Characterization of the D_w-Laguerre–Hahn Functionals

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We give some characterization theorems for the D_w Laguerre Hahn linear functionals and we extend the concept of the class of the usual Laguerre Hahn functionals to the D_w Laguerre Hahn functionals, recovering the classic results when w tends to zero. Moreover, we show that some transformations carried out on the D_w Laguerre Hahn linear functionals lead to new D_w Laguerre Hahn linear functionals. Finally, we analyze the class of the resulting functionals and we give some applications relative to the first associated Charlier, Meixner, Krawtchouk and Hahn orthogonal polynomials.

Keywords: Regular linear functionals; Orthogonal polynomials; Stieltjes functions; Riccati difference equation; Laguerre Hahn class; D_w Laguerre Hahn class

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INTRODUCTION

Let $\mathscr{U}$ be a linear functional on the linear space $\mathscr{P}$ of polynomials with complex coefficients. $\mathscr{U}$ is said to be regular (or quasi-definite [2]) if there exists a sequence of monic polynomials $(P_n)_n$, orthogonal with respect to $\mathscr{U}$ i.e.

i) $P_n(x) \quad x^n +$ lower degree terms,
ii) $\langle \mathscr{U}, P_n P_m \rangle \quad k_n \delta_{n,m},\ k_n \neq 0,\ n \quad 0, 1, 2, \ldots$

Here, $\langle \cdot, \cdot \rangle$ means the duality bracket.

Let $(P_n)_n$ be a sequence of monic polynomials, orthogonal with respect to the regular linear functional $\mathscr{U}$. It satisfies a three-term recurrence relation

$$P_{n+1}(x) \quad (x \quad \beta_n)P_n(x) \quad \gamma_n P_{n \quad 1}(x), \quad n \geq 1,$$
$$P_0(x) \quad 1,\ P_1(x) \quad x \quad \beta_0, \tag{1}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0, \forall n$. We assume that the linear functionals used in this paper are normalized by: $\langle \mathscr{U}, P_0^2 \rangle \quad \gamma_0 \quad 1$.

Given a regular linear functional $\mathscr{U}$ and the corresponding sequence of monic polynomials $(P_n)_n$ satisfying Eq. (1), we define the co-recursive [1,4] $(P_n^{[\mu]})_n$ of $(P_n)_n$ and the first associated $(P_n^{(1)})_n$ of $(P_n)_n$ as the two families of monic polynomials defined, respectively, by the following three-term recurrence relations

$$P_{n+1}^{[\mu]} = (x \quad \beta_n)P_n^{[\mu]} \quad \gamma_n P_{n \quad 1}^{[\mu]}, \qquad n \geq 1, \quad P_0^{[\mu]} = 1, \quad P_1^{[\mu]} = x \quad \beta_0 \quad \mu,$$
$$P_{n+1}^{(1)} = (x \quad \beta_{n+1})P_n^{(1)} \quad \gamma_{n+1} P_{n \quad 1}^{(1)}, \quad n \geq 1, \quad P_0^{(1)} = 1, \qquad P_1^{(1)} = x \quad \beta_1, \tag{2}$$

where μ is a complex number. By Favard theorem [2,6] these families are orthogonal, and we denote by $\mathscr{U}^{[\mu]}$ and $\mathscr{U}^{(1)}$, respectively, the regular normalized functionals associated with these orthogonal polynomial families. The first associated $(P_n^{(1)})_n$ of $(P_n)_n$ can also be defined [2] by

$$P_n^{(1)}(t) \quad \left\langle \mathscr{U}, \frac{P_{n+1}(x) \quad P_{n+1}(t)}{x \quad t} \right\rangle, \tag{3}$$

where the regular linear functional $\mathscr{U}$ acts on the variable x.

A regular linear functional $\mathscr{U}$ belongs to the Laguerre Hahn class if the Stieltjes function $S(\mathscr{U}) \quad S$ satisfies a Riccati differential equation [3,5,10]

$$\phi S' \quad BS^2 + CS + D, \tag{4}$$

where $\phi \neq 0$, B, C and D are polynomials with $D \quad (\mathscr{D}\mathscr{U})\theta_0\phi + \mathscr{U}\theta_0 C \quad \mathscr{U}^2\theta_0^2 B$. Here $\mathscr{D}(\quad d/\mathrm{d}x)$ is the usual derivative operator. θ_0 and the right multiplication by a polynomial are introduced in "Preliminaries and notations". In 1988 Dini [3] (see also Ref. [11]) obtained a characterization theorem in terms of the functional equation satisfied by $\mathscr{U}$

$$\mathscr{D}(\phi\mathscr{U}) + B(x \quad ^1\mathscr{U}^2) + \psi\mathscr{U} \quad 0, \quad \psi \quad (\phi' + C), \tag{5}$$

as well as the structure relation fulfilled by the corresponding orthogonal polynomials $(P_n)_n$ and the corresponding first associated sequence $(P_n^{(1)})_n$) as follows:

$$\phi P'_{n+1} \quad BP_n^{(1)} \quad \sum_{i=n \quad s}^{n+d} \zeta_{n,i} P_i, \quad n > s, \quad \zeta_{n,n \quad s} \neq 0. \tag{6}$$

In Ref. [11] Marcellán and Prianes have introduced the notion of the class of a given Laguerre Hahn linear functional, as was done for the semi-classical linear functional [14]. They have also given a necessary and sufficient condition for the reducibility of the functional equation and they have used these conditions to determine the class of some Laguerre Hahn linear functionals which are obtained by a perturbation of a given Laguerre Hahn linear functional.

The notion of discrete Laguerre Hahn linear functional has been considered by several authors [7 9]. In the Doctoral Dissertation by Guerfi [9], the D_w-Laguerre Hahn linear functional is defined as the one for which the Stieltjes function $S(\mathscr{U})$ satisfies

$$A(z)D_w S(\mathscr{U})(z) \quad B(z)S(\mathscr{U})(z)S(\mathscr{U})(z+w) + C(z)S(\mathscr{U})(z) + D(z),$$

where $A \neq 0$, B,C, and D are polynomials and D_w is the operator defined by

$$D_w f(x) \quad (f(x+w) \quad f(x)/w.$$

An equivalent definition given by Foupouagnigni *et al.* [7,8] is stated as follows.

A regular linear functional $\mathscr{U}$ belongs to the D_w-Laguerre Hahn class if the corresponding Stieltjes function $S(\mathscr{U})$ satisfies a D_w-Riccati difference equation

$$\phi(z+w)D_wS(\mathscr{U})(z) \quad G(z)S(\mathscr{U})(z)S(\mathscr{U})(z+w)+E(z)S(\mathscr{U})(z)$$
$$+F(z)S(\mathscr{U})(z+w)+H(z),$$

where $\phi \neq 0$, E, F, G, and H are polynomials. This definition allowed us to derive a fourth-order difference equation satisfied by the associated polynomials $(P_n^{(k)})$ of any positive integer order k corresponding to a Laguerre Hahn linear functional of a discrete variable [7,8].

The aim of this work is to give a characterization theorem for a D_w-Laguerre Hahn linear functional, define the concept of the class of a D_w-Laguerre Hahn linear functional and, finally, to give necessary and sufficient conditions in order to the class of a given D_w-Laguerre Hahn linear functional is a nonnegative integer s.

In the second section we give the preliminaries and some previous results needed for this work. The third section is devoted to the characterization theorem of the D_w-Laguerre Hahn linear functionals. The fourth section gives the definition of the class of the D_w-Laguerre Hahn linear functional and provides necessary and sufficient conditions for the characterization of the class of a D_w-Laguerre Hahn linear functional. In the fifth section some applications are presented. In particular, we determine the class of a D_w-Laguerre Hahn linear functional obtained by some perturbations of a D_w-Laguerre Hahn linear functional.

PRELIMINARIES AND NOTATIONS

Let $\mathscr{U}$ be a regular linear functional on the linear space $\mathscr{P}$ of polynomials with complex coefficients and let $S(\mathscr{U})(z)$ be its Stieltjes function defined by

$$S(\mathscr{U})(z) \qquad \sum_{n\geq 0}\frac{(\mathscr{U})_n}{z^{n+1}},$$

where $(\mathscr{U})_n \quad \langle \mathscr{U}, x^n\rangle$ are the moments of $\mathscr{U}$ and $\langle\cdot,\cdot\rangle$ is the duality bracket. Let $\mathscr{P}'$ be the algebraic dual space of $\mathscr{P}$ and $\mathscr{E}$ the linear space generated by $\{\delta^{(n)}\}_{n\geq 0}$, where $\delta^{(n)}$ means the nth derivative of the Dirac delta in the origin i.e.

$$\langle \delta^n, p\rangle \quad (\;\;1)^n p^{(n)}(0) \quad (\;\;1)^n \frac{d^n p}{\mathrm{d}x^n}(0), \quad p \in \mathscr{P}.$$

- Consider the isomorphism $\mathscr{F} : \mathscr{E} \to \mathscr{P}$ defined as follows [14]: For

$$\mathscr{U} \quad \sum_{n=0}^{k} (\mathscr{U})_n \frac{(\;\;1)^n}{n!} \delta^{(n)}, \quad \mathscr{F}(\mathscr{U})(z) \quad \sum_{n=0}^{k} (\mathscr{U})_n z^n.$$

  This isomorphism yields the relation

$$S(\mathscr{U})(z) \qquad \frac{1}{z}\mathscr{F}(\mathscr{U})\left(\frac{1}{z}\right), \quad \forall \mathscr{U} \in \mathscr{P}'. \tag{7}$$

- Given $f \in \mathscr{P}$ and $\mathscr{U} \in \mathscr{P}'$, $f\mathscr{U}$ and $x^{\;\;1}\mathscr{U}$ will denote, respectively, the linear functionals defined by:

$$\langle f\mathscr{U}, p\rangle \quad \langle \mathscr{U}, fp\rangle, \quad \langle x^{\;\;1}\mathscr{U}, p\rangle \quad \langle \mathscr{U}, \theta_0 p\rangle, \quad \forall p \in \mathscr{P}, \tag{8}$$

  where $\theta_a p(x) \quad (p(x) \quad p(a))/(x \quad a)$ and a is a complex number.
- For $f \in \mathscr{P}$ and $\mathscr{U} \in \mathscr{P}'$, the product $\mathscr{U}f$ is the polynomial

$$\mathscr{U}f(x) \quad \sum_{i=0}^{n} \left(\sum_{j=i}^{n} (\mathscr{U})_{j \;\; i} f_j \right) x^i,$$

  where

$$f(x) \quad \sum_{i=0}^{n} f_i x^i.$$

- The product $\mathscr{U}\mathscr{V}$ of two linear functionals $\mathscr{U}$ and $\mathscr{V}$ is the linear functional

$$\langle \mathscr{U}\mathscr{V}, p\rangle \quad \langle \mathscr{U}, \mathscr{V}p\rangle, \forall p \in \mathscr{P}.$$

  The product defined as before is commutative [14] i.e.

$$\mathscr{U}\mathscr{V} \quad \mathscr{V}\mathscr{U}, \quad \forall \mathscr{U}, \mathscr{V} \in \mathscr{P}'.$$

- The operators D_w and $\mathscr{T}_w$ (where w is a complex number) are defined in the linear space $\mathscr{P}$ of polynomials in the following way

$$D_w P(x) \quad \frac{P(x+w) \quad P(x)}{w}, \quad \mathscr{T}_w P(x) \quad P(x+w), \quad P \in \mathscr{P}.$$

When w is equal to 1, $\mathscr{T}_{1} \quad \mathscr{T}$ and $D_1 \quad \Delta$. Also, $D_{\ 1} \quad \nabla$. Δ and ∇ are, respectively, the backward and the forward difference operators.

By duality, the image of a linear functional using these operators D_w and $\mathscr{T}_w$ is a linear functional such that

$$\langle D_w \mathscr{U}, p\rangle \qquad \langle \mathscr{U}, D_{\ w} p\rangle, \quad \langle \mathscr{T}_w \mathscr{U}, p\rangle \quad \langle \mathscr{U}, \mathscr{T}_{\ w} p\rangle, \forall p \in \mathscr{P}.$$

The following known results (see Refs. [3,7,11,17]) will be useful for our work. We summarize them in

Lemma 1 *For $p, q \in \mathscr{P}$ and for $\mathscr{U}, \mathscr{V} \in \mathscr{P}'$, we have*:

i) $x(x^{\ 1}\mathscr{U}) \quad \mathscr{U}, \quad x^{\ 1}(x\mathscr{U}) \quad \mathscr{U} \quad (\mathscr{U})_0\delta,$
ii) $x^{\ 1}(p\mathscr{U}) \quad p(x^{\ 1}\mathscr{U}) \quad \langle \mathscr{U}, \theta_0 p\rangle\delta,$
$x^{\ 1}(\mathscr{U}\mathscr{V}) \quad (x^{\ 1}\mathscr{U})\mathscr{V} \quad \mathscr{U}(x^{\ 1}\mathscr{V})$
iii) $\theta_a(\theta_b p) \quad \theta_b(\theta_a p), \quad \theta_a(\mathscr{U}p) \quad \mathscr{U}(\theta_a p),$
iv) $\mathscr{U}(pq) \quad (p\mathscr{U})q + xq\mathscr{U}\theta_0 p, \quad p(\mathscr{U}\mathscr{V}) \quad (p\mathscr{V})\mathscr{U} + x(\mathscr{V}\theta_0 p)\mathscr{U},$
v) $q(\mathscr{U}\theta_0 p) \quad \mathscr{U}\theta_0(qp) \qquad \theta_0[(p\mathscr{U})q], \quad (x^{\ 1}\mathscr{U})f \quad \mathscr{U}(\theta_0 f),$
vi) $S(\mathscr{U}\mathscr{V})(z) \qquad zS(\mathscr{U})(z)S(\mathscr{V})(z),$
vii) $S(x^{\ 1}\mathscr{U})(z) \quad (1/z)S(\mathscr{U})(z), \quad S(p\mathscr{U}) \quad pS(\mathscr{U}) + \mathscr{U}\theta_0 p,$
viii) $D_w S(\mathscr{U}) \quad S(D_w\mathscr{U}), \quad \mathscr{T}_w S(\mathscr{U}) \quad S(\mathscr{T}_w\mathscr{U}).$

Furthermore,

Lemma 2 *Let $\mathscr{U}$ be a linear functional, a a complex number, f a polynomial and $(P_n)_n$ a family of polynomials orthogonal with respect to $\mathscr{U}$. Then*

i) $x^{\ 1}\delta_a \quad (x \quad a)^{\ 1}\delta,$
ii) $(D_w f)(a \quad w) \quad (\theta_a f)(a \quad w),$
iii) $(\mathscr{U}\theta_\alpha f)(0) \quad (\mathscr{U}\theta_0 f)(a),$
iv) $D_w(\mathscr{U}f)(a \quad w) \quad (\mathscr{U}\theta_a f)(a \quad w),$
v) $\mathscr{U}\theta_0(fP_{n+1}) \quad fP_n^{(1)}, n+1 \geq \deg f,$
vi) $D_w(\mathscr{U}\theta_0 f) \quad (D_w\mathscr{U})\theta_0\mathscr{T}_w f + \mathscr{U}\theta_0 D_w f.$

Proof (i) We have $\langle x^{\ 1}\delta_a, x^n\rangle \quad \langle \delta_a, x^{n\ 1}\rangle \quad a^{n\ 1}, n \geq 1.$

On the other hand

$$\langle (x \quad a)^{\ 1}\delta, x^n\rangle \quad \langle \delta, \theta_a x^n\rangle \quad \left\langle \delta, \sum_{i=0}^{n\ 1} a^{n\ i\ 1} x^i \right\rangle \quad a^{n\ 1}, \quad n \geq 1.$$

Finally, for $n \quad 0$, $\langle (x \quad a)^{\ 1}\delta, 1\rangle \quad 0 \quad \langle x^{\ 1}\delta_a, 1\rangle$.

(ii) $(D_w f)(a \quad w) \quad D_w((x \quad a)(\theta_a f)(x) + f(a))(a \quad w)$

$$[(x + w \quad a)D_w(\theta_a f)(x) + \theta_a f)](a \quad w)$$

$$(\theta_a f)(a \quad w).$$

(iii) Using (i) and Lemma 1 we get

$$
\begin{aligned}
(\mathscr{U}\theta_a f)(0) \quad & \langle \delta, \mathscr{U}\theta_a f\rangle \\
& \langle \delta, \theta_a(\mathscr{U}f)\rangle \\
& \langle (x \quad a) \quad {}^1\delta, \mathscr{U}f\rangle \\
& \langle x \quad {}^1\delta_a, \mathscr{U}f\rangle \\
& \langle \delta_a, \theta_0(\mathscr{U}f)\rangle \\
& \langle \delta_a, \mathscr{U}(\theta_0 f)\rangle \\
& \mathscr{U}(\theta_0 f)(a).
\end{aligned}
$$

(iv) Let n and p be two nonnegative integers. We have

$$\delta^{(p)}x^n \quad \begin{cases} 0, & \text{if} \quad n < p, \\ (\quad 1)^p p! x^{n \quad p}, & \text{if} \quad n \geq p. \end{cases} \tag{9}$$

Since

$$D_w x^n \quad \sum_{i=0}^{n \quad 1} \binom{n}{i} w^{n \quad i \quad 1} x^i, \tag{10}$$

From Eqs. (9) and (10) we deduce that

$$D \quad {}_w(\delta^{(p)}x^n)(a) = \begin{cases} 0, & \text{if} \quad n \leq p, \\ (\quad 1)^p p! \displaystyle\sum_{i=0}^{n \quad p \quad 1} \binom{n \quad p}{i} (\quad w)^{n \quad p \quad i \quad 1} a^i, & \text{if} \quad n > p. \end{cases} \tag{11}$$

On the other hand,

$$\theta_a(\delta^{(p)}x^n)(a \quad w) = \begin{cases} 0, & \text{if} \quad n \leq p, \\ (\quad 1)^p p! \displaystyle\sum_{i=0}^{n \quad p \quad 1} a^{n \quad p \quad i \quad 1}(a \quad w)^i, & \text{if} \quad n > p. \end{cases} \tag{12}$$

The two relations

$$(a \quad w)^n \quad \sum_{i=0}^{n} \binom{n}{i} (\quad w)^{n \quad i} a^i, \qquad \binom{n+k+1}{k} \quad \sum_{i=0}^{k} \binom{n+i}{n} \tag{13}$$

after some changes of variable in the summations give for $n > p$,

$$\sum_{i=0}^{n \quad p \quad 1} \binom{n \quad p}{i} a^{n \quad p \quad i \quad 1} (a \quad w)^i \qquad \sum_{k=0}^{n \quad p \quad 1} \binom{n \quad p}{k} (\quad w)^{n \quad p \quad k \quad 1} a^k.$$

Therefore using the relation $(D \quad_w f)(a) \quad (D_w f)(a \quad w)$ we deduce

$$D_w(\delta^{(p)} x^n)(a \quad w) \quad \theta_a(\delta^{(p)} x^n)(a \quad w), \quad \forall n, p \in \mathcal{N}.$$

To conclude, we first introduce topologies in the spaces $\mathscr{P}$ and $\mathscr{P}'$ and use the continuity of some applications. To do this, we introduce in the vector space $\mathscr{P}$ the strict inductive limit topology of the vector spaces of polynomials of degree at most n, $\mathscr{P}_n$ i.e.

$$\mathscr{P}_n \subset \mathscr{P}_{n+1}, \quad n \geq 0, \quad \mathscr{P} \quad \bigcup_{n=0}^{\infty} \mathscr{P}_n.$$

Here P_n is endowed with its natural topology, which make it a Banach space. The dual $\mathscr{P}'$ of $\mathscr{P}$ is equipped with the topology defined by the system of semi-norms:

$$\|\mathscr{U}\|_n \quad \sup_{k \leq n} |(\mathscr{U})_k|, \quad \forall \mathscr{U} \in \mathscr{P}', \quad n \geq 0.$$

Since the applications $(\mathscr{U}, f) \rightarrow \mathscr{U} f$, $P \rightarrow D_w P$ and $P \rightarrow \theta_a P$ are continuous where $\mathscr{P}$ and $\mathscr{P}'$ are equipped with the topologies defined above [9,14], we use the decomposition [14]

$$\mathscr{U} \quad \sum_{n \geq 0} \frac{(\quad 1)^n}{n!} (\mathscr{U})_n \delta^{(n)}$$

to deduce that

$$D_w(\mathscr{U} x^n)(a \quad w) \quad \theta_a(\mathscr{U} x^n)(a \quad w).$$

Thus our result is valid for every polynomial f.

(v) We use the relation (iii) to get

$$\begin{aligned} \mathscr{U}\theta_0(fP_{n+1})(a) \quad & \mathscr{U}\theta_a(fP_{n+1})(0) \\ & \langle \delta, \mathscr{U}\theta_a(fP_{n+1})\rangle \\ & \langle \mathscr{U}, \theta_a(fP_{n+1})\rangle \\ & \langle \mathscr{U}, f(a)\theta_a P_{n+1} + P_{n+1}\theta_a f\rangle \\ & f(a)\langle \mathscr{U}, \theta_a P_{n+1}\rangle + \langle \mathscr{U}, P_{n+1}\theta_a f\rangle \\ & f(a)P_n^{(1)}(a), \quad \text{for } n+1 \geq \deg f, \end{aligned}$$

for every complex number a.

It should be mentioned that the relation (v) of Lemma 2 has already been established in Ref. [3] but with the condition $n \geq \deg G$. But since $n >$ s $\quad$ class$(\mathscr{U}) \Rightarrow n+1 \geq \deg G$, we need to check if this result is still valid for $n+1 \geq \deg G$. This checking allows us to obtain the extension mentioned above. Notice that here, the polynomial G is one of the five coefficients of the Eq. (14).

(vi) The proof follows from the relation [7,8]

$$D_w(fS(\mathscr{U}) \quad \mathscr{T}_w f D_w S(\mathscr{U}) + D_w f S(\mathscr{U})$$

and (vii) of Lemma 1. $\square$

CHARACTERIZATION OF D_W-LAGUERRE–HAHN LINEAR FUNCTIONALS

Definition 1 A linear functional $\mathscr{U}$ on the linear space $\mathscr{P}$ belongs to the D_w-Laguerre Hahn class if its Stieltjes function $S(\mathscr{U})$ satisfies a D_w-Riccati difference equation

$$\begin{aligned} \phi(z+w)D_w S(\mathscr{U})(z) \quad & G(z)S(\mathscr{U})(z)S(\mathscr{U})(z+w) + E(z)S(\mathscr{U})(z) \\ & + F(z)S(\mathscr{U})(z+w) + H(z), \end{aligned} \tag{14}$$

where ϕ, E, F, G, and H are polynomials with

$$\phi \neq 0, G \neq 0, \tag{15}$$

and

$$H(z) \quad \mathscr{U}\theta_0 E + (T_w\mathscr{U})\theta_0 F \quad (D_w\mathscr{U})\theta_0\mathscr{T}_w\phi \quad (\mathscr{U}\mathscr{T}_w\mathscr{U})\theta_0^2 G.$$

Remark 1 In Ref. [7], some examples of D_w-Laguerre Hahn polynomials are given in terms of classical discrete orthogonal polynomials. It should be noticed that the previous equation is not mentioned in previous works; therefore, definition 1 is more general and improve those given in Ref. [7 9].

Remark 2 When $G \quad 0$, the Stieltjes function $S(\mathscr{U})$ satisfies a linear difference equation

$$\phi(z+w)D_w S(\mathscr{U})(z) \quad C(z)S(\mathscr{U})(z) + D(z),$$

with $\phi \neq 0$.

The corresponding orthogonal polynomials are called affine D_w-- Laguerre Hahn orthogonal polynomials [7]. More precisely, they are D_w-semi-classical (see Ref. [9]).

THEOREM 3 *Let $\mathscr{U}$ be a regular linear functional and $(P_n)_n$ the corresponding sequence of monic orthogonal polynomials. The following statements are equivalen*t:

i) $\mathscr{U}$ belongs to the D_w-Laguerre Hahn class.

ii) $\mathscr{U}$ satisfies a D_w-functional equation

$$D_w(\phi\mathscr{U}) + G(x \quad ^1\mathscr{U}\mathscr{T}_w\mathscr{U}) + \psi\mathscr{U} \quad F\mathscr{T}_w\mathscr{U} \quad 0, \qquad (16)$$

with $\phi \neq 0$, $G \neq 0$, and $\psi \quad (D_w\phi + E)$, where the polynomial coefficients ϕ, E, F, G, and H are those given in the Eq. (14).

iii) $\mathscr{U}$ satisfies a D_w-functional equation

$$D_w((x \quad w)\phi\mathscr{U}) + G(\mathscr{U}\mathscr{T}_w\mathscr{U}) + (x\psi \quad \phi)\mathscr{U} \quad xF\mathscr{T}_w\mathscr{U} \quad 0, \quad (17)$$

with $\phi \neq 0$, $G \neq 0$, and

$$\langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G\rangle + \langle \mathscr{U}, \psi\rangle \quad \langle \mathscr{U}, \mathscr{T} \ _w F\rangle \quad 0. \qquad (18)$$

iv) There exists a sequence of complex numbers $\zeta_{n,j}$ such that

$$\phi D \ _w P_{n+1} + \mathscr{T} \ _w(FP_{n+1} \quad GP_n^{(1)}) = \sum_{j=n \ s}^{n+d} \zeta_{n,j}P_j, \quad n > s, \quad \zeta_{n,n \ s} \neq 0. \qquad (19)$$

Here, if $t \quad \deg \phi$, $r \quad \deg G$, $\rho \quad \max(\deg \psi, \deg F) \geq 1$ then $d \quad \max(r, t)$ and $s \quad \max(d \quad 2, \rho \quad 1)$.

Proof (i) $\Rightarrow$ (ii). Using the relations (v), (vii), and (viii) of Lemma 1 the Eq. (14) becomes

$$S(\mathscr{T}_w \phi D_w \mathscr{U}) + S(x \quad {}^1 G\mathscr{U}\mathscr{T}_w\mathscr{U}) \quad S(E\mathscr{U}) \quad S(F\mathscr{T}_w\mathscr{U})$$

$$D_w\mathscr{U}\theta_0\mathscr{T}_w\phi + (\mathscr{U}\mathscr{T}_w\mathscr{U})\theta_0^2 G \quad \mathscr{U}\theta_0 E \quad \mathscr{T}_w\mathscr{U}\theta_0 F + H.$$

On the other hand, taking into account the polynomial components in the previous relation we get

$$S(\mathscr{T}_w \phi D_w \mathscr{U}) + S(x \quad {}^1 G\mathscr{U}\mathscr{T}_w\mathscr{U}) \quad S(E\mathscr{U}) \quad S(F\mathscr{T}_w\mathscr{U}) \quad 0, \tag{20}$$

$$H \quad \mathscr{U}\theta_0 E + \mathscr{T}_w\mathscr{U}\theta_0 F \quad (D_w\mathscr{U})\theta_0\mathscr{T}_w\phi \quad (\mathscr{U}\mathscr{T}_w\mathscr{U})\theta_0^2 G. \tag{21}$$

From Eq. (20) and the relation (see [7])

$$D_w(f\mathscr{U}) \quad \mathscr{T}_w f D_w\mathscr{U} + D_w f\mathscr{U}, \quad f \in \mathscr{P}, \tag{22}$$

we get

$$D_w(\phi\mathscr{U}) + G(x \quad {}^1\mathscr{U}\mathscr{T}_w\mathscr{U}) + \psi\mathscr{U} \quad F\mathscr{T}_w\mathscr{U} \quad 0, \tag{23}$$

with $\psi \quad (D_w\phi + E)$.

(ii) $\Rightarrow$ (iii). First we multiply both members of Eq. (16) by x. Using the relations (22) and (i) of Lemma 1, we get

$$D_w((x \quad w)\phi\mathscr{U}) + G(\mathscr{U}\mathscr{T}_w\mathscr{U}) + (x\psi \quad \phi)\mathscr{U} \quad xF\mathscr{T}_w\mathscr{U} \quad 0.$$

Moreover, application of the relation (16) to the constant polynomial $p(x) \equiv 1$ gives $\langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G\rangle + \langle \mathscr{U}, \psi\rangle \quad \langle \mathscr{U}, \mathscr{T} \quad {}_w F\rangle \quad 0$.

(iii) $\Rightarrow$ (iv). Since $\phi D \quad {}_w P_{n+1} \quad \mathscr{T} \quad {}_w(FP_n + \mathscr{U}\theta_0(GP_{n+1}))$ is a polynomial of degree at most $n + d$ and $(P_n)_n$ constitutes a basis of $\mathscr{P}$, there exists a family of complex numbers $\zeta_{n,j}$ such that

$$\phi D \quad {}_w P_{n+1} \quad \mathscr{T} \quad {}_w(FP_n + \mathscr{U}\theta_0(GP_{n+1})) \quad \sum_{j=0}^{n+d} \zeta_{n,j} P_j. \tag{24}$$

According to Eq. (22), Eq. (17) is equivalent to

$$xD_w(\phi\mathscr{U}) + G(\mathscr{U}\mathscr{T}_w\mathscr{U}) + x\psi\mathscr{U} \quad xF\mathscr{T}_w\mathscr{U} \quad 0. \tag{25}$$

In a first step, we apply the previous equation to $\theta_0(P_{n+1}\mathscr{T}_w P_m)$ and we get

$$\langle x^{\;\;1}(xD_w(\phi\mathscr{U})), P_{n+1}\mathscr{T}_w P_m\rangle + \langle x^{\;\;1}(G(\mathscr{U}\mathscr{T}_w\mathscr{U})), P_{n+1}\mathscr{T}_w P_m\rangle$$

$$+\langle x^{\;\;1}(x\psi\mathscr{U}), P_{n+1}\mathscr{T}_w P_m\rangle \quad \langle x^{\;\;1}(xF\mathscr{T}_w\mathscr{U}), P_{n+1}\mathscr{T}_w P_m\rangle \quad 0. \tag{26}$$

In a second step we use the previous relation, Eq. (18), and relation (ii) of Lemma 1 to get

$$\langle D_w(\phi\mathscr{U}), P_{n+1}\mathscr{T}_w P_m\rangle + \langle G(x^{\;\;1}\mathscr{U}\mathscr{T}_w\mathscr{U}), P_{n+1}\mathscr{T}_w P_m\rangle$$

$$+\langle \psi\mathscr{U}, P_{n+1}\mathscr{T}_w P_m\rangle \quad \langle F\mathscr{T}_w\mathscr{U}, P_{n+1}\mathscr{T}_w P_m\rangle \quad 0. \tag{27}$$

In the third step, some straightforward computations lead to

$$\langle G(x^{\;\;1}\mathscr{U}\mathscr{T}_w\mathscr{U}), P_{n+1}\mathscr{T}_w P_m\rangle \quad \langle \mathscr{U}, P_m\mathscr{T}_{\;\;w}[\mathscr{U}\theta_0(GP_{n+1})]\rangle$$

$$+\langle \mathscr{U}, GP_{n+1}(\mathscr{T}_w\mathscr{U})\theta_0(\mathscr{T}_w P_m)\rangle. \tag{28}$$

Next, we use Eqs. (27) and (28) and the relations [7]

$$D_w(fg) \quad \mathscr{T}_w f D_w g + g D_w f, \quad D_{\;\;w}(\mathscr{T}_w f) \quad D_w f$$

to get

$$\langle \mathscr{U}, \phi P_m D_{\;\;w} P_{n+1}\rangle + \langle \mathscr{U}, \mathscr{T}_{\;\;w}(FP_{n+1})P_m\rangle$$

$$\langle \mathscr{U}, P_m\mathscr{T}_{\;\;w}[\mathscr{U}\theta_0(GP_{n+1})]\rangle$$

$$\langle \mathscr{U}, GP_{n+1}(\mathscr{T}_w\mathscr{U})\theta_0(\mathscr{T}_w P_m)\rangle$$

$$+\langle \mathscr{U}, \psi P_{n+1}\mathscr{T}_w P_m\rangle \quad \langle \mathscr{U}, \phi P_{n+1} D_w P_m\rangle. \tag{29}$$

Finally, we multiply both sides of Eq. (24) by P_m, then apply the linear functional $\mathscr{U}$ and use the orthogonality of $(P_n)_n$ with respect to $\mathscr{U}$. Thus, we get

$$\langle \mathscr{U}, \phi P_m D_{\;\;w} P_{n+1}\rangle + \langle \mathscr{U}, \mathscr{T}_{\;\;w}(FP_{n+1})P_m\rangle$$

$$\langle \mathscr{U}, P_m\mathscr{T}_{\;\;w}[\mathscr{U}\theta_0(GP_{n+1})]\rangle$$

$$\zeta_{n,m}\langle \mathscr{U}, P_m P_m\rangle. \tag{30}$$

The combination of the last two equations leads to

$$\zeta_{n,m}\langle \mathscr{U}, P_m P_m\rangle \quad \langle \mathscr{U}, P_{n+1}[G(\mathscr{T}_w \mathscr{U})\theta_0(\mathscr{T}_w P_m) + \psi \mathscr{T}_w P_m \quad \phi D_w P_m]\rangle. \tag{31}$$

Since $\deg(\psi \mathscr{T}_w P_m) \leq \rho + m \leq m + s + 1$, $\deg(\phi D_w P_m) \leq t + m$ $1 \leq m + d \quad 1 \leq m + s + 1$, and $\deg(G(\mathscr{T}_w \mathscr{U})\theta_0(\mathscr{T}_w P_m) \leq r + m$ $1 \leq m + d \quad 1 \leq m + s + 1$, we have

$$\deg(\psi \mathscr{T}_w P_m + G(\mathscr{T}_w \mathscr{U})\theta_0(\mathscr{T}_w P_m) \quad \phi D_w P_m) \leq m + s + 1. \tag{32}$$

Therefore we conclude from Eqs. (31) and (32) that for $n > s$, $\zeta_{n,m} \quad 0$ when $m < n \quad s$.

As a consequence

$$\phi D \;\, _w P_{n+1} + \mathscr{T} \;\, _w(F P_{n+1} \quad \mathscr{U}\theta_0(G P_{n+1})) \quad \sum_{j=n \;\; s}^{n+d} \zeta_{n,j} P_j, \quad n > s. \tag{33}$$

The relation $\zeta_{n,n \;\; s} \neq 0$ is obtained thanks to the regularity of $\mathscr{U}$.

Using the relation

$$\mathscr{U}\theta_0(G P_{n+1}) \quad G P_n^{(1)}, \quad n + 1 \geq \deg G, \tag{34}$$

(see v) in Lemma 2 we get Eq. (19).

(iv) ⇒ (i). Let us consider the linear functional $\mathscr{V}$ defined by

$$\mathscr{V} \quad D_w(\phi \mathscr{U}) + G(x \quad ^1 \mathscr{U} \mathscr{T}_w \mathscr{U}) \quad F \mathscr{T}_w \mathscr{U} + \left(\sum_{j=0}^{s+1} A_j x^j\right) \mathscr{U}, \tag{35}$$

where the coefficients A_j are complex numbers. We obtain

$$\langle \mathscr{V}, P_n\rangle \quad \langle \mathscr{U}, \quad \phi D \;\, _w P_n + \mathscr{T} \;\, _w(\mathscr{U}\theta_0(G P_n) \quad F P_{n \;\; 1})\rangle + \left\langle \mathscr{U}, P_n \sum_{j=0}^{s+1} A_j x^j \right\rangle.$$

From Eq. (19) the previous equation becomes

$$\langle \mathscr{V}, P_n\rangle \qquad \left\langle \mathscr{U}, \sum_{j=n \;\; s \;\; 1}^{n+d \;\; 1} \zeta_{n,j} P_j \right\rangle + \left\langle \mathscr{U}, P_n \sum_{j=0}^{s+1} A_j x^j \right\rangle.$$

From the orthogonality condition of (P_n) with respect to $\mathscr{U}$ we get $\langle \mathscr{V}, P_n \rangle \quad 0$ for $n > s+1$.

In order to get $\langle \mathscr{V}, P_n \rangle \quad 0$, $\forall n$, we shall choose coefficients A_j, $j \quad 0, \ldots, s+1$, such that $\langle \mathscr{V}, P_j \rangle \quad 0$, $j \quad 0, \ldots, s+1$.

These coefficients A_j are determined in a unique way.

Thus, we have deduced the existence of a polynomial

$$\psi \quad \sum_{j=0}^{s+1} A_j x^j$$

such that

$$\mathscr{V} \quad D_w(\phi\mathscr{U}) + G(x \quad ^1\mathscr{U}\mathscr{T}_w\mathscr{U}) \quad F\mathscr{T}_w\mathscr{U} + \psi\mathscr{U} \quad 0.$$

Then,

$$\mathscr{V} \quad 0 \Rightarrow S(D_w(\phi\mathscr{U}) + G(x \quad ^1\mathscr{U}\mathscr{T}_w\mathscr{U}) \quad F\mathscr{T}_w\mathscr{U} + \psi\mathscr{U}) \quad 0.$$

Taking into account Lemma 1 the last equation yields

$$\phi(z+w)D_wS(\mathscr{U})(z) \quad G(z)S(\mathscr{U})(z)S(\mathscr{U})(z+w) + E(z)S(\mathscr{U})(z) + F(z)S(\mathscr{U}) \times (z+w) + H(z),$$

with $E \quad (\psi + D_w\phi)$ and

$$H(z) \quad \mathscr{U}\theta_0 E + (T_w\mathscr{U})\theta_0 F \quad (D_w\mathscr{U})\theta_0\mathscr{T}_w\phi \quad (\mathscr{U}\mathscr{T}_w\mathscr{U})\theta_0^2 G. \qquad \square$$

THE CLASS OF A D_W-LAGUERRE–HAHN LINEAR FUNCTIONAL

In the distributional characterization of D_w-Laguerre Hahn functionals given in Eq. (16), there does not exist uniqueness in the representation for the polynomial coefficients. In fact, it is enough to multiply by any polynomial both members of the equation. On the other hand, uniqueness can be obtained if we assume a minimality condition as we will discuss below.

Definition 2 *Given a regular linear functional $\mathscr{U}$ satisfying*

$$D_w(\phi\mathscr{U}) + G(x \quad ^1\mathscr{U}\mathscr{T}_w\mathscr{U}) \quad F\mathscr{T}_w\mathscr{U} + \psi\mathscr{U} \quad 0, \tag{36}$$

with $\phi \neq 0$ and $G \neq 0$, we define the class of $\mathscr{U}$, which we will denote *class*($\mathscr{U}$), *as*

$$\text{class}(\mathscr{U}) = \min\{\max\{\max(\deg\psi, \deg F) - 1, \max(\deg\phi, \deg G) - 2\}\}, \tag{37}$$

where the minimum is taken among all polynomials ϕ, G, F, *and* ψ *satisfying Eq. (36).*

THEOREM 4 *Let* $\mathscr{U}$ *be a regular linear functional satisfying Eq. (36). Then the class of the* D_w*-Laguerre–Hahn linear functional* $\mathscr{U}$, *class*($\mathscr{U}$), *is equal to s with*

$$s = \max\{\max(\deg\psi, \deg F) - 1, \max(\deg\phi, \deg G) - 2\} \tag{38}$$

if and only if

$$\prod_{a \in Z_\phi} \{|\langle \mathscr{U}, \psi_{a-w}\rangle + \langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G_{a-w}\rangle - \langle \mathscr{T}_w\mathscr{U}, F_{a-w}\rangle| + |e_{a-w}| + |f_{a-w}| + |g_{a-w}|\} \neq 0, \tag{39}$$

where Z_ϕ *is the set of zeros of* ϕ. *The polynomials* ψ_{a-w}, G_{a-w} *and* F_{a-w} *as well as the complex numbers* e_{a-w}, f_{a-w} *and* g_{a-w} *are defined by the expressions*

$$\begin{aligned} \phi(x) &= (x-a)\phi_a(x), \quad \psi(x) + \phi_a(x) = (x+w-a)\psi_{a-w}(x) + e_{a-w}, \\ G(x) &= (x+w-a)G_{a-w}(x) + g_{a-w}, \\ F(x) &= (x+w-a)F_{a-w}(x) + f_{a-w}. \end{aligned} \tag{40}$$

Proof Let a be a zero of ϕ. From Lemma 1, Eqs. (22) and (40) we deduce that relation (36) is equivalent to

$$\begin{aligned} D_w(\phi_a\mathscr{U}) &+ G_{a-w}(x^{-1}\mathscr{U}\mathscr{T}_w\mathscr{U}) - F_{a-w}\mathscr{T}_w\mathscr{U} + \psi_{a-w}\mathscr{U} \\ &= (\langle \mathscr{U}, \psi_{a-w}\rangle + \langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G_{a-w}\rangle - \langle \mathscr{T}_w\mathscr{U}, F_{a-w}\rangle)\delta_{a-w} \\ &\quad - g_{a-w}(x+w-a)^{-1}(x^{-1}\mathscr{U}\mathscr{T}_w\mathscr{U}) - e_{a-w}(x+w-a)^{-1}\mathscr{U} \\ &\quad + f_{a-w}(x+w-a)^{-1}\mathscr{T}_w\mathscr{U}. \end{aligned} \tag{41}$$

If

$$\langle \mathscr{U}, \psi_{a \ \ w} \rangle + \langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G_{a \ \ w} \rangle \quad \langle \mathscr{T}_w\mathscr{U}, F_{a \ \ w} \rangle = e_{a \ \ w} = f_{a \ \ w} = g_{a \ \ w} = 0,$$

then $\mathscr{U}$ satisfies

$$D_w(\phi_a \mathscr{U}) + G_{a \ \ w}(x^{\ \ 1}\mathscr{U}\mathscr{T}_w\mathscr{U}) \quad F_{a \ \ w}\mathscr{T}_w\mathscr{U} + \psi_{a \ \ w}\mathscr{U} \quad 0. \tag{42}$$

Furthermore

$$\max\{\max(\deg \psi_{a \ \ w}, \deg F_{a \ \ w}) \quad 1, \max(\deg \phi_{a \ \ w}, \deg G_{a \ \ w}) \quad 2\}$$

$$\max\{\max(\deg \psi, \deg F) \quad 1, \max(\deg \phi, \deg G) \quad 2\} \quad 1.$$

Thus, we conclude that $\text{class}(\mathscr{U}) \le s \quad 1 < s$.

Conversely, we assume that $\mathscr{U}$ satisfies Eq. (42). We will prove that

$$\langle \mathscr{U}, \psi_{a \ \ w} \rangle + \langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G_{a \ \ w} \rangle \quad \langle \mathscr{T}_w\mathscr{U}, F_{a \ \ w} \rangle = e_{a \ \ w} = f_{a \ \ w} = g_{a \ \ w} = 0,$$

where a is a zero of ϕ.

Since Eq. (36) is equivalent to Eq. (41), we deduce

$$\begin{aligned} \mathscr{V} \quad & (\langle \mathscr{U}, \psi_{a \ \ w} \rangle + \langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G_{a \ \ w} \rangle \\ & \langle \mathscr{T}_w\mathscr{U}, F_{a \ \ w} \rangle)\delta_{a \ \ w} \quad g_{a \ \ w}(x + w \quad a)^{\ \ 1}(x^{\ \ 1}\mathscr{U}\mathscr{T}_w\mathscr{U}) \\ & e_{a \ \ w}(x + w \quad a)^{\ \ 1}\mathscr{U} + f_{a \ \ w}(x + w \quad a)^{\ \ 1}\mathscr{T}_w\mathscr{U} \quad 0. \end{aligned}$$

Then we get

$$\langle \mathscr{V}, 1 \rangle \quad 0 \Rightarrow \langle \mathscr{U}, \psi_{a \ \ w} \rangle + \langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G_{a \ \ w} \rangle \quad \langle \mathscr{T}_w\mathscr{U}, F_{a \ \ w} \rangle \quad 0.$$

$$\langle \mathscr{V}, x + w \quad a \rangle \quad 0 \Rightarrow e_{a \ \ w} \quad f_{a \ \ w}.$$

$$\langle \mathscr{V}, (x + w \quad a)^2 \rangle \quad 0 \Rightarrow g_{a \ \ w} \quad we_{a \ \ w}.$$

$$\langle \mathscr{V}, (x + w \quad a)^4 \rangle \quad 0 \Rightarrow we_{a \ \ w}((\mathscr{U})_1^2 \quad (\mathscr{U})_2) \quad 0.$$

Since $\mathscr{U}$ is regular and $w \neq 0$, from the last equation we get $e_{a \ \ w} \quad 0$. Thus, our statement follows.

We shall now establish an equivalent result to Theorem 4 where the condition about the class will be given in terms of polynomials ϕ, G, E, F, and H defined in Eq. (16). □

THEOREM 5 *Let $\mathscr{U}$ be a regular linear functional of the D_w-Laguerre Hahn class verifying Eq. (14). A necessary and sufficient condition for $\mathscr{U}$ to be of class s with*

$$s \quad \max\{\max(\deg\psi, \deg F) \quad 1, \max(\deg\phi, \deg G) \quad 2\}$$

is

$$\prod_{a\in Z_\phi}\{|G(a \quad w)| + |E(a \quad w)| + |F(a \quad w)| + |H(a \quad w)|\} \neq 0, \tag{43}$$

where Z_ϕ is the set of zeros of ϕ and

$$\phi(x) \quad (x \quad a)\phi_a(x), \quad \psi(x) + \phi_a(x) \quad (x+w \quad a)\psi_{a\ \ w}(x) + e_{a\ \ w},$$

$$G(x) \quad (x+w \quad a)G_{a\ \ w}(x) + g_{a\ \ w},$$

$$F(x) \quad (x+w \quad a)F_{a\ \ w}(x) + f_{a\ \ w}. \tag{44}$$

Proof Using the above relations and relation (ii) of Lemma 2 we get

$$e_{a\ \ w} \quad E(a \quad w), \quad g_{a\ \ w} \quad G(a \quad w), \quad f_{a\ \ w} \quad F(a \quad w).$$

First, using the relation (iii) of Lemma 2, we get

$$\langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G_{a\ \ w}\rangle \quad \langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0\theta_{a\ \ w}G\rangle \quad (\mathscr{U}\mathscr{T}_w\mathscr{U}\theta_0^2 G)(a \quad w),$$

$$\langle \mathscr{T}_w\mathscr{U}, F_{a\ \ w}\rangle \quad ((\mathscr{T}_w\mathscr{U})\theta_0 F)(a \quad w). \tag{45}$$

Second, using the relations (44), (ii), and (iii) in Lemma 2 we get

$$\begin{aligned}\langle \mathscr{U}, \psi_{a\ \ w}\rangle \quad & \langle \mathscr{U}, \theta_{a\ \ w}((x+w \quad a)\psi_{a\ \ w})\rangle \\ & \langle \mathscr{U}, \theta_{a\ \ w}(\psi + \phi_a)\rangle \\ & \langle \mathscr{U}, \theta_{a\ \ w}\psi\rangle + \langle \mathscr{U}, \theta_{a\ \ w}\phi_a\rangle \\ & (\mathscr{U}\theta_0\psi)(a \quad w) + \theta_a(\mathscr{U}\theta_0\phi)(a \quad w) \\ & (\mathscr{U}\theta_0\psi)(a \quad w) + D_w(\mathscr{U}\theta_0\phi)(a \quad w).\end{aligned} \tag{46}$$

Since $E \quad (\psi + D_w \phi)$, using Eq. (44), and (vi) of Lemma 2, we deduce

$$
\begin{aligned}
H(a \quad w) \quad & (\mathscr{U}\theta_0 E + \mathscr{T}_w \mathscr{U}\theta_0 F \quad (D_w\mathscr{U})\theta_0\mathscr{T}_w\phi \quad \mathscr{U}\mathscr{T}_w\mathscr{U}\theta_0^2 G)(a \quad w) \\
& (\quad \mathscr{U}\theta_0\psi \quad \mathscr{U}\theta_0 D_w\phi \quad (D_w\mathscr{U})\theta_0\mathscr{T}_w\phi + \mathscr{T}_w\mathscr{U}\theta_0 F \\
& \quad \mathscr{U}\mathscr{T}_w\mathscr{U}\theta_0^2 G)(a \quad w) \\
& (\mathscr{U}\theta_0\psi + D_w(\mathscr{U}\theta_0\phi) \quad \mathscr{T}_w\mathscr{U}\theta_0 F + \mathscr{U}\mathscr{T}_w\mathscr{U}\theta_0^2 G)(a \quad w).
\end{aligned}
\tag{47}
$$

From Eqs. (45) (47) we conclude that

$$
H(a \quad w) \quad \langle \mathscr{U}, \psi_{a \quad w}\rangle + \langle \mathscr{U}\mathscr{T}_w\mathscr{U}, \theta_0 G_{a \quad w}\rangle \quad \langle \mathscr{T}_w\mathscr{U}, F_{a \quad w}\rangle,
$$

and our result follows from Theorem 4. □

COROLLARY 6

i) *If for every zero a of ϕ,*

$$
|G(a \quad w)| + |E(a \quad w)| + |F(a \quad w)| + |H(a \quad w)| \neq 0,
$$

then Eq. (14) as well as Eq. (16) cannot be simplified. They are said to be not reducible.

ii) *If there exists a zero a of ϕ such that*

$$
G(a \quad w) \quad E(a \quad w) \quad F(a \quad w) \quad H(a \quad w) \quad 0,
$$

then Eq. (14) as well as Eq. (16) can be simplified. They are said to be reducible with respect to the zero a of ϕ.

More precisely, after simplifications Eqs. (14) and (16) become, respectively, (with $S \quad S(\mathscr{U})$):

$$
\begin{aligned}
\phi_a(z+w)D_w S(z) \quad & G_{a \quad w}(z)S(z)S(z+w) + E_{a \quad w}(z)S(z) \\
& + F_{a \quad w}(z)S(z+w) + H_{a \quad w}(z),
\end{aligned}
$$

$$
D_w(\phi_a\mathscr{U}) + G_{a \quad w}(x^{\quad 1}\mathscr{U}\mathscr{T}_w\mathscr{U}) \quad F_{a \quad w}\mathscr{T}_w\mathscr{U} + \psi_{a \quad w}\mathscr{U} \quad 0,
$$

with $H(z) \quad (x + w \quad a)H_{a \quad w}(z)$.

iii) *According to the above proposition in order to obtain the class of a D_w-Laguerre Hahn linear functional, one must simplify the Riccati* *difference equation satisfied by the Stieltjes function of this linear functional, and deduce the class when the simplification is not more possible*.

APPLICATIONS

In this section, we shall determine the class of some perturbations of a D_w-Laguerre Hahn functional.

The Co-recursive of The D_w-Laguerre–Hahn Functionals

Let μ be a complex number and $\mathscr{U}$ a regular linear functional. Then we have:

PROPOSITION 7 *If $\mathscr{U}$ is a D_w-Laguerre Hahn linear functional of class s, then $\mathscr{U}^{[\mu]}$ is a D_w-Laguerre Hahn linear functional of the same class, s.*

Proof We use Eq. (14) satisfied by the Stieltjes function S($\mathscr{U}$) S of $\mathscr{U}$ and the relation linking S($\mathscr{U}$) and S($\mathscr{U}^{[\mu]}$) S_μ [17]

$$S(\mathscr{U}^{[\mu]}) \quad \frac{S(\mathscr{U})}{1+\mu S(\mathscr{U})}$$

to get

$$\begin{aligned}\phi(z+w)D_w S_\mu(z) \quad & G^*(z)S_\mu(z)S_\mu(z+w)+E^*(z)S_\mu(z)\\ &+F^*(z)S_\mu(z+w)+H^*(z),\end{aligned} \tag{48}$$

with

$$\begin{aligned}&G^*(z) \quad G(z) \quad \mu(E(z)+F(z))+\mu^2 H(z), \quad E^*(z) \quad E(z) \quad \mu H(z)\\ &F^*(z) \quad F(z) \quad \mu H(z), \quad H^*(z) \quad H(z).\end{aligned} \tag{49}$$

$\mathscr{U}^{[\mu]}$ is then a D_w-Laguerre Hahn linear functional.

With respect to the class, we use results of Theorem 5 and get for every zero a of ϕ:

If $H(a-w)\neq 0$, then $H^*(a-w)\neq 0$ and Eq. (48) is not reducible.

We suppose that $H(a-w)=0$. Then if $E(a-w)\neq 0$ or $F(a-w)\neq 0$,

$$|E^*(a-w)|+|F^*(a-w)|=|E(a-w)|+|F(a-w)|\neq 0$$

and Eq. (48) is still not reducible.

If $H(a-w)=E(a-w)=F(a-w)=0$, then $G^*(a-w)=G(a-w)\neq 0$. From Theorem 5, we conclude that

$$|G^*(a-w)|+|E^*(a-w)|+|F^*(a-w)|+|H^*(a-w)|\neq 0. \qquad \square$$

Addition of a Dirac Mass to a D_w-Laguerre–Hahn Linear Functional

Let $\mathscr{U}$ be a D_w-Laguerre–Hahn linear functional, μ and c two complex numbers. The linear functional $\mathscr{V}=\mathscr{U}+\mu\delta_c$ is regular up to a countable set of values of μ [15].

PROPOSITION 8 *Let $\mathscr{U}$ be a regular D_w-Laguerre–Hahn linear functional and $\mathscr{V}=\mathscr{U}+\mu\delta_c$, $\mu\neq 0$. Then we have:*

i) $\mathscr{V}$ is a D_w-Laguerre–Hahn linear functional.
ii) If $\mathscr{V}$ is regular, then the class $\bar{s}$ of $\mathscr{V}$ satisfies

$$s-2\leq \bar{s}\leq s+2, \tag{50}$$

where s is the class of $\mathscr{U}$.

Proof We assume that the Stieltjes function $S(\mathscr{U})=S$ satisfies Eq. (14).

In the first step, using the relation $\langle \mathscr{V},x^n\rangle=\langle \mathscr{U}+\mu\delta_c,x^n\rangle=(\mathscr{U})_n+\mu c^n, n\geq 0$, we get (see Ref. [11] $S(\mathscr{U})=S(\mathscr{V})+(\mu/x-c)$.

Next, from last relation and Eq. (14) we get that $S(\mathscr{V})=\bar{S}$ satisfies a D_w-Riccati difference equation

$$\bar{\phi}(z+w)D_w\bar{S}(z)=\bar{G}(z)\bar{S}(z)\bar{S}(z+w)+\bar{E}(z)\bar{S}(z)$$
$$+\bar{F}(z)\bar{S}(z+w)+\bar{H}(z), \tag{51}$$

with

$$
\begin{aligned}
\bar{\phi}(x) &= (x-c-w)(x-c)\phi(x), \quad \bar{G}(x) = (x-c)(x+w-c)G(x),\\
\bar{E}(x) &= (x-c)(x+w-c)E(x) + \mu(x-c)G(x),\\
\bar{F}(x) &= (x-c)(x+w-c)F(x) + \mu(x+w-c)G(x),\\
\bar{H}(x) &= (x-c)(x+w-c)H(x) + \mu(x+w-c)E(x) + \mu(x-c)F(x)\\
&\quad + \mu^2 G(x) - \mu\phi(x+w).
\end{aligned}
\tag{52}
$$

From Theorem 3, the linear functional $\mathscr{V}$ satisfies

$$
D_w(\bar{\phi}U) + \bar{G}(x^{-1}UT_wU) - \bar{F}\mathscr{T}_wU + \bar{\psi}U = 0, \tag{53}
$$

with $\bar{\psi} = -(\bar{E} + D_w\bar{\phi})$.

Second, we assume that $\mathscr{U}$ is regular as well as $s = \max(d-2, p-1)$ where $d = \max(\deg G, \deg \phi)$ and $p = \max(\deg \psi, \deg F) \geq 1$. Using the inequalities $d \leq s+2$ and $p \leq s+1$ we get

$$
\begin{aligned}
\bar{d} &= \max(\deg \bar{G}, \deg \bar{\phi}) \leq d+2 \leq s+4,\\
\deg(\bar{F}) &\leq \max(\deg F + 2, \deg(G)+1) \leq \max(p+2, d+1) \leq s+3,\\
\deg(\bar{E}) &\leq \max(\deg E + 2, \deg(G)+1) \leq \max(p+2, d+1) \leq s+3,\\
\deg \bar{\psi} &\leq \max(\deg \bar{E}, \deg \bar{\phi} - 1) \leq \max(\deg \bar{E}, \bar{d} - 1) \leq s+3.
\end{aligned}
$$

Then we conclude that $\bar{p} = \max(\deg \bar{\psi}, \deg \bar{F}) \leq s+3$ and, finally,

$$
\bar{s} = \max(\bar{d}-2, \bar{p}-1) \leq s+2.
$$

On the second hand, since $\mathscr{U} = \mathscr{V} - \mu\delta_c$ we have $s \leq \bar{s}+2$. □

Proposition 9 *Let $\mathscr{U}$ be a D_w-Laguerre Hahn linear functional satisfying Eq. (14). Then for every zero a of ϕ different from c and $c+w$, Eq. (53) satisfied by $\mathscr{V} = \mathscr{U} + \mu\delta_c$ is not reducible with respect to a.*

Proof We assume that Eq. (14) is not reducible with respect to every zero a of ϕ. Let a be a zero of ϕ different from c and $c-w$. From Eq. (52), we have:

If $G(a \quad w) \neq 0$, then $\bar{G}(a \quad w) \quad (a \quad w \quad c)(a \quad c)G(a \quad w) \neq 0$. Equation (51) is therefore not reducible with respect to a.

We suppose that $G(a \quad w) \quad 0$. If $E(a \quad w) \neq 0$ or $F(a \quad w) \neq 0$, then $\bar{E}(a \quad w) \quad (a \quad w \quad c)(a \quad c)E(a \quad w) \neq 0$ or $\bar{F}(a \quad w) \quad (a \quad w \quad c)(a \quad c)F(a \quad w) \neq 0$. Again, Eq. (51) is not reducible.

We suppose that $G(a \quad w) \quad E(a \quad w) \quad F(a \quad w) \quad 0$. Since $|G(a \quad w)| + |E(a \quad w)| + |F(a \quad w)| + |H(a \quad w)| \neq 0$, we deduce that $H(a \quad w) \neq 0$.

Then, $\bar{H}(a \quad w) \quad (a \quad w \quad c)(a \quad c)H(a \quad w) \neq 0$. Equation (51) is still not reducible with respect to a and our statement follows. $\square$

Next, we analyze the class of the functional $\mathscr{V}$ when $\mathscr{U}$ is the first associated of the classical orthogonal polynomial of a discrete variable. We state the following known result [8]:

LEMMA 10 *If $\mathscr{U}$ is a classical regular linear functional satisfying the functional equation $\Delta(\phi\mathscr{U}) \quad \psi\mathscr{U}$ where ϕ is a polynomial of degree at most two and ψ a polynomial of degree one, the first associated $\mathscr{U}^{(1)} \quad \mathscr{U}_1$ of $\mathscr{U}$ is a Δ-Laguerre Hahn linear functional. $\mathscr{U}_1$ and the Stieltjes function $S(\mathscr{U}_1) \quad S_1$ satisfy, respectively, the following functional and Riccati difference equation*

$$\Delta(\phi\mathscr{U}_1) + G_1(x \quad {}^1\mathscr{U}_1\mathscr{T}\mathscr{U}_1) + \psi_1\mathscr{U}_1 \quad F_1\mathscr{T}\mathscr{U}_1 \quad 0,$$

$$\phi(x+1)\Delta S_1(x) \quad G_1(x)S_1(x)S_1(x+1) + E_1(x)S_1(x)$$
$$+ F_1(x)S_1(x+1) + H_1(x),$$

where

$$G_1(x) \quad \psi' \quad \phi''/2, \quad E_1(x) \quad (\psi' \quad \phi''/2)\left(x+1+\frac{\psi(0)}{\psi'}\right),$$

$$F_1(x) \quad (\psi' \quad \phi''/2)\left(x+\frac{\psi(0)}{\psi'}\right) \quad \psi(x) + \Delta\phi(x),$$

$$\psi_1(x) \quad (E_1(x) + \Delta\phi(x)),$$

$$H_1(x) \quad \phi(x+1) \quad \left(x+1+\frac{\psi(0)}{\psi'}\right)(\psi(x) \quad \Delta\phi(x))$$
$$+ (\psi' \quad \phi''/2)\left(x+\frac{\psi(0)}{\psi'}\right)\left(x+1+\frac{\psi(0)}{\psi'}\right). \tag{54}$$

In the following table, we give the above coefficients for the first associated Charlier, Meixner, Krawtchouk and Hahn [16].

	ϕ	ψ	G_1	E_1	F_1	H_1	ψ_1
Charlier	x	$\lambda \quad x, \lambda > 0$	1	$\lambda \quad x \quad 1$	1	λ	$x \quad \lambda$
Meixner	x	$(\lambda \quad 1)x + \lambda\nu, 0 \leq \lambda \leq 1, \nu > 0$	$\lambda \quad 1$	$(\lambda \quad 1)x + \lambda \quad 1 + \lambda\nu$	1	$\lambda\nu/\lambda \quad 1$	$(1 \quad \lambda)x \quad \lambda(\nu \quad 1)$
Krawtchouk	x	$(1/q)((1 \quad q)N \quad x), 0 \leq q \leq 1$	$(1/q)$	$(x/q) \quad ((1 \quad N + Nq)/(q))$	1	$N(q \quad 1)$	$(x/q) + ((q \quad 1)(N \quad 1))/q$

For the first associated Hahn polynomials we have:

$$\phi(x) = x(N + \alpha - x), \quad \psi(x) = (\beta + 1)(N - 1) - (\alpha + \beta + 2)x, \quad \alpha > -1, \quad \beta > -1,$$

$$G_1(x) = -(\alpha + \beta + 1), E_1(x) = -(\alpha + \beta + 1)x - \frac{(\beta + 1 - \beta(N - 1) + \alpha)(\alpha + \beta + 1)}{\alpha + \beta + 2},$$

$$F_1(x) = -x + \frac{2N + \alpha + N\alpha + N\beta + \alpha^2 + \alpha\beta - 3 - \beta - \beta(N - 1)}{\alpha + \beta + 2},$$

$$H_1(x) = -\frac{(\beta(N - 1) + 1)(-\beta(N - 1) + 2N + 2\alpha + \alpha^2 + \alpha\beta - 1 + N\alpha + N\beta)}{(\alpha + \beta + 2)^2},$$

$$\psi_1(x) = (\alpha + \beta + 3)x - \frac{(2N - \alpha - \alpha\beta - \beta^2 + \beta(N - 1)\alpha + \beta(N - 1)\beta + N\alpha + N\beta + \beta(N - 1) - 3 - 3\beta)}{\alpha + \beta + 2}.$$

According to the definition, and from the above coefficients, the class of the Laguerre Hahn regular linear functional $\mathscr{U}_1$ is $s \quad$ class$(\mathscr{U}_1) \quad 0$.

We shall now study the class of the functional $\mathscr{V}$ such that $\mathscr{V} \quad \mathscr{U}_1 + \mu\delta_c$ (with $\mu \neq 0$). It is known from the Proposition 8 that the Stieltjes function $S(\mathscr{V}) \quad \bar{S}$ satisfies the equation

$$\bar{\phi}(z + 1)\Delta\bar{S}(z) \quad \bar{G}(z)\bar{S}(z)\bar{S}(z + 1) + \bar{E}(z)\bar{S}(z) + \bar{F}(z)\bar{S}(z + 1) + \bar{H}(z), \quad (55)$$

with

$$\bar{\phi}(x) \quad (x \quad c \quad 1)(x \quad c)\phi(x), \quad \bar{G}(x) \quad (x \quad c)(x + 1 \quad c)G_1(x),$$

$$\bar{E}(x) \quad (x \quad c)(x + 1 \quad c)E_1(x) + \mu(x \quad c)G_1$$

$$\bar{F}(x) \quad (x \quad c)(x + 1 \quad c)F_1(x) + \mu(x + 1 \quad c)G_1(x),$$

$$\bar{H}(x) \quad (x \quad c)(x + 1 \quad c)H_1(x) + \mu(x + 1 \quad c)E_1(x) + \mu(x \quad c)F_1(x) + \mu^2 G_1(x) \quad \mu\phi(x + 1).$$

Taking into account the fact that for the four families of D_w-classical polynomials (Charlier, Meixner, Krawtchouk and Hahn)

$$(|\bar{E}(c)| + |\bar{F}(c)|)(|\bar{E}(c - 1)| + |\bar{F}(c - 1)|) \neq 0,$$

and does not depend of c, from Proposition 8 we deduce that when $\mathscr{U}_1$ is the first associated classical orthogonal polynomial of a discrete variable, Eq. (55) is not reducible and the class $\bar{s}$ of $\mathscr{V} = \mathscr{U}_1 + \mu\delta_c$ is given by

$$\bar{s} = \max(\max(\deg \bar{G}, \deg \bar{\phi}) - 2, \quad \max(\deg \bar{\psi}, \deg \bar{F}) - 1) - 2,$$

where the polynomial $\bar{\psi}$ is defined by $\bar{\psi}(x) = -(\bar{E}(x) + \Delta\bar{\phi}(x))$.

Study of the Linear Functional $\mathscr{U}$ Such That $(x - c)\mathscr{U} = \mu\mathscr{W}$, Where the $\mathscr{W}$ Is a D_w-Laguerre–Hahn Linear Functional

PROPOSITION 11 *Let $\mathscr{U}$ and $\mathscr{W}$ be two linear functionals related by $(x - c)\mathscr{U} = \mu\mathscr{W}$, where μ and c are complex numbers. Then we have,*

i) *$\mathscr{W}$ is a D_w-Laguerre–Hahn linear functional if and only if $\mathscr{U}$ is a D_w-Laguerre–Hahn linear functional.*
ii) *If $\mathscr{W}$ is a D_w-Laguerre–Hahn linear functional of class s, then the class $\tilde{s}$ of $\mathscr{U}$ satisfies $s - 1 \leq \tilde{s} \leq s + 2$.*

Proof (i) If $\mathscr{W}$ is a D_w-Laguerre–Hahn regular linear functional and $S(\mathscr{W}) = S$ is the corresponding Stieltjes function satisfying Eq. (14), then under certain conditions concerning c and μ, $\mathscr{U}$ is regular [13].

From the relation $\mu(\mathscr{W})_n = (\mathscr{U})_{n+1} - c(\mathscr{U})_n$, we get the link between S and the Stieltjes function $\tilde{S} = S(\mathscr{U})$ of $\mathscr{U}$

$$S(x) = \frac{1}{\mu}((x - c)\tilde{S}(x) + 1). \tag{56}$$

Substitution of Eq. (56) in Eq. (14) allows to conclude that $\tilde{S}$ satisfies

$$\tilde{\phi}(x + w)D_w\tilde{S}(x) = \tilde{G}(z)\tilde{S}(x)\tilde{S}(x + w) + \tilde{E}(x)\tilde{S}(x) + \tilde{F}(x)\tilde{S}(x + w) + \tilde{H}(x), \tag{57}$$

with

$$\begin{aligned}
\tilde{\phi}(x) &\quad \mu(x \quad c)\phi(x),\\
\tilde{G}(x) &\quad (x \quad c)(x+w \quad c)G(x),\\
\tilde{H}(x) &\quad G(x)+\mu(E(x)+F(x))+\mu^2 H(x),\\
\tilde{E}(x) &\quad (x \quad c)(G(x)+\mu E(x)) \quad \mu\phi(x+w),\\
\tilde{F}(x) &\quad (x+w \quad c)(G(x)+\mu F(x)).
\end{aligned} \tag{58}$$

Moreover, $\mathscr{U}$ satisfies

$$D_w(\tilde{\phi}\mathscr{U})+\tilde{G}(x \quad {}^1\mathscr{U}\mathscr{T}_w\mathscr{U}) \quad \tilde{F}\mathscr{T}_w\mathscr{U}+\tilde{\psi}\mathscr{U} \quad 0, \tag{59}$$

with $\tilde{\psi} \quad (\tilde{E}+D_w\tilde{\phi})$.

Conversely, from the relation (56), we deduce that if $\tilde{S} \quad S(\mathscr{U})$ satisfies

$$\tilde{\phi}(z+w)D_w\tilde{S}(z) \quad \tilde{G}(z)\tilde{S}(z)\tilde{S}(z+w)+\tilde{E}(z)\tilde{S}(z)+\tilde{F}(z)\tilde{S}(z+w)+\tilde{H}(z),$$

then $S \quad S(\mathscr{W})$ satisfies

$$\phi(z+w)D_wS(z) \quad G(z)S(z)S(z+w)+E(z)S(z)+F(z)S(z+w)+H(z)$$

with

$$\begin{aligned}
\phi(x) &\quad \mu(x \quad c)\tilde{\phi}(x),\\
E(x) &\quad \mu\tilde{\phi}(x+w) \quad \mu G(x)+\mu(x+w \quad c)\tilde{E}(x),\\
F(x) &\quad\quad \mu\tilde{G}(x)+\mu(x \quad c)\tilde{F}(x), \quad G(x) \quad \mu^2\tilde{G}(x),\\
H(x) &\quad\quad \tilde{\phi}(z+w)+\tilde{G}(x) \quad (x+w \quad c)\tilde{E}(x)\\
&\quad\quad (x \quad c)\tilde{F}(x)+(x \quad c)(x+w \quad c)\tilde{H}(x).
\end{aligned} \tag{60}$$

(ii) Secondly, we assume that $s \quad \max(d \quad 2, p \quad 1)$ where $d \quad \max(\deg G, \deg\phi)$ and $p \quad \max(\deg\psi, \mathrm{degs}\, F) \geq 1$. Using the inequalities $d \leq s+2$ and $p \leq s+1$ we get

$$\tilde{d} \quad \max(\deg \tilde{G}, \deg \tilde{\phi}) \le d + 2 \le s + 4,$$

$$\deg(\tilde{F}) \le \max(\deg F + 1, \deg(G) + 1) \le \max(p + 1, d + 1) \le s + 3,$$

$$\deg(\tilde{E}) \le \max(\deg E + 1, \deg(G) + 1) \le \max(p + 1, d + 1) \le s + 3,$$

$$\deg \tilde{\psi} \le \max(\deg \tilde{E}, \deg \tilde{\phi} \quad 1) \le \max(\deg \tilde{E}, \tilde{d} \quad 1) \le s + 3.$$

Then $\tilde{p} \quad \max(\deg \tilde{\psi}, \deg \tilde{F}) \le s + 3$ and, finally,

$$\tilde{s} \quad \max(\tilde{d} \quad 2, \tilde{p} \quad 1) \le s + 2.$$

Secondly, Eq. (60) allows us to conclude (using the same process as above) that if $\mathscr{U}$ is a D_w-Laguerre Hahn linear functional of class $\tilde{s}$, with $\tilde{s} \quad \max(\tilde{d} \quad 2, \tilde{p} \quad 1)$, $\tilde{d} \quad \max(\deg \tilde{G}, \deg \tilde{\phi})$ and $\tilde{p} \quad \max(\deg \tilde{\psi}, \deg \tilde{F})$, then the class s of $\mathscr{W}$ defined by $s \quad \max(d \quad 2, p \quad 1)$ where $d \quad \max(\deg G, \deg \phi)$ and $p \quad \max(\deg \psi, \deg F) \ge 1$ verifies $s \le \tilde{s} + 1$. Therefore we conclude that $s \quad 1 \le \tilde{s} \le s + 2$. □

PROPOSITION 12 *Let $\mathscr{W}$ be a D_w-Laguerre Hahn linear functional and $S(\mathscr{W}) \quad S$ the corresponding Stieltjes function satisfying Eq. (14). Then Eq. (57) satisfied by the Stieltjes function $\tilde{S}$ of $\mathscr{U}$, where $(x \quad c)\mathscr{U} \quad \mu\mathscr{W}$, is not reducible with respect to any zero a of ϕ different from c and $c + w$.*

Proof Let a be a zero of ϕ different from c and $c + w$. We have:

If $G(a \quad w) \ne 0$, then $\tilde{G}(a \quad w) \quad (a \quad c)(a \quad c \quad w)G(a \quad w) \ne 0$ and Eq. (57) is not reducible.

If $G(a \quad w) \quad 0$ and $E(a \quad w) \ne 0$ or $F(a \quad w) \ne 0$, then $\tilde{E}(a \quad w) \quad \mu(a \quad w \quad c) \times E(a \quad w) \ne 0$ or $\tilde{F}(a \quad w) \quad \mu(a \quad c)F(a \quad w) \ne 0$. Thus, Eq. (57) is not reducible with respect to a.

Finally if $G(a \quad w) \quad E(a \quad w) \quad F(a \quad w) \quad 0$, then $\tilde{H}(a \quad w) \quad \mu^2 H(a \quad w) \ne 0$ since $|G(a \quad w)| + |E(a \quad w)| + |F(a \quad w)| + |H(a \quad w)| \ne 0$. □

Examples

Here we suppose that $\mathscr{W} \quad \mathscr{U}^{(1)} \quad \mathscr{U}_1$ and analyze the class of the Δ-Laguerre Hahn linear functional $\mathscr{U}$ defined by $(x \quad c)\mathscr{U} \quad \mu\mathscr{W}$, with $\mu \ne 0$. According to the Proposition 11, $\mathscr{W}$ and the Stieltjes function

$S(\mathscr{W}) \quad \tilde{S}$ of $\mathscr{W}$ satisfy, respectively, the functional and the Riccati difference equation

$$\Delta(\tilde{\phi}\mathscr{W}) + \tilde{G}(x \quad {}^{1}\mathscr{W}\mathscr{T}\mathscr{W}) \quad \tilde{F}\mathscr{T}\mathscr{W} + \tilde{\psi}\mathscr{W} \quad 0,$$

$$\tilde{\phi}(x+1)\Delta\tilde{S}(x) = \tilde{G}(x)\tilde{S}(x)\tilde{S}(x+1) + \tilde{E}(x)\tilde{S}(x) + \tilde{F}(x)\tilde{S}(x+1) + \tilde{H}(x), \quad (61)$$

with

$$\tilde{\phi}(x) \quad \mu(x \quad c)\phi(x),$$

$$\tilde{G}(x) \quad (x \quad c)(x+1 \quad c)G_1(x),$$

$$\tilde{H}(x) \quad G_1(x) + \mu(E_1(x) + F_1(x)) + \mu^2 H_1(x),$$

$$\tilde{E}(x) \quad (x \quad c)(G_1(x) + \mu E_1(x)) \quad \mu\phi(x+1),$$

$$\tilde{F}(x) \quad (x+1 \quad c)(G_1(x) + \mu F_1(x)),$$

$$\tilde{\psi}(x) \qquad (\tilde{E}(x) + \Delta\tilde{\phi}(x)),$$

where the coefficients E_1, F_1, G_1, and H_1 are those given by Eq. (54).

Taking into account the fact that for the three families of polynomials (Charlier, Meixner, Krawtchouk)

$$(|\tilde{E}(c)| + |\tilde{F}(c)|)(|\tilde{E}(c \quad 1)| + |\tilde{F}(c \quad 1)|) \neq 0,$$

and it does not depend on c, we use Proposition 12 to deduce that when $\mathscr{U}_1$ is the first associated classical orthogonal polynomial of a discrete variable (Charlier, Meixner, Krawtchouk), Eq. (61) is not reducible and the class $\bar{s}$ of $\mathscr{U}$, defined by $(x \quad c)\mathscr{U} \quad \mu\mathscr{W}$ is

$$\tilde{s} \quad \max(\max(\deg\tilde{G}, \deg\tilde{\phi}) \quad 2, \max(\deg\tilde{\psi}, \deg\tilde{F}) \quad 1) \quad 1,$$

where

$$\tilde{\psi}(x) \qquad (\tilde{E}(x) + \Delta\tilde{\phi}(x)).$$

CONCLUDING REMARKS

i) Since $\lim_{w\to 0} D_w \quad \mathscr{D}$, we recover, by a limit process ($w \to 0$) and from the results obtained in this paper, the results already known for the Laguerre Hahn orthogonal polynomials of a continuous variable [11,17]. In this case the polynomial $E+F$ is replaced by the polynomial C of the Riccati differential equation satisfied by S (see

Eqs. (4) (14)):

$$\phi S' \quad BS^2 + CS + D.$$

ii) In this paper we have given a characterization theorem for Laguerre Hahn orthogonal polynomials of a discrete variable, extending some previous work by Marcellán and Prianes [11,17] for Laguerre Hahn orthogonal polynomials of a continuous variable to the Laguerre Hahn orthogonal polynomials of a discrete variable. We have used this characterization to define the notion of the class of the Laguerre Hahn orthogonal polynomials of a discrete variable and we have analyzed the class of regular linear functionals obtained by some perturbations of a given Laguerre Hahn linear functional of a discrete variable. We have proved that the first associated classical discrete orthogonal polynomials are Δ-Laguerre Hahn orthogonal polynomials of class $s \quad 0$. Finally, the addition of a Dirac delta functionals mass to the first associated of a classical regular linear functional of a discrete variable gives a Laguerre Hahn linear functional of class at most $s \quad 2$. Notice that the above cases are examples of homographic transformations in a Riccati equation. This equation is invariant under such a kind of transformatons.

Acknowledgements

The work of the second author (FM) was supported by Ministerio de Ciencia y Tecnología (Dirección General de Investigación) of Spain under grant BFM 2000-0206-C04-01 and the INTAS project INTAS 2000-272. We also express our gratitude to N.G. ANDJIGA and the anonymous referees; the first for providing computer facilities to the first author (MF) and the second for their comments and suggestions in order to improve the manuscript.

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