynomials was stated in [1].

The structure of this paper is as follows. In Section 2 we will introduce orthonormal matrix polynomials on the real line and discuss some basic properties which we will need in the following sections. In Section 3, we deduce a relation between two truncated Jacobi matrices, when the corresponding matrix measures are related by the addition of a Dirac delta matrix measure. Section 4 is devoted to findin an explicit form for the perturbed matrix coefficient in the three-term recurrence relation and their asymptotic behavior. Finally, in Section 5 we give some examples.

2. Orthogonal matrix polynomials: some basic properties

We consider in the linear space of the polynomials $\mathbb{C}^{N\times N}[t]$ in the variable t with coefficient in $\mathbb{C}^{N\times N}$ a bilinear form

$$\langle P, Q\rangle_{\mathrm{d}\alpha} \stackrel{\text{def}}{=} \int_{\mathbb{R}} P(x)\mathrm{d}\alpha(t)Q^*(x), \tag{2.1}$$

where $\mathrm{d}\alpha$ is a positive definite matrix measure with infinit support and $\int \mathrm{d}\alpha(x) = I_N$. Furthermore, we require that $\int P(x)\,\mathrm{d}\alpha(x) P^*(x)$ is non-singular for every matrix polynomial with non-singular leading coefficient

This bilinear form satisfie

1. $\langle P, Q\rangle_{\mathrm{d}\alpha} = \langle Q, P\rangle_{\mathrm{d}\alpha}^*$.
2. $\langle xP, Q\rangle_{\mathrm{d}\alpha} = \langle P, xQ\rangle_{\mathrm{d}\alpha}$.
3. $\langle P, P\rangle_{\mathrm{d}\alpha}$ is a non-negative definite matrix. If $\det P \neq 0$, it is a positive definite matrix.

Using the generalized Gram–Schmidt orthonormalization procedure [7] for the set $\{I_N, xI_N, x^2 I_N, \ldots\}$ with respect to (2.1) we will obtain a set of orthonormal matrix polynomials $\{P_n(x, \mathrm{d}\alpha)\}_{n\in\mathbb{N}}$, i.e.,

$$\langle P_n(x, \mathrm{d}\alpha), P_m(x, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} = \int_{\mathbb{R}} P_n(x, \mathrm{d}\alpha)\mathrm{d}\alpha(x) P_m^*(x, \mathrm{d}\alpha) = \delta_{n,m} I_N, \tag{2.2}$$

where I_N means the identity matrix in $\mathbb{C}^{N\times N}$[1,6].

Notice that the set $\{U_n P_n(x, \mathrm{d}\alpha)\}$ is also a set of orthonormal matrix polynomials for every sequence $(U_n)_{n\in\mathbb{N}}$ of unitary matrices. If we impose the fact that the leading coefficien is positive definite, then the uniqueness of the sequence of orthonormal matrix polynomials follows from the uniqueness of the polar decomposition [1,9].

As in the scalar case, an orthonormal matrix polynomial $P_n(x, \mathrm{d}\alpha)$ for the matrix measure $\mathrm{d}\alpha$ is orthogonal to every matrix polynomial of degree less than n and the orthonormal polynomials satisfy a three-term recurrence relation

$$xP_n(x, \mathrm{d}\alpha) = D_{n+1}(\mathrm{d}\alpha) P_{n+1}(x, \mathrm{d}\alpha) + E_n(\mathrm{d}\alpha) P_n(x, \mathrm{d}\alpha) + D_n^*(\mathrm{d}\alpha) P_{n-1}(x, \mathrm{d}\alpha), \quad n \geqslant 0, \tag{2.3}$$

where $P_{-1} = 0$, $P_0 = \int \mathrm{d}\alpha = I_N$, $D_n(\mathrm{d}\alpha)$ are the non-singular matrices and $E_n(\mathrm{d}\alpha)$ are Hermitian. Furthermore, definin

$$K_{n+1}(x, y, \mathrm{d}\alpha) \overset{\text{def}}{=} \sum_{j=0}^{n} P_j^*(y, \mathrm{d}\alpha) P_j(x, \mathrm{d}\alpha), \tag{2.4}$$

we get the Christoffel–Darboux formula [16]

$$(x-y)K_{n+1}(x, y, \mathrm{d}\alpha) = P_n^*(y, \mathrm{d}\alpha) D_{n+1}(\mathrm{d}\alpha) P_{n+1}(x, \mathrm{d}\alpha) - P_{n+1}^*(y, \mathrm{d}\alpha) D_{n+1}^*(\mathrm{d}\alpha) P_n(x, \mathrm{d}\alpha). \tag{2.5}$$

By means of a straightforward computation we get the confluen formula

$$K_{n+1}(x, x, \mathrm{d}\alpha) = P_{n+1}^*(x, \mathrm{d}\alpha)' D_{n+1}^*(\mathrm{d}\alpha) P_n(x, \mathrm{d}\alpha) - P_n^*(x, \mathrm{d}\alpha)' D_{n+1}(\mathrm{d}\alpha) P_{n+1}(x, \mathrm{d}\alpha). \tag{2.6}$$

The matrix polynomial $K_n(x, y, \mathrm{d}\alpha)$ is called the nth reproducing kernel because of the following property. For every matrix polynomial $\Pi_m(x)$ of degree $m \leqslant n-1$, we have

$$\langle \Pi_m(x), K_n(x, y, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} = \int_{\mathbb{R}} \Pi_m(x)\, \mathrm{d}\alpha(x) K_n^*(x, y, \mathrm{d}\alpha) = \Pi_m(y). \qquad (2.7)$$

Definition 2.1. Given two matrices D and E, with E Hermitian, a sequence of matrix polynomials $\{P_n(x, \mathrm{d}\alpha)\}$ satisfying (2.3) belongs to the Nevai matrix class $M(D, E)$ if

$$\lim_{n\to\infty} D_n(\mathrm{d}\alpha) = D \quad \text{and} \quad \lim_{n\to\infty} E_n(\mathrm{d}\alpha) = E. \qquad (2.8)$$

We say that a positive definite matrix measure $\mathrm{d}\alpha$ belongs to the Nevai matrix class $M(D, E)$ if there exists a sequence of orthonormal matrix polynomials which belongs to $M(D, E)$.

It is shown in [2] that the support of the matrix measure $\mathrm{d}\alpha$ ($\mathrm{supp}(\mathrm{d}\alpha) = \mathrm{supp}(\mathrm{d}\tau\alpha) = \mathrm{supp}(\mathrm{d}\alpha_{1,1} + \mathrm{d}\alpha_{2,2} + \cdots + \mathrm{d}\alpha_{N,N})$; $\mathrm{d}\tau\alpha$ is the trace measure of $\mathrm{d}\alpha$ [3,13]) when $\mathrm{d}\alpha$ belongs to the Nevai matrix class $M(D, E)$ is uniquely determined by its sequence of orthonormal matrix polynomials, and the ratio asymptotics between the $(n-1)$th and nth orthonormal matrix polynomials with respect to the positive matrix measure $\mathrm{d}\alpha$ in the Nevai matrix class $M(D, E)$ with D non-singular, is given by

$$\begin{aligned}&\lim_{n\to\infty} P_{n-1}(z, \mathrm{d}\alpha) P_n^{-1}(z, \mathrm{d}\alpha) D_n^{-1}(\mathrm{d}\alpha)\\ &\quad = \int \frac{\mathrm{d}W_{D,E}(t)}{z-t} \stackrel{\text{def}}{=} \mathscr{F}_{D,E}(z), \quad z \in \mathbb{C}\backslash\Gamma,\end{aligned} \qquad (2.9)$$

where Γ is a certain bounded set of real numbers containing $\mathrm{supp}(\mathrm{d}\alpha)$ and $W_{D,E}$ is the positive matrix measure such that the matrix polynomials $\{R_n(t)\}$ satisfying the three-term matrix recurrence relation

$$t R_n(t) = D R_{n+1}(t) + E R_n(t) + D^* R_{n-1}(t), \quad n \geqslant 0,$$

are orthonormal with respect to it. Unless otherwise stated, we will denote $\mathscr{F}(z) = \mathscr{F}_{D,E}(z)$ for the sake of simplicity in notation.

The asymptotic results obtained by Nevai [12] when some mass points are added to a measure such that the corresponding Jacobi matrix is a compact perturbation of the infinit tridiagonal matrix

$$\begin{pmatrix} b & a & & & \\ a & b & a & & \\ & a & b & a & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

have been extended recently (see [15]) by the authors to the matrix case.

Let $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ be two matrix measures and $\hat{\Gamma}$ be the smallest closed interval which contains the support of $\mathrm{d}\alpha$. We assume that $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ are related by

$$\mathrm{d}\beta(u) = \mathrm{d}\alpha(u) + M\delta(u - c), \tag{2.10}$$

where M is a positive definite matrix, δ is the Dirac delta matrix measure supported in a point c outside $\hat{\Gamma}$ containing the support of $\mathrm{d}\alpha$.

In the following, we consider the sequence of matrix polynomials $\{P_n(x, \mathrm{d}\alpha) = A_n(\mathrm{d}\alpha)x^n + B_n(\mathrm{d}\alpha)x^{n-1}+$ lower degree terms$\}$ which satisfie (2.3) and belongs to $M(D, E)$ with D non-singular. Let $\{P_n(x, \mathrm{d}\beta)\}$ be the sequence of orthonormal matrix polynomials with respect to the perturbed matrix measure $\mathrm{d}\beta$, for which $[A_n(\mathrm{d}\beta)A_n^{-1}(\mathrm{d}\alpha)]^*$ are lower triangular matrices with positive diagonal elements (see [15]), and

$$\Lambda(c) \stackrel{\text{def}}{=} \lim_{n\to\infty} \left[A_n(\mathrm{d}\beta)A_n^{-1})(\mathrm{d}\alpha)\right]^* . \tag{2.11}$$

The following theorem has been proved.

Theorem 2.1 [15]. *Let $\{P_n(x, \cdot)\}$ be a sequence of orthonormal matrix polynomials in $\mathbb{C}^{N\times N}[t]$ with respect to the matrix measures* $\mathrm{d}\alpha$ *and* $\mathrm{d}\beta$, *related by* (2.10). *Assume that $\{P_n(x, \mathrm{d}\alpha)\}$ belongs to $M(D, E)$ with D non-singular and $\{P_n(x, \mathrm{d}\beta)\}$ is define as above. Then*

1. $$\lim_{n\to\infty} \left[A_n(\mathrm{d}\beta)A_n(\mathrm{d}\alpha)^{-1}\right]^* \left[A_n(\mathrm{d}\beta)A_n(\mathrm{d}\alpha)^{-1}\right] = I_N + \mathscr{F}(c)\mathscr{F}'(c)^{-1}\mathscr{F}(c),$$

2. $$\lim_{n\to\infty} P_n(x, \mathrm{d}\beta)P_n^{-1}(x, \mathrm{d}\alpha) = \Lambda(c)^{-1} + \frac{1}{c - x}\left\{\Lambda(c)^* - \Lambda(c)^{-1}\right\}\left\{\mathscr{F}(c)^{-*} - \mathscr{F}(x)^{-1}\right\}$$

   for $x \in \mathbb{R}\backslash\{\hat{\Gamma} \cup \{c\}\}$.

Notice that the above formulas in the scalar case agree with the Lemma 16 in [12, p. 132] taking into account that

$$\Lambda(c) = \frac{1}{|\varphi(c)|}, \quad \mathscr{F}(c) = \frac{c}{\varphi(c)}, \quad \varphi(c) \stackrel{\text{def}}{=} c + \sqrt{c^2 - 1}.$$

3. Perturbed Jacobi block matrix and eigenvalue problem

Consider the $(2N + 1)$ banded infinit Hermitian matrix $\mathbb{J}(\cdot)$ constructed from the sequence of matrices $D_n(\cdot)$, $E_n(\cdot)$ of (2.3) in the following way:

$$\mathbb{J}(\cdot) = \begin{pmatrix} E_0(\cdot) & D_1(\cdot) & 0 & \\ D_1^*(\cdot) & E_1(\cdot) & D_2(\cdot) & \ddots \\ 0 & D_2^*(\cdot) & E_2(\cdot) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.1}$$

We call this matrix the *N*-Jacobi block matrix associated with the matrix polynomials $\{P_n(x,\cdot)\}$.

Using the recurrence relation (2.3), we can show that the zeros of $P_n(x, \mathrm{d}\alpha)$ i.e., the zeros of $\det P_n(x, \mathrm{d}\alpha)$, are the eigenvalues of the truncated *N*-Jacobi matrix $\mathbb{J}_n(\mathrm{d}\alpha)$ of dimension nN (see [14]), with the same multiplicity (see [5]).

In the following we start by findin an explicit form of the perturbed truncated *N*-Jacobi matrix $\mathbb{J}_n(\mathrm{d}\beta)$, when the associated matrix measure is define by (2.10). To do this, we need the following lemma.

Lemma 3.1. *Let* $\mathrm{d}\alpha$ *and* $\mathrm{d}\beta$ *be two matrix measures, and* M *be a positive semi-definite matrix such that* $\mathrm{d}\beta(u) = \mathrm{d}\alpha(u) + M\delta(u-c)$, *where* c *is a real number. Then*

$$P_n(x, \mathrm{d}\beta) = \mathscr{M}_n[P_n(x, \mathrm{d}\alpha) - \mathscr{V}_n M K^*_{n+1}(c, x, \mathrm{d}\alpha)], \tag{3.2}$$

where

$$\begin{cases} \mathscr{M}_n = A_n^{-*}(\mathrm{d}\beta) A_n^*(\mathrm{d}\alpha), \\ \mathscr{V}_n = P_n(c, \mathrm{d}\alpha)(I_N + M K_{n+1}(c, c, \mathrm{d}\alpha))^{-1}. \end{cases} \tag{3.3}$$

Proof. See [15]. □

Proposition 3.1. *Let* $\mathbb{J}_n(\cdot)$ *be the truncated N-Jacobi block matrix of dimension* $(n+1)N$, *associated with the sequence of orthonormal matrix polynomials* $\{P_n(x,\cdot)\}$ *with respect to the matrix measures* $\mathrm{d}\alpha$ *and* $\mathrm{d}\beta$, *respectively. Then* $\mathbb{J}_n(\mathrm{d}\beta)$ *is an N-rank perturbation of* $\mathbb{J}_n(\mathrm{d}\alpha)$.

Proof. Using Lemma 3.1, (2.4) and (2.7) we have

$$\begin{aligned} P_n(x, \mathrm{d}\beta) &= \mathscr{M}_n P_n(x, \mathrm{d}\alpha) - \sum_{j=0}^{n} \{\mathscr{M}_n \mathscr{V}_n M\} P_j^*(c, \mathrm{d}\alpha) P_j(x, \mathrm{d}\alpha) \\ &= \mathscr{M}_n^{-*} P_n(x, \mathrm{d}\alpha) - \sum_{j=0}^{n-1} \{\mathscr{M}_n \mathscr{V}_n M\} P_j^*(c, \mathrm{d}\alpha) P_j(x, \mathrm{d}\alpha). \end{aligned}$$

We denote by W_j and V_j^n two matrices in $\mathbb{C}^{N\times N}$ define by

$$\begin{cases} W_j = A_j(\mathrm{d}\beta) A_j^{-1}(\mathrm{d}\alpha), \\ V_j^n = \mathscr{M}_n \mathscr{V}_n M P_j^*(c, \mathrm{d}\alpha), \end{cases}$$

then

$$P_n(x, d\beta) = \sum_{j=0}^{n-1} V_j^n P_j(x, d\alpha) + W_n P_n(x, d\alpha). \tag{3.4}$$

In matrix form

$$\begin{pmatrix} P_0(x, d\beta) \\ P_1(x, d\beta) \\ P_2(x, d\beta) \\ \vdots \\ P_n(x, d\beta) \end{pmatrix} = \begin{pmatrix} W_0 & 0 & \cdots & \cdots & 0 \\ V_0^1 & W_1 & \ddots & & \vdots \\ V_0^2 & V_1^2 & W_2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ V_0^n & V_1^n & \cdots & V_{n-1}^n & W_n \end{pmatrix} \begin{pmatrix} P_0(x, d\alpha) \\ P_1(x, d\alpha) \\ P_2(x, d\alpha) \\ \vdots \\ P_n(x, d\alpha) \end{pmatrix} \tag{3.5}$$

or equivalently,

$$\mathbb{P}_n(x, d\beta) = \mathbb{T}_n \cdot \mathbb{P}_n(x, d\alpha), \tag{3.6}$$

where $\mathbb{P}_n(x, d\beta)$, $\mathbb{P}_n(x, d\alpha)$ denote the $(n+1)N$ dimensional column vectors of (3.5), and $\mathbb{T}_n$ denotes the $(n+1)N \times (n+1)N$ dimensional block-triangular matrix of (3.5).

On the other hand, using the three-term recurrence relation (2.3), we get

$$x\mathbb{P}_n(x, d\beta) = \mathbb{J}_n(d\beta)\mathbb{P}_n(x, d\beta) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{pmatrix} D_{n+1}(d\beta) P_{n+1}(x, d\beta), \tag{3.7}$$

$$x\mathbb{P}_n(x, d\alpha) = \mathbb{J}_n(d\alpha)\mathbb{P}_n(x, d\alpha) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{pmatrix} D_{n+1}(d\alpha) P_{n+1}(x, d\alpha), \tag{3.8}$$

where

$$\mathbb{J}_n(\cdot) = \begin{pmatrix} E_0(\cdot) & D_1(\cdot) & 0 & \cdots & \cdots & 0 \\ D_1^*(\cdot) & E_1(\cdot) & D_2(\cdot) & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_{n-1}^*(.) & E_{n-1}(.) & D_n(\cdot) \\ 0 & \cdots & \cdots & 0 & D_n^*(\cdot) & E_n(\cdot) \end{pmatrix}$$

is the $(n+1)N \times (n+1)N$ dimensional Hermitian block-tridiagonal matrix with respect to the matrix measures $d\beta$ and $d\alpha$, respectively.

Substituting (3.6) into (3.7), we get

$$x\mathbb{T}_n.\mathbb{P}_n(x, \mathrm{d}\alpha)$$

$$= \mathbb{J}_n(\mathrm{d}\beta)\mathbb{T}_n.\mathbb{P}_n(x, \mathrm{d}\alpha) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{pmatrix} D_{n+1}(\mathrm{d}\beta) P_{n+1}(x, \mathrm{d}\beta).$$

But from (3.4)

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{pmatrix} D_{n+1}(\mathrm{d}\beta) P_{n+1}(x, \mathrm{d}\beta)$$

$$= \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ D_{n+1}(\mathrm{d}\beta) V_0^{n+1} & \cdots & D_{n+1}(\mathrm{d}\beta) V_n^{n+1} & D_{n+1}(\mathrm{d}\beta) W_{n+1} \end{pmatrix}$$

$$\times \begin{pmatrix} \mathbb{P}_n(x, \mathrm{d}\alpha) \\ P_{n+1}(x, \mathrm{d}\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ D_{n+1}(\mathrm{d}\beta) V_0^{n+1} & \cdots & D_{n+1}(\mathrm{d}\beta) V_n^{n+1} \end{pmatrix} \mathbb{P}_n(x, \mathrm{d}\alpha)$$

$$+ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{pmatrix} D_{n+1}(\mathrm{d}\beta) W_{n+1} P_{n+1}(x, \mathrm{d}\alpha).$$

Then

$$x\mathbb{T}_n.\mathbb{P}_n(x, \mathrm{d}\alpha)$$

$$= \left[\mathbb{J}_n(\mathrm{d}\beta)\mathbb{T}_n + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ D_{n+1}(\mathrm{d}\beta) V_0^{n+1} & \cdots & D_{n+1}(\mathrm{d}\beta) V_n^{n+1} \end{pmatrix} \right] \mathbb{P}_n(x, \mathrm{d}\alpha)$$

$$+\begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{pmatrix} D_{n+1}(\mathrm{d}\beta) W_{n+1} P_{n+1}(x, \mathrm{d}\alpha).$$

Since the square matrix $\mathbb{T}_n$ is block-triangular with non-singular blocks W_i ($i = 0, \ldots, n$) on the main (block) diagonal, it follows that $\mathbb{T}_n$ is non-singular ($\det \mathbb{T}_n = \prod_{k=0}^{n} \det W_i$). Furthermore,

$$\mathbb{T}_n^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ W_n^{-1} \end{pmatrix},$$

hence

$$\begin{aligned} & x\mathbb{P}_n(x, \mathrm{d}\alpha) \\ &= \left[\mathbb{T}_n^{-1} \mathbb{J}_n(\mathrm{d}\beta) \mathbb{T}_n + \mathbb{T}_n^{-1} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ D_{n+1}(\mathrm{d}\beta) V_0^{n+1} & \cdots & D_{n+1}(\mathrm{d}\beta) V_n^{n+1} \end{pmatrix} \right] \\ &\quad \times \mathbb{P}_n(x, \mathrm{d}\alpha) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ W_n^{-1} \end{pmatrix} D_{n+1}(\mathrm{d}\beta) W_{n+1} P_{n+1}(x, \mathrm{d}\alpha) \\ &= \mathbb{T}_n^{-1} \left[\mathbb{J}_n(\mathrm{d}\beta) + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ D_{n+1}(\mathrm{d}\beta) V_0^{n+1} & \cdots & D_{n+1}(\mathrm{d}\beta) V_n^{n+1} \end{pmatrix} \mathbb{T}_n^{-1} \right] \mathbb{T}_n \\ &\quad \times \mathbb{P}_n(x, \mathrm{d}\alpha) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_N \end{pmatrix} A_n(\mathrm{d}\alpha) A_{n+1}^{-1}(\mathrm{d}\alpha) P_{n+1}(x, \mathrm{d}\alpha). \end{aligned}$$

Taking into account (3.8), we deduce that

$$
\begin{aligned}
&\mathbb{J}_n(\mathrm{d}\alpha)\\
&= \mathbb{T}_n^{-1}\left[\mathbb{J}_n(\mathrm{d}\beta) + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ D_{n+1}(\mathrm{d}\beta)V_0^{n+1} & \cdots & D_{n+1}(\mathrm{d}\beta)V_n^{n+1} \end{pmatrix} \mathbb{T}_n^{-1}\right]\mathbb{T}_n\\
&= \mathbb{T}_n^{-1}\left[\mathbb{J}_n(\mathrm{d}\beta) + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ H_0^n & \cdots & H_n^n \end{pmatrix}\right]\mathbb{T}_n,
\end{aligned}
$$

where H_i^n $(i = 0, \cdots, n)$ can be generated as follows:

$$
D_{n+1}(\mathrm{d}\beta)V_r^{n+1} = \begin{cases} H_n^n W_n & \text{if } r = n, \\ H_r^n W_r + \displaystyle\sum_{i=r+1}^{n} H_i^n V_r^i & \text{if } 0 \leqslant r \leqslant n-1. \end{cases}
$$

This means

$$
\begin{aligned}
H_n^n &= D_{n+1}(\mathrm{d}\beta)V_n^{n+1}W_n^{-1},\\
H_{n-1}^n &= \left[D_{n+1}(\mathrm{d}\beta)V_{n-1}^{n+1} - H_n^n V_{n-1}^n\right]W_{n-1}^{-1},\\
H_{n-2}^n &= \left[D_{n+1}(\mathrm{d}\beta)V_{n-2}^{n+1} - H_{n-1}^n V_{n-2}^{n-1} - H_n^n V_{n-2}^n\right]W_{n-2}^{-1},\\
&\vdots\\
H_0^n &= \left[D_{n+1}(d\beta)V_0^{n+1} - \sum_{i=1}^{n} H_i^n V_0^i\right]W_0^{-1}.
\end{aligned}
$$

Thus, $\mathbb{J}_n(\mathrm{d}\alpha)$ is, essentially, an N-rank perturbation of the matrix $\mathbb{J}_n(\mathrm{d}\beta)$. □

4. Asymptotics for the perturbed matrix coefficients

Let $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ be two positive definite matrix measures which are related by

$$
\mathrm{d}\beta(u) = \mathrm{d}\alpha(u) + M\delta(u - c), \quad c \in \mathbb{R}\backslash\hat{\Gamma}, \tag{4.1}
$$

where M is a positive definite matrix. We assume that the matrix polynomial sequence $\{P_n(x, \mathrm{d}\alpha)\}$ for the matrix measure $\mathrm{d}\alpha$ satisfying (2.3) belongs to $M(D, E)$, where D is the non-singular matrix.

In this section, we will fin an explicit expression for the matrix parameters of a Nevai matrix class which contains the perturbed matrix measure $\mathrm{d}\beta$.

Theorem 4.1. *Let* $\mathrm{d}\alpha$ *and* $\mathrm{d}\beta$ *be two matrix measures related by* (4.1). *Assume that*

$$\{P_n(x, \mathrm{d}\alpha)\} \in M(D, E),$$

where D *is non-singular. Then there exists* $\{P_n(x, \mathrm{d}\beta)\}$ *such that*

$$\{P_n(x, \mathrm{d}\beta)\} \in M(\tilde{D}, \tilde{E}),$$

where

$$\begin{aligned} \tilde{D} &= \Lambda^*(c) D \Lambda^{-*}(c), \\ \tilde{E} &= \Lambda^*(c) E \Lambda(c) - c(\Lambda^*(c)\Lambda(c) - I_N) \\ &\quad + \Lambda^*(c) D^* \mathscr{F}(c) D \Lambda(c) - \Lambda^{-1}(c) D^* \mathscr{F}(c) D \Lambda^{-*}(c). \end{aligned} \tag{4.2}$$

Notice that $\tilde{E}$ is a Hermitian matrix.

Proof. Let $\{P_n(x, \mathrm{d}\alpha)\}$ be a sequence of orthonormal matrix polynomials with respect to the matrix measure $\mathrm{d}\alpha$, and $\{P_n(x, \mathrm{d}\beta)\}$ introduced in Section 2. Then writing

$$\begin{aligned} A_{n-1}(\mathrm{d}\beta) A_n^{-1}(\mathrm{d}\beta) &= (A_{n-1}(\mathrm{d}\beta) A_{n-1}^{-1}(\mathrm{d}\alpha)) \\ &\quad \times (A_{n-1}(\mathrm{d}\alpha) A_n^{-1}(\mathrm{d}\alpha))(A_n(\mathrm{d}\beta) A_n^{-1}(\mathrm{d}\alpha))^{-1}, \end{aligned} \tag{4.3}$$

taking into account that from (2.3) and (2.2)

$$\begin{aligned} D_{n+1}(\mathrm{d}\alpha) &= \langle x P_n(x, \mathrm{d}\alpha), P_{n+1}(x, \mathrm{d}\alpha) \rangle_{\mathrm{d}\alpha} \\ &= \int \left[A_n(\mathrm{d}\alpha) x^{n+1} + \cdots \right] \mathrm{d}\alpha(x) P_{n+1}^*(x, \mathrm{d}\alpha) \\ &= A_n(\mathrm{d}\alpha) A_{n+1}^{-1}(\mathrm{d}\alpha) \int \left[A_{n+1}(\mathrm{d}\alpha) x^{n+1} + \cdots \right] \mathrm{d}\alpha(x) P_{n+1}^*(x, \mathrm{d}\alpha) \\ &= A_n(\mathrm{d}\alpha) A_{n+1}^{-1}(\mathrm{d}\alpha), \end{aligned} \tag{4.4}$$

and using (2.11), we deduce

$$\lim_{n \to \infty} D_n(\mathrm{d}\beta) = \Lambda^*(c) D \Lambda^{-*}(c).$$

Now, using (2.2), (2.3) and (4.1), we have

$$\begin{aligned} E_n(\mathrm{d}\beta) &= \langle x P_n(x, \mathrm{d}\beta), P_n(x, \mathrm{d}\beta) \rangle_{\mathrm{d}\beta} \\ &= \langle x P_n(x, \mathrm{d}\beta), P_n(x, \mathrm{d}\beta) \rangle_{\mathrm{d}\alpha} + c P_n(c, \mathrm{d}\beta) M P_n^*(c, \mathrm{d}\beta). \end{aligned} \tag{4.5}$$

But using (3.2) we have

$$\begin{aligned} &\langle x P_n(x, \mathrm{d}\beta), P_n(x, \mathrm{d}\beta) \rangle_{\mathrm{d}\alpha} \\ &\quad = \langle \mathscr{M}_n x P_n(x, \mathrm{d}\alpha) - \mathscr{M}_n \mathscr{V}_n M x K_{n+1}(x, c, \mathrm{d}\alpha), \\ &\qquad \mathscr{M}_n P_n(x, \mathrm{d}\alpha) - \mathscr{M}_n \mathscr{V}_n M K_{n+1}(x, c, \mathrm{d}\alpha) \rangle_{\mathrm{d}\alpha} \end{aligned}$$

$$
\begin{aligned}
&= \mathscr{M}_n \langle x P_n(x, \mathrm{d}\alpha), P_n(x, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} \mathscr{M}_n^* \\
&\quad - \mathscr{M}_n \mathscr{V}_n M \langle x K_{n+1}(x, c, \mathrm{d}\alpha), P_n(x, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} \mathscr{M}_n^* \qquad (4.6a) \\
&\quad - \mathscr{M}_n \langle x P_n(x, \mathrm{d}\alpha), K_{n+1}(x, c, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} M \mathscr{V}_n^* \mathscr{M}_n^* \\
&\quad + \mathscr{M}_n \mathscr{V}_n M \langle x K_{n+1}(x, c, \mathrm{d}\alpha), K_{n+1}(x, c, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} M \mathscr{V}_n^* \mathscr{M}_n^*. \qquad (4.6b)
\end{aligned}
$$

Then the term given in (4.6a) is equal to

$$
\begin{aligned}
&- \mathscr{M}_n \mathscr{V}_n M \langle x P_n^*(c, \mathrm{d}\alpha) P_n(x, \mathrm{d}\alpha) \\
&\quad + x P_{n-1}^*(c, \mathrm{d}\alpha) P_{n-1}(x, \mathrm{d}\alpha), P_n(x, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} \mathscr{M}_n^* \\
&= - \mathscr{M}_n \mathscr{V}_n M [P_n^*(c, \mathrm{d}\alpha) E_n(\mathrm{d}\alpha) + P_{n-1}^*(c, \mathrm{d}\alpha) D_n(\mathrm{d}\alpha)] \mathscr{M}_n^* \\
&= - \mathscr{M}_n \mathscr{V}_n M [c P_n^*(c, \mathrm{d}\alpha) - P_{n+1}^*(c, \mathrm{d}\alpha) D_{n+1}^*(\mathrm{d}\alpha)] \mathscr{M}_n^*.
\end{aligned}
$$

Using (2.7) and (2.5), we have

$$
\begin{aligned}
&\langle x K_{n+1}(x, c, \mathrm{d}\alpha), K_{n+1}(x, c, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} \\
&= \langle (x - c) K_{n+1}(x, c, \mathrm{d}\alpha), K_{n+1}(x, c, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} \\
&\quad + c K_{n+1}(c, c, \mathrm{d}\alpha) \\
&= \langle P_n^*(c, \mathrm{d}\alpha) D_{n+1}(\mathrm{d}\alpha) P_{n+1}(x, \mathrm{d}\alpha) \\
&\qquad - P_{n+1}^*(c, \mathrm{d}\alpha) D_{n+1}^*(\mathrm{d}\alpha) P_n(x, \mathrm{d}\alpha), K_{n+1}(x, c, \mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha} \\
&\quad + c K_{n+1}(c, c, \mathrm{d}\alpha) \\
&= - P_{n+1}^*(c, \mathrm{d}\alpha) D_{n+1}^*(\mathrm{d}\alpha) P_n(c, \mathrm{d}\alpha) + c K_{n+1}(c, c, \mathrm{d}\alpha).
\end{aligned}
$$

From the Christoffel–Darboux formula (2.5) when $x = y = c$, we get that $P_{n+1}^*(c, \mathrm{d}\alpha) D_{n+1}^*(\mathrm{d}\alpha) P_n(c, \mathrm{d}\alpha)$ is Hermitian, and the term given in (4.6b) is equal to

$$
\begin{aligned}
&c \mathscr{M}_n \mathscr{V}_n M K_{n+1}(c, c, \mathrm{d}\alpha) M \mathscr{V}_n^* \mathscr{M}_n^* \\
&\quad - \mathscr{M}_n \mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha) D_{n+1}(\mathrm{d}\alpha) P_{n+1}(c, \mathrm{d}\alpha) M \mathscr{V}_n^* \mathscr{M}_n^*.
\end{aligned}
$$

From Lemma 3.1

$$
\begin{aligned}
P_n(c, \mathrm{d}\beta) &= \mathscr{M}_n [P_n(c, \mathrm{d}\alpha) - \mathscr{V}_n M K_{n+1}(c, c, \mathrm{d}\alpha)] \\
&= \mathscr{M}_n P_n(c, \mathrm{d}\alpha) \\
&\quad \times \left[I_N - (I_N + M K_{n+1}(c, c, \mathrm{d}\alpha))^{-1} M K_{n+1}(c, c, \mathrm{d}\alpha) \right] \\
&= \mathscr{M}_n P_n(c, \mathrm{d}\alpha) (I_N + M K_{n+1}(c, c, \mathrm{d}\alpha))^{-1} \\
&= \mathscr{M}_n \mathscr{V}_n.
\end{aligned}
$$

Then (4.5) becomes

$$
\begin{aligned}
E_n(\mathrm{d}\beta) &= \mathscr{M}_n E_n(\mathrm{d}\alpha) \mathscr{M}_n^* \\
&\quad - \mathscr{M}_n \mathscr{V}_n M \left[c P_n^*(c, \mathrm{d}\alpha) - P_{n+1}^*(c, \mathrm{d}\alpha) D_{n+1}^*(\mathrm{d}\alpha) \right] \mathscr{M}_n^*
\end{aligned}
$$

$$
\begin{aligned}
&-\mathscr{M}_n\left[cP_n(c,\mathrm{d}\alpha)-D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c,\mathrm{d}\alpha)\right]M^*\mathscr{V}_n^*\mathscr{M}_n^*\\
&+c\mathscr{M}_n\mathscr{V}_nMK_{n+1}(c,c,\mathrm{d}\alpha)M\mathscr{V}_n^*\mathscr{M}_n^*\\
&-\mathscr{M}_n\mathscr{V}_nMP_n^*(c,\mathrm{d}\alpha)D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c,\mathrm{d}\alpha)M\mathscr{V}_n^*\mathscr{M}_n^*\\
&+c\mathscr{M}_n\mathscr{V}_nM\mathscr{V}_n^*\mathscr{M}_n^*\\
=\;&\mathscr{M}_n\left\{E_n(\mathrm{d}\alpha)-\mathscr{V}_nMP_n^*(c,\mathrm{d}\alpha)D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c,\mathrm{d}\alpha)\,M\mathscr{V}_n^*\right.\\
&\qquad+\mathscr{V}_nMP_{n+1}^*(c,\mathrm{d}\alpha)D_{n+1}^*(\mathrm{d}\alpha)\\
&\qquad\left.+D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c,\mathrm{d}\alpha)M\mathscr{V}_n^*\right\}\mathscr{M}_n^*\\
&-c\mathscr{M}_n\left\{\mathscr{V}_nMP_n^*(c,\mathrm{d}\alpha)+P_n(c,\mathrm{d}\alpha)M\mathscr{V}_n^*\right. &\text{(4.7a)}\\
&\qquad\left.-\mathscr{V}_nMK_{n+1}(c,c,\mathrm{d}\alpha)M\mathscr{V}_n^*-\mathscr{V}_nM\mathscr{V}_n^*\right\}M_n^*. &\text{(4.7b)}
\end{aligned}
$$

Notice that from (3.2) and (3.3)

$$
\begin{aligned}
&I_N-P_n(c,\mathrm{d}\alpha)M(I_N+K_{n+1}(c,c,\mathrm{d}\alpha)M)^{-1}P_n^*(c,\mathrm{d}\alpha)\\
&\quad=\langle P_n(x,\mathrm{d}\alpha),\,\mathscr{M}_n^{-1}P_n(x,\mathrm{d}\beta)\rangle_{\mathrm{d}\alpha}\\
&\quad=\langle P_n(x,\mathrm{d}\alpha),\,P_n(x,\mathrm{d}\beta)\rangle_{\mathrm{d}\alpha}\mathscr{M}_n^{-*}\\
&\quad=\langle P_n(x,\mathrm{d}\alpha),\,A_n(\mathrm{d}\beta)A_n(\mathrm{d}\alpha)^{-1}P_n(x,\mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha}\mathscr{M}_n^{-*}\\
&\quad=\langle P_n(x,\mathrm{d}\alpha),\,P_n(x,\mathrm{d}\alpha)\rangle_{\mathrm{d}\alpha}\mathscr{M}_n^{-1}\mathscr{M}_n^{-*}\\
&\quad=(\mathscr{M}_n^*+\mathscr{M}_n)^{-1}. &\text{(4.8)}
\end{aligned}
$$

Then using (4.8), the terms given in (4.7a) and (4.7b) become

$$
\begin{aligned}
&\mathscr{V}_nMP_n^*(c,\mathrm{d}\alpha)+P_n(c,\mathrm{d}\alpha)M\mathscr{V}_n^*-\mathscr{V}_nMK_{n+1}(c,c,\mathrm{d}\alpha)M\mathscr{V}_n^*-\mathscr{V}_nM\mathscr{V}_n^*\\
&\quad=-\mathscr{V}_nM\left[\mathscr{V}_n^*+K_{n+1}(c,c,\mathrm{d}\alpha)M\mathscr{V}_n^*-P_n^*(c,\mathrm{d}\alpha)\right]+P_n^*(c,\mathrm{d}\alpha)M\mathscr{V}_n^*\\
&\quad=-\mathscr{V}_nM\big[(I_N+K_{n+1}(c,c,\mathrm{d}\alpha)M)^{-1}P_{n+1}^*(c,\mathrm{d}\alpha)\\
&\qquad\qquad+K_{n+1}(c,c,\mathrm{d}\alpha)M(I_N+K_{n+1}(c,c,\mathrm{d}\alpha)M)^{-1}P_n^*(c,\mathrm{d}\alpha)\\
&\qquad\qquad-P_n^*(c,\mathrm{d}\alpha)\big]+P_n(c,\mathrm{d}\alpha)M\mathscr{V}_n^*\\
&\quad=P_n(c,\mathrm{d}\alpha)M\mathscr{V}_n^*\\
&\quad=P_n(c,\mathrm{d}\alpha)M(I_N+K_{n+1}(c,c,\mathrm{d}\alpha)M)^{-1}P_n^*(c,\mathrm{d}\alpha)\\
&\quad=I_N-(\mathscr{M}_n^*\mathscr{M}_n)^{-1}.
\end{aligned}
$$

Hence, substituting in (4.7a) and (4.7b), (4.5) becomes

$$
\begin{aligned}
E_n(\mathrm{d}\beta)=\;&cI_N-c\mathscr{M}_n\mathscr{M}_n^*\\
&+\mathscr{M}_n\left\{E_n(\mathrm{d}\alpha)-\mathscr{V}_nMP_n^*(c,\mathrm{d}\alpha)D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c,\mathrm{d}\alpha)M\mathscr{V}_n^*\right.\\
&\qquad+\mathscr{V}_nMP_{n+1}^*(c,\mathrm{d}\alpha)D_{n+1}^*(\mathrm{d}\alpha)\\
&\qquad\left.+D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c,\mathrm{d}\alpha)M\mathscr{V}_n^*\right\}\mathscr{M}_n^*.
\end{aligned}
$$

To get the asymptotic expression $\tilde{E}$ for $E_n(\mathrm{d}\beta)$, we take into account the following results (see [2,15]) for $c \in \mathbb{R})\backslash\hat{\Gamma}$

- $\lim_{n\to\infty} \mathscr{M}_n^{-1} = \Lambda(c)$,
- $\lim_{n\to\infty} P_{n-1}(c, \mathrm{d}\alpha)P_n^{-1}(c, \mathrm{d}\alpha)D_n^{-1}(\mathrm{d}\alpha) = \int \frac{\mathrm{d}W_{D,E}(t)}{c-t} \stackrel{\text{def}}{=} \mathscr{F}(c).$

Then writing

$$
\begin{aligned}
E_n(\mathrm{d}\beta) &= cI_N - c\mathscr{M}_n\mathscr{M}_n^* \\
&\quad + \mathscr{M}_n\big\{E_n(\mathrm{d}\alpha) - (\mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha))D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c, \mathrm{d}\alpha) \\
&\qquad \times P_n^{-1}(c, \mathrm{d}\alpha)(P_n(c, \mathrm{d}\alpha)M\mathscr{V}_n^*) \\
&\qquad + (\mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha))P_n^{-*}(c, \mathrm{d}\alpha)P_{n+1}^*(c, \mathrm{d}\alpha)D_{n+1}^*(\mathrm{d}\alpha) \\
&\qquad + D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c, \mathrm{d}\alpha)P_n^{-1}(c, \mathrm{d}\alpha)(P_n(c, \mathrm{d}\alpha)M\mathscr{V}_n^*)\big\}\mathscr{M}_n^* \\
&= cI_N - c\mathscr{M}_n\mathscr{M}_n^* \\
&\quad + \mathscr{M}_n\big\{E_n(\mathrm{d}\alpha) - (I_N - \mathscr{M}_n^{-1}\mathscr{M}_n^{-*}) \\
&\qquad \times D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c, \mathrm{d}\alpha)P_n^{-1}(c, \mathrm{d}\alpha)(I_N - \mathscr{M}_n^{-1}\mathscr{M}_n^{-*}) \\
&\qquad + (I_N - \mathscr{M}_n^{-1}\mathscr{M}_n^{-*})P_n^{-*}(c, \mathrm{d}\alpha)P_{n+1}^*(c, \mathrm{d}\alpha)D_{n+1}^*(\mathrm{d}\alpha) \\
&\qquad + D_{n+1}(\mathrm{d}\alpha)P_{n+1}(c, \mathrm{d}\alpha)P_n^{-1}(c, \mathrm{d}\alpha)(I_N - \mathscr{M}_n^{-1}\mathscr{M}_n^{-*})\big\}\mathscr{M}_n^*,
\end{aligned}
$$

we get

$$
\begin{aligned}
\tilde{E} &= cI_N - c\Lambda(c)^{-1}\Lambda(c)^{-*} \\
&\quad + \Lambda(c)^{-1}\big\{E - (I_N - \Lambda(c)\Lambda(c)^*)\mathscr{F}(c)^{-1}(I_N - \Lambda(c)\Lambda(c)^*) \\
&\qquad + (I_N - \Lambda(c)\Lambda(c)^*)\mathscr{F}(c)^{-*} \\
&\qquad + \mathscr{F}(c)^{-1}(I_N - \Lambda(c)\Lambda(c)^*)\big\}\Lambda(c)^{-*} \\
&= \Lambda(c)^{-1}\big\{c(\Lambda(c)\Lambda(c)^* - I_N) \\
&\qquad + E - (I_N - \Lambda(c)\Lambda(c)^*)\mathscr{F}(c)^{-1}(I_N - \Lambda(c)\Lambda(c)^*) \\
&\qquad + (I_N - \Lambda(c)\Lambda(c)^*)\mathscr{F}(c)^{-*} \\
&\qquad + \mathscr{F}(c)^{-1}(I_N - \Lambda(c)\Lambda(c)^*)\big\}\Lambda(c)^{-*} \\
&= \Lambda^{-1}(c)\big\{E + c(\Lambda(c)\Lambda(c)^* - I) \\
&\qquad - \Lambda(c)\Lambda(c)^* + \mathscr{F}(c)^{-1} + \Lambda(c)\Lambda(c)^* + \mathscr{F}(c)^{-1}\big\}\Lambda^{-*}(c) \\
&= \Lambda^{-1}(c)\big\{E - cI_N + \mathscr{F}(c)^{-1}\big\}\Lambda^{-*}(c) \\
&\quad + cI_N - \Lambda^*(c)\mathscr{F}(c)^{-1}\Lambda(c).
\end{aligned}
\tag{4.9a}
$$

But from (2.3), we have

$$
D_n^*(\mathrm{d}\alpha)P_{n-1}(z, \mathrm{d}\alpha)P_n(z, \mathrm{d}\alpha)^{-1}D_n(\mathrm{d}\alpha)^{-1})D_n(\mathrm{d}\alpha)
$$

$$\times (P_n(z, \mathrm{d}\alpha) P_{n+1}(z, \mathrm{d}\alpha)^{-1} D_{n+1}(\mathrm{d}\alpha)^{-1})(E_n(\mathrm{d}\alpha) - zI_N)$$
$$\times (P_n(z, \mathrm{d}\alpha) P_{n+1}(z, \mathrm{d}\alpha)^{-1} D_{n+1}(\mathrm{d}\alpha)^{-1}) + I_N) = 0$$

for $z \in \mathbb{C} \backslash \Gamma$. Taking into account (2.9) we get

$$D^* \mathscr{F}(z) D \mathscr{F}(z) + (E - zI_N)\mathscr{F}(z) + I_N = 0.$$

In particular,

$$D^* \mathscr{F}(c) D \mathscr{F}(c) + (E - cI_N)\mathscr{F}(c) + I_N = 0, \tag{4.10}$$

i.e.,

$$D^* \mathscr{F}(c) D + (E - cI_N) + \mathscr{F}(c)^{-1} = 0.$$

Then (4.9a) becomes

$$\begin{aligned}\tilde{E} &= -\Lambda^{-1}(c) D^* \mathscr{F}(c) D \Lambda^{-*}(c) + cI_N \\ &\quad + \Lambda^*(c)\{D^* \mathscr{F}(c) D + E - cI_N\}\Lambda(c) \\ &= \Lambda^*(c) E \Lambda(c) - c(\Lambda^*(c)\Lambda(c) - I_N) \\ &\quad + \Lambda^*(c) D^* \mathscr{F}(c) D^* \Lambda(c) - \Lambda^{-1}(c) D^* \mathscr{F}(c) D \Lambda^{-*}(c). \qquad \square\end{aligned}$$

Now, from (4.8) we have

$$\begin{aligned}&\left[A_n(\mathrm{d}\beta) A_n^{-1}(\mathrm{d}\alpha)\right]^* \left[A_n(\mathrm{d}\beta) A_n^{-1}(\mathrm{d}\alpha)\right] \\ &\quad = I_N - P_n(c, \mathrm{d}\alpha) M (I_N + K_{n+1}(c, c, \mathrm{d}\alpha) M)^{-1} P_n^*(c, \mathrm{d}\alpha).\end{aligned} \tag{4.11}$$

Then using the Cholesky factorization of the positive definite matrix given in the right-hand side of (4.11), there is a unique lower triangular matrix $L_n(c, \mathrm{d}\alpha)$ with positive diagonal elements such that

$$\begin{aligned}&L_n(c, \mathrm{d}\alpha) L_n^*(c, \mathrm{d}\alpha) \\ &\quad = I_N - P_n(c, \mathrm{d}\alpha)(I_N + M K_{n+1}(c, c, \mathrm{d}\alpha))^{-1} M P_n^*(c, \mathrm{d}\alpha)\end{aligned}$$

and thus

$$\left[A_n(\mathrm{d}\beta) A_n^{-1}(\mathrm{d}\alpha)\right]^* = L_n(c, \mathrm{d}\alpha). \tag{4.12}$$

Proposition 4.1. *Consider two matrix measures* $\mathrm{d}\alpha$ *and* $\mathrm{d}\beta$ *which are related by* (2.10). *Let* $\mathbb{J}(\mathrm{d}\alpha)$ *be the N-Jacobi block matrix define by* (3.1). *Then the perturbed N-Jacobi block matrix* $\mathbb{J}(\mathrm{d}\beta)$ *associated to* $\{P_n(x, \mathrm{d}\beta)\}$ *is determined by its matrix entries*

$$\begin{aligned}D_n(\mathrm{d}\beta) &= L_{n-1}^*(c, \mathrm{d}\alpha) D_n(\mathrm{d}\alpha) L_n^{-*}(c, \mathrm{d}\alpha), \\ E_n(\mathrm{d}\beta) &= L_n^*(c, \mathrm{d}\alpha) A_n(\mathrm{d}\alpha)(\Pi_n(c, \mathrm{d}\alpha) \\ &\qquad - \Pi_{n+1}(c, \mathrm{d}\alpha)) A_n^{-1}(\mathrm{d}\alpha) L_n^{-*}(c, \mathrm{d}\alpha),\end{aligned} \tag{4.13}$$

where

$$
\begin{aligned}
\Pi_n(c, \mathrm{d}\alpha) = {} & A_n^{-1}(\mathrm{d}\alpha) B_n(\mathrm{d}\alpha) \\
& - A_n^{-1}(\mathrm{d}\alpha) \left\{ P_n^{-*}(c, \mathrm{d}\alpha) M^{-1} \mathscr{V}_n^{-1} - I_N \right\}^{-1} \\
& \times (P_{n-1}(c, \mathrm{d}\alpha) P_n^{-1}(c, \mathrm{d}\alpha))^* A_{n-1}(\mathrm{d}\alpha). \qquad (4.14)
\end{aligned}
$$

Proof. From (4.3), (4.12) and (4.4), we deduce

$$
D_n(\mathrm{d}\beta) = L_{n-1}^*(c, \mathrm{d}\alpha) D_n(\mathrm{d}\alpha) L_n^{-*}(c, \mathrm{d}\alpha).
$$

Using the three-term recurrence relation (2.3) and the orthogonality property (2.2), we have

$$
\begin{aligned}
E_n(\mathrm{d}\beta) &= \langle x P_n(x, \mathrm{d}\beta), P_n(x, \mathrm{d}\beta) \rangle_{\mathrm{d}\beta} \\
&= \int [A_n(\mathrm{d}\beta) x^{n+1} + B_n(\mathrm{d}\beta) x^n + \cdots] \mathrm{d}\beta(x) P_n^*(x, \mathrm{d}\beta) \\
&= A_n(\mathrm{d}\beta) A_{n+1}^{-1}(\mathrm{d}\beta) \\
&\quad \times \int [A_{n+1}(\mathrm{d}\beta) x^{n+1} + B_{n+1}(\mathrm{d}\beta) x^n + \cdots] \mathrm{d}\beta(x) P_n^*(x, \mathrm{d}\beta) \\
&\quad - A_n(\mathrm{d}\beta) A_{n+1}^{-1}(\mathrm{d}\beta) \int [B_{n+1}(\mathrm{d}\beta) x^n + \cdots] \mathrm{d}\beta(x) P_n^*(x, \mathrm{d}\beta) \\
&\quad + B_n(\mathrm{d}\beta) A_n^{-1}(\mathrm{d}\beta) \int [A_n(\mathrm{d}\beta) x^n + \cdots] \mathrm{d}\beta(x) P_n^*(x, \mathrm{d}\beta) \\
&= A_n(\mathrm{d}\beta)(A_n^{-1}(\mathrm{d}\beta) B_n(\mathrm{d}\beta) - A_{n+1}^{-1}(\mathrm{d}\beta) B_{n+1}(\mathrm{d}\beta)) A_n^{-1}(\mathrm{d}\beta) \quad (4.15a) \\
&= L_n^*(c, \mathrm{d}\alpha) A_n(\mathrm{d}\alpha)(A_n^{-1}(\mathrm{d}\beta) B_n(\mathrm{d}\beta) \\
&\quad - A_{n+1}^{-1}(\mathrm{d}\beta) B_{n+1}(\mathrm{d}\beta)) A_n^{-1}(\mathrm{d}\alpha) L_n^{-*}(c, \mathrm{d}\alpha).
\end{aligned}
$$

To get the second part of (4.13), it is sufficien to prove $A_n^{-1}(\mathrm{d}\beta) B_n(\mathrm{d}\beta) = \Pi_n(c, \mathrm{d}\alpha)$. In fact from Lemma 3.1 and (4.8)

$$
\begin{aligned}
A_n(\mathrm{d}\beta) &= \mathscr{M}_n [A_n(\mathrm{d}\alpha) - \mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha) A_n(\mathrm{d}\alpha)] \\
&= \mathscr{M}_n [I_N - \mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha)] A_n(\mathrm{d}\alpha) \\
&= \mathscr{M}_n^{-*} A_n(\mathrm{d}\alpha) \qquad (4.16)
\end{aligned}
$$

as well as

$$
\begin{aligned}
B_n(\mathrm{d}\beta) &= \mathscr{M}_n \left[B_n(\mathrm{d}\alpha) - \mathscr{V}_n M (P_n^*(c, \mathrm{d}\alpha) B_n(\mathrm{d}\alpha) - P_{n-1}^*(c, \mathrm{d}\alpha) A_{n-1}(\mathrm{d}\alpha)) \right] \\
&= \mathscr{M}_n \left[I_N - \mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha) \right] B_n(\mathrm{d}\alpha) \\
&\quad - \mathscr{M}_n \mathscr{V}_n M P_{n-1}^*(c, \mathrm{d}\alpha) A_{n-1}(\mathrm{d}\alpha) \\
&= \mathscr{M}_n^{-*} B_n(\mathrm{d}\alpha) - \mathscr{M}_n \mathscr{V}_n M P_{n-1}^*(c, \mathrm{d}\alpha) A_{n-1} \mathrm{d}\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathscr{M}_n^{-*} B_n(\mathrm{d}\alpha) \\
&\quad - \mathscr{M}_n^{-*} \{ I_N - \mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha) \}^{-1} \\
&\quad \times \mathscr{V}_n M P_{n-1}^*(c, \mathrm{d}\alpha) A_{n-1}(\mathrm{d}\alpha).
\end{aligned} \tag{4.17}
$$

Using (4.16) and (4.17)

$$
\begin{aligned}
A_n^{-1}(\mathrm{d}\beta) B_n(\mathrm{d}\beta) = {} & A_n^{-1}(\mathrm{d}\alpha) B_n(\mathrm{d}\alpha) \\
& - A_n^{-1}(\mathrm{d}\alpha) \{ P_n^{-*}(c, \mathrm{d}\alpha) M^{-1} \mathscr{V}_n^{-1} - I_N \}^{-1} \\
& \times (P_{n-1}(c, \mathrm{d}\alpha) P_n^{-1}(c, \mathrm{d}\alpha))^* A_{n-1}(\mathrm{d}\alpha). \qquad \square
\end{aligned}
$$

Corollary 4.1. *Under the hypothesis of Theorem* 4.1 *we have*

$$
\begin{aligned}
\tilde{D} &= \Lambda^*(c) D \Lambda^{-*}(c), \\
\tilde{E} &= \Lambda^*(c) \cdot \Big\{ E + D\big[\Lambda^{-*}(c).\Lambda^{-1}(c) - I_N\big] D^* \mathscr{F}(c) \\
&\qquad - \big[\Lambda^{-*}(c) \cdot \Lambda^{-1}(c) - I_N\big] D^* \mathscr{F}(c) D \Big\} \Lambda^{-*}(c).
\end{aligned} \tag{4.18}
$$

Proof. From (4.14)

$$
\begin{aligned}
& A_n(\mathrm{d}\alpha)[\Pi_n(c, \mathrm{d}\alpha) - \Pi_{n+1}(c, \mathrm{d}\alpha)] A_n^{-1}(\mathrm{d}\alpha) \\
&\quad = A_n(\mathrm{d}\alpha)[A_n^{-1}(\mathrm{d}\alpha) B_n(\mathrm{d}\alpha) - A_{n+1}^{-1}(\mathrm{d}\alpha) B_{n+1}(\mathrm{d}\alpha)] A_n^{-1}(\mathrm{d}\alpha) \\
&\quad\quad - \Omega_n(c, \mathrm{d}\alpha) A_{n-1}(\mathrm{d}\alpha) A_n^{-1}(\mathrm{d}\alpha) + A_n(\mathrm{d}\alpha) A_{n+1}^{-1}(\mathrm{d}\alpha) \Omega_{n+1}(c, \mathrm{d}\alpha),
\end{aligned} \tag{4.19}
$$

where

$$
\begin{aligned}
\Omega_n(c, \mathrm{d}\alpha) = {} & \{ P_n^{-*}(c, \mathrm{d}\alpha) M^{-1} \mathscr{V}_n^{-1} - I_N \}^{-1} \\
& \times (P_{n-1}(c, \mathrm{d}\alpha) P_n^{-1}(c, \mathrm{d}\alpha))^*.
\end{aligned} \tag{4.20}
$$

From (4.8), the expression in (4.20) becomes

$$
\begin{aligned}
\Omega_n(c, \mathrm{d}\alpha) &= \{ I_N - \mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha) \}^{-1} \\
&\quad \times \mathscr{V}_n M P_n^*(c, \mathrm{d}\alpha)(P_{n-1}(c, \mathrm{d}\alpha) P_n^{-1}(c, \mathrm{d}\alpha))^* \\
&= \mathscr{M}_n^* \mathscr{M}_n (I_N - \mathscr{M}_n^{-1} \mathscr{M}_n^{-*})(P_{n-1}(c, \mathrm{d}\alpha) P_n^{-1}(c, \mathrm{d}\alpha))^* \\
&= (\mathscr{M}_n^* \mathscr{M}_n - I_N) D_n^*(\mathrm{d}\alpha)(P_{n-1}(c, \mathrm{d}\alpha) P_n^{-1}(c, \mathrm{d}\alpha) D_n^{-1}(\mathrm{d}\alpha))^*.
\end{aligned}
$$

Then

$$
\lim_{n \to \infty} \Omega_n(c, \mathrm{d}\alpha) = (\Lambda^{-*}(c) \Lambda^{-1}(c) - I_N) D^* \mathscr{F}(c).
$$

From (4.13), (4.19) and taking into account (4.15a) and (4.4)

$$E_n(\mathrm{d}\beta) = L_n^*(c, \mathrm{d}\alpha)\big\{E_n(\mathrm{d}\alpha) + D_{n+1}(\mathrm{d}\alpha)\Omega_{n+1}(c, \mathrm{d}\alpha) \\ - \Omega_n(\mathrm{d}\alpha)D_n(\mathrm{d}\alpha)\big\}L_n^{-*}(c, \mathrm{d}\alpha).$$

Hence

$$\tilde{E} = \Lambda^*(c) \cdot \big\{E + D[\Lambda^{-*}(c) \cdot \Lambda^{-1}(c) - I_N]D^*\mathscr{F}(c) \\ -[\Lambda^{-*}(c) \cdot \Lambda^{-1}(c) - I_N]D^*\mathscr{F}(c)D\big\}\Lambda^{-*}(c). \qquad \square$$

Remark 4.1.

(i) The perturbed orthonormal matrix polynomials $\{P_n(x, \mathrm{d}\beta)\}$ do not belong, in general, to $M(D, E)$. Even more as Example 1 shows, we can easily verify that E and $\tilde{E}$ are not unitarity equivalent.

(ii) The expression of $\tilde{E}$ in (4.2) is equal to that given in (4.18).

To show it, let P and Q be the right-hand side expressions in (4.2) and (4.18). Then

$$P = \Lambda^*(c)E\Lambda(c) - c\Lambda^*(c)\Lambda(c)cI_N \\ +\Lambda^*(c)D^*\mathscr{F}(c)D\Lambda(c) - \Lambda^{-1}(c)D^*\mathscr{F}(c)D\Lambda^{-*}(c). \tag{4.21}$$

From Theorem 2.1,

$$\Lambda(c)\Lambda^*(c) = I_N + \mathscr{F}(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c).$$

Then (4.21) becomes

$$\begin{aligned} P &= \Lambda^*(c)E\big\{I_N + \mathscr{F}(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c)\big\}\Lambda^{-*}(c) \\ &\quad -c\Lambda^*(c)\big\{I_N + \mathscr{F}(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c)\big\}\Lambda^{-*}(c) + cI_N \\ &\quad +\Lambda^*(c)D^*\mathscr{F}(c)D\big\{I_N + \mathscr{F}(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c)\big\}\Lambda^{-*}(c) \\ &\quad -\Lambda^{-1}(c)D^*\mathscr{F}(c)D\Lambda^{-*}(c) \\ &= \Lambda^*(c)E\Lambda^{-*}(c) \\ &\quad +\Lambda^*(c)\big\{[E - cI_N + D^*\mathscr{F}(c)D]\mathscr{F}(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c)\big\}\Lambda^{-*}(c) \\ &\quad +\Lambda^*(c)D^*\mathscr{F}(c)D\Lambda^{-*}(c) - \Lambda^{-1}(c)D^*\mathscr{F}(c)D\Lambda^{-*}(c), \end{aligned}$$

and taking into account (4.10), we have

$$P = \Lambda^*(c)E\Lambda^{-*}(c) - \Lambda^*(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c)\Lambda^{-*}(c) \\ +\Lambda^*(c)D^*\mathscr{F}(c)D\Lambda^{-*}(c) - \Lambda^{-1}(c)D^*\mathscr{F}(c)D\Lambda^{-*}(c).$$

From (4.18)

$$Q = \Lambda^*(c)E\Lambda^{-*}(c) + \Lambda^*(c)D\Lambda^{-*}(c)\Lambda^{-1}(c)D^*\mathscr{F}(c)\Lambda^{-*}(c)$$

$$-\Lambda^*(c)DD^*\mathscr{F}(c)\Lambda^{-*}(c) - \Lambda^{-1}(c)D^*\mathscr{F}(c)D\Lambda^{-*}(c)$$
$$+\Lambda^*(c)D^*\mathscr{F}(c)D\Lambda^{-*}(c).$$

Finally, to prove $P = Q$, it is sufficien to show that

$$(\mathscr{F}'(c))^{-1} + D\Lambda^{-*}(c)\Lambda^{-1}(c)D^* - DD^* = 0. \tag{4.22}$$

Taking derivatives in (4.10) at the point c, we get

$$D^*\mathscr{F}'(c)D\mathscr{F}(c) + \{D^*\mathscr{F}(c)D + E - cI_N\}\mathscr{F}'(c) = \mathscr{F}(c),$$

i.e.,

$$\begin{aligned}
&\mathscr{F}(c)\mathscr{F}'(c))^{-1}\mathscr{F}(c)\{D^*\mathscr{F}'(c)D - I_N\} = I_N \\
&\quad\Leftrightarrow\quad \mathscr{F}(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c)D^* \\
&\qquad\qquad = D^{-1}(\mathscr{F}'(c))^{-1} + \mathscr{F}(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c)D^{-1}(\mathscr{F}'(c))^{-1} \\
&\qquad\qquad = \{I_N + \mathscr{F}(c)(\mathscr{F}'(c))^{-1}\mathscr{F}(c)\}D^{-1}(\mathscr{F}'(c))^{-1} \\
&\qquad\qquad = \Lambda(c)\Lambda^*(c)D^{-1}(\mathscr{F}'(c))^{-1} \\
&\quad\Leftrightarrow\quad \Lambda(c)\Lambda^*(c)D^* - D^* = \Lambda(c)\Lambda^*(c)D^{-1}(\mathscr{F}'(c))^{-1} \\
&\quad\Leftrightarrow\quad (\mathscr{F}'(c))^{-1} + D\Lambda^{-*}(c)\Lambda^{-1}(c)D^* - DD^* = 0.
\end{aligned}$$

Hence (4.22) is true and the second part of Remark 1 holds.

5. Examples

Example 1. Consider

$$D = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Writing

$$\mathscr{F}(x) = \int \frac{\mathrm{d}W_{D,E}}{x - t},$$

from (2.3) and (2.9) we deduce that this analytic matrix function satisfie the matrix equation

$$D^*\mathscr{F}(x)D\mathscr{F}(x) + (E - xI_2)\mathscr{F}(x) + I_2 = 0 \quad x \in \mathbb{R}\backslash\Gamma. \tag{5.1}$$

Since D is a positive definite matrix, the explicit expression for $\mathscr{F}(x)$ (see [2]) is

$$\begin{aligned}
\mathscr{F}(x) = &\frac{1}{2}D^{-1}(xI_2 - E)D^{-1} \\
&-\frac{1}{2}D^{-1/2}[D^{-1/2}(E - xI_2)D^{-1}(E - xI_2)D^{-1/2} - 4I_2]^{1/2}D^{-1/2},
\end{aligned} \tag{5.2}$$

where $x \notin \text{supp}(\mathrm{d}W_{D,E}) = \{x \in \mathbb{R};\ D^{-1/2}(E - xI_2)D^{-1/2}$ has at least an eigenvalue in $[-2, 2]\}$.

The matrix square root given in (5.2) is define in the natural way, i.e., using the diagonal form of the matrix and applying the square root to its eigenvalue w which satisfie $|w - \sqrt{w^2 - 4}| < 2$.

A straightforward computation yields

$$\frac{1}{2}D^{-1}(cI_2 - E)D^{-1} = \begin{pmatrix} \frac{-1+c}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{-1+c}{4} \end{pmatrix}$$

and

$$D^{-1/2}(E - cI_2)D^{-1}(E - cI_2)D^{-1/2} - 4I_2 = \begin{pmatrix} \frac{-6-2c+c^2}{2} & 1-c \\ 1-c & \frac{-6-2c+c^2}{2} \end{pmatrix}.$$

According to the explicit form given in (5.2) we get

$$\mathscr{F}(c) = \frac{1}{8}\begin{pmatrix} x_1(c) & x_2(c) \\ x_2(c) & x_1(c) \end{pmatrix},$$

where

$$\begin{cases} x_1(c) = -2 + 2c - \sqrt{-8 + c^2} - \sqrt{-4 - 4c + c^2}, \\ x_2(c) = -2 + \sqrt{-8 + c^2} - \sqrt{-4 - 4c + c^2}. \end{cases}$$

$$\begin{aligned} \text{supp}(\mathrm{d}W_{D,E}) &= \left[-2\sqrt{2}, 2\sqrt{2}\right] \cup \left[\sqrt{2}(-2 + \sqrt{2}), \sqrt{2}(2 + \sqrt{2})\right] \\ &= \left[-2\sqrt{2}, \sqrt{2}(2 + \sqrt{2})\right], \end{aligned}$$

and the square roots are chosen such that $\mathscr{F}(z)$ is analytic in $z \in \mathbb{R}\backslash[-2\sqrt{2}, \sqrt{2}(2 + \sqrt{2})]$.

Computing the terms given in the right-hand side of the firs part of Theorem 2.1 and using (2.11), we get

$$\Lambda(c) \cdot \Lambda^*(c) = \frac{1}{8}\begin{pmatrix} y_1(c) & y_2(c) \\ y_2(c) & y_1(c) \end{pmatrix}, \tag{5.3}$$

where

$$\begin{cases} y_1(c) = -4 + 2c^2 + 2\sqrt{-4 - 4c + c^2} \\ \qquad\qquad -c\left(4 + \sqrt{-8 + c^2} + \sqrt{-4 - 4c + c^2}\right), \\ y_2(c) = c\left(-4 + \sqrt{-8 + c^2} - \sqrt{-4 - 4c + c^2}\right) \\ \qquad\qquad + 2\left(2 + \sqrt{-4 - 4c + c^2}\right). \end{cases}$$

Using the Cholesky factorization of the positive definite matrix given in the right-hand side of (5.3), we obtain

$$\Lambda(c) = \frac{1}{2}\begin{pmatrix} z_1(c) & 0 \\ z_2(c) & z_3(c) \end{pmatrix},$$

where

$$z_1(c)=\sqrt{-2-2c+c^2-\frac{(-2+c)\sqrt{-4+(-4+c)c}}{2}-\frac{c\sqrt{-8+c^2}}{2}},$$

$$z_2(c)=\frac{4-4c-(-2+c)\sqrt{-4+(-4+c)c}+c\sqrt{-8+c^2}}{\sqrt{-2(-2+c)\sqrt{-4+(-4+c)c}+4(-2+(-2+c)c)-2c\sqrt{-8+c^2}}},$$

$$z_3(c)=4\sqrt{2}\sqrt{\frac{1}{(-2+c)\sqrt{-4+(-4+c)c}+2(-2+(-2+c)c)+c\sqrt{-8+c^2}}}.$$

Finally, according to (4.2), the limits of the perturbed matrix coefficient are given by

$$\tilde{D} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

and

$$\tilde{E} = \begin{pmatrix} \dfrac{2c\sqrt{-8+c^2}-w(c)}{2(-2+(-2+c)c)} & \dfrac{8u(c)}{v(c)} \\ \dfrac{8u(c)}{v(c)} & \dfrac{2(4+c(-c+\sqrt{-8+c^2}}{w(c)} \end{pmatrix},$$

where

$$\begin{cases} u(c) = \Big[(-2+c)\sqrt{-4+(-4+c)c} \\ \qquad\qquad +2(-2+(-2+c)+c)+c\sqrt{-8+c^2}\Big]^{-1/2}, \\ v(c) = \Big[-\big((-2+c)\sqrt{-4+(-4+c)c}\big) \\ \qquad\qquad +2(-2+(-2+c)c)-c\sqrt{-8+c^2}\Big]^{1/2}, \\ w(c) = 4-2(-2+c)c+(-2+c)\sqrt{-4+(-4+c)c}+c\sqrt{-8+c^2}. \end{cases}$$

Example 2. Consider

$$D = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A straightforward computation yields

$$\frac{1}{2}D^{-1}(cI_3 - E)D^{-1} = \begin{pmatrix} \frac{-1+c}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{-1+c}{6} & 0 \\ 0 & 0 & \frac{-1+c}{6} \end{pmatrix}$$

and

$$\begin{aligned} &D^{-1/2}(E - cI_3)D^{-1}(E - cI_3)D^{-1/2} - 4I_3 \\ &= \begin{pmatrix} \frac{-7-2c+c^2}{3} & \frac{-4(-1+c)}{3} & 0 \\ \frac{-4(-1+c)}{3} & \frac{-7-2c+c^2}{3} & 0 \\ 0 & 0 & \frac{-11-2c+c^2}{3} \end{pmatrix}. \end{aligned}$$

Using (5.2) we get

$$\mathscr{F}(c) = \frac{1}{12}\begin{pmatrix} x_1(c) & x_2(c) & 0 \\ x_2(c) & x_1(c) & 0 \\ 0 & 0 & x_3(c) \end{pmatrix},$$

where

$$\begin{aligned} x_1(c) &= -2 + 2c - (-3 + (-6 + c)c)^{1/2} - (-11 + c(2 + c))^{1/2}, \\ x_2(c) &= -4 - (-3 + (-6 + c)c)^{1/2} + (-11 + c(2 + c))^{1/2}, \\ x_3(c) &= 2\left(-1 + c - (-11 + (-2 + c)c)^{1/2}\right) \end{aligned}$$

and

$$\begin{aligned} \operatorname{supp}(\mathrm{d}W_{D,E}) &= \left[\frac{-6+\sqrt{3}}{\sqrt{3}}, \frac{6+\sqrt{3}}{\sqrt{3}}\right] \cup \left[\sqrt{3}(-2\sqrt{3}), \sqrt{3}(2+\sqrt{3})\right] \\ &\cup \left[-(\sqrt{3}(2+\sqrt{3})), -(\sqrt{3}(-2+\sqrt{3}))\right] \\ &= \left[-(\sqrt{3}(2+\sqrt{3})), (\sqrt{3}(2+\sqrt{3}))\right]. \end{aligned}$$

The square roots are chosen such that $\mathscr{F}(z)$ is analytic in

$$z \in \mathbb{R}\Big\backslash\left[-\left(\sqrt{3}\left(2+\sqrt{3}\right)\right), \left(\sqrt{3}\left(2+\sqrt{3}\right)\right)\right].$$

Computing the terms given in the right-hand side of the firs part of Theorem 2.1 and using (2.11), we get

$$\Lambda(c) \cdot \Lambda^*(c) = \frac{1}{2}\begin{pmatrix} y_1(c) & y_2(c) & 0 \\ y_2(c) & y_1(c) & 0 \\ 0 & 0 & y_3(c) \end{pmatrix}, \tag{5.4}$$

where

$$
\begin{aligned}
y_1(c) &= -\left((-3+c)\sqrt{-3+(-6+c)c}\right)+2(-1+(-2+c)c) \\
&\quad -(1+c)\sqrt{-11+c(2+c)}, \\
y_2(c) &= -8(-1+c)-(-3+c)\sqrt{-3+(-6+c)c} \\
&\quad +(1+c)\sqrt{-11+c(2+c)}, \\
y_3(c) &= 2\left(-5-2c+c^2-(-1+c)\sqrt{-11+(-2+c)c}\right),
\end{aligned}
$$

Using the Cholesky factorization of the positive definite matrix given in the right-hand side of (5.4), we obtain

$$
\Lambda(c) = \frac{1}{2\sqrt{3}} \begin{pmatrix} z_1(c) & 0 & 0 \\ z_2(c) & z_3(c) & 0 \\ 0 & 0 & z_4(c) \end{pmatrix},
$$

where

$$
\begin{cases}
z_1(c) = \Big[-\left((-3+c)\sqrt{-3+(-6+c)c}\right) \\
\qquad\qquad +2(-1+(-2+c)c)-(1+c)\sqrt{-11+c(2+c)}\Big]^{1/2}, \\
z_2(c) = -\dfrac{\sqrt{3}(8(-1+c)+(-3+c)\sqrt{-3+(-6+c)c}-(1+c)\sqrt{-11+c(2+c)})}{\sqrt{-3(-3+c)\sqrt{-3+(-6+c)c}+6(-1+(-2+c)c)-3(1+c)\sqrt{-11+c(2+c)}}}, \\
z_3(c) = 12\sqrt{\dfrac{1}{(-3+c)\sqrt{-3+(-6+c)c}+2(-1+(-2+c)c)+(1+c)\sqrt{-11+c(2+c)}}}, \\
z_4(c) = \sqrt{2}\sqrt{-5-2c+c^2-(-1+c)\sqrt{-11+(-2+c)c}}.
\end{cases}
$$

Finally, according to (4.2), the limits of the perturbed matrix coefficient are given by

$$
\tilde{D} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}
$$

and

$$
\tilde{E} = \begin{pmatrix} u(c) & w(c) & 0 \\ w(c) & v(c) & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

where

$$
\begin{cases}
u(c) = \dfrac{-1-2c+c^2-(-3+c)\sqrt{-3+(-6+c)c}+(1+c)\sqrt{-11+c(2+c)}}{-1+(-2+c)c}, \\
v(c) = \dfrac{-1-2c+c^2+(-3+c)\sqrt{-3+(-6+c)c}-(1+c)\sqrt{-11+c(2+c)}}{-1+(-2+c)c}, \\
w(c) = \dfrac{24\sqrt{\frac{1}{(-3+c)\sqrt{-3+(-6+c)c}+2(-1+(-2+c)c)+(1+c)\sqrt{-11+c(2+c)}}}}{\sqrt{-((-3+c)\sqrt{-3+(-6+c)c})+2(-1+(-2+c)c)-(1+c)\sqrt{-11+c(2+c)}}}.
\end{cases}
$$

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