J. of Inequal. & Appl., 2002, Vol. 7(5), pp. 673–699

On Fourier Series of Jacobi-Sobolev Orthogonal Polynomials

F. MARCELLÁN[1], B. P. OSILENKER[2], and I. A. ROCHA[3]

[1] *Dpto. de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 20, 28911 Leganés, Madrid (SPAIN)* <pacomarc@ing.uc3m.es> [2] *Dep. of Math., Moscow State Civil Engineering University, Moscow (RUSSIA)* <b_osilenker@mail.ru>; [3] *Dpto. de Matemática Aplicada, E.U.I.T. Telecomunicación, Universidad Politécnica de Madrid, Ctra. de Valencia Km. 7, 28031 Madrid (SPAIN)* <igalvar@euitt.upm.es>

(Received on October 16 2000; Revised February 26 2001)

Let μ be the Jacobi measure on the interval $[-1,1]$ and introduce the discrete Sobolev-type inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + Mf(c)g(c) + Nf'(c)g'(c)$$

where $c \in (1,\infty)$ and M, N are non negative constants such that $M+N>0$. The main purpose of this paper is to study the behaviour of the Fourier series in terms of the polynomials associated to the Sobolev inner product. For an appropriate function f, we prove here that the Fourier-Sobolev series converges to f on the interval $(-1,\ 1)$ as well as to $f(c)$ and the derivative of the series converges to $f'(c)$. The term appropriate means here, in general, the same as we need for a function $f(x)$ in order to have convergence for the series of $f(x)$ associated to the standard inner product given by the measure μ. No additional conditions are needed.

Keywords: Orthogonal polynomials; Sobolev inner product; Fourier series.

Classification: *AMS Subject Classification*: 42C05

ISSN 1025-5834 print; ISSN 1029-242X.
DOI: 10.1080/1025583021000022450

1 INTRODUCTION

Let μ be a finite positive Borel measure on the interval $[-1, 1]$ such that *supp* μ is an infinite set and let c be a real number on $(1, \infty)$. For f and g in $L^2(\mu)$ such that there exits the first derivative in c we can introduce the Sobolev-type inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + Mf(c)g(c) + Nf'(c)g'(c) \tag{1}$$

where M and N are non-negative real numbers with $M + N > 0$. Let $(\hat{B}_k(x))_{k=0}^{\infty}$ the sequence of orthonormal polynomials with respect to this inner product

$$\langle \hat{B}_n(x), \hat{B}_k(x)\rangle = \delta_{n,k} \quad k, n = 0, 1, \ldots$$

For every function f such that $\langle f, \hat{B}_k\rangle$ exists for $k = 0, 1, \ldots$ we introduce the associated Fourier–Sobolev series

$$\sum_{k=0}^{\infty} \langle f, \hat{B}_k\rangle \hat{B}_k(x)$$

The main purpose of this paper is the proof of the relation

$$\sum_{k=0}^{\infty} \langle f, \hat{B}_k\rangle \hat{B}_k(x) = f(x), \quad x \in (-1, 1) \cup \{c\}, \quad \sum_{k=0}^{\infty} \langle f, \hat{B}_k\rangle \hat{B}'_k(c) = f'(c)$$

for the Jacobi measure $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx, \alpha > -1, \beta > -1$, under standard sufficient conditions for f. The precise terms of this result are given in Section 4.

In order to obtain this result we need previously some estimates for $\hat{B}_k(x)$ in $[-1, 1] \cup \{c\}$ and also for $\hat{B}'_k(c)$. They are obtained in Section 3 not only for the Jacobi measure but for any measure μ which belongs to Szegö class. We start with a representation of $\hat{B}_k(x)$ in terms of the polynomials $(q_n(x))_{n=0}^{\infty}$ orthonormal with respect to the measure $(c-x)^2 d\mu(x)$. In Section 2 we prove that

$$\hat{B}_n(x) = A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x)$$

and that the constants A_n, B_n, C_n have limit points provided that the measure μ has ratio asymptotics. One consequence of this result is the asymptotics for the polynomials $\hat{B}_k(x)$ and the asymptotic behaviour of their zeros. This is well known from papers by G. López, F. Marcellán and W. Van Assche ([2], [4]) where they solved this problem using a different representation of the polynomials $\hat{B}_k(x)$.

The fact that the point c is outside the interval $[-1, 1]$ plays an important role in the whole paper because it allows the function $1/(x-c)^2$ to be continuous in the interval and the Sobolev space behaves as a vector space with two real components and the other on $L^2(\mu)$. Notice that some estimates of polynomials $\hat{B}_k$ when $c = 1$ have been obtained in [1]. It remains open the problem of the estimates when $c \in (-1, 1)$.

2 ASYMPTOTIC FORMULAS

We will denote by $(p_n(x))_{n=0}^{\infty}$ the sequence of orthonormal polynomials with respect to $d\mu(x)$ and by $(\tilde{q}_n(x))_{n=0}^{\infty}$ and $(q_n(x))_{n=0}^{\infty}$ the orthonormal sequences with respect to $(c-x)d\mu(x)$ and $(c-x)^2 d\mu(x)$ respectively. We will also denote by $k(\pi_n)$ the leading coefficient of any polynomial $\pi_n(x)$ and $\hat{B}_n(x)$ the orthonormal polynomials with respect to the inner product (1) as it was said.

Since there are important differences for the different choices of M and N, we will start with $M > 0$ and $N > 0$ and, in the next subsection, cases $N = 0$ and $M = 0$ will be studied separately.

2.1 Case $M > 0, N > 0$

In this paragraph, we assume that $\mu' > 0$ a.e., i.e., the polynomials $P_n(x)$, $\tilde{q}_n(x)$ and $q_n(x)$ have ratio asymptotics.

THEOREM 2.1 *If $M > 0$ and $N > 0$, there are real constants A_n, B_n* and C_n *such that*

$$\hat{B}_n(x) = A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x), \quad n = 0, 1, 2 \ldots$$

Moreover

$$\lim_{n\to\infty} A_n = \frac{c-\sqrt{c^2-1}}{2}, \quad \lim_{n\to\infty} B_n = -1, \quad \lim_{n\to\infty} C_n = \frac{c+\sqrt{c^2-1}}{2}$$

Proof Since $\hat{B}_n(x) = \sum_{j=0}^{n} a_{n,j}q_j(x)$ and

$$\begin{aligned} a_{n,j} &= \int_{-1}^{1} \hat{B}_n(x)q_j(x)(x-c)^2 d\mu(x) \\ &= \langle \hat{B}_n(x), (x-c)^2 q_j(x)\rangle = 0, \quad j = 0, 1, \ldots n-3, \end{aligned}$$

then

$$\begin{aligned} \hat{B}_n(x) &= a_{n,n}\, q_n(x) + a_{n,n-1}q_{n-1}(x) + a_{n,n-2}\, q_{n-2}(x) \\ &= A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x). \end{aligned}$$

On the other hand, since

$$\begin{aligned} A_n^2 + B_n^2 + C_n^2 &= \int_{-1}^{1} \hat{B}_n^2(x)(x-c)^2 d\mu(x) \\ &\leq (c+1)^2 \int_{-1}^{1} \hat{B}_n^2(x) d\mu(x) \leq (C+1)^2, \end{aligned}$$

the coefficients A_n, B_n and C_n are bounded.

Denoting by $k(q_n)$ and $k(\hat{B}_n)$ the leading coefficients of $q_n(x)$ and $\hat{B}_n(x)$ respectively, we get

$$A_n = \frac{k(\hat{B}_n)}{k(q_n)},$$

and

$$\begin{aligned} C_n &= \int_{-1}^{1} \hat{B}_n(x)q_{n-2}(x)(x-c)^2 d\mu(x) = \langle \hat{B}_n(x), (x-c)^2 q_{n-2}(x)\rangle \\ &= \frac{k(q_{n-2})}{k(\hat{B}_n)} = \frac{k(q_{n-2})}{k(q_n)} \frac{1}{A_n}. \end{aligned}$$

Because $1/x-c$ and $1/(x-c)^2$ are continuous functions on $[-1,1]$ and, in particular, they belong to $L^2((x-c)^2 d\mu(x))$,

$$\lim_{n\to\infty}\int_{-1}^{1} q_n(x)d\mu(x) = \lim_{n\to\infty}\int_{-1}^{1} \frac{q_n(x)}{(x-c)^2}(x-c)^2 d\mu(x) = 0$$

and

$$\lim_{n\to\infty}\int_{-1}^{1} q_n(x)(x-c)d\mu(x) = \lim_{n\to\infty}\int_{-1}^{1} \frac{q_n(x)}{x-c}(x-c)^2 d\mu(x) = 0.$$

Then

$$\lim_{n\to\infty}\int_{-1}^{1} q_n(x)\hat{B}_n(x)d\mu(x) = \lim_{n\to\infty}\int_{-1}^{1} q_n(x)[\hat{B}_n(c) + \hat{B}_n'(c)(x-c)]d\mu(x) = 0 \qquad (2)$$

$$\lim_{n\to\infty}\int_{-1}^{1} q_n(x)\hat{B}_{n+1}(x)d\mu(x) = \lim_{n\to\infty}\int_{-1}^{1} q_n(x)[\hat{B}_{n+1}(c) + \hat{B}_{n+1}'(c)(x-c)]d\mu(x) = 0 \qquad (3)$$

because $\hat{B}_n(c)$ and $\hat{B}_n'(c)$ are bounded from the orthonormality of $\hat{B}_n(x)$.

Let Λ be a family of non negative integers such that $\lim_{n\in\Lambda} A_n = a$ and $\lim_{n\in\Lambda} B_n = b$. As it is well known, (see [5] and [6]), if $\mu' > 0$ a.e. then $\lim_{n\in\Lambda} C_n = 1/4a$ (notice that $a > 0$ because C_n are bounded) and

$$\lim_{n\to\infty}\int_{-1}^{1} \frac{q_n(x)q_{n+k}(x)}{(x-c)^2}(x-c)^2 d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{T_k(x)}{(x-c)^2}\frac{dx}{\sqrt{1-x^2}}$$

where $T_k(x) = \cos(k\theta)$, $x = \cos\theta$, are the Tchebichef polynomials of the first kind.

As a consequence, from (2) and (3) and Theorem 2.1 we obtain

$$0 = \lim_{n\in\Lambda} \int_{-1}^{1} q_n(x)\hat{B}_n(x)d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{a + bT_1(x) + 1/4a\ T_2(x)}{(x-c)^2}\frac{dx}{\sqrt{1-x^2}}, \tag{4}$$

$$\begin{aligned} 0 &= \lim_{n\in\Lambda} \int_{-1}^{1} q_n(x)\hat{B}_{n+1}(x)d\mu(x) \\ &= \frac{1}{\pi}\int_{-1}^{1} \frac{aT_1(x) + b + 1/4a\ T_1(x)}{(x-c)^2}\frac{dx}{\sqrt{1-x^2}}. \end{aligned} \tag{5}$$

Denoting

$$\Pi_1(x) = (4a^2+1)T_1(x) + 4ab = (4a^2+1)x + 4ab,$$

$$\Pi_2(x) = T_2(x) + 4abT_1(x) + 4a^2 = 2x^2 + 4abx + 4a^2 - 1,$$

and

$$\omega(z) = \frac{-1}{\sqrt{z^2-1}} = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{x-z}\frac{dx}{\sqrt{1-x^2}},$$

(5) becomes

$$\begin{aligned} 0 &= \frac{1}{\pi}\int_{-1}^{1} \frac{\Pi_1(x)}{(x-c)^2}\frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi}\int_{-1}^{1} \frac{\Pi_1(c) + \Pi_1'(c)(x-c)}{(x-c)^2}\frac{dx}{\sqrt{1-x^2}} \\ &= (\Pi_1\omega)'(c) = \frac{4abc + 4a^2 + 1}{(c^2-1)^{3/2}} \end{aligned}$$

which means that $b = -(1/c)(a + (1/4a))$.

Analogously, (4) becomes

$$0 = (\Pi_2\omega)'(c) + 2$$

and it gives

$$4a^2 = (\varphi^-(c))^2 =: (c - \sqrt{c^2-1})^2$$

Since $a > 0$, $a = \varphi^{-}(c)/2$ and $b = -1$. As a consequence the only limit points of A_n and B_n are $\varphi^{-}(c)/2$ and -1 respectively and the theorem is proved ■

As a straightforward consequence of Theorem 2.1 one obtains the strong (resp. ratio) asymptotics for the polynomials $\hat{B}_n(x)$ provided that μ belongs to Szegö (resp. Nevai) class. These results were obtained by G. López, F. Marcellán and W. Van Assche in [2] and [4] using a different representation for $\hat{B}_n(x)$.

COROLLARY 2.1 *With the previous conditions we have*

i)

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{q_n(x)} = \frac{\varphi^{-}(c)}{2}(1 - \frac{\varphi(c)}{\varphi(x)})^2$$

uniformly on compact sets of $C \setminus [-1, 1]$ *and* $\varphi(x) = x + \sqrt{x^2 - 1}$.

ii) $n - 2$ *zeros of* $\hat{B}_n(x)$ *are in* $[-1, 1]$ *and the other 2 zeros tend to* c.

iii)

$$\lim_{n\to\infty} \frac{\hat{B}_{n+1}(x)}{\hat{B}_n(x)} = x + \sqrt{x^2 - 1}$$

uniformly on compact sets of $C \setminus ([-1, 1] \cup \{c\})$.

iv) If $\int_{-1}^{1} \log \mu'(x) dx/\sqrt{1 - x^2} > -\infty$ *then*

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{(x + \sqrt{x^2 - 1})^n} = \frac{\varphi^{-}(c)}{2}\left(1 - \frac{\varphi(c)}{\varphi(x)}\right)^2 S(x)$$

uniformly on compact sets of $\mathbb{C} \setminus [-1, 1]$, *where* $S(x)$ *is the Szegö function of* $(x - c)^2 \mu'(x)$ *(see [8], Th. 12.1.2 as well as the definition in page 276)* ■

Item ii) is a consequence of the fact that $\int_{-1}^{1} \hat{B}_n(x)(x - c)^2 x^k d\mu(x) = 0$ for $k = 0, 1, \ldots, n - 3$ and the asymptotic formula i).

The Sobolev polynomials satisfy a five term recurrence relation and its coefficients behave as the ones of standard orthogonal polynomials.

THEOREM 2.2 *There are constants $\alpha_n, \beta_n, \gamma_n$ such that*

$$(x-c)^2\hat{B}_n(x) = \alpha_n\hat{B}_{n+2}(x) + \beta_n\hat{B}_{n+1}(x) + \gamma_n\hat{B}_n(x) + \beta_{n-1}\hat{B}_{n-1}(x) + \alpha_{n-2}\hat{B}_{n-2}(x), \quad n \geq 0.$$

$$\alpha_{-1} = \alpha_{-2} = \beta_{-1} = 0.$$

Moreover, if $\mu' > 0$ a.e. then

$$\lim_{n\to\infty} \alpha_n = \frac{1}{4}, \quad \lim_{n\to\infty} \beta_n = -c, \quad \lim_{n\to\infty} \gamma_n = c^2 + \frac{1}{2}.$$

Proof Recurrence relation is a straightforward consequence of the fact that

$$\langle (x-c)^2 f(x), g(x)\rangle = \langle f(x), (x-c)^2 g(x)\rangle.$$

For the asymptotic behaviour of the coefficients we have

$$\alpha_n = \frac{k(\hat{B}_n)}{k(\hat{B}_{n+2})} = \frac{k(\hat{B}_n)}{k(q_n)}\frac{k(q_n)}{k(q_{n+2})}\frac{k(q_{n+2})}{k(\hat{B}_{n+2})}$$

and, if $\mu' > 0$ *a.e.*, $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty} k(q_n)/k(q_{n+2}) = \frac{1}{4}$.

$$\gamma_n = \langle (x-c)^2\hat{B}_n(x), \hat{B}_n(x)\rangle = \int_{-1}^{1}(x-c)^2\hat{B}_n^2(x)d\mu(x) = A_n^2 + B_n^2 + C_n^2$$

where $\hat{B}_n(x) = A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x)$. Then

$$\lim_{n\to\infty}\gamma_n = \left(\frac{\varphi^-(c)}{2}\right)^2 + 1 + \left(\frac{\varphi(c)}{2}\right)^2 = c^2 + \frac{1}{2}.$$

Finally, from

$$\begin{aligned}\beta_n &= \langle (x-c)^2 \hat{B}_n(x), \hat{B}_{n+1}(x)\rangle \\ &= \int_{-1}^{1} (x-c)^2 \hat{B}_n(x)\hat{B}_{n+1}(x)d\mu(x) = A_n B_{n+1} + B_n C_{n+1},\end{aligned}$$

we get

$$\lim_{n\to\infty} \hat{B}_n = -\left(\frac{\varphi(c)}{2} + \frac{\varphi^{-}(c)}{2}\right) = -c.$$

THEOREM 2.3 *If* $\mu' > 0$ *a.e. on* $[-1, 1]$ *then*

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\hat{B}_n(x)\hat{B}_{n+k}(x)d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} f(x)T_k(x)\frac{dx}{\sqrt{1-x^2}}$$

for any continuous function on $[-1, 1]$.

Proof

$$\begin{aligned}&\int_{-1}^{1} f(x)\hat{B}_n(x)\hat{B}_{n+k}(x)d\mu(x) \\ &\quad= \int_{-1}^{1} \frac{f(x)}{(x-c)^2}[A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x)]\cdot \\ &\qquad\cdot\ [A_{n+k}q_{n+k}(x) + B_{n+k}q_{n+k-1}(x) + C_{n+k}q_{n+k-2}(x)](x-c)^2 d\mu(x).\end{aligned}$$

From the properties of $q_n(x)$ and taking into account that $A_n \to \varphi^-(c)/2$, $B_n \to -1$, $C_n \to \varphi(c)/2$ we have for $k \geq 2$

$$\lim_{n\to\infty}\int_{-1}^{1} f(x)\hat{B}_n(x)\hat{B}_{n+k}(x)d\mu(x)$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{(x-c)^2}\left[\left(c^2+\frac{1}{2}\right)T_k(x)\right.$$

$$+\frac{1}{4}(T_{k+2}(x)+T_{k-2}(x)) - c(T_{k+1}(x).$$

$$\left.+\,T_{k-1}(x))\right]\frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{(x-c)^2}(x^2-2cx+c^2)T_k(x)\frac{dx}{\sqrt{1-x^2}}.$$

In the same way, this relation holds for $k = 0$ and $k = 1$.

2.2 Cases $N = 0$ and $M = 0$

Because our goal is to study Fourier series in Jacobi–Sobolev polynomials, from now on we will assume that the measure μ belongs to the Szegö class, i.e.,

$$\int_{-1}^{1} \log\mu'(x)\frac{dx}{\sqrt{1-x^2}} > -\infty.$$

In case $N = 0$ this is not important because the same proof given in Theorem 2.1 works here. However, in case $M = 0$, the proof given here needs strong asymptotics for the polynomials $q_n(x)$.

THEOREM 2.4 *i) If we assume $N = 0$ in the inner product (1), then there are real constants A_n and B_n such that*

$$\hat{B}_n(x) = A_n\tilde{q}_n(x) + B_n\tilde{q}_{n-1}(x) \quad n = 0, 1\ldots$$

where $\tilde{q}_n(x)$ *are the orthonormal polynomials with respect to* $(c-x)d\mu(x)$. *Moreover*

$$\lim_{n\to\infty} A_n = \left(\frac{c-\sqrt{c^2-1}}{2}\right)^{1/2}, \quad \lim_{n\to\infty} B_n = -\left(\frac{c+\sqrt{c^2-1}}{2}\right)^{1/2}.$$

ii) When $M=0$ *in the inner product (1), there are constants* A_n, B_n *and* C_n *such that*

$$\hat{B}_n(x) = A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x), \quad n = 0, 1, \dots$$

where $q_n(x)$ *are the orthonormal polynomials with respect to* $(c-x)^2 d\mu(x)$. *Moreover*

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} C_n = \frac{1}{2}, \quad \lim_{n\to\infty} B_n = -c.$$

Proof

i) Since

$$\int_{-1}^{1} \hat{B}_n(x)\tilde{q}_j(x)(c-x)d\mu(x) = \langle \hat{B}_n(x), (c-x)\tilde{q}_j(x)\rangle$$
$$= 0, \; j = 0, 1, \dots, n-2,$$

we have

$$\hat{B}_n(x) = A_n\tilde{q}_n(x) + B_n\tilde{q}_{n-1}(x).$$

Since $\hat{B}_n(c)$ are bounded because of the orthonormality condition, and $\tilde{q}_n(c)$ behaves like $(c+\sqrt{c^2-1})^n$,

$$\lim_{n\to\infty} \frac{\hat{B}_n(c)}{\tilde{q}_{n-1}(c)} = \lim_{n\to\infty}\left(A_n \frac{\tilde{q}_n(c)}{\tilde{q}_{n-1}(c)} + B_n\right) = 0. \tag{6}$$

Taking into account that $A_n = k(\hat{B}_n)/k(\tilde{q}_n)$, $B_n = -k(\tilde{q}_{n-1})/k(q_n)1/A_n$ as well as A_n and B_n are bounded, from (6) we deduce

$$0 = \lim_{n\in\Lambda} \frac{\hat{B}_n(c)}{\tilde{q}_{n-1}(c)} = A(c + \sqrt{c^2 - 1}) - \frac{1}{2A}$$

for a family of non-negative integers Λ, where A is a limit point of A_n. Then $A^2 = c - \sqrt{c^2 - 1}/2$ and i) is proved.

In case ii) we have

$$\begin{aligned}\int_{-1}^{1} \hat{B}_n(x) q_j(x)(c - x)^2 d\mu(x) &= \langle \hat{B}_n(x), (x - c)^2 q_j(x)\rangle \\ &= 0,\ j = 0, \ldots, n - 3\end{aligned}$$

This yields $\hat{B}_n(x) = A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x)$. Moreover $C_n = k(q_{n-2})/k(q_n)1/A_n$. From the boundedness of $\hat{B}'_n(c)$ and the asymptotic properties of $q_n(x)$ we have

$$0 = \lim_{n\to\infty} \frac{\hat{B}'_n(c)}{q'_{n-2}(c)} = \lim_{n\to\infty}\left(A_n \frac{q'_n(c)}{q'_{n-2}(c)} + B_n \frac{q'_{n-1}(c)}{q'_{n-2}(c)} + \frac{k(q_{n-2})}{k(q_n)}\frac{1}{A_n}\right).$$

Let Λ be a sequence of non negative integers such that $\lim_{n\in\Lambda} A_n = A$ and $\lim_{n\in\Lambda} B_n = B$ which exist because A_n, B_n and C_n are bounded. Then

$$0 = A\varphi^2(c) + B\varphi(c) + \frac{1}{4A} \tag{7}$$

where $\varphi(c) = c + \sqrt{c^2 - 1}$.

On the other hand,

$$|-N\hat{B}'_n(c)q'_{n-1}(c)| = \left|\int_{-1}^{1} \hat{B}_n(x) q_{n-1}(x) d\mu(x)\right| \leq \frac{1}{c-1}$$

and hence $|\hat{B}_n'(c)| \leq K/n(\varphi(c))^{n-1}$ for some constant K. As a consequence, $\lim_{n\to\infty} \hat{B}_n'(c)q_{n-1}(c) = 0$. Taking into account that

$$\int_{-1}^{1} \hat{B}_n(x)q_{n-1}(x)(c-x)d\mu(x) = \langle \hat{B}_n(x), (c-x)q_{n-1}(x)\rangle + N\hat{B}_n'(c)q_{n-1}(c),$$

the last relation yields

$$\begin{aligned}\lim_{n\in\Lambda}\int_{-1}^{1} \hat{B}_n(x)q_{n-1}(x)(c-x)d\mu(x) &= \lim_{n\in\Lambda}\langle \hat{B}_n(x), (c-x)q_{n-1}(x)\rangle \\ &= -\lim_{n\in\Lambda}\langle \hat{B}_n(x), xq_{n-1}(x)\rangle \\ &= -\lim_{n\in\Lambda}\frac{k(q_{n-1})}{k(\hat{B}_n)} = -\lim_{n\in\Lambda}\frac{k(q_{n-1})}{k(q_n)}\frac{1}{A_n} \\ &= -\frac{1}{2A}.\end{aligned}$$

But

$$\lim_{n\in\Lambda}\int_{-1}^{1} \hat{B}_n(x)q_{n-1}(x)(c-x)d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{(A+1/4A)T_1(x)+B}{c-x}\frac{dx}{\sqrt{1-x^2}},$$

and thus we have

$$\frac{1}{\pi}\int_{-1}^{1} \frac{(A+1/4A)T_1(x)+B}{c-x}\frac{dx}{\sqrt{1-x^2}} = -\frac{1}{2A}. \tag{8}$$

After some calculations, equations (7) and (8) give $A = \frac{1}{2}$ and $B = -c$.

When the measure belongs to the Szegö class we have the following consequences

COROLLARY 2.2 *For $N = 0$,*

i)

$$\lim_{n\to\infty}\frac{\hat{B}_n(x)}{\tilde{q}_n(x)} = \left(\frac{\varphi^-(c)}{2}\right)^{1/2}\left(1-\frac{\varphi(c)}{\varphi(x)}\right)$$

uniformly on compact sets of $C \setminus [-1, 1]$. *Moreover,* $n-1$ *zeros of* $\hat{B}_n(x)$ *lie on* $(-1, 1)$ *and the other one tends to* c.

ii)

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{\varphi(x)^n} = \left(\frac{\varphi^-(c)}{2}\right)^{1/2} \left(1 - \frac{\varphi(c)}{\varphi(x)}\right) S(x),$$

where $S(x)$ *is the Szegö function of* $(c-x)\mu'(x)$, *and the convergence is uniform on compact sets of* $C \setminus [-1, 1]$.

For $M = 0$,

iii)

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{\tilde{q}_n(x)} = \frac{1}{2}\left(1 - \frac{\varphi(c)}{\varphi(x)}\right)\left(1 - \frac{\varphi^-(c)}{\varphi(x)}\right)$$

uniformly on compact sets of $C \setminus [-1, 1]$. $n-2$ *zeros of* $\hat{B}_n(x)$ *lie on* $(-1, 1)$ *one more tends to* c *and the other tends to* $[-1, 1]$.

iv)

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{\varphi^n(x)} = \frac{1}{2}\left(1 - \frac{\varphi(c)}{\varphi(x)}\right)\left(1 - \frac{\varphi^-(c)}{\varphi(x)}\right) S(x)$$

where $S(x)$ *is the Szegö function of* $(c-x)^2\mu'(x)$, *and the convergence is uniform on compact sets of* $C \setminus [-1, 1]$.

The following is also straightforward from the theorem,

COROLLARY 2.3 *i) When* $N = 0$,

$$x\hat{B}_n(x) = \alpha_n \hat{B}_{n+1}(x) + \beta_n \hat{B}_n(x) + \alpha_{n-1}\hat{B}_{n-1}(x)$$

and $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$, $\lim_{n\to\infty} \beta_n = 0$.

ii) When $M = 0$,

$$\begin{aligned}(x-c)^2\hat{B}_n(x) = \alpha_n \hat{B}_{n+2}(x) + \beta_n \hat{B}_{n+1}(x) + \gamma_n \hat{B}_n(x) \\ + \beta_{n-1}\hat{B}_{n-1}(x) + \alpha_{n-2}\hat{B}_{n-2}(x)\end{aligned}$$

and $\lim_{n\to\infty}\alpha_n=\frac{1}{4}$, $\lim_{n\to\infty}\beta_n=-c$, $\lim_{n\to\infty}\gamma_n=c^2+\frac{1}{2}$ ■

In both cases we get

COROLLARY 2.4 *If $f(x)$ is a continuous function in $[-1,1]$ then*

$$\lim_{n\to\infty}\int_{-1}^{1} f(x)\hat{B}_n(x)\hat{B}_{n+k}(x)d\mu(x)=\frac{1}{\pi}\int_{-1}^{1} f(x)T_k(x)\frac{dx}{\sqrt{1-x^2}}$$ ■

Finally, we include here the maximality of the polynomials in the Sobolev space for the different possible choices of M and N.

THEOREM 2.5 *i) The family $\phi=\{(x-c)^{2n}\}_{n=1}^{\infty}$ is maximal in the space $L^2(\mu)$.*

ii) $\phi\cup\{1\}$ is maximal in $L^2(\mu+M\delta(x-c))$.

iii) When $N>0$, the family $\phi\cup\{1,x-c\}$ is maximal in the Hilbert space associated with the Sobolev product given in (1).

Proof *If* $\int_{-1}^{1}(x-c)^{2n}f(x)d\mu(x)=0$ for $n=1,2,\ldots$ *and for a function* $f\in L^2(\mu)\ \cup L^2(\mu+M\delta(x-c))$ *or in the Sobolev space, one has*

$$\begin{aligned}0&=\int_{[(c-1)^2,(c+1)^2]} t^n f(c+\sqrt{t})d\mu(c+\sqrt{t})\\&=\int_{[(c-1)^2,(c+1)^2]} t^n f(c+\sqrt{t})dv(t)\end{aligned}$$

which means that $tf(c+\sqrt{t})=0\ v-a.e.$ on $[(c-1)^2,(c+1)^2]$. Thus $(x-c)^2f(x)=0\mu-a.e.$ on $[-1,1]$ and $f(x)=0\ \mu-a.e.$ also holds. So we have i).

If, moreover, $0=\int_{-1}^{1}f(x)d\mu(x)+Mf(c)$, then $Mf(c)=0$ and ii) follows. Finally, when $0=\langle f(x),(x-c)\rangle=\int_{-1}^{1}f(x)(x-c)d\mu(x)+Nf'(c)$ also holds, then $f'(c)=0$ and we have iii) ■

We need a Christoffel-Darboux type formula which was proved in [3].

THEOREM 2.6 *When $N > 0$, the Sobolev polynomials $\hat{B}_n(x)$ satisfy the following Christoffel-Darboux type formula*

$$
\begin{aligned}
&[(x-c)^2 - (y-c)^2] \sum_{j=0}^{n} \hat{B}_j(x)\hat{B}_j(y) \\
&\quad = \alpha_n[\hat{B}_{n+2}(x)\hat{B}_n(y) - \hat{B}_n(x)\hat{B}_{n+2}(y)] + \alpha_{n-1}[\hat{B}_{n+1}(x)\hat{B}_{n-1}(y) \\
&\qquad - \hat{B}_{n-1}(x)\hat{B}_{n+1}(y)] \\
&\qquad + \beta_n[\hat{B}_{n+1}(x)\hat{B}_n(y) - \hat{B}_n(x)\hat{B}_{n+1}(y)]
\end{aligned}
$$

where α_n and β_n are the coefficients of the five term recurrence relation of $\hat{B}_n(x)$ ■

In case $N = 0$, we have standard orthogonality as well as the standard Christoffel-Darboux formula.

3 ESTIMATES FOR SOBOLEV POLYNOMIALS

Because of

$$\int_{-1}^{1} \tilde{q}_n(x)p_j(x)d\mu(x) = p_j(c)\int_{-1}^{1} \tilde{q}_n(x)d\mu(x), \quad j = 0, 1, \ldots, n,$$

one can write

$$\tilde{q}_n(x) = \lambda_n \sum_{k=0}^{n} p_k(c)p_k(x) = \lambda_n\alpha_n \frac{p_{n+1}(x)p_n(c) - p_{n+1}(c)p_n(x)}{x-c}$$

where α_n are the coefficients of the recurrence relation of $p_n(x)$ and $\lambda_n = \int_{-1}^{1} \tilde{q}_n(x)d\mu(x)$. If μ belongs to the Szegö class, for every $x \in [-1, 1]$,

$$
\begin{aligned}
|\tilde{q}_n(x)| &\leq \frac{|\lambda_n\alpha_n p_n(c)|}{|x-c|}\left(|p_{n+1}(x)| + \frac{|p_{n+1}(c)|}{|p_n(c)|}|p_n(x)|\right) \\
&\leq K_1(|p_{n+1}(x)| + |p_n(x)|)
\end{aligned}
$$

for some constant K_1 and for n large enough, because

$$\begin{aligned}\lambda_n p_n(c) &= \int_{-1}^{1} \tilde{q}_n(x) p_n(c) d\mu(x) \\ &= \int_{-1}^{1} \tilde{q}_n(x)(p_n(c) + (x-c)\sum_{k=1}^{n} p_n^{(k)}(c)(x-c)^{k-1}) d\mu(x) \\ &= \int_{-1}^{1} \tilde{q}_n(x) p_n(x) d\mu(x) = \frac{k(\tilde{q}_n)}{k(p_n)}\end{aligned}$$

and $k(\tilde{q}_n)/k(p_n), p_{n+1}(c)/p_n(c)$ and α_n have limit points. So we have

COROLLARY 3.1 *Let* μ be a measure on $[-1, 1]$ *which belongs to the Szegö class. Then, for every* $x \in [-1, 1]$,

$$|\tilde{q}_n(x)| \leq K_1(|p_{n+1}(x)| + |p_n(x)|)$$

for a constant K_1 *which does not depend on* x *and for* n *large enough.* ■

Taking into account that $(q_n(x))_{n=0}^{\infty}$ are the orthonormal polynomials with respect to $(c-x)(c-x)d\mu(x)$, writting $q_n(x)$ in terms of $\tilde{q}_k(x)$, in the same way as before we have

COROLLARY 3.2 *If* μ *belongs to the Szegö class then for every* $x \in [-1, 1]$,

$$|q_n(x)| \leq K_2(|p_{n+2}(x)| + |p_{n+1}(x)| + |p_n(x)|)$$

for n *large enough and for some positive real constant* K_2 *which does not depend on* x. ■

Using now Theorems 2.1 and 2.4, if $N > 0$ then

$$\hat{B}_n(x) = A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x)$$

where A_n, B_n, C_n have limit points. In case $N = 0$ then

$$\hat{B}_n(x) = A_n \tilde{q}_n(x) + B_n \tilde{q}_{n-1}(x)$$

from Theorem 2.4. Then

COROLLARY 3.3 *Let μ be a measure in the Szegö class. Then, for every $x \in [-1, 1]$,*

i) when $N > 0$,

$$|\hat{B}_n(x)| \leq K_3(|p_{n+3}(x)| + |p_{n+2}(x)| + |p_{n+1}(x)| + |p_n(x)| + |p_{n-1}(x)|)$$

for n large enough and for some positive real constant K_3 independent of x.

ii) when $N = 0$ there is a constant K_3^* such that

$$|\hat{B}_n(x)| \leq K_3^*(|p_{n+1}(x)| + |p_n(x)| + |p_{n-1}(x)|)$$

for n large enough.

COROLLARY 3.4 *If μ belongs to the Szegö class and there is a function $h(x)$ such that the orthonormal polynomials $p_n(x)$* satisfy $|p_n(x)| \leq h(x)$, $x \in [-1, 1]$, *then there is a constant K such that*

$$|\hat{B}_n(x)| \leq Kh(x)$$

for n large enough and for every $x \in [-1, 1]$. ∎

In particular, if μ is the Jacobi measure we know the function $h(x)$ and it will be very useful for the study of Fourier series.

Also in order to study Fourier series of Sobolev polynomials, we need estimates for $\hat{B}_n(c)$ and $\hat{B}'_n(c)$.

For $M > 0$, one has $0 = \langle \hat{B}_n, 1 \rangle = \int_{-1}^{1} \hat{B}_n(x) d\mu(x) + M\hat{B}_n(c)$, so

$$|\hat{B}_n(c)| = \frac{1}{M}\left|\int_{-1}^{1} \hat{B}_n(x) d\mu(x)\right|.$$

In the same way, when $N > 0$,

$$|\hat{B}'_n(c)| = \frac{1}{N}\left|\int_{-1}^{1} (c - x)\hat{B}_n(x) d\mu(x)\right|.$$

Taking into account that

$$\hat{B}_n(x) = A_n q_n(x) + B_n q_{n-1}(x) + C_n q_{n-2}(x)$$

where, as before, $q_n(x)$ are the orthonormal polynomials with respect to $(x-c)^2 d\mu(x)$, in order to estimate $\hat{B}_n(c)$ and $\hat{B}'_n(c)$ we only have to estimate $\int_{-1}^{1} q_n(x)d\mu(x)$, and $\int_{-1}^{1} q_n(x)(x-c)d\mu(x)$ respectively.

Moreover

$$\int_{-1}^{1} q_n(x)(x-c)d\mu(x) = \frac{1}{q_n(c)}\int_{-1}^{1} q_n(x)q_n(c)(x-c)d\mu(x)$$
$$= \frac{1}{q_n(c)}\int_{-1}^{1} q_n(x)(q_n(c) + (x-c)\pi_{n-1}(x))(x-c)d\mu(x)$$

for any polynomial $\pi_{n-1}(x)$ of degree at most $n-1$. Then

$$\int_{-1}^{1} q_n(x)(x-c)d\mu(x) = \frac{1}{q_n(c)}\int_{-1}^{1} (x-c)q_n^2(x)d\mu(x)$$

Analogously,

$$\int_{-1}^{1} q_n(x)d\mu(x) = \frac{1}{q_n(c)}\int_{-1}^{1} q_n(x)(q_n(c) + q'_n(c)(x-c))d\mu(x)$$
$$- \frac{q'_n(c)}{q_n(c)}\int_{-1}^{1} q_n(x)(x-c)d\mu(x)$$
$$= \frac{1}{q_n(c)}\int_{-1}^{1} q_n^2(x)d\mu(x) + \frac{q'_n(c)}{q_n^2(c)}\int_{-1}^{1} q_n^2(x)(c-x)d\mu(x).$$

Then, for a measure in Szegö class, we get

$$\int_{-1}^{1} q_n(x)(x-c)d\mu(x) = O((c-\sqrt{c^2-1})^n), \int_{-1}^{1} q_n(x)d\mu(x)$$
$$= O(n(c-\sqrt{c^2-1})^n).$$

So, we have

THEOREM 3.1 *If μ is a measure on $[-1,1]$ which belongs to the Szegö class and $M>0$, $N>0$, then there are constants K_1 and K_2 such that*
i) $|\hat{B}_n(c)| \leq K_1 n r_o^n$,
ii) $|\hat{B}'_n(c)| \leq K_2 r_0^n$
where $0 < r_0 = c - \sqrt{c^2-1} < 1$ ■

In the same way we get

THEOREM 3.2 *Let μ be a measure in Szegö class.*
i) When $N=0$, there is a constant K such that $|\hat{B}_n(c)| \leq K r_0^n$.
ii) When $M=0$, there is a constant K such that $|\hat{B}'_n(c)| \leq K r_0^n$
where $r_0 = c - \sqrt{c^2-1} < 1$ ■

4 FOURIER SERIES

Since $L^2(\mu)$ is a Hilbert space, it is clear that the space S given by

$$S = \{f(x) : \int_{-1}^{1} |f(x)|^2 d\mu(x) < \infty, \textit{and } f'(c) \textit{ exists}\}$$

with the associated norm $\|\cdot\|_s$ derived from the Sobolev inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + Mf(c)g(c) + Nf'(c)g'(c)$$

is also a Hilbert space because $\|f(x)\|_s^2 = \|f(x)\|_\mu^2 + Mf^2(c) + N(f'(c)^2)$ and a Cauchy sequence in S is a Cauchy sequence in $L^2(\mu)$ and in the point c. Moreover, the maximality of the polynomials was seen in Theorem 2.5. So, $S_n(x;f) \to f(x)$ in S for any function of S, where

$$S_n(x;f) = \sum_{j=0}^{n} \langle \hat{B}_j(t), f(t) \rangle \hat{B}_j(x)$$

is the partial sum of the Fourier-Sobolev series of f. In particular, it means that

$$\lim_{n\to\infty} S_n(c;f) = f(c), \ \lim_{n\to\infty} S'_n(c;f) = f'(c).$$

If we apply this to the functions $f_1(x)$ and $f_2(x)$ defined by

$$f_1(x) = 0 \textit{ if } x \in |-1, 1|, \ f_1(c) = 1, \ f_1'(c) = 0,$$
$$f_2(x) = 0 \textit{ if } x \in |-1, 1|, \ f_2(c) = 0, \ f_2'(c) = 1,$$

their Fourier-Sobolev series are $M\sum_{k=0}^{\infty}\hat{B}_k(c)\hat{B}_k(x)$ and $N\sum_{k=0}^{\infty}\hat{B}_k'(c)\hat{B}_k(x)$ respectively. Thus, we get

$$M\sum_{k=0}^{n}\hat{B}_k(c)\hat{B}_k(x) \to f_1(x) \quad \text{in} \quad S$$
$$M\sum_{k=0}^{n}\hat{B}_k(c)\hat{B}_k(x) \to f_2(x) \quad \text{in} \quad S$$

which means that

$$c1 : \sum_{k=0}^{n}\hat{B}_k(c)\hat{B}_k(x) \to 0 \quad \text{in} \quad L_2(\mu),$$

$$c2 : \sum_{k=0}^{n}\hat{B}_k'(c)\hat{B}_k(x) \to 0 \quad \text{in} \quad L_2(\mu),$$

$$c3 : \sum_{k=0}^{\infty}(\hat{B}_k(c))^2 = 1/M,$$

$$c4 : \sum_{k=0}^{\infty}(\hat{B}_k'(c))^2 = 1/N,$$

$$c5 : \sum_{k=0}^{\infty}\hat{B}_k(c)\hat{B}_k'(c) = 0.$$

From now on, the Jacobi measure, $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, $\alpha > -1$, $\beta > -1$, will be considered, and the behaviour of the corresponding Fourier–Sobolev series will be studied.

We know that the Jacobi orthonormal polynomials $p_n^{(\alpha,\beta)}(x)$ satisfy (see [7], Theorem 3.14 in page 101)

$$(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}|p_n^{(\alpha,\beta)}(x)| \leq C \quad x \in [-1, 1].$$

Then, for the corresponding Sobolev orthonormal polynomials $\hat{B}_n^{(\alpha,\beta)}(x)$, Corollary 3.4 yields the uniform bound

$$|\hat{B}_n^{(\alpha,\beta)}(x)| \leq \frac{K}{(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}} = h^{(\alpha,\beta)}(x) \quad for\; x \in (-1,1) \qquad (9)$$

for some constant K and for n large enough (we will continue denoting by $\hat{B}_n(x)$ the polynomials $\hat{B}_n^{(\alpha,\beta)}(x)$). From Theorem 3.1 and 3.2, for k large enough, every term of the series $\sum_{k=0}^{\infty} \hat{B}_k(c)\hat{B}_k(x)$ has the majorant $K^*k(c-\sqrt{c^2-1})^k$ for some constant K^* in closed subsets of $(-1,1)$. Then $\sum_{k=0}^{\infty} \hat{B}_k(c)\hat{B}_k(x)$ converges for $x \in (-1,1)$ and uniformly in any compact set $[-1+\epsilon, 1-\epsilon]$, $0<\epsilon<1$. Hence, $\sum_{k=0}^{\infty} \hat{B}_k(c)\hat{B}_k(x)$ is a continuous function for $x \in (-1,1)$ which, from condition c1, equals zero $\mu - a.e.$ in $[-1,1]$ provided that $M>0$. As a consequence, $\sum_{k=0}^{\infty} \hat{B}_k(c)\hat{B}_k(x) = 0$, $x \in (-1,1)$. In the same way, Theorems 3.1 and 3.2 and condition c2 give $\sum_{k=0}^{\infty} \hat{B}'_k(c)\hat{B}_k(x) = 0$, $x \in (-1,1)$ provided that $N > 0$.

THEOREM 4.1 *Let $\hat{B}_n(x)$ be the orthonormal polynomials with respect to the Sobolev inner product associated with the Jacobi measure. Then*

i) When $M>0$, $\sum_{k=0}^{\infty} \hat{B}_k(c)\hat{B}_k(x) = 0$ for every $x \in (-1,1)$.

ii) When $N>0$, $\sum_{k=0}^{\infty} \hat{B}'_k(c)\hat{B}_k(x) = 0$ for every $x \in (-1,1)$

Now, we can prove the pointwise convergence of $S_n(x;f)$ to $f(x)$ when one has standard sufficient conditions for the function $f(x)$.

THEOREM 4.2 *Let $x_0 \in (-1,1)$ and let f be a function with derivative in c such that $(f(x_0)-f(t))/(x_0-t) \in L^2(\mu)$ where μ is the Jacobi measure. Then*

i) $\sum_{k=0}^{\infty} c_k\hat{B}_k(x_0) = f(x_0)$,

ii) $\sum_{k=0}^{\infty} c_k\hat{B}_k(c) = f(c)$ *if* $M>0$,

iii) $\sum_{k=0}^{\infty} c_k\hat{B}'_k(c) = f'(c)$ *if* $N>0$

where $c_k = \langle f, \hat{B}_k\rangle$.

Proof Because of $f \in L^2(\mu)$ when $(f(x_0 - f(t))/(x_0 - t) \in L^2(\mu)$, ii) and iii) are proved, so we only have to prove i). Let us denote $D_n(x,t) = \sum_{k=0}^{n} \hat{B}_k(x)\hat{B}_k(t)$. Since

$$f(x_0) - S_n(x_0;f) = \langle f(x_0) - f(t), D_n(x_0,t)\rangle$$
$$= \int_{-1}^{1} (f(x_0) - f(t))D_n(x_0,t)d\mu(t) + M(f(x_0)$$
$$- f(c))D_n(x_0,c) - f'(c)\frac{\partial D_n}{\partial t}(t,x_0)\bigg|_{t=c},$$

Theorem 4.1 yields

$$\lim_{n\to\infty}(f(x_0) - S_n(x_0;f)) = \lim_{n\to\infty}\int_{-1}^{1}(f(x_0) - f(t))D_n(x_0,t))d\mu(t)$$

On the other hand, Christoffel-Darboux formula gives

$$\left|\int_{-1}^{1}(f(x_0) - f(t))D_n(x_0,t)d\mu(t)\right|$$
$$\leq \alpha_n|\hat{B}_{n+2}(x_0)|\left|\int_{-1}^{1}\frac{f(x_0) - f(t)}{(x_0 - c)^2 - (t-c)^2}\hat{B}_n(t)d\mu(t)\right|$$
$$+ \alpha_n|\hat{B}_n(x_0)|$$
$$\cdot\left|\int_{-1}^{1}\frac{f(x_0) - f(t)}{(x_0 - c)^2 - (t-c)^2}\hat{B}_{n+2}(t))d\mu(t)\right| + \alpha_{n-1}|\hat{B}_{n+1}(x_0)|$$
$$\cdot\left|\int_{-1}^{1}\frac{f(x_0) - f(t)}{(x_0 - c)^2 - (t-c)^2}\hat{B}_{n-1}(t))d\mu(t)\right| + \alpha_{n-1}|\hat{B}_{n-1}(x_0)|$$
$$\cdot\left|\int_{-1}^{1}\frac{f(x_0) - f(t)}{(x_0 - c)^2 - (t-c)^2}\hat{B}_{n+1}(t)d\mu(t)\right| + |\beta_n||\hat{B}_{n+1}(x_0)|$$
$$\cdot\left|\int_{-1}^{1}\frac{f(x_0) - f(t)}{(x_0 - c)^2 - (t-c)^2}\hat{B}_n(t)d\mu(t)\right| + |\beta_n|$$
$$\cdot\left|\hat{B}_n(x_0)\int_{-1}^{1}\frac{f(x_0) - f(t)}{(x_0 - c)^2 - (t-c)^2}\hat{B}_{n+1}(t))d\mu(t)\right|$$

Furthermore, α_n and β_n are bounded according to Theorem 2.2 and Corollary 2.3. $|\hat{B}_n(x_0)| \leq h^{(\alpha,\beta)}(x_0) < \infty$ also holds, and since $|(x_0 - c)^2 - (t - c)^2| \geq 2|x_0 - t|(c - 1)$ when x_0, $t \in [-1, 1]$, the function $f(x_0) - f(t)/(x_0 - c)^2 - (t - c)^2$ belongs to $L^2(\mu)$ when $f(x_0) - f(t)/x_0 - t \in L^2(\mu)$. Hence, it also belongs to $L^2((t - c)^2 d\mu(t))$ and

$$\lim_{n\to\infty} \int_{-1}^{1} \frac{f(x_0) - f(t)}{(x_0 - c)^2 - (t - c)^2} \hat{B}_n(t) d\mu(t) = 0$$

follows from Theorems 2.1 and 2.4. As a consequence, each term in the last sum tends to zero and the theorem is proved.

THEOREM 4.3 *Let $f(x)$ be a function with first derivative in c satisfying a Lipschitz condition of order $\eta < 1$ uniformly in $[-1, 1]$, i.e. $|f(x + h) - f(x)| \leq K|h|^\eta$ for $|h| < \delta$ for some $\delta > 0$. If $c_k = \langle f, \hat{B}_k \rangle$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} c_k \hat{B}_k(x) = f(x), \quad x \in (-1, 1)$$

and the convergence is uniform in $[-1 + \epsilon, 1 - \epsilon]$ for every ϵ such that $0 < \epsilon < 1$. Moreover $\sum_{k=0}^{\infty} c_k \hat{B}_k(c) = f(c)$ when $M > 0$ and $\sum_{k=0}^{\infty} c_k \hat{B}'_k(c) = f'(c)$ if $N > 0$.

Proof In the same way as before, we only have to prove that $\int_{-1}^{1} f(t) D_n(x, t) d\mu(t)$ converges to $f(x)$. Moreover,

$$\begin{aligned} \left| f(x) - \int_{-1}^{1} f(t) D_n(x, t) d\mu(t) \right| &= \left| \int_{-1}^{1} (f(x) - f(t)) D_n(x, t) d\mu(t) \right| \\ &\leq \left| \int_{|x-t|\geq\delta} (f(x) - f(t)) D_n(x, t) d\mu(t) \right| \\ &\quad + \left| \int_{|x-t|<\delta} (f(x) - f(t)) D_n(x, t) d\mu(t) \right| \\ &= I_n^{(1)}(x) + I_n^{(2)}(x). \end{aligned}$$

Since $f(x) - f(t)/(x-c)^2 - (t-c)^2(1 - \chi_{(x-\delta,x+\delta)}(t))$, where $\chi_{(x-\delta,x+\delta)}(t)$ is the characteristic function of the interval, belongs to $L^2(\mu)$, using the Christoffel-Darboux formula and with the same procedure as in the last Theorem, the term $I_n^{(1)}(x)$ tends to zero uniformly in closed subintervals of $(-1, 1)$. For $I_n^2(x)$ we can write

$$\begin{aligned} I_n^{(2)}(x) \le \alpha_n |\hat{B}_{n+2}(x)| &\left| \int_{|x-t|<\delta} \frac{f(x)-f(t)}{(x-c)^2-(t-c)^2} \hat{B}_n(t) d\mu(t) \right| \\ + \alpha_n |\hat{B}_n(x)| &\left| \int_{|x-t|<\delta} \frac{f(x)-f(t)}{(x-c)^2-(t-c)^2} \hat{B}_{n+2}(t)) d\mu(t) \right| \\ + \alpha_{n-1} |\hat{B}_{n+1}(x)| &\left| \int_{|x-t|<\delta} \frac{f(x)-f(t)}{(x-c)^2-(t-c)^2} \hat{B}_{n-1}(t)) d\mu(t) \right| \\ + \alpha_{n-1} |\hat{B}_{n-1}(x)| &\left| \int_{|x-t|<\delta} \frac{f(x)-f(t)}{(x-c)^2-(t-c)^2} \hat{B}_{n+1}(t)) d\mu(t) \right| \\ + |\beta_n| |\hat{B}_{n+1}(x)| &\left| \int_{|x-t|<\delta} \frac{f(x)-f(t)}{(x-c)^2-(t-c)^2} \hat{B}_n(t) d\mu(t) \right| \\ + |\beta_n| |\hat{B}_n(x)| &\left| \int_{|x-t|<\delta} \frac{f(x)-f(t)}{(x-c)^2-(t-c)^2} \hat{B}_{n+1}(t)) d\mu(t) \right|. \end{aligned}$$

Lipschitz condition gives

$$\begin{aligned} &\left| \int_{|x-t|<\delta} \frac{f(x)-f(t)}{(x-c)^2-(t-c)^2} \hat{B}_n(t) d\mu(t) \right| \\ &\quad \le \int_{|x-t|<\delta} \frac{K|\hat{B}_n(t)|}{|x-t|^{1-\eta}|x+t-2c|} d\mu(t) \\ &\quad \le \frac{K(h^{(\alpha,\beta)}(x) + O(1))}{(c-1)^2} \int_{|x-t|<\delta} \frac{d\mu(t)}{|x-t|^{1-\eta}} \end{aligned}$$

Hence,

$$I_n^{(2)} = O\left(\int_{|x-t|<\delta} \frac{d\mu(t)}{|x-t|^{1-\eta}} \right)$$

and, as a consequence, $\int_{-1}^{1} (f(x) - f(t)) D_n(x,t) d\mu(t)$ tends to zero uniformly in any closed subinterval of $(-1, 1)$.

Let us denote, as usual,

$$\omega(\delta) = \omega(\delta, f) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in [-1, 1], |x_1 - x_2| \le \delta\}$$

the modulus of continuity of a function $f(x)$ in $[-1, 1]$.

THEOREM 4.4 *Let $f(x)$ be a function such that its modulus of continuity $w(\delta)$ satisfies the condition*

$$\omega(\delta) = O\left(\log^{-(1+\epsilon)} \frac{1}{\delta}\right)$$

for $\epsilon > 0$, and with first derivative in c. If $c_k = \langle f, \hat{B}_k \rangle$, $\sum_{k=0}^{\infty} c_k \hat{B}_k(x) = f(x)$ a.e. in$[-1, 1]$. Moreover, $\sum_{k=0}^{\infty} c_k \hat{B}_k(c) = f(c)$ provided that $M > 0$, and $\sum_{k=0}^{\infty} c_k \hat{B}'_k(c) = f'(c)$ when $N > 0$.

Proof It is clear that the modulus of continuity of the function $f(x)/(x-c)^2$ satisfies the condition

$$\omega\left(\delta, \frac{f(x)}{(x-c)^2}\right) = O\left(\log^{-(1+\epsilon)} \frac{1}{\delta}\right)$$

Let $d_k = \int_{-1}^{1} f(x)/(x-c)^2 q_k(x)(x-c)^2 d\mu(x)$. By Jackson's Approximation Theorem (see [7] Chap I), there is a polynomial $\pi_n(x)$ such that

$$\left|\frac{f(x)}{(x-c)^2} - \pi_n(x)\right| = O\left(\frac{1}{\log^{1+\epsilon} n}\right)$$

whence

$$\sum_{k=n}^{\infty} d_k^2 = \int_{-1}^{1} \left(\frac{f(x)}{(x-c)^2} - \pi_n(x)\right)^2 (x-c)^2 d\mu(x) = O\left(\frac{1}{\log^{2+2\epsilon} n}\right).$$

Taking into account that, from Theorems 2.1 and 2.4,

$$c_k = \langle f, \hat{B}_k \rangle = A_k d_k + B_k d_{k-1} + C_k d_{k-2} + Mf(c)\hat{B}_k(c) + Nf'(c)\hat{B}'_k(c),$$

as well as $\sum_{k=n}^{\infty} d_k d_{k-1} \leq (\sum_{k=n}^{\infty} d_k^2)^{1/2}(\sum_{k=n}^{\infty} d_{k-1}^2)^{1/2}$ and the estimates for $\hat{B}_k(c)$ and for $\hat{B}'_k(c)$, we get

$$\sum_{k=n}^{\infty} c_k^2 = O\left(\frac{1}{\log^{2+2\epsilon} n}\right)$$

As a consequence (see Theorem 3.3 in pag. 137 of [7]), $\sum_{k=0}^{\infty} c_k^2 \log^2 k < \infty$ and it yields (see Theorem 2.5 in pag. 126 of [7]), $\sum_{k=0}^{n} c_k \hat{B}_k(x)$ converges *a.e.* $x \in [-1, 1]$ (here one has to take into account that $\int_{-1}^{1} g^2(x) d\mu(x) \leq \langle g, g \rangle$ for every function g in S. But, since $f(x)$ is a continuous function, $\sum_{k=0}^{n} c_k \hat{B}_k(x)$ converges to $f(x)$ in the Sobolev space. Then

$$\sum_{k=0}^{\infty} c_k \hat{B}_k(x) = f(x), \quad a.e. x \in [-1, 1],$$

as well as $\sum_{k=0}^{\infty} c_k \hat{B}_k(c) = f(c)$ and $\sum_{k=0}^{\infty} c_k \hat{B}'_k(c) = f'(c)$.

Acknowledgements

The work of F. Marcellán was supported by a grant of Dirección General de Investigación (Ministerio de Ciencia y Tecnología) of Spain BFM-2000-0206-C04-01, and INTAS project INTAS 2000-272

References

[1] Alfaro M., Marcellán F., Rezola M. L., *Estimates for Jacobi-Sobolev type orthogonal polynomials*. Applic. Analysis **67** (1997) 157–174

[2] López G., Marcellán F., Van Assche W. *Relative asymptotics for polynomials orthogonal with respect to a discrete Sobolev inner product*, Constr. Approx. **11** (1995) 107–137.

[3] Marcellán F., Pérez T. E., Piñar M. *Regular Sobolev type orthogonal polynomials: the Bessel case*, Rocky Mount. J. of Math., **25** (4) (1995) 1431–1457.

[4] Marcellán F., Van Assche W. *Relative asymptotics for orthogonal polynomials with a Sobolev inner product*, J. Approx. Theory **72** (1992) 192–209.

[5] Maté A., Nevai P., Totik V. *Asymptotics for the leading coefficients of orthogonal polynomials*, Constr. Approx. **1** (1985) 63–69.

[6] Maté A., Nevai P., Totik V. *Strong and weak convergence of orthogonal polynomials on the unit circle*, Amer. J. Math. **109** (1987) 239–282.

[7] Osilenker B. *Fourier Series in Orthogonal Polynomials*, World Scientific, Singapore 1999.

[8] Szegö G. *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Pub. **23** Amer. Math. Soc. Providence, RI, 1975 (4th edition).