Second kind functionals for the Laguerre-Hahn affine class on the unit circle

A. Cachafeiro* F. Marcellán† C. Pérez*

Abstract

The aim of this paper is the study of some transformations in the Laguerre-Hahn affine class that do not preserve the class. Under very general conditions, we establish that the second kind functional associated with a Laguerre-Hahn affine functional does not belong to the Laguerre-Hahn affine class. The transformations related to the associated polynomials and quadratic decomposition of a sequence of orthogonal polynomials are also considered.

1 Introduction

Semiclassical linear functionals have been introduced as an extension of linear functionals associated with a weight function w supported on the real line satisfying a Pearson equation $(\phi w)' = \psi w$ where ϕ, ψ are polynomials and $\deg \psi \geq 1$. Some extra conditions concerning the behavior of $x^n w$ in the end points of the support of w are required (see [HenvR]).

Notice that every weight function w defines a linear functional u in the linear space of complex polynomials $\mathbb{P}$ in the following way

$$\langle u, p \rangle = \int_{\mathbb{R}} p(x)w(x)dx.$$

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If we denote $u_n = \langle u, x^n \rangle$ then we can associate with a linear functional a formal series

$$S(z) = -\sum \frac{u_n}{z^{n+1}},$$

which is the z-transform of the sequence of the moments $\{u_n\}$. Such a function S is called the Stieltjes function of the linear functional u (see [Mar1]). For a weight function w, S means the Cauchy transform of w.

If the linear functional u satisfies a distributional Pearson equation $D(\phi u) = \psi u$, then the corresponding Stieltjes function satisfies a linear first order differential equation

$$A(z)S'(z) = B(z)S(z) + C(z) \tag{1}$$

where A, B, C are polynomials related to the polynomials ϕ, ψ which appear in the Pearson equation.
Conversely, if the Stieltjes function of a linear functional satisfies (1) then a Pearson distributional equation for u holds.

A linear functional u is said to be a Laguerre-Hahn affine linear functional if (1) holds. Thus the concepts of Laguerre-Hahn affine and semiclassical linear functionals are the same. A nice survey about them is [Mar1]. As examples of Laguerre-Hahn affine linear functionals we get the classical ones (Jacobi, Laguerre, Hermite, and Bessel) as well as some perturbations of them.

A linear functional u is said to be a Laguerre-Hahn linear functional when S satisfies a Riccati equation

$$M(z)S'(z) = N(z)S^2(z) + P(z)S(z) + R(z),$$

where $M \neq 0, N \neq 0, P$ and R are polynomials. Laguerre-Hahn linear functionals are related to Riccati acceleration of Jacobi continued fractions.

In [Mar1] second degree linear functionals are introduced. Its Stieltjes function satisfies a quadratic equation with polynomial coefficients

$$T(z)S^2(z) + U(z)S(z) + V(z) = 0$$

where $T \neq 0, U^2 - 4TV \neq 0$ and V is a certain function of T and U.
Every second degree linear functional belongs to the Laguerre-Hahn affine class. Some examples of second degree linear functionals are considered in [Mar2]. The simplest one is the Tchebychev linear functional of second kind.

In the last years some work was done in the analysis of similar problems for weight functions supported on the unit circle. In [CaPe] the Laguerre-Hahn affine functionals on the unit circle have been studied. There, it is shown that there exist Laguerre-Hahn functionals which are not semiclassical. Moreover some modifications of functionals that preserve the Laguerre-Hahn affine character are analyzed there. Now we are interested in studying transformations which do not preserve the Laguerre-Hahn affine character.

Using the second degree functionals we establish, under very general conditions, that the second kind linear functional of a Laguerre-Hahn affine linear functional is not Laguerre-Hahn affine. We also present the transformation corresponding to the associated polynomials of order N which, in general, does not preserve the class.

The organization of the paper is the following.

In section 2 we recall the definition and some properties of the Laguerre-Hahn affine functionals on the unit circle, and we also introduce the second kind polynomials as well as the associated polynomials of order N, where N is a nonnegative integer.

In section 3 we study the functionals of second degree and we present some relevant examples. Moreover we prove that Laguerre-Hahn affine functionals which are not rational and satisfy a Riccati equation must be second degree functionals.

Finally, in the last section, we prove that for Laguerre-Hahn affine functionals, which are not second degree functionals and not rational and satisfy an additional condition, the associated second kind functionals do not belong to the Laguerre-Hahn affine class. Other transformations in the Laguerre-Hahn affine class which may not preserve the class correspond to the associated polynomials and quadratic decomposition of a sequence of orthogonal polynomials. We conclude our work with some illustrative examples.

2 Laguerre-Hahn affine polynomials.

Let Λ be the linear space of the Laurent polynomials, i.e., $\Lambda = \text{span}\left\{z^k\,;\,k \in \mathbb{Z}\right\}$. Given a sequence of complex numbers $\{c_n\}_{n\in\mathbb{Z}}$ we consider a linear functional $\mathcal{L}$: $\Lambda \longrightarrow \mathbb{C}$ such that $\mathcal{L}\left(z^n\right) = c_n$ for every $n \in \mathbb{Z}$.

The linear functional $\mathcal{L}$ is said to be hermitian if $c_{-n} = \overline{c_n}, \quad \forall n \geq 0$. If we define in the space of the algebraic polynomials $\mathbb{P}$ the indefinite inner product

$$\langle P(z)\,,\,Q(z)\rangle_{\mathcal{L}} = \mathcal{L}\left(P(z)\overline{Q}\left(\frac{1}{z}\right)\right)\ ,$$

then the Gram matrix with respect to the canonical basis of $\mathbb{P}$, $\{z^n; n \in \mathbb{N}\}$, is the following Toeplitz matrix.

$$M = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & c_1 & \cdots & c_{n-1} & \cdots \\ c_{-2} & c_{-1} & c_0 & \cdots & c_{n-2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & c_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We denote by M_n the principal submatrix of order $n+1$ of the above infinite matrix M and we write $\Delta_n = \det M_n$.

The functional $\mathcal{L}$ is said to be quasi-definite (positive definite) if $\Delta_n \neq 0$ ($\Delta_n > 0$) for every $n \geq 0$. When $\mathcal{L}$ is positive definite it is well-known (see [Ger]) that there exists a positive Borel measure μ with infinite support on $[0\,,\,2\pi]$ such that

$$c_n = \int_0^{2\pi} z^n\, d\,\mu(\theta), \quad \forall n \geq 0,\ z = e^{i\theta}\ .$$

Next we recall some definitions:

Definition 1. *([Ger])*
Let £ be a hermitian quasi-definite linear functional. A sequence of polynomials $\{P_n(z)\}_{n\in\mathbb{N}}$ *is said to be a sequence of orthogonal polynomials with respect to £ if :*

1. $\forall n \geq 0,\ \ deg\, P_n = n.$

2. $\forall\, n,\, m \geq 0 \quad \langle P_n(z)\,,\, P_m(z)\rangle_{£} = k_n\delta_{n,m}$, *with* $k_n \neq 0$.

If the leading coefficient of each polynomial is 1*, we say that it is a sequence of monic orthogonal polynomials (MOPS).*

Definition 2. *([CaPe])*
Let £ be a hermitian quasi-definite linear functional. We denote by $\mathbb{G}(z)$ *the formal series associated with £*

$$\mathbb{G}(z) = \sum_{k=-\infty}^{+\infty} \frac{c_k}{z^k}\,.$$

$\mathbb{G}$ *is said to be rational if there exist polynomials* C *and* D *with* $C \neq 0$ *such that* $C\,\mathbb{G} = D$.

We introduce the following formal series $\mathbb{F}(z)$ associated with £

$$\mathbb{F}(z) = c_0 + 2\sum_{k=1}^{\infty} \overline{c_k} z^k\ .$$

Notice that $\mathbb{F}$ is rational if and only if $\mathbb{G}$ is rational. Furthermore

$$\mathbb{G}(z) = \frac{\mathbb{F}(z) + \overline{\mathbb{F}}\left(\frac{1}{z}\right)}{2}\ .$$

In the positive definite case $\mathbb{F}$ is the Carathéodory function associated with the measure of orthogonality. It is the Herglotz transform of such a measure ([Ger]).

Definition 3. *([CaPe])*
A quasi-definite and hermitian linear functional is said to be a Laguerre-Hahn affine functional if there exist polynomials A*,* B*, and* H *with* $A \neq 0$ *such that the formal series* $\mathbb{G}(z)$ *and its formal derivative* $\mathbb{G}^{'}(z)$ *satisfy*

$$A(z)\mathbb{G}^{'}(z) + B(z)\mathbb{G}(z) + H(z) = 0\ . \tag{2}$$

When $H = 0$ *in (2) £ is said to be a semiclassical linear functional.*

Notice that the differential equation satisfied by $\mathbb{G}$ is not unique.
In [CaPe] it is shown that on the unit circle the class of semiclassical functionals and the class of Laguerre-Hahn affine functionals are different and some examples of Laguerre-Hahn affine functionals which are not semiclassical are given. One of the simplest example is obtained as follows. Let L be a semiclassical linear functional with non rational formal series G satisfying $A(z)G'(z) + B(z)G(z) = 0$ with A and B polynomials, $A \neq 0, B \neq 0$ and such that $z^2\overline{A}(\frac{1}{z})B(z) + A(z)\overline{B}(\frac{1}{z}) = 0$. This is

the case of some Jacobi linear functionals. If L_1 is the functional associated with the normalized Lebesgue measure then $\pounds = L + L_1$ is a Laguerre-Hahn affine functional which is not semiclassical.

Next we present some properties concerning the Laguerre-Hahn affine linear functionals. For the real case analogous results can be found in [PriM].

Proposition 1. *If $\pounds$ is a Laguerre-Hahn affine linear functional such that $\mathbb{G}$ satisfies (2), then there exists a polynomial D such that $\mathbb{F}$ satisfies*

$$A(z)\mathbb{F}'(z) + B(z)\mathbb{F}(z) + 2H(z) + D(z) = 0\ , \tag{3}$$

and

$$A(z)\left(\overline{\mathbb{F}}\left(\frac{1}{z}\right)\right)' + B(z)\overline{\mathbb{F}}\left(\frac{1}{z}\right) - D(z) = 0\ . \tag{4}$$

Let us denote by p, q, r, and s the degrees of A, B, $2H + D$, and D respectively and by α, β, ρ, and σ the multiplicities of 0 as a zero of A, B, $2H + D$, and D, respectively. Then

(i) Either $\mathbb{F}$ is rational

(ii) or

$$\frac{-A(z)}{z^2\overline{A}\left(\frac{1}{z}\right)} = \frac{B(z)}{\overline{B}\left(\frac{1}{z}\right)} = -\frac{2H(z)+D(z)}{\overline{D}\left(\frac{1}{z}\right)} = -\frac{D(z)}{2\overline{H}\left(\frac{1}{z}\right)+\overline{D}\left(\frac{1}{z}\right)} \tag{5}$$

and $p + \alpha - 2 = q + \beta = s + \rho = r + \sigma = k$.

Moreover, in case (ii) $2H + D$ and D have the same zeros of modulus 1, and there exist a natural number n_0 with $0 \le n_0 \le s - \sigma$ and a complex χ with $|\chi| = 1$ such that

$$\omega(z) = -\frac{z^{p-2}A(z)}{A^*(z)} = \frac{z^q B(z)}{B^*(z)} = -\frac{z^s\left(2H(z)+D(z)\right)}{D^*(z)} = -\frac{z^r D(z)}{(2H+D)^*(z)}\ , \tag{6}$$

where

$$\omega(z) = \begin{cases} \chi\, z^k & \text{if } n_0 = 0 \\ \chi \displaystyle\prod_{j=1}^{n_0} \frac{z-\beta_j}{1-\overline{\beta_j}z} z^k & \text{if } n_0 > 0\ . \end{cases}$$

In the last case, β_j are common zeros of A, B, $2H + D$, and D such that $\beta_j \ne 0$ and $|\beta_j| \ne 1$ for $j = 1, 2, \ldots, n_0$.

Finally, if g.c.d.$\{A, B, H, D\} = 1$ then $\omega(z) = \chi z^k$ and the representation (2) is said to be minimal.

(Notice that for $\deg(P) = n$, the reversed polynomial is defined by $P^*(z) = z^n\overline{P}\left(\frac{1}{z}\right)$, see [Ger]).

Proof. See [CaPe]. ∎

Next we recall the definition and some properties of the second kind polynomials and the associated polynomials (see [Pe]).

Definition 4.
Let £ be a hermitian and quasi-definite linear functional with formal series $\mathbb{F}$ and let $\{\phi_n(z)\}_{n\in\mathbb{N}}$ be the MOPS with respect to £. We say that $\{\Omega_n(z)\}_{n\in\mathbb{N}}$ is the sequence of monic polynomials of second kind with respect to £ if $\{\Omega_n(z)\}_{n\in\mathbb{N}}$ satisfies the recurrence relation

$$\Omega_n(z) = z\Omega_{n-1}(z) - \phi_n(0)\Omega^*_{n-1}(z) \ \ \textit{for}\ n \geq 1,$$

with $\Omega_0(z) = 1$, i.e.,

$$\Omega_n(0) = -\phi_n(0), \ \ \forall n \geq 1, \quad \textit{and}\ \ \Omega_0(z) = 1.$$

It is well-known that $\{\Omega_n(z)\}_{n\in\mathbb{N}}$ is the MOPS with respect to a linear functional L with formal series $F = \frac{1}{\mathbb{F}}$.

Definition 5.
Let £ be a hermitian and quasi-definite linear functional and let $\{\phi_n(z)\}_{n\in\mathbb{N}}$ be the corresponding MOPS.
$\left\{\phi^N_n(z)\right\}_{n\in\mathbb{N}}$ is said to be the sequence of associated polynomials of order N if

$$\phi^N_n(z) = z\phi^N_{n-1}(z) + \phi_{n+N}(0)(\phi^N_{n-1}(z))^*, \ \ n \geq 1,$$

with $\phi^N_0(z) = 1$. In other words

$$\phi^N_n(0) = \phi_{n+N}(0), \qquad \forall n \geq 1\ .$$

The MOPS $\{\phi_n(z)\}_{n\in\mathbb{N}}$ is said to be self-associated of order N if

$$\phi^N_n(0) = \phi_{n+N}(0) \qquad \forall n \geq 1.$$

Proposition 2. *Let $\{\phi_n(z)\}_{n\in\mathbb{N}}$ be a MOPS with respect to £ and let $\left\{\phi^N_n(z)\right\}_{n\in\mathbb{N}}$ be the sequence of associated polynomials of order N. If we denote by $\{\Omega_n(z)\}_{n\in\mathbb{N}}$ and $\left\{\Omega^N_n(z)\right\}_{n\in\mathbb{N}}$ their corresponding sequences of second kind polynomials, respectively, then the following relations hold:*

$$\phi^N_n(z) = \frac{\Delta_{N-1}}{\Delta_N}\left[\frac{(\Omega_N(z) + \Omega^*_N(z))\,\phi_{n+N}(z) + (\phi^*_N(z) - \phi_N(z))\,\Omega_{n+N}(z)}{2z^N}\right],$$

and

$$\Omega^N_n(z) = \frac{\Delta_{N-1}}{\Delta_N}\left[\frac{(\Omega^*_N(z) - \Omega_N(z))\,\phi_{n+N}(z) + (\phi_N(z) + \phi^*_N(z))\,\Omega_{n+N}(z)}{2z^N}\right].$$

Moreover, if we denote by $\mathbb{F}$ and $\mathbb{F}_N$ the Carathéodory functions associated with $\{\phi_n(z)\}_{n\in\mathbb{N}}$ and $\left\{\phi^N_n(z)\right\}_{n\in\mathbb{N}}$, respectively, then:

$$\mathbb{F}_N(z) = \frac{\Omega_N(z) - \Omega_N^*(z) + (\phi_N(z) + \phi_N^*(z))\ \mathbb{F}(z)}{\Omega_N(z) + \Omega_N^*(z) + (\phi_N(z) - \phi_N^*(z))\ \mathbb{F}(z)} \ .$$

In particular, if $\{\phi_n(z)\}_{n\in\mathbb{N}}$ *is a MOPS with respect to* $\pounds$ *which is self-associated of order* N*, then the Carathéodory function* $\mathbb{F}_N = \mathbb{F}$ *satisfies the following quadratic equation:*

$$\left(\phi_N - \phi_N^*\right)(z)\mathbb{F}^2(z) + \left(\Omega_N + \Omega_N^* - \phi_N - \phi_N^*\right)(z)\mathbb{F}(z) - \left(\Omega_N - \Omega_N^*\right)(z) = 0 \ . \quad (7)$$

Proof. See [Pe]. ∎

3 Second degree functionals.

In [CaPe] we have studied some modifications of a linear functional preserving the Laguerre-Hahn affine class. Now we are interested to study transformations in the Laguerre-Hahn affine class which do not preserve this class. In order to do it, first we study second degree linear functionals. In the real case the second degree functionals have been studied in [Mar2] and [MPri].

Definition 6.
Let $\pounds$ *be a hermitian and quasi-definite linear functional such that its formal series* $\mathbb{F}$ *is not rational. Then*

- $\pounds$ *is a linear functional of second degree if there exist polynomials* $\alpha \neq 0$, β, *and* $\gamma \neq 0$ *such that*

  $$\alpha(z)\mathbb{F}^2(z) + \beta(z)\mathbb{F}(z) + \gamma(z) = 0 \ , \quad (8)$$

  with $\beta^2 - 4\alpha\gamma \neq r^2$*, where* $r \in \mathbb{P}$.

- $\pounds$ *has rational square if there exist polynomials* $m \neq 0$ *and* $p \neq 0$ *such that*

  $$m(z)\mathbb{F}^2(z) + p(z) = 0 \ . \quad (9)$$

  (For the real case, see [Mar2]).

It is clear that if $\pounds$ satisfies (9) then $\pounds$ also satisfies (8). It is also evident that every Laguerre-Hahn affine functional $\pounds$ of rational square satisfying (9) is such that the corresponding second kind linear functional L with formal series F has rational square and satisfies $p(z)F^2(z) + m(z) = 0$.

Remark 1. *Let* $\pounds$ *be a second degree functional such that (8) holds. Since, by definition, its formal series is not rational, then the following conditions hold:*

1.

$$(2\alpha\mathbb{F} + \beta)^2 = \beta^2 - 4\alpha\gamma = \Delta. \quad (10)$$

2. *Since* $\mathbb{F}(z) = \frac{-\beta(z)\pm\Delta^{\frac{1}{2}}}{2\alpha(z)}$, *if* $\beta = 0$, *then there does not exist any polynomial* $\rho \neq 0$ *such that* $\frac{-\gamma}{\alpha} = \rho^2$.

3. *The second kind linear functional* L *is a second degree functional with*

$$\gamma(z)F^2(z) + \beta(z)F(z) + \alpha(z) = 0 \; .$$

Next we present some examples of second degree functionals.

Example 1. *Let us consider the MOPS* $\{\phi_n(z)\}_{n\in\mathbb{N}}$ *such that* $\phi_n(0) = a, \;\; \forall n \geq 1$, *where* a *is a complex number with* $0 < |a| < 1$.
It is known ([Ger]) that its Carathéodory function $\mathbb{F}$ *is not a rational function. Indeed*

$$\mathbb{F}(z) = \frac{\overline{a}\,z + a - \xi(z)}{(a-1) - (\overline{a}-1)z} \; , \quad \textit{with} \;\; \xi(z) = \sqrt{z^2 + 2(2|a|^2 - 1)z + 1} \; .$$

Since $\phi_{n+1}(0) = \phi_n(0), \;\; \forall n \geq 1$, *the sequence* $\{\phi_n(z)\}$ *is self-associated of order 1. Taking into account that* $\phi_1(z) = z + a$, $\phi_1^*(z) = 1 + \overline{a}z$, $\Omega_1(z) = z - a$, *and* $\Omega_1^*(z) = 1 - \overline{a}z$ *we get that* $\mathbb{F}$ *satisfies the quadratic equation (7):*

$$[(a-1) + (1-\overline{a})z]\mathbb{F}^2(z) - 2(a+\overline{a}z)\mathbb{F}(z) + (a+1) - (1+\overline{a})z = 0 \; .$$

Hence we have a second degree linear functional.

Example 2. *Let us consider the sequence of monic orthogonal polynomials* $\{\phi_n(z)\}$ *such that* $\phi_{2n-1}(0) = a$ *and* $\phi_{2n}(0) = b$ *where* a *and* b *are complex numbers such that* $a \neq b$, $|a| < 1$ *and* $|b| < 1$.
Since $\phi_{n+2}(0) = \phi_n(0), \;\; \forall n \geq 1$, *the sequence* $\{\phi_n(z)\}$ *is self-associated of order 2. The corresponding Carathéodory function, which is not rational, is given by*

$$\mathbb{F}(z) = \frac{\overline{b}z^2 + (a+\overline{a})z + b - \psi(z)}{H(z\,;\,a,b)} \; ,$$

where
$\psi(z) = \left\{[\overline{b}z^2 + (a+\overline{a})z + b]^2 + H(z\,;\,a,b)H(-z\,;\,a,-b)\right\}^{\frac{1}{2}}$ *and*
$H(z\,;\,a,b) = (1-\overline{b})z^2 + [a(1-\overline{b}) - \overline{a}(1-b)]z + b - 1$, *(see [Can]).*
Moreover the quadratic equation (7) becomes

$$H(z\,;\,a,b)\mathbb{F}^2(z) - 2[\,\overline{b}\,z^2 + (a+\overline{a})z + b]\mathbb{F}(z) - H(-z\,;\,a,-b) = 0 \; .$$

Thus we have another example of second degree linear functional.

Proposition 3. *Let £ be a hermitian and quasi-definite linear functional such that its formal series* $\mathbb{F}$ *is not rational. If £ is a second degree linear functional satisfying (8) then* $\mathbb{F}$ *satisfies:*

$$\widetilde{A}(z)\,\mathbb{F}'(z) + \widetilde{B}(z)\mathbb{F}(z) + \widetilde{H}(z) = 0 \; , \tag{11}$$

with

$$\begin{aligned} \widetilde{A} &= \alpha(\beta^2 - 4\alpha\gamma), \\ \widetilde{B} &= 2\alpha(\alpha\gamma' - \alpha'\gamma) + \beta(\alpha'\beta - \alpha\beta'), \\ \widetilde{H} &= 2\gamma(\alpha'\beta - \alpha\beta') + \beta(\alpha\gamma' - \alpha'\gamma). \end{aligned} \tag{12}$$

Proof. Taking derivatives in (8) we get

$$2\alpha(z)\mathbb{F}(z)\mathbb{F}'(z) + \alpha'(z)\mathbb{F}^2(z) + \beta'(z)\mathbb{F}(z) + \beta(z)\mathbb{F}'(z) + \gamma'(z) = 0\ ,$$

and since $\mathbb{F}^2(z) = -\dfrac{\beta(z)\mathbb{F}(z) + \gamma(z)}{\alpha(z)}$ we get

$$\begin{aligned} \alpha(z)\left[2\alpha(z)\mathbb{F}(z) + \beta(z)\right]\mathbb{F}'(z) + \left[\alpha(z)\beta'(z) - \alpha'(z)\beta(z)\right]\mathbb{F}(z) \\ +\alpha(z)\gamma'(z) - \alpha'(z)\gamma(z) = 0\ . \end{aligned}$$

If we multiply by $2\alpha\mathbb{F} + \beta$, from (8) and (10) our statement follows.

■

Remark 2. *The second degree linear functional in Example* 1 *is a Laguerre-Hahn affine functional with non rational formal series.*

Indeed,

$$\{[(a-1) - (\overline{a}-1)z]\mathbb{F}(z)\}' = \overline{a} - \frac{z + 2|a|^2 - 1}{\sqrt{z^2 + 2(2|a|^2 - 1)z + 1}},$$

or equivalently

$$\begin{aligned} [z^2 + 2(2|a|^2 - 1)z + 1]\{[(a-1) - (\overline{a}-1)z]\mathbb{F}(z)\}' = \overline{a}[z^2 + 2(2|a|^2 - 1)z + 1] + \\ (z + 2|a|^2 - 1)\{[(a-1) - (\overline{a}-1)z]\mathbb{F}(z) - \overline{a}z - a\}. \end{aligned}$$

So $\mathbb{F}(z)$ satisfies the differential equation $\widetilde{A}(z)\mathbb{F}'(z) + \widetilde{B}(z)\mathbb{F}(z) + \widetilde{H}(z) = 0$, with
$\widetilde{A}(z) = [z^2 + 2(2|a|^2 - 1)z + 1][(a-1) - (\overline{a}-1)z]$,
$\widetilde{B}(z) = -(\overline{a}-1)[(2|a|^2 - 1)z + 1] - (a-1)(z + 2|a|^2 - 1)$,
$\widetilde{H}(z) = -\overline{a}[z^2 + 2(2|a|^2 - 1)z + 1] + (\overline{a}z + a)(z + 2|a|^2 - 1)$,
and it holds that $z^2\overline{\widetilde{A}}(\frac{1}{z})\widetilde{B}(z) + \widetilde{A}(z)\overline{\widetilde{B}}(\frac{1}{z}) = 0$. Therefore the functional is Laguerre-Hahn affine, (see [CaPe]).

Remark 3. *The second degree linear functional in Example* 2 *is a Laguerre-Hahn affine functional with non rational formal series.*

Proceeding in the same way as in the preceding remark it is easy to obtain that

$$(H(z;a,b)\mathbb{F}(z))' = 2\overline{b}z + a + \overline{a} - \Psi'(z).$$

If we denote $\Psi(z) = (V(z))^{\frac{1}{2}}$, then

$$H'(z;a,b)\mathbb{F}(z) + H(z;a,b)\mathbb{F}'(z) = 2\overline{b}z + a + \overline{a} - \frac{V'(z)}{2\sqrt{V(z)}}.$$

If we multiply by $V(z)$ and take into account the expression of $\mathbb{F}(z)$ we get

$$V(z)[H'(z;a,b)\mathbb{F}(z)+H(z;a,b)\mathbb{F}'(z)] = V(z)(2\overline{b}z+a+\overline{a})+ \frac{1}{2}V'(z)[H(z;a,b)\mathbb{F}(z)-\overline{b}z^2-(a+\overline{a})z-b].$$

Hence $\mathbb{F}(z)$ satisfies the differential equation $\widetilde{A}(z)\mathbb{F}'(z)+\widetilde{B}(z)\mathbb{F}(z)+\widetilde{H}(z)=0$, with
$\widetilde{A}(z)=V(z)H(z;a,b)$,
$\widetilde{B}(z)=V(z)H'(z;a,b)-\frac{1}{2}V'(z)H(z;a,b)$,
$\widetilde{H}(z)=-(2\overline{b}z+a+\overline{a})V(z)+\frac{1}{2}V'(z)[\overline{b}z^2+(a+\overline{a})z+b]$.

Taking into account that $\overline{H}(\frac{1}{z};a,b)=-\frac{H(z;a,b)}{z^2}$, $\overline{H}(-\frac{1}{z};a,-b)=-\frac{H(-z;a,-b)}{z^2}$ and $\overline{V}(\frac{1}{z})=\frac{V(z)}{z^4}$, we deduce that $z^2\overline{\widetilde{A}}(\frac{1}{z})\widetilde{B}(z)+\widetilde{A}(z)\overline{\widetilde{B}}(\frac{1}{z})=0$, which implies that the functional is Laguerre-Hahn affine, (see [CaPe]).

Proposition 4. *Let £ be a Laguerre-Hahn affine linear functional with non rational formal series $\mathbb{F}$ which satisfies (3), i.e.*

$$A(z)\mathbb{F}'(z)+B(z)\mathbb{F}(z)+2H(z)+D(z)=0\,.$$

If £ is a second degree linear functional satisfying (8) then

$$\frac{\alpha(z)\Delta(z)}{A(z)}=\frac{\alpha'(z)\Delta(z)-\frac{1}{2}\alpha(z)\Delta'(z)}{B(z)}=\frac{\frac{1}{2}\beta'(z)\Delta(z)-\frac{1}{4}\beta(z)\Delta'(z)}{2H(z)+D(z)}. \tag{13}$$

Proof. Taking into account the previous proposition, from (3) and (11) we get:

$$\left[B(z)\widetilde{A}(z)-A(z)\widetilde{B}(z)\right]\mathbb{F}(z)+[2H(z)+D(z)]\,\widetilde{A}(z)-A(z)\widetilde{H}(z)=0\,.$$

Since $\mathbb{F}$ is not rational then $B\widetilde{A}-A\widetilde{B}=0$ and $(2H+D)\widetilde{A}-A\widetilde{H}=0$.

Thus

$$\frac{\widetilde{A}(z)}{A(z)}=\frac{\widetilde{B}(z)}{B(z)}=\frac{\widetilde{H}(z)}{2H(z)+D(z)}\,. \tag{14}$$

From the definition of Δ in (10) $\Delta'=2\beta\beta'-4(\alpha'\gamma+\alpha\gamma')$ and it is straightforward to prove that

$$2\alpha(\alpha\gamma'-\alpha'\gamma)+\beta(\alpha'\beta-\alpha\beta')=-\tfrac{3}{2}\alpha\Delta'+(\alpha\Delta)'\,,$$
$$2\gamma(\alpha'\beta-\alpha\beta')+\beta(\alpha\gamma'-\alpha'\gamma)=\tfrac{1}{4}(2\beta'\Delta-\beta\Delta')\,.$$

Finally, using (12) and the previous relations, from (14) we deduce (13). ∎

Proposition 5. *Let £ be a Laguerre-Hahn affine functional with non rational formal series $\mathbb{F}$.*
If the formal series $\mathbb{F}$ satisfies a Riccati differential equation with polynomial coefficients, then £ is a second degree linear functional.

Proof. Assume that $\mathbb{F}$ satisfies a Riccati differential equation with polynomial coefficients:

$$m(z)\mathbb{F}'(z) + n(z)\mathbb{F}(z) + p(z)\mathbb{F}^2(z) + q(z) = 0 \ ,$$

with $m \neq 0$ and $p \neq 0$.
From the Laguerre-Hahn affine character of $\pounds$ we have that

$$A(z)\mathbb{F}'(z) + B(z)\mathbb{F}(z) + 2H(z) + D(z) = 0 \qquad \text{with } A \neq 0 \ .$$

Thus, if we eliminate $\mathbb{F}'(z)$ from these two last relations we obtain

$$\begin{aligned} A(z)p(z)\mathbb{F}^2(z) + [A(z)n(z) - B(z)m(z)]\mathbb{F}(z) \\ + A(z)q(z) - [2H(z) + D(z)]m(z) = 0 \ , \end{aligned}$$

with $A\,p \neq 0$.
Since $\mathbb{F}$ is not rational we get $A\,q - (2H + D)m \neq 0$ and $(A\,n - B\,m)^2 - 4A\,p[A\,q - (2H + D)m] \neq r^2$, with $r \in \mathbb{P}$. Then $\pounds$ is a second degree linear functional.

∎

4 The second kind linear functional.

Next we consider a Laguerre-Hahn affine linear functional which is not a second degree functional and not rational and such that the polynomial D introduced in Proposition 1 is $D \not\equiv 0$. In these conditions the corresponding second kind linear functional is not Laguerre-Hahn affine.

Proposition 6. *Let $\pounds$ be a Laguerre-Hahn affine functional such that the formal series $\mathbb{F}$ is not rational and satisfies (3) and (4) with $D \neq 0$.*
Let L be the corresponding second kind linear functional with formal series $F(z) = \dfrac{1}{\mathbb{F}(z)}$. If L is Laguerre-Hahn affine, then $\pounds$ must be a second degree linear functional.

Proof. Since $\mathbb{F}$ is not rational and satisfies (3) and (4) then $F(z) = \dfrac{1}{\mathbb{F}(z)}$ is not rational and satisfies:

$$-A(z)F'(z) + B(z)F(z) + [2H(z) + D(z)]\,F^2(z) = 0 \ , \tag{15}$$

$$-A(z)\left[\overline{F}\left(\frac{1}{z}\right)\right]' + B(z)\overline{F}\left(\frac{1}{z}\right) - D(z)\overline{F}^2\left(\frac{1}{z}\right) = 0 \ . \tag{16}$$

If L is a Laguerre-Hahn affine linear functional such that the formal series $G(z) = \dfrac{F(z) + \overline{F}(\frac{1}{z})}{2}$ satisfies:

$$a(z)G'(z) + b(z)G(z) + h(z) = 0, \qquad \text{with } \ a \neq 0 \ ,$$

from Proposition 1 there exists a polynomial d such that:

$$a(z)F'(z) + b(z)F(z) + 2h(z) + d(z) = 0 \quad \text{and} \tag{17}$$

$$a(z)\left[\overline{F}\left(\frac{1}{z}\right)\right]' + b(z)\overline{F}\left(\frac{1}{z}\right) - d(z) = 0\,. \tag{18}$$

From (15) and (17) we eliminate $F'(z)$ as well as from (16) and (18) we eliminate $\left[\overline{F}\left(\frac{1}{z}\right)\right]'$.
Thus

$$A(z)[2h(z) + d(z)] + [B(z)\,a(z) + A(z)\,b(z)]F(z) + a(z)[2H(z) + D(z)]F^2(z) = 0\,,$$

and

$$-A(z)d(z) + [B(z)\,a(z) + A(z)\,b(z)]\overline{F}\left(\frac{1}{z}\right) - a(z)D(z)\left[\overline{F}\left(\frac{1}{z}\right)\right]^2 = 0\,.$$

Since $a \neq 0$, $A \neq 0$, and F is not rational then the following relations hold:

$$\begin{gathered} 2H + D \neq 0 \quad \text{if and only if} \quad 2h + d = 0 \quad \text{and} \\ D \neq 0 \quad \text{if and only if} \quad d \neq 0\,. \end{gathered}$$

Since $D \neq 0$, from (5) we get that $2H + D \neq 0$. Therefore

$$\begin{gathered} (B\,a + A\,b)^2 - 4a\,D\,A\,d \neq r^2 \qquad \text{with } r \in \mathbb{P} \qquad \text{and} \\ (B\,a + A\,b)^2 - 4a(2H + D)A(2h + d) \neq s^2 \qquad \text{with } s \in \mathbb{P}\,. \end{gathered}$$

Thus $\mathcal{L}$ must be a second degree linear functional because the following relations hold:

$$A(z)[2h(z)+d(z)]\mathbb{F}^2(z)+[a(z)B(z)+A(z)b(z)]\mathbb{F}(z)+a(z)[2H(z)+D(z)] = 0\,, \tag{19}$$

$$-A(z)d(z)\overline{\mathbb{F}}^2\left(\frac{1}{z}\right) + [a(z)B(z) + A(z)b(z)]\overline{\mathbb{F}}\left(\frac{1}{z}\right) - a(z)D(z) = 0\,. \tag{20}$$

■

Example 3. *Let us consider the Jacobi measure* $d\,\mu(\theta) = \cos\left(\frac{\theta}{2}\right)\left|\sin\left(\frac{\theta}{2}\right)\right|\frac{d\,\theta}{2\pi}$. *Its Carathéodory function is*

$$\mathbb{F}(z) = (1 - z^2)\,{}_2F_1\left[1,\, \frac{1}{2}\,;\, \frac{3}{2}\,;\, z^2\right] = \frac{1 - z^2}{2z}\log\frac{1 + z}{1 - z}\,,$$

which is not rational and satisfies (3) and (4) with

$$A(z) = z(z^2 - 1), \;\; B(z) = -(z^2 + 1), \;\; H(z) = 0, \;\; \textit{and } D(z) = \pi^{-1}(1 - z^2).$$

(see [Mag] and [Gol]).
Since it is not a second degree linear functional, then the corresponding second kind linear functional is not Laguerre-Hahn affine.

Another transformation in the Laguerre-Hahn affine class which may not preserve the class corresponds to the associated polynomials of order N. Let us consider this transformation.

Let $\mathcal{L}$ be a linear functional which is not a second degree functional. Let $\{\phi_n(z)\}_{n\in\mathbb{N}}$ be the MOPS with respect to $\mathcal{L}$ with a non rational Carathéodory function $\mathbb{F}$ and let $\left\{\phi_n^N(z)\right\}_{n\in\mathbb{N}}$ be the associated polynomials of order N. It is known that $\left\{\phi_n^N(z)\right\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}_N$ with Carathéodory function $\mathbb{F}_N$. Applying Proposition 2 we get

$$\mathbb{F}_N(z) = \frac{A_{1,N}(z) + B_{1,N}(z)\mathbb{F}(z)}{A_{2,N}(z) + B_{2,N}(z)\mathbb{F}(z)} \ , \tag{21}$$

with $A_{i,N}^*(z) = -A_{i,N}(z)$ and $B_{i,N}^*(z) = -B_{i,N}(z)$ for $i \in \{1\,,\,2\}$.

The derivative $(\mathbb{F}_N)'$ is

$$(\mathbb{F}_N)' = \frac{C_1 + C_2\mathbb{F} + C_3\mathbb{F}' + C_4\mathbb{F}^2}{\left(A_{2,N} + B_{2,N}\mathbb{F}\right)^2} \ , \tag{22}$$

with

$$\begin{aligned}
C_1 &= A_{1,N}' A_{2,N} - A_{1,N} A_{2,N}' \ , \\
C_2 &= A_{1,N}' B_{2,N} - A_{1,N} B_{2,N}' + B_{1,N}' A_{2,N} - B_{1,N} A_{2,N}' \ , \\
C_3 &= B_{1,N} A_{2,N} - A_{1,N} B_{2,N} \ , \\
C_4 &= B_{1,N}' B_{2,N} - B_{1,N} B_{2,N}' \ .
\end{aligned}$$

If $\mathcal{L}_N$ belongs to the Laguerre-Hahn affine class then there exist polynomials a, b, and $2h+d$ with $a \neq 0$ such that

$$a\ (\mathbb{F}_N)' + b\,\mathbb{F}_N + 2h + d = 0 \ . \tag{23}$$

Taking into account (21) and (22) $\mathbb{F}$ satisfies the following differential equation

$$m\,\mathbb{F}' + b\,\mathbb{F} + p\,\mathbb{F}^2 + q = 0 \ , \tag{24}$$

with

$$\begin{aligned}
m &= a\,C_3 \ , \\
n &= a\,C_2 + b\ (A_{1,N}B_{2,N} + B_{1,N}A_{2,N}) + (2h+d)\,2A_{2,N}B_{2,N} \ , \\
p &= a\,C_4 + b\,B_{1,N}B_{2,N} + (2h+d)\,B_{2,N}^2 \ , \\
q &= a\,C_1 + b\,A_{1,N}A_{2,N} + (2h+d)\,A_{2,N}^2 \ .
\end{aligned}$$

If $m \neq 0$ and $p \neq 0$ then (24) is a Riccati equation and applying Proposition 5 we get that $\mathcal{L}$ must be a second degree linear functional. Therefore $\mathcal{L}_N$ is not Laguerre-Hahn affine.

Notice that in our Example 1, $\mathbb{F}_N$ is Laguerre-Hahn affine for every N. Indeed $\mathbb{F}_N = \mathbb{F}$. In general, if $\mathbb{F}$ is of second degree then $\mathbb{F}_N$ is of second degree too.

Next we present an example of a particular case, $(N = 1)$, of the previous situation.

Example 4. *Let us consider the particular Jacobi functional introduced in Example 3 which belongs to the Laguerre-Hahn affine class and it is not a second degree functional.*

We consider the associated polynomials of order 1, $\{\phi_n^1(z)\}_{n=0}^{\infty}$, with Carathéodory function

$$\mathbb{F}_1(z) = \frac{z-1+(z+1)\mathbb{F}(z)}{z+1+(z-1)\mathbb{F}(z)} \, .$$

If we assume that $\pounds_1$ belongs to the Laguerre-Hahn affine class then there exist polynomials a, b, and $2h+d$ with $a \neq 0$ such that

$$a \, (\mathbb{F}_1)' + b\,\mathbb{F}_1 + 2h + d = 0 \, .$$

Therefore there exist polynomials m, n, p, and q such that

$$m\,\mathbb{F}' + n\,\mathbb{F} + p\,\mathbb{F}^2 + q = 0 \, ,$$

with

$$\begin{aligned}
&m(z) = 4z\,a(z) \, , \\
&n(z) = 2\,(z^2+1)\,b(z) + 2(z^2-1)\,(2h(z)+d(z)) \, , \\
&p(z) = -2a(z) + (z^2-1)\,b(z) + (z-1)^2(2h(z)+d(z)) \, , \\
&q(z) = 2a(z) + (z^2-1)b(z) + (z+1)^2(2h(z)+d(z)) \, .
\end{aligned}$$

Notice that $m \neq 0$ and next we prove that $p \neq 0$.

Indeed, if $p = 0$, then

$$z(z^2-1)\mathbb{F}'(z) - (z^2+1)\,\mathbb{F}(z) + \pi^{-1}\,(1-z^2) = 0 \, ,$$

$$m(z)\mathbb{F}'(z) + n(z)\mathbb{F}(z) + q(z) = 0 \, .$$

Since $\mathbb{F}$ is not rational, then from (5)

$$\begin{aligned}
\frac{4a(z)}{z^2-1} &= \frac{2(z^2+1)b(z) + 2(z^2-1)(2h+d)(z)}{-\,(z^2+1)} \\
&= \frac{2a(z) + (z^2-1)b(z) + (z+1)^2(2h+d)(z)}{\pi^{-1}(1-z^2)} \, .
\end{aligned}$$

From the above relations, if $p = 0$ then we obtain an homogeneous linear system of three equations in the variables $a(z), b(z)$, and $(2h+d)(z)$ with a unique solution $a = b = 2h+d = 0$. Since $a \neq 0$, then $p \neq 0$ and therefore $\pounds_1$ is not a Laguerre-Hahn affine functional.

Finally we consider a transformation related with the quadratic decomposition of a sequence of orthogonal polynomials, (see [CanM]).

Let $\mathcal{L}$ be a Laguerre-Hahn affine functional with Carathéodory function $\mathbb{F}$ which is not rational, and let $\{\phi_n(z)\}$ be the MOPS($\mathcal{L}$). Let us consider the functional $\widehat{\mathcal{L}}$ with MOPS $\{\widehat{\phi}_n(z)\}$ such that $\widehat{\phi}_{2n}(z) = \phi_n(z^2)$ and $\widehat{\phi}_{2n+1}(z) = z\phi_n(z^2)$. It is clear that $\widehat{\mathcal{L}}$ is Laguerre-Hahn affine and its Carathéodory function is given by $\widehat{\mathbb{F}}(z) = \mathbb{F}(z^2)$. We assume that $\widehat{\mathbb{F}}$ is not rational and $\widehat{\mathcal{L}}$ is not a second degree functional.

Now we consider the sequence $\{\psi_n(z)\}$

$$\psi_{2n}(z) = \phi_n(z^2) + zB_{n-1}(z^2),$$

$$\psi_{2n+1}(z) = z\phi_n(z^2) + D_n(z^2),$$

for $n \geq 0$, with $B_{-1} = 0$ and $\deg(B_{n-1}) \leq n-1$, $\deg(D_n) \leq n$.

Necessary and sufficient conditions for the orthogonality of this sequence are given in [CanM]. In such a case, the sequence is orthogonal with respect to a functional $\widetilde{\mathcal{L}}$ with formal series $\widetilde{\mathbb{F}}$

$$\widetilde{\mathbb{F}}(z) = \frac{V(z)\widehat{\mathbb{F}}(z) - U(z)}{-S(z)\widehat{\mathbb{F}}(z) + R(z)},$$

where V, U, S, and R are polynomials, (see [CanM]).

If we assume that $\widetilde{\mathcal{L}}$ belongs to the Laguerre-Hahn affine class then there exist polynomials a, b, and $2h+d$ with $a \neq 0$ such that $a\widetilde{\mathbb{F}}' + b\widetilde{\mathbb{F}} + 2h + d = 0$. Then there exist polynomials m, n, p, and q such that $m\widehat{\mathbb{F}}'(z) + n\widehat{\mathbb{F}}(z) + p\widehat{\mathbb{F}}^2(z) + q = 0$, and it is easy to see that $m \neq 0$. Then when $p \neq 0$, the formal series $\widehat{\mathbb{F}}(z)$ satisfies a Riccati differential equation. Since $\widehat{\mathcal{L}}$ is Laguerre-Hahn affine (see [CaPe]), thus $\widehat{\mathcal{L}}$ must be a second degree linear functional, which contradicts our hypothesis.

Next we present an example of this situation.

Example 5. *Let us consider the particular Jacobi functional introduced in Example* 3 *which belongs to the Laguerre-Hahn affine class and has a non rational Carathéodory function. If we take $D_1(0) \neq 0$ and $D_n(0) = 0$ for $n \neq 1$, then the sequence $\{\psi_n(z)\}$ is orthogonal with respect to the functional $\widetilde{\mathcal{L}}$ studied before, (see [CanM]). In this case $\widetilde{\mathcal{L}}$ is not Laguerre-Hahn affine.*

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Alicia Cachafeiro
Departamento de Matemtica Aplicada I
E.T.S. Ingenieros Industriales
Universidad de Vigo
36280 Vigo, Spain *email :* acachafe@uvigo.es

Francisco Marcellán
Departamento de Matemáticas
Escuela Politécnica Superior
Universidad Carlos III de Madrid
Avda. de la Universidad 30, 28911 Leganés, Spain *email :* pacomarc@ing.uc3m.es

Carlos Pérez
Departamento de Matemtica Aplicada I
Universidad de Vigo
36280 Vigo, Spain *email :* carlosp@uvigo.es