The Moments of the $M/M/s$ Queue Length Process

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Abstract. A representation for the moments of the number of customers in a $M/M/s$ queueing system is deduced from the Karlin and McGregor representation for the transition probabilities. This representation allows us to study the limit behavior of the moments as time tends to infinity. We study some consequences of the representation for the mean.

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1. Introduction

It is easy to calculate the moments of the steady-state distribution of the number of customers in an $M/M/s$ queueing system. However, in some occasions, it is hardly realistic to assume that the system has reached the steady state situation, and representations will be needed in order to compute these moments at time t. For the first two moments several representations are known in the case of one server, see, for instance, [2,3], or [16], but little is known for the general case.

When the traffic intensity $\rho = \lambda/s\mu$, is greater than or equal to 1, the steady-state distribution does not exist and we will be interested in the limit behaviour of the moments, especially of the mean and of the variance. This behaviour is known for the first two moments in the case of one server, see [5,13], but it is unknown when there is more than one server, or for moments greater than second order.

In this paper we will obtain a representation for the moments of the $M/M/s$ queue length process, from the spectral representation for the transition probabilities. To this end, we will extend the representation given in [2] for the first two moments when we consider one server and $\rho < 1$.

From the representation obtained, we will deduce the limit behaviour of the moments. When $\rho > 1$ this behaviour will be related to the birth and death process on all the integers without barriers, of constant rates $\lambda_n = \lambda$ and $\mu_n = s\mu$. When $\rho = 1$ we will only study the limit behaviour of the mean and the variance.

The mean has a simple representation. From this representation we will deduce a representation in terms of the distribution function, some bounds, and the distance of the mean to its asymptote in the $L^1(\mathbb{R}^+)$ norm. Representations for the mean of the number of customers in the queue and of busy servers are given.

This paper is organized as follows. In section 2 we will deal with the Karlin–McGregor representation for the transition probabilities of a birth and death process, especially of the $M/M/s$ queue length process. In section 3 we will obtain a representation for the moments of the $M/M/s$ queue length process. In section 4 we will study the limit behaviour of these moments. In section 5 we will particularize for the cases $M/M/1$ and $M/M/2$. Finally, we will deal with the mean in section 6.

2. Preliminaries

Let $X(t)$ be a birth and death process on the state space $\{0, 1, \ldots\}$, with birth rates $\{\lambda_n,\ n \geqslant 0\}$ and death rates $\{\mu_n,\ ,n \geqslant 0\}$, all strictly positive, except for μ_0 which may be equal to 0.

The transition probabilities $P_{ij}(t) = P(X(t) = j \mid X(0) = i)$ satisfy the backward equations

$$\begin{aligned} P'_{ij}(t) &= \mu_i P_{i-1\,j}(t) - (\lambda_i + \mu_i)P_{ij}(t) + \lambda_i P_{i+1\,j}(t), \quad i \geqslant 1,\ j \geqslant 0, \\ P'_{0j}(t) &= -(\lambda_0 + \mu_0)P_{0j}(t) + \lambda_0 P_{1j}(t), \quad j \geqslant 0. \end{aligned} \tag{1}$$

The potential coefficients are

$$\pi_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}, \quad n \geqslant 1, \quad \pi_0 = 1.$$

If the rates satisfy the condition

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n} \sum_{i=0}^{n} \pi_i = \infty, \tag{2}$$

then the transition probabilities satisfy also the forward equations

$$\begin{aligned} P'_{ij}(t) &= \lambda_{j-1}P_{ij-1}(t) - (\lambda_j + \mu_j)P_{ij}(t) + \mu_{j+1}P_{ij+1}(t), \quad j \geqslant 1,\ i \geqslant 0, \\ P'_{i0}(t) &= -(\lambda_0 + \mu_0)P_{i0}(t) + \mu_1 P_{i1}(t), \quad i \geqslant 0. \end{aligned} \tag{3}$$

The birth–death polynomials $\{Q_n(x)\}$ are a sequence of polynomials defined by the recurrence relations

$$\begin{aligned} -xQ_n(x) &= \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \quad n \geqslant 0, \\ Q_0(x) &= 1, \qquad Q_{-1}(x) = 0. \end{aligned} \tag{4}$$

Karlin and McGregor [9] proved that if a birth and death process satisfies the condition (2) then its transition probabilities can be represented as

$$P_{ij}(t) = \pi_j \int_0^\infty Q_i(x) Q_j(x) \mathrm{e}^{-xt} \, \mathrm{d}\psi(x), \quad i \geqslant 0,\ j \geqslant 0,\ t \geqslant 0, \tag{5}$$

where $\psi(x)$ is a nondecreasing, left-continuous function, with $\lim_{x\to\infty} \psi(x) = 1$, and $\psi(x) = 0$ for $x \leqslant 0$, such that

$$\pi_j \int_0^\infty Q_i(x) Q_j(x) \, \mathrm{d}\psi(x) = \delta_{ij}. \tag{6}$$

When $\mu_0 = 0$ the dual polynomials of $\{Q_n(x)\}$ are defined by

$$\begin{aligned} -x Q_n^*(x) &= \lambda_n Q_{n-1}^*(x) - (\mu_{n+1} + \lambda_n) Q_n^*(x) + \mu_{n+1} Q_{n+1}^*(x), \quad n \geqslant 0, \\ Q_0^*(x) &= 1, \qquad Q_{-1}^*(x) = 0. \end{aligned} \tag{7}$$

In the determination of $\psi(x)$ one frequently encounters the so-called associated polynomials of the first kind (see [4]), defined by

$$\begin{aligned} -x Q_n^{(0)}(x) &= \mu_n Q_{n-1}^{(0)}(x) - (\lambda_n + \mu_n) Q_n^{(0)}(x) + \lambda_n Q_{n+1}^{(0)}(x), \quad n \geqslant 1, \\ Q_0^{(0)}(x) &= 0, \qquad Q_1^{(0)}(x) = -\frac{1}{\lambda_0}. \end{aligned}$$

The number of customers in a $M/M/s$ queueing system is a birth and death process with rates

$$\lambda_n = \lambda, \quad n \geqslant 0, \qquad \mu_n = \begin{cases} n\mu, & n \leqslant s-1, \\ s\mu, & n \geqslant s. \end{cases} \tag{8}$$

In the following, we will suppose these values of the rates. Condition (2) is satisfied. The potential coefficients are

$$\pi_n = \begin{cases} \dfrac{\rho^n s^n}{n!}, & n \leqslant s-1, \\ \dfrac{\rho^n s^s}{s!}, & n \geqslant s, \end{cases} \qquad \text{where } \rho = \frac{\lambda}{s\mu}.$$

$X(t)$ is ergodic if and only if $\rho < 1$. In this case the steady-state distribution is

$$p_0 = \left(\sum_{n=0}^\infty \pi_n \right)^{-1} = \left(\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!(1-\rho)} \right)^{-1}, \qquad p_n = \pi_n p_0, \quad n \geqslant 0. \tag{9}$$

Karlin and McGregor [11] gave a representation for the polynomials $Q_n(x)$ and they determined the function $\psi(x)$ in the representation (5), for these values of the rates. Afterwards, they applied the resulting representation to the study of the characteristics of a $M/M/s$ queue; length of the busy period, number of customers served during a busy period, etc. Van Doorn [17] studied in detail the function $\psi(x)$ and applied the representation to the study of the stochastic monotonicity and of the exponential ergodicity of the $M/M/s$ queue length process.

The birth and death polynomials can be represented as (see [11])

$$\begin{aligned} Q_n(x) &= c_n\left(\frac{x}{\mu}, \frac{\lambda}{\mu}\right), \quad 0 \leqslant n \leqslant s, \\ Q_{s+n}(x) &= \left(\frac{1}{\sqrt{\rho}}\right)^n \left(Q_s(x)U_n\big(\alpha(x)\sqrt{\rho}\big) - \frac{1}{\sqrt{\rho}}Q_{s-1}(x)U_{n-1}\big(\alpha(x)\sqrt{\rho}\big)\right), \quad n \geqslant 1, \end{aligned} \tag{10}$$

where

$$\alpha(x) = \frac{1}{2}\left(1 - \frac{x}{\lambda} + \frac{1}{\rho}\right),$$

$c_n(x, a)$ are the Poisson–Charlier polynomials, defined by the recurrence relations

$$\begin{aligned} -xc_n(x,a) &= nc_{n-1}(x,a) - (n+a)c_n(x,a) + ac_{n+1}(x,a), \quad n \geqslant 0, \\ c_0(x,a) &= 1, \qquad c_{-1}(x,a) = 0, \end{aligned}$$

and $U_n(x)$ are the Chebyshev polynomials of the second kind, defined by the recurrence relations

$$\begin{aligned} 2xU_n(x) &= U_{n-1}(x) + U_{n+1}(x), \quad n \geqslant 0, \\ U_0(x) &= 1, \qquad U_{-1}(x) = 0. \end{aligned}$$

In [4] or [8] the reader can find the explicit expression of the Poisson–Charlier and of the Chebyshev polynomials, i.e.

$$\begin{aligned} c_n(x,a) &= \sum_{r=0}^{n} (-1)^r \binom{n}{r}\binom{x}{r}\frac{r!}{a^r}, \\ U_n(\cos\theta) &= \frac{\sin(n+1)\theta}{\sin\theta}. \end{aligned} \tag{11}$$

The function $\psi(x)$ satisfies (see [17])

(i)

$$\psi'(x) = \frac{\sqrt{\rho - \rho^2\alpha^2(x)}}{\pi\lambda\pi_s(Q_s^2(x) - Q_{s-1}(x)Q_{s+1}(x))}, \quad \lambda\left(1-\frac{1}{\sqrt{\rho}}\right)^2 \leqslant x \leqslant \lambda\left(1+\frac{1}{\sqrt{\rho}}\right)^2. \tag{12}$$

(ii) If $x \neq \lambda(1 - 1/\sqrt{\rho})^2$ is a zero of $D(z) = C(z)Q_{s-1}(z) - Q_s(z)$, where $C(z)$ is the principal value of $\alpha(z) - \sqrt{\alpha^2(z) - 1/\rho}$, then $\psi(x)$ has a jump in x of magnitude

$$\Delta\psi(x) = \frac{C(x)Q_{s-1}^{(0)}(x) - Q_s^{(0)}(x)}{D'(x)} > 0.$$

Except in these points, and outside the interval $(\lambda(1-1/\sqrt{\rho})^2, \lambda(1+1/\sqrt{\rho})^2)$, we have $\psi'(x) = 0$.

(iii) If $\rho \geqslant 1$ then $\psi(x)$ is continuous.

(iv) Is $\rho < 1$ then $\psi(x)$ has i jumps where i is one of the integers $1, 2, \ldots, s-1$. All these jumps lie on the interval $[0, \lambda(1-1/\sqrt{\rho})^2)$, and one jump always occurs at $x = 0$.

In this paper, using the representation (5), we will study the moments of the $M/M/s$ queue length process, especially the mean and the variance

$$m_i(t) = E\big[X(t) \mid X(0) = i\big], \qquad V_i(t) = E\Big[\big(X(t) - m_i(t)\big)^2 \mid X(0) = i\Big].$$

We always assume the value (8) for the rates, but some of the obtained results are valid, or can easily be generalized to birth–death processes in general, if the rates are constant from some state n onwards.

First, we obtain a representation for the factorial moments

$$M_k^{(i)}(t) = E\big[X(t)\big(X(t)-1\big)\cdots\big(X(t)-k+1\big) \mid X(0) = i\big].$$

When $\rho < 1$, let m, V, and M_k denote respectively the mean, the variance and the factorial moment of order k, of the steady-state distribution. Let denote $a_{(k)} = a(a-1)\cdots(a-k+1)$, $a_{(0)} = 1$. For $s = 1$ by convention $\sum_{n=0}^{s-2} = 0$. In order to simplify the notation we introduce the functions

$$\eta_{k,n}(x) = \left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \frac{\lambda z^{n+1} Q_{n-1}^*(x) - s\mu z^n Q_n^*(x)}{\lambda z^2 + (x - \lambda - s\mu)z + s\mu}\right]_{z=1}, \tag{13}$$

$$\Phi_{k,n}(x) = x^k \big(n_{(k)} Q_{n-1}^*(x) - k\eta_{k,n}(x)\big), \quad x \neq 0. \tag{14}$$

In an appendix we will summarize some equalities for the polynomials $Q_n(x)$, and for their duals $Q_n^*(x)$, which can be deduced from their recurrence relations, (4) and (7).

3. Representation of the factorial moments

From the next lemma, the existence and differentiability of the moments $M_k^{(i)}(t)$, for every $t \geqslant 0$, follows. This lemma is also valid for birth–death processes with constant birth rates.

Lemma 1. For each $A > 0$ and every $d = 0, 1, \ldots$

$$\sum_{j=0}^{\infty} A^j P_{ij}^{(d)}(t) \text{ converges uniformly on every finite interval } 0 \leqslant t \leqslant T.$$

Proof. Let $A > 0$ and $T > 0$. It is known that $Q_j(x)$ has j positive zeros, $x_1, \ldots, x_j$ (see [4]). Then $Q_j(-\lambda A) = \lambda^{-j}(\lambda A + x_1)(\lambda A + x_2)\cdots(\lambda A + x_j) > A^j$. From

[9, theorem 6], we get that $\sum_{j=0}^{\infty} Q_j(-\lambda A)P_{ij}(t)$ converges uniformly on $0 \leqslant t \leqslant T$, and then

$$\sum_{j=0}^{\infty} A^j P_{ij}(t) \text{ converges uniformly on } 0 \leqslant t \leqslant T.$$

From the backward equations (1), and using an induction the lemma follows. □

Now we can deduce the main result.

Theorem 1. If $\rho \neq 1$ then

$$M_k^{(i)}(t) = h_{k,i}(t) + \int_a^b Q_i(x)\frac{q_k(x)}{x^k}\mathrm{e}^{-xt}\,\mathrm{d}\psi(x), \quad i \geqslant 0,\ t \geqslant 0, \tag{15}$$

where

(i) a is the first nonzero point in the support of ψ and $b = \lambda(1 + 1/\sqrt{\rho})^2$ the last one.

(ii) $q_k(x)$ is a polynomial of degree less than or equal to $s + k - 2$ and is given by

$$q_k(x) = kx^k\left(\eta_{k,s-1}(x) - \sum_{j=0}^{s-2} j_{(k-1)}Q_j^*(x)\right). \tag{16}$$

(iii) If $\rho < 1$ then $h_{k,i}(t) = M_k$.

(iv) If $\rho > 1$ then $h_{k,i}(t)$ is a polynomial

$$h_{k,i}(t) = (\lambda - s\mu)^k t^k + \frac{\gamma_{k,k-1}(i)}{(k-1)!}t^{k-1} + \cdots + \frac{\gamma_{k,2}(i)}{2!}t^2 + \gamma_{k,1}(i)t + \gamma_{k,0}(i),$$

where

$$\gamma_{k,0}(i) = i_{(k)} - \int_0^{\infty} Q_i(x)\frac{q_k(x)}{x^k}\,\mathrm{d}\psi(x), \quad i \geqslant 0, \tag{17}$$

and $\gamma_{k,n}(i)$ for $n \geqslant 1$ are given by the recurrence relations

$$\gamma_{k,n}(i) = \mu_i\gamma_{k,n-1}(i-1) - (\lambda + \mu_i)\gamma_{k,n-1}(i) + \lambda\gamma_{k,n-1}(i+1), \quad i \geqslant 1,$$
$$\gamma_{k,n}(0) = -\lambda\gamma_{k,n-1}(0) + \lambda\gamma_{k,n-1}(1), \quad n = 1, \ldots, k-1.$$

Moreover, for every $\rho > 0$

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}M_k^{(i)}(t) = (\lambda - s\mu)^k k! + \int_0^{\infty} Q_i(x)(-1)^k\mathrm{e}^{-xt}q_k(x)\,\mathrm{d}\psi(x), \quad i \geqslant 0,\ t \geqslant 0. \tag{18}$$

Proof. Using the Karlin and McGregor representation (5) and (A.11) we obtain

$$\sum_{j=0}^{N} j_{(k)}P_{ij}^{(k+1)}(t) = \sum_{j=0}^{N} j_{(k)}\pi_j\int_0^{\infty} Q_i(x)Q_j(x)(-x)^{k+1}\mathrm{e}^{-xt}\,\mathrm{d}\psi(x)$$

$$= \int_0^\infty Q_i(x)(-x)^{k+1} \mathrm{e}^{-xt} \sum_{j=0}^{N} j_{(k)} \pi_j Q_j(x)\, \mathrm{d}\psi(x)$$

$$= \int_0^\infty Q_i(x)(-1)^{k+1} x\, \mathrm{e}^{-xt} \Bigg(x^k \sum_{j=0}^{s-2} j_{(k)} \pi_j Q_j(x) - \Phi_{k,s-1}(x) + \Phi_{k,N+1}(x) \Bigg) \mathrm{d}\psi(x).$$

Let $q_k(x) = x^k \sum_{j=0}^{s-2} j_{(k)} \pi_j Q_j(x) - \Phi_{k,s-1}(x)$. Notice, in the expression (A.12) for $\Phi_{k,n}(x)$, that $q_k(x)$ is a polynomial of degree less than or equal to $s + k - 2$. From (14) and (A.9) we obtain $q_k(x) = kx^k(\eta_{k,s-1}(x) - \sum_{j=0}^{s-2} j_{(k-1)} Q_j^*(x))$. Using $q_k(x)$, the above equality can be written as

$$\sum_{j=0}^{N} j_{(k)} P_{ij}^{(k+1)}(t) = \int_0^\infty Q_i(x)(-1)^{k+1} x\, \mathrm{e}^{-xt} \big(q_k(x) + \Phi_{k,N+1}(x)\big)\, \mathrm{d}\psi(x). \tag{19}$$

On the other hand, from (A.2) and (5), for every $d = 0, 1, \ldots$

$$\int_0^\infty x Q_i(x) x^d Q_N^*(x)\, \mathrm{e}^{-xt}\, \mathrm{d}\psi(x) = \int_0^\infty \lambda \pi_N x^d Q_i(x) \big(Q_N(x) - Q_{N+1}(x)\big)\, \mathrm{e}^{-xt}\, \mathrm{d}\psi(x)$$
$$= (-1)^d \big(\lambda P_{iN}^{(d)}(t) - \mu_{N+1} P_{iN+1}^{(d)}(t)\big).$$

From lemma 1 we get that $\lim_{N\to\infty} A^N P_{iN}^{(d)}(t) = 0$, $A > 0$, $d = 0, 1, \ldots$. Hence

$$\lim_{N\to\infty} A^N \int_0^\infty x Q_i(x) x^d Q_N^*(x)\, \mathrm{e}^{-xt}\, \mathrm{d}\psi(x) = 0, \quad A > 0,\ d = 0, 1, \ldots.$$

From this limit and the expression (A.12) for $\Phi_{k,n}(x)$, we get

$$\lim_{N\to\infty} \int_0^\infty x Q_i(x) \Phi_{k,N}(x)\, \mathrm{e}^{-xt}\, \mathrm{d}\psi(x) = 0, \quad i \geqslant 0,\ t \geqslant 0.$$

Hence from (19) we obtain

$$\sum_{j=0}^{\infty} j_{(k)} P_{ij}^{(k+1)}(t) = \int_0^\infty Q_i(x)(-1)^{(k+1)} x\, \mathrm{e}^{-xt} q_k(x)\, \mathrm{d}\psi(x). \tag{20}$$

This representation for the $k + 1$ derivative of the factorial moment of order k is valid for every $\rho > 0$. If $\rho \neq 1$ there is a first nonzero point in the support of ψ. In this case, let a the first nonzero point in the support of ψ and b the last one. Then

$$\sum_{j=0}^{\infty} j_{(k)} P_{ij}^{(k+1)}(t) = \int_a^b Q_i(x)(-1)^{(k+1)} x\, \mathrm{e}^{-xt} q_k(x)\, \mathrm{d}\psi(x).$$

If $\rho \neq 1$, using lemma 1 and integrating $k+1$ times we get

$$M_k^{(i)}(t) = \sum_{n=0}^{k} \frac{\gamma_{k,n}(i)}{n!} t^n + \int_a^b Q_i(x) \frac{q_k(x)}{x^k} \mathrm{e}^{-xt} \, \mathrm{d}\psi(x), \tag{21}$$

where

$$\gamma_{k,n}(i) = \sum_{j=0}^{\infty} j_{(k)} P_{ij}^{(n)}(0) - \int_a^b Q_i(x)(-1)^n \frac{q_k(x)}{x^{k-n}} \, \mathrm{d}\psi(x), \quad n = 0, 1, \ldots, k. \tag{22}$$

Let $\rho < 1$. From (5) and (9) it follows that $P_{ij}(t) = \pi_i^{-1} \pi_j P_{ji}(t) \leqslant \pi_i^{-1} p_j \times \sum_{l=0}^{\infty} \pi_l$. So for $N > 0$

$$\left| \sum_{j=0}^{\infty} P_{ij}(t) j_{(k)} - \sum_{j=0}^{\infty} p_j j_{(k)} \right| \leqslant \sum_{j=0}^{N} \left| P_{ij}(t) - p_j \right| j_{(k)} + \left(1 + \frac{\sum_{l=0}^{\infty} \pi_l}{\pi_i} \right) \sum_{j=N}^{\infty} j_{(k)} p_j .$$

It follows that $\lim_{t \to \infty} M_k^{(i)}(t) = M_k$, $i \geqslant 0$. Hence from (21) we obtain $\gamma_{k,k}(i) = \gamma_{k,k-1}(i) = \cdots = \gamma_{k,1}(i) = 0$, $\gamma_{k,0}(i) = M_k$, $i \geqslant 0$. This proves (iii).

Let now $\rho > 1$. Using the Karlin and McGregor representation (5), the orthogonality property (6), and (A.11) we obtain that for $N > \max\{i+n, s\}$

$$\begin{aligned}
\sum_{j=0}^{\infty} j_{(k)} P_{ij}^{(n)}(0) &= \sum_{j=0}^{\infty} j_{(k)} \pi_j \int_a^b Q_i(x) Q_j(x) (-x)^n \, \mathrm{d}\psi(x) \\
&= \sum_{j=0}^{N-1} j_{(k)} \pi_j \int_a^b Q_i(x) Q_j(x) (-x)^n \, \mathrm{d}\psi(x) \\
&= \int_a^b Q_i(x)(-x)^n \left(\frac{q_k(x)}{x^k} + \frac{\Phi_{k,N}(x)}{x^k} \right) \mathrm{d}\psi(x).
\end{aligned} \tag{23}$$

Then from (22) we have

$$\gamma_{k,n}(i) = \int_a^b Q_i(x)(-x)^n \frac{\Phi_{k,N}(x)}{x^k} \, \mathrm{d}\psi(x), \quad N > \max\{i+n, s\}. \tag{24}$$

From (A.2) and the orthogonality property (6), we have $\int_a^b x x^l Q_N^*(x) \, \mathrm{d}\psi(x) = 0$, $l \geqslant 0$, $N > l$. Using this equality, (A.3), (6), (A.12), and for N large enough, we obtain

$$\begin{aligned}
\gamma_{k,k}(i) &= \int_a^b Q_i(x)(-1)^k \Phi_{k,N}(x) \, \mathrm{d}\psi(x) \\
&= \int_a^b Q_i(x)(\lambda - s\mu)^{k-1} k! \big(\lambda Q_{N-1}^*(x) - s\mu Q_N^*(x) \big) \, \mathrm{d}\psi(x) \\
&= (\lambda - s\mu)^k k!.
\end{aligned} \tag{25}$$

Using (24) and the recurrence relations for $\{Q_n(x)\}$, we obtain that $\{\gamma_{k,n}(i)\}$ satisfies the recurrence relations given in (iv). Taking $n = 0$ in (22) we obtain the representation of $\gamma_{k,0}(i)$ given in (iv).

It remains to prove (18) for every $\rho > 0$. Integrating both sides of the equality (20), we get that for some constant $\gamma(i)$

$$\sum_{j=0}^{\infty} j_{(k)} P_{ij}^{(k)}(t) = \gamma(i) + \int_0^{\infty} Q_i(x)(-1)^k \,\mathrm{e}^{-xt} q_k(x)\,\mathrm{d}\psi(x).$$

In the same way as we obtained (24), we obtain $\gamma(i) = \int_0^{\infty} Q_i(x)(-1)^k \Phi_{k,N}(x)\,\mathrm{d}\psi(x)$, for N large enough, and in an analog way as in (25), we get $\gamma(i) = (\lambda - s\mu)^k k!$. This completes the proof. □

Remark. $\gamma_{k,0}(i)$ may be calculated from (17), using the equalities (see [10, appendix A])

$$\begin{aligned}
&\int_0^{\infty} \frac{Q_i(x)Q_j(x)}{x}\,\mathrm{d}\psi(x) = \int_0^{\infty} \frac{Q_i(x)}{x}\,\mathrm{d}\psi(x) = \frac{1}{\lambda}\sum_{n=i}^{\infty}\frac{1}{\pi_n}, \quad i \geqslant j,\ \rho \geqslant 1,\\
&\int_0^{\infty} \frac{Q_i(x)}{x^r}\,\mathrm{d}\psi(x) = \frac{1}{\lambda}\sum_{n=i}^{\infty}\frac{1}{\pi_n}\sum_{l=0}^{n}\pi_l \int_0^{\infty}\frac{Q_l(x)}{x^{r-1}}\,\mathrm{d}\psi(x), \qquad \rho \geqslant 1.
\end{aligned} \tag{26}$$

In section 5 the polynomials $q_k(x)$ and $h_{k,i}(t)$ are given explicitly for some values of the number of servers s and of the order of the moment k.

The representation given in (15) extends the representations given in [2] for the first two moments of the $M/M/1$ queue length process in the ergodic case.

Integrating (18), we can get representations for the moments. For example, for $\rho \geqslant 1$.

$$m_i(t) = (\lambda - s\mu)t + i + \int_0^{\infty} Q_i(x) q_1(x) \frac{\mathrm{e}^{-xt} - 1}{x}\,\mathrm{d}\psi(x). \tag{27}$$

From (16) we have

$$q_1(x) = \lambda Q_{s-2}^*(x) - s\mu Q_{s-1}^*(x) - x\sum_{j=0}^{s-2} Q_j^*(x). \tag{28}$$

From (A.6) and (A.7) we get

$$q_1(x) = -\mu\sum_{j=0}^{s-1} Q_j^*(x) = \mu\frac{s\mu Q_s^*(x) - \lambda Q_{s-1}^*(x)}{x - \mu}. \tag{29}$$

4. Limit behaviour

From theorem 1, we deduce that in the ergodic case ($\rho < 1$) the difference between the factorial moment and the factorial moment of the steady-state distribution decreases exponentially

$$M_k^{(i)}(t) = M_k + \mathrm{O}\big(\mathrm{e}^{-at}\big), \quad t \to \infty,\ i \geqslant 0,$$

where a is the first nonzero point in the support of $\psi(x)$, called the decay parameter (see [17]). If $\Delta\psi(a) = 0$ it could be written $\mathrm{o}(\mathrm{e}^{-at})$ instead of $\mathrm{O}(\mathrm{e}^{-at})$.

Now we will deal with the limit behaviour in the transient case ($\rho > 1$) as well as in the null recurrent case ($\rho = 1$).

4.1. Case $\rho > 1$

In this case, from theorem 1, we get for $t \to \infty$

$$M_k^{(i)}(t) = (\lambda - s\mu)^k t^k + \frac{\gamma_{k,k-1}(i)}{(k-1)!} t^{k-1} + \cdots + \gamma_{k,1}(i)t + \gamma_{k,0}(i) + \mathrm{o}\big(\mathrm{e}^{-at}\big).$$

In particular, the mean becomes

$$m_i(t) = (\lambda - s\mu)t + \gamma_{1,0}(i) + \mathrm{o}\big(\mathrm{e}^{-at}\big), \quad i \geqslant 0. \tag{30}$$

For the variance we have

$$\begin{aligned} V_i(t) &= M_2^{(i)}(t) + m_i(t) - m_i(t)^2 = h_{2,i}(t) + h_{1,i}(t) - h_{1,i}^2(t) + \mathrm{o}\big(t\mathrm{e}^{-at}\big) \\ &= \big[\gamma_{2,1}(i) + (\lambda - s\mu)\big(1 - 2\gamma_{1,0}(i)\big)\big]t + \gamma_{2,0}(i) + \gamma_{1,0}(i) - \gamma_{1,0}^2(i) + \mathrm{o}\big(t\mathrm{e}^{-at}\big). \end{aligned}$$

Using (24), (A.12), (A.3), and (6), we obtain that for N large enough

$$\gamma_{2,1}(i) - 2(\lambda - s\mu)\gamma_{1,0}(i) = -\int_a^b Q_i(x)\left(\frac{\Phi_{2,N}(x)}{x} + 2(\lambda - s\mu)\frac{\Phi_{1,N}(x)}{x}\right) \mathrm{d}\psi(x)$$

$$= \int_a^b Q_i(x)(2\mu N s - xN_{(2)})Q_{N-1}^*(x) + 2s\mu(1-N)Q_N^*(x)\,\mathrm{d}\psi(x) = 2s\mu.$$

Therefore, when $t \to \infty$

$$V_i(t) = (\lambda + s\mu)t + v_i + \mathrm{o}\big(t\mathrm{e}^{-at}\big), \quad i \geqslant 0, \tag{31}$$

where $v_i = \gamma_{2,0}(i) + \gamma_{1,0}(i) - \gamma_{1,0}^2(i)$.

When $\rho > 1$ and $t \to \infty$, the number of customers in the queue tends to increase. So, if in some specific moment there is a large number i of customers in the queue, the probability of the queue emptying, from that moment, is rather small. Therefore, t units of time after that, the number of customers will have approximately the same distribution as $Y_i(t) = i + A_t - S_t$, where A_t and S_t are independent random variables, with Poisson distribution of mean λt and $s\mu t$, respectively. In fact, using [10, theorem 10] it would be easy to prove $|P_{ij}(t) - P(Y_i(t) = j)| \leqslant 2/\rho^{i-s+1}$, $i > s$. Note that the mean of $Y_i(t)$ is $(\lambda - s\mu)t + i$ and the variance is $(\lambda + s\mu)t$ (for more information about $Y_i(t)$

see [6]). This partially explains the limit behavior obtained for the mean and variance when $\rho > 1$. The following theorem gives more precise information about the relation between the moments of $X(t)$ and the moments of $Y_i(t)$.

Theorem 2. Let $\rho > 1$. If $i \to \infty$ and $t \to \infty$ then

(i) $M_k^{(i)}(t) = E\big[Y_i(t)_{(k)}\big] + \mathrm{o}\Big(\dfrac{1}{\rho^{i/2}}\Big)t^{k-1} + \cdots + \mathrm{o}\Big(\dfrac{1}{\rho^{i/2}}\Big)t + \mathrm{o}\Big(\dfrac{1}{\rho^{i/2}}\Big) + R_1(t,i)$,

(ii) $V_i(t) = (\lambda + s\mu)t + \mathrm{o}\Big(\dfrac{1}{\rho^{i/2}}\Big) + tR_2(t,i)$

where $R_j(t,i) = \mathrm{o}(1/\rho^{i/2})$ and $R_j(t,i) = \mathrm{o}(\mathrm{e}^{-at})$, $j = 1, 2$.

Proof. If $f(x)$ is a continuous function on $[a,b]$, then $c_i = \pi_i^{1/2}\int_a^b Q_i(x)f(x)\,\mathrm{d}\psi(x)$ are the Fourier coefficients of $f(x)$ relative to the orthonormal system $\{\pi_i^{1/2}Q_i(x)\}$, and then $\lim_{i\to\infty} c_i = 0$. Since $\pi_i = \rho^i s^s/s!$, $i \geqslant s$, we have $\int_a^b Q_i(x)f(x)\,\mathrm{d}\psi(x) = \mathrm{o}(1/\rho^{i/2})$.

Consequently, from (22)

$$\gamma_{k,n}(i) = \sum_{j=0}^{\infty} j_{(k)}P_{ij}^{(n)}(0) - \int_a^b Q_i(x)(-1)^n\frac{q_k(x)}{x^{k-n}}\,\mathrm{d}\psi(x) = \sum_{j=0}^{\infty} j_{(k)}P_{ij}^{(n)}(0) + \mathrm{o}\Big(\frac{1}{\rho^{i/2}}\Big) \tag{32}$$

and

$$\int_a^b Q_i(x)\frac{q_k(x)}{x^k}\,\mathrm{e}^{-xt}\,\mathrm{d}\psi(x) = \mathrm{o}\Big(\frac{1}{\rho^{i/2}}\Big). \tag{33}$$

Let $d_{k,n} = \sum_{j=0}^{\infty} j_{(k)}P_{ij}^{(n)}(0)$. From the backward equations it follows that

$$\begin{aligned} &d_{k,n}(i) = s\mu d_{k,n-1}(i-1) - (\lambda + s\mu)d_{k,n-1}(i) + \lambda d_{k,n-1}(i+1), \quad i \geqslant s,\\ &d_{k,0}(i) = i_{(k)}. \end{aligned} \tag{34}$$

On the other hand, the factorial moment of $Y_i(t)$

$$E\big[Y_i(t)_{(k)}\big] = \left(\frac{\mathrm{d}^k}{\mathrm{d}z^k}E\big[z^{Y_i(t)}\big]\right)_{z=1} = \left(\frac{\mathrm{d}^k}{\mathrm{d}z^k}z^i\,\mathrm{e}^{t\lambda(z-1)+s\mu t(1/z-1)}\right)_{z=1},$$

is a polynomial of degree k. If we denote

$$E\big[Y_i(t)_{(k)}\big] = \frac{\beta_{k,k}(i)}{k!}t^k + \frac{\beta_{k,k-1}(i)}{(k-1)!}t^{k-1} + \cdots + \beta_{k,0}(i),$$

then $\beta_{k,n}(i)$ satisfies the recurrence relations

$$\begin{aligned} &\beta_{k,n}(i) = s\mu\beta_{k,n-1}(i-1) - (\lambda + s\mu)\beta_{k,n-1}(i) + \lambda\beta_{k,n-1}(i+1), \quad i \in \mathbb{Z},\\ &\beta_{k,0}(i) = i_{(k)}. \end{aligned} \tag{35}$$

From (34) and (35) it follows that $\beta_{k,n}(i) = d_{k,n}(i) = \sum_{j=0}^{\infty} j_{(k)} P_{ij}^{(n)}(0)$, $i \geqslant s+n-1$, and then from (32)

$$\gamma_{k,n}(i) = \beta_{k,n}(i) + \mathrm{o}\left(\frac{1}{\rho^{i/2}}\right).$$

Hence (i) follows from theorem 1 and (33). Taking into account that the variance of $Y_i(t)$ is $(\lambda + s\mu)t$ and with a similar calculation as we realised to deduce (31), (ii) follows. □

Corollary 1. Let $\rho > 1$. When $i \to \infty$

$$M_k^{(i)}(t) = E\big[Y_i(t)_{(k)}\big] + \mathrm{o}\left(\frac{1}{\rho^{i/2}}\right).$$

4.2. Case $\rho = 1$

In the following two theorems, we will obtain, from the representations (27) and (18), the limit behavior of the mean and of the variance for this case.

Theorem 3. Let $\rho = 1$. When $t \to \infty$

$$m_i(t) = \sqrt{\frac{4\lambda t}{\pi}} + D + \frac{d_i}{\sqrt{t}} + \mathrm{O}\left(\frac{1}{t\sqrt{t}}\right), \quad i \geqslant 0,$$

where

$$D = s - \frac{1}{2} - \frac{1}{\pi_s}\sum_{j=0}^{s-1}\pi_j, \qquad d_0 = \frac{1}{\sqrt{\lambda\pi}}\left(\frac{1}{8} - s - \sum_{l=0}^{s-2}\frac{1}{\pi_l}\sum_{j=0}^{l}\pi_j + \left(\frac{1}{\pi_s}\sum_{j=0}^{s-1}\pi_j\right)^2\right) \tag{36}$$

and d_i for $i \geqslant 1$ are given by the recurrence relations

$$\mu_i d_{i-1} - (\lambda + \mu_i)d_i + \lambda d_{i+1} = \sqrt{\frac{\lambda}{\pi}}, \quad i \geqslant 0. \tag{37}$$

Proof. Let $\rho = 1$. The representation for the mean (27) in this case is

$$m_i(t) = i + \int_0^{4\lambda} Q_i(x)q_1(x)\frac{\mathrm{e}^{-xt} - 1}{x}\psi'(x)\,\mathrm{d}x.$$

Hence, observing the value of $\psi'(x)$ in (12) we have $Q_i(x)q_1(x)\psi'(x)\sqrt{x} = f_i(x)$ for $0 < x < 4\lambda$, where

$$f_i(x) = Q_i(x)q_1(x)\frac{\sqrt{1 - x/(4\lambda)}}{\sqrt{\lambda\pi}\,Q(x)} \quad \text{and} \quad Q(x) = \frac{\lambda\pi_s(Q_s^2(x) - Q_{s-1}(x)Q_{s+1}(x))}{x}.$$

From (A.8)

$$Q(x) = Q_s^*(x)Q_s(x) - Q_{s-1}^*(x)Q_{s+1}(x). \tag{38}$$

Hence

$$m_i(t) = i + \int_0^{4\lambda} f_i(x)\frac{e^{-xt}-1}{x\sqrt{x}}\,dx$$
$$= i + \int_0^{4\lambda} \frac{f_i(0)-f_i(x)}{x\sqrt{x}}\,dx + \int_0^{4\lambda} \frac{f_i(x)-f_i(0)}{x\sqrt{x}}\,e^{-xt}\,dx$$
$$+ \int_0^{4\lambda} f_i(0)\frac{e^{-xt}-1}{x\sqrt{x}}\,dx.$$

Since $Q_s^2(x) - Q_{s-1}(x)Q_{s+1}(x)$ has no zeros on $(0, 4\lambda]$ (see [17, p. 52]) and $Q_s^*(0)Q_s(0) - Q_{s-1}^*(0)Q_{s+1}(0) = \pi_s \neq 0$ (see (A.5)), it follows that $f_i(x)$ is bounded on $[0, 4\lambda]$ and analytic in $x = 0$. So using the dominated convergence theorem we obtain

$$\lim_{t\to\infty} t\sqrt{t}\left(\int_0^{4\lambda} \frac{f_i(x)-f_i(0)}{x\sqrt{x}}\,e^{-xt}\,dx - \frac{f_i'(0)\sqrt{\pi}}{\sqrt{t}}\right)$$
$$= \lim_{t\to\infty} t\left(\int_0^{4\lambda t} \frac{f_i(u/t)-f_i(0)}{u/t}\frac{e^{-u}}{\sqrt{u}}\,du - \int_0^{\infty} \frac{f_i'(0)\,e^{-u}}{\sqrt{u}}\,du\right)$$
$$= \lim_{t\to\infty} \int_0^{4\lambda t} \frac{f_i(u/t)-f_i(0)-(u/t)f_i'(0)}{(u/t)^2}\sqrt{u}\,e^{-u}\,du + tf_i'(0)\int_{4\lambda t}^{\infty} \frac{e^{-u}}{\sqrt{u}}\,du$$
$$= \frac{f_i''(0)\sqrt{\pi}}{2}.$$

Moreover

$$\int_0^{4\lambda} \frac{e^{-xt}-1}{x\sqrt{x}}\,dx = \frac{1}{\sqrt{\lambda}}\left(1-e^{-4\lambda t}\right) - 2\sqrt{\pi t} + 2\sqrt{t}\int_{4\lambda t}^{\infty}\frac{e^{-u}}{\sqrt{u}}\,du$$
$$= -2\sqrt{\pi t} + \frac{1}{\sqrt{\lambda}} + O\left(\frac{1}{t\sqrt{t}}\right).$$

Therefore

$$m_i(t) = -2f_i(0)\sqrt{\pi t} + a_i + \frac{d_i}{\sqrt{t}} + O\left(\frac{1}{t\sqrt{t}}\right),$$

where

$$a_i = i + \frac{f_i(0)}{\sqrt{\lambda}} + \int_0^{4\lambda} \frac{f_i(0)-f_i(x)}{x\sqrt{x}}\,dx \quad \text{and} \quad d_i = f_i'(0)\sqrt{\pi}. \tag{39}$$

Taking $\rho = 1$ in (29) and using (A.1), we get $q_1(x) = \mu\lambda\pi_s Q_s(x)/(x-\mu)$. Then from (A.5), $q_1(0) = -\lambda\pi_s$ and $q_1'(0) = -\lambda\pi_s(1/\mu + Q_s'(0))$. Thus we easily obtain $f_i(0) = -\sqrt{\lambda}/\pi$, and with a simple computation using (38), (A.2), and (A.5), the equality given in (36) for d_0 follows.

From the recurrence relations (4) for $\{Q_n(x)\}$ it follows that

$$\mu_i f_{i-1}(x) - (\lambda+\mu_i)f_i(x) + \lambda f_{i+1}(x) = -xf_i(x), \quad i \geqslant 0. \tag{40}$$

From which we obtain that $d_i = f_i'(0)\sqrt{\pi}$ satisfies the recurrence relations (37). From the backward equations (1) we get $m_i'(0) = \lambda - \mu_i$. Then using (39), (40), and (18) we obtain

$$\mu_i a_{i-1} - (\lambda + \mu_i) a_i + \lambda a_{i+1} = \lambda - \mu_i + \int_0^{4\lambda} \frac{f_i(x)}{\sqrt{x}}\,\mathrm{d}x$$
$$= \lambda - \mu_i + \int_0^{4\lambda} Q_i(x) q_1(x) \psi'(x)\,\mathrm{d}x = 0,$$

for $i \geqslant 0$. Hence $a_i = a_0$, $i \geqslant 0$. It remains to determine a_0.

From (39)

$$\begin{aligned}
a_0 &= -\frac{1}{\pi} + \int_0^{4\lambda} \frac{-\sqrt{\lambda}/\pi - f_0(x)}{x\sqrt{x}}\,\mathrm{d}x \\
&= -\frac{1}{\pi} - \int_0^{4\lambda} \frac{f_0(x) + (\sqrt{\lambda}/\pi)\sqrt{1 - x/(4\lambda)} + \sqrt{\lambda}/\pi - (\sqrt{\lambda}/\pi)\sqrt{1 - x/(4\lambda)}}{x\sqrt{x}}\,\mathrm{d}x \\
&= -\frac{1}{2} - \int_0^{4\lambda} \left(\frac{q_1(x)}{\sqrt{\lambda}\pi\, Q(x)} + \frac{\sqrt{\lambda}}{\pi} \right) \frac{\sqrt{1 - x/(4\lambda)}}{x\sqrt{x}}\,\mathrm{d}x \\
&= -\frac{1}{2} - \int_0^{\infty} \frac{q_1(x) + \lambda Q(x)}{x}\,\mathrm{d}\psi(x).
\end{aligned}$$

Using (28), (38), (A.4) and (7) we obtain

$$\frac{q_1(x) + \lambda Q(x)}{x} = -\sum_{j=0}^{s-1} Q_j^*(x) + Q_{s-1}^*(x)\frac{Q_s^*(x)}{\pi_s} - \pi_s Q_s(x) \sum_{j=0}^{s-1} \frac{Q_j^*(x)}{\pi_j}.$$

Hence, from (A.3) and (6)

$$a_0 = -\frac{1}{2} + s - \frac{1}{\pi_s} \int_0^{\infty} Q_{s-1}^*(x) Q_s^*(x)\,\mathrm{d}\psi(x) = s - \frac{1}{2} - \frac{1}{\pi_s}\sum_{j=0}^{s-1} \pi_j. \qquad \square$$

Theorem 4. Let $\rho = 1$. When $t \to \infty$

$$V_i(t) = 2\lambda\left(1 - \frac{2}{\pi}\right) t + D - D^2 - 4\sqrt{\frac{\lambda}{\pi}} d_i + b_i + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right), \quad i \geqslant 0,$$

where

$$b_0 = \frac{B}{\sqrt{\lambda}} + \int_0^{4\lambda} \frac{B\sqrt{x} - q_2(x)\psi'(x)}{x^2}\,\mathrm{d}x, \qquad B = \frac{2\sqrt{\lambda}}{\pi}\left(2 - s + \frac{\sum_{j=0}^{s-2}\pi_j}{\pi_s}\right) \quad (41)$$

and b_i for $i \geqslant 1$, are given by

$$\mu_i b_{i-1} - (\lambda + \mu_i) b_i + \lambda b_{i+1} = 2\lambda, \quad i \geqslant 0.$$

Proof. From the representation (18)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} M_2^{(i)}(t) = \int_0^{4\lambda} Q_i(x)\,\mathrm{e}^{-xt} q_2(x)\psi'(x)\,\mathrm{d}x.$$

From (16), we have $q_2(x) = xp(x)$, where

$$p(x) = 2\lambda\big[(s-1)Q_{s-2}^*(x) + (2-s)Q_{s-1}^*(x)\big] - 2x\sum_{j=0}^{s-2} jQ_j^*(x). \tag{42}$$

Integrating

$$\frac{\mathrm{d}}{\mathrm{d}t} M_2^{(i)}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} M_2^{(i)}(t)\right)_{t=0} + \int_0^{4\lambda} Q_i(x)\big(1-\mathrm{e}^{-xt}\big)p(x)\psi'(x)\,\mathrm{d}x.$$

With a similar calculation as we realised in (23), we obtain that if $N > \max\{i+1, s\}$, then

$$\sum_{j=0}^{\infty} j_{(2)} P'_{ij}(0) = -\int_0^{4\lambda} Q_i(x)p(x)\psi'(x)\,\mathrm{d}x - \int_0^{4\lambda} Q_i(x)\frac{\Phi_{2,N}(x)}{x}\,\mathrm{d}\psi(x).$$

From (14), (A.1), (A.3) and (6), we obtain that for N large enough

$$\begin{aligned}\int_0^{4\lambda} Q_i(x)\frac{\Phi_{2,N}(x)}{x}\,\mathrm{d}\psi(x) &= \int_0^{4\lambda} Q_i(x)\big[Q_{N-1}^*(x)(xN_{(2)} - 2\lambda N) \\ &\qquad + 2\lambda(N-1)Q_N^*(x)\big]\,\mathrm{d}\psi(x) \\ &= -2\lambda.\end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} M_2^{(i)}(t) = 2\lambda - \int_0^{4\lambda} Q_i(x)\,\mathrm{e}^{-xt} p(x)\psi'(x)\,\mathrm{d}x. \tag{43}$$

Integrating again

$$M_2^{(i)}(t) = 2\lambda t + i(i-1) + \int_0^{4\lambda} g_i(x)\frac{\mathrm{e}^{-xt}-1}{x\sqrt{x}}\,\mathrm{d}x,$$

where

$$g_i(x) = Q_i(x)p(x)\frac{\sqrt{1-x/(4\lambda)}}{\sqrt{\lambda}\pi(Q_s^*(x)Q_s(x) - Q_{s-1}^*(x)Q_{s+1}(x))}.$$

Using the same techniques as in the proof of theorem 3 we get

$$M_2^{(i)}(t) = 2\lambda t - 2g_i(0)\sqrt{\pi t} + b_i + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right), \tag{44}$$

where

$$b_i = i(i-1) + \frac{g_i(0)}{\sqrt{\lambda}} + \int_0^{4\lambda} \frac{g_i(0) - g_i(x)}{x\sqrt{x}}\,\mathrm{d}x.$$

From (42) we obtain $g_i(0) = B$, where B is given in (41).

From the recurrence relations (4) for $\{Q_n\}$, the backward equations (1) and (43) we obtain

$$\mu_i b_{i-1} - (\lambda + \mu_i)b_i + \lambda b_{i+1} = \left(\frac{\mathrm{d}}{\mathrm{d}t} M_2^{(t)}(t)\right)_{t=0} + \int_0^{4\lambda} Q_i(x)p(x)\psi'(x)\,\mathrm{d}x = 2\lambda.$$

From the limit behavior of the factorial moment of second order (44) and the limit behavior of the mean given in theorem 3 we obtain the limit behavior of the variance given in the theorem. □

5. Examples: $M/M/1$ and $M/M/2$

Let $A = \rho - 1$. Some values of $q_k(x)$ and $h_{k,i}(t)$ in theorem 1 are:

- $M/M/1$

$$\begin{aligned} q_1(x) &= -\mu,\ h_{1,i}(t) = (\lambda-\mu)t + i + \frac{1}{\rho^i A}, \quad \rho > 1. \\ q_2(x) &= 2\mu(x + \lambda - \mu), \\ h_{2,i}(t) &= (\lambda-\mu)^2 t^2 + \left(2(\lambda-\mu)i + 2\mu + 2\mu\frac{1}{\rho^i}\right)t \\ &\quad + i(i-1) - 2\frac{1}{A\rho^i}\left(i + 2 + \frac{1}{A}\right), \quad \rho > 1. \end{aligned} \tag{45}$$

  and

$$h_{k,i}(t) = \frac{k!\rho^k}{(1-\rho)^k}, \quad \rho < 1,\ k \geqslant 1.$$

- $M/M/2$

$$\begin{aligned} q_1(x) &= x - \lambda - 2\mu,\ h_{1,i}(t) = (\lambda - 2\mu)t + i + \frac{\rho+1}{\rho^i 2A} - \delta_{i0}\frac{A}{2\rho}, \quad \rho > 1. \\ q_2(x) &= (8\mu - 2\lambda)x + 2\lambda^2 - 8\mu^2, \\ h_{2,i}(t) &= (\lambda - 2\mu)^2 t^2 + \left(4\mu + 2(\lambda-2\mu)i + \frac{\lambda+2\mu}{\rho^i} - \delta_{i0}\frac{(\lambda-2\mu)^2}{\lambda}\right)t \\ &\quad + i(i-1) - \frac{1}{2A^2\rho^{i+1}}\big(7A^2 + 10A + 4 + \big(2A^3 + 6A^2 + 4A\big)i\big) \\ &\quad - \delta_{i0}\frac{5A+4}{2\rho^2}, \quad \rho > 1. \end{aligned} \tag{46}$$

and

$$h_{k,i}(t) = \frac{2k!\rho^k}{(1+\rho)(1-\rho)^k}, \quad \rho < 1,\ k \geqslant 1.$$

5.1. *Trigonometric integral representations*

For a $M/M/1$ system with $\rho \neq 1$, theorem 1 gives the representations

$$M_k^{(i)}(t) = h_{k,i}(t) + \frac{1}{\pi}\int_{(\sqrt{\lambda}-\sqrt{\mu})^2}^{(\sqrt{\lambda}+\sqrt{\mu})^2} Q_i(x)\frac{q_k(x)}{x^{k+1}}\,\mathrm{e}^{-xt}\sqrt{\rho-\rho^2\alpha^2(x)}\,\mathrm{d}x.$$

Making the change of variable $x = \lambda + \mu - 2\sqrt{\lambda\mu}\cos\theta$ and using (10) and (11) we obtain

$$M_k^{(i)}(t) = h_{k,i}(t) + \frac{2\lambda}{\pi\rho^{i/2}}\int_0^{\pi}\left(\sin(i+1)\theta - \frac{\sin i\theta}{\sqrt{\rho}}\right)\frac{q_k(x)}{x^{k+1}}\sin\theta\mathrm{e}^{-xt}\,\mathrm{d}\theta.$$

This representation for the special case of the mean ($k = 1$), is given in [15], and is proposed by Abate and Whitt [3] to calculate the mean numerically. For the special case $k = 2$ is given in [16].

For a $M/M/2$ system with $\rho > 1/9$ and $\rho \neq 1$, theorem 1 gives the representation

$$M_k^{(i)}(t) = h_{k,i}(t) + \frac{2\mu\lambda}{\pi}\int_{(\sqrt{\lambda}-\sqrt{2\mu})^2}^{(\sqrt{\lambda}+\sqrt{2\mu})^2} Q_i(x)\frac{q_k(x)}{x^{k+1}}\,\mathrm{e}^{-xt}\frac{\sqrt{\rho-\rho^2\alpha^2(x)}}{\lambda^2-2x\lambda-x\mu+x^2+2\lambda\mu}\,\mathrm{d}x. \tag{47}$$

Making the change of variable $x = \lambda + 2\mu - 2\sqrt{2\lambda\mu}\cos\theta$, we obtain

$$M_k^{(i)}(t) = h_{k,i}(t) + \frac{2\lambda}{\pi\rho^{i/2}}\int_0^{\pi} u_i\frac{q_k(x)}{x^{k+1}}\,\mathrm{e}^{-xt}\,\mathrm{d}\theta, \tag{48}$$

where

$$u_i = \frac{\sin\theta[(w-1)\sin(i-1)\theta + (2\rho^{-1/2} - 4\cos\theta)\sin(i-2)\theta]}{1+w}, \quad i \geqslant 1,$$

$$u_0 = \frac{2\sin^2\theta}{1+w}, \qquad w = \frac{1}{\rho} - 6\frac{1}{\sqrt{\rho}}\cos\theta + 8\cos^2\theta.$$

If $\rho \leqslant 1/9$ then $\psi(x)$ has a jump in $a = (2\lambda + \mu + c)/2$, of magnitude $\Delta\psi(a) = \lambda(3c - \mu)/2ac$, where $c = \sqrt{\mu(\mu - 4\lambda)}$. Therefore if $\rho \leqslant 1/9$, we must add $Q_i(a)q_k(a)\,\mathrm{e}^{-at}\Delta\psi(a)/a^k$ in the right side of the equalities (47) and (48).

5.2. *Limit behaviour*

From theorem 1, using the values given in (45) for $q_k(x)$ and $h_{k,i}(t)$, we obtain that for a $M/M/1$ system with $\rho > 1$, when $t \to \infty$

$$m_i(t) = (\lambda - \mu)t + i + \frac{1}{\rho^i(\rho-1)} + \mathrm{o}\big(\mathrm{e}^{-(\sqrt{\lambda}-\sqrt{\mu})^2 t}\big), \quad i \geqslant 0, \tag{49}$$

$$V_i(t) = (\lambda+\mu)t + \frac{1-4Ai-3\rho-\rho^{-i}}{A^2\rho^i} + \mathrm{o}\big(t\,\mathrm{e}^{-(\sqrt{\lambda}-\sqrt{\mu})^2 t}\big), \quad i \geqslant 0. \tag{50}$$

From theorems 3 and 4, using the expression of $\psi(x)$ given in (12) we deduce when $\rho = 1$ and $t \to \infty$

$$m_i(t) = \sqrt{\frac{4\lambda t}{\pi}} - \frac{1}{2} + \frac{(2i+1)^2}{8\sqrt{\pi\lambda t}} + \mathrm{O}\Big(\frac{1}{t\sqrt{t}}\Big), \qquad i \geqslant 0, \tag{51}$$

$$V_i(t) = 2\lambda\Big(1-\frac{2}{\pi}\Big)t + i^2 + i + \frac{1}{4} - \frac{(2i+1)^2}{2\pi} + \mathrm{O}\Big(\frac{1}{\sqrt{t}}\Big), \quad i \geqslant 0. \tag{52}$$

The limit behaviour of the mean (49) and (51) can be found in [5] and the coefficients of t in the equalities (50) and (52) for the limit behaviour of the variance, can be found in [13].

From theorem 1, with the values given in (46) for $q_k(x)$ and $h_{k,i}(t)$, we obtain that for a $M/M/2$ system with $\rho > 1$, when $t \to \infty$

$$m_i(t) = (\lambda-2\mu)t + i + \frac{\rho+1}{2\rho^i(\rho-1)} - \delta_{i0}\frac{A}{2\rho} + \mathrm{o}\big(\mathrm{e}^{-(\sqrt{\lambda}-\sqrt{2\mu})^2 t}\big),$$
$$V_i(t) = (\lambda+2\mu)t + v_i + \mathrm{o}\big(t\mathrm{e}^{-(\sqrt{\lambda}-\sqrt{2\mu})^2 t}\big),$$

where

$$v_i = \frac{8(1-\rho^2)i + 2\rho^2 - 14\rho + 6 - 2\rho^{-1} - (\rho+1)^2\rho^{-i}}{4A^2\rho^i} + \delta_{i0}\frac{4\mu^2-\lambda^2-8\lambda\mu}{4\lambda^2}.$$

From theorems 3 and 4, using the expression of $\psi(x)$ given in (12) we obtain that if $\rho = 1$ and $t \to \infty$

$$m_i(t) = \sqrt{\frac{4\lambda t}{\pi}} + \frac{(2i)^2 - 1 - 4\delta_{i0}}{8\sqrt{\lambda\pi t}} + \mathrm{O}\Big(\frac{1}{t\sqrt{t}}\Big), \qquad i \geqslant 1,$$
$$V_i(t) = 2\lambda\Big(1-\frac{2}{\pi}\Big)t - \frac{(2i)^2-1-4\delta_{i0}}{2\pi} + i^2 - \delta i0 + \mathrm{O}\Big(\frac{1}{\sqrt{t}}\Big), \quad i \geqslant 1.$$

6. The mean

Theorem 1 with the expression (29) for $q_1(x)$, gives the following representation for the mean

$$m_i(t) = h_{1,i}(t) + \mu\int_a^b Q_i(x)\frac{s\mu Q_s^*(x) - \lambda Q_{s-1}^*(x)}{x(x-\mu)}\,\mathrm{e}^{-xt}\,\mathrm{d}\psi(x),$$
$$i \geqslant 0,\ t \geqslant 0,\ \rho \neq 1, \tag{53}$$

where $h_{1,i}(t) = m$ if $\rho < 1$ and $h_{1,i}(t) = (\lambda - s\mu)t + \gamma_{1,0}(i)$ if $\rho > 1$.

Using (17), (28) and (26) we obtain

$$\gamma_{1,0}(i) = \begin{cases} i, & i > s-2 \\ s-1, & i \leqslant s-2 \end{cases} + (s\mu - \lambda) \sum_{n=0}^{s-2} \sum_{l=\max(i,n)}^{\infty} \frac{\pi_n}{\lambda \pi_l} + s\mu \sum_{l=\max(i,s-1)}^{\infty} \frac{\pi_{s-1}}{\lambda \pi_l}.$$

6.1. *The mean of the number of customers in the queue and of busy servers*

The expected number of customers in the system, $m_i(t)$, can be expressed as the sum of the expected number in queue, $c_i(t)$, and the expected number of busy servers, $b_i(t)$. In an ergodic $M/M/s$ system, the expected number in queue and the expected number of busy servers of the steady-state distribution are respectively $C = p_s\rho/(1-\rho)^2$ and $B = s\rho$ (see [12]). From the Karlin and McGregor representation (5), and using (A.3) and (A.7), it is easy to obtain a representation of $b_i(t)$

$$\begin{aligned} b_i(t) &= \sum_{j=0}^{s-1} j P_{ij}(t) + s \sum_{j=s}^{\infty} P_{ij}(t) = s - \sum_{j=0}^{s-1} \sum_{l=0}^{j} P_{il}(t) \\ &= s - \sum_{j=0}^{s-1} \int_0^{\infty} Q_i(x)\, \mathrm{e}^{-xt} Q_j^*(x)\, \mathrm{d}\psi(x) \\ &= s + \int_0^{\infty} Q_i(x) \frac{s\mu Q_s^*(x) - \lambda Q_{s-1}^*(x)}{x-\mu}\, \mathrm{e}^{-xt}\, \mathrm{d}\psi(x) \\ &= B_i + \int_a^b Q_i(x) \frac{s\mu Q_s^*(x) - \lambda Q_{s-1}^*(x)}{x-\mu}\, \mathrm{e}^{-xt}\, \mathrm{d}\psi(x), \end{aligned}$$

for some constant B_i. By letting $t \to \infty$, we see that $B_i = B$ if $\rho < 1$ and $B_i = s$ if $\rho > 1$. Since $c_i(t) = m_i(t) - b_i(t)$ and (53), we get

$$c_i(t) = l_i(t) + \int_a^b Q_i(x) \frac{\lambda Q_{s-1}^*(x) - s\mu Q_s^*(x)}{x}\, \mathrm{e}^{-xt}\, \mathrm{d}\psi(x), \quad i \geqslant 0,\ t \geqslant 0,\ \rho \neq 1,$$

where $l_i(t) = C$ if $\rho < 1$ and $l_i(t) = (\lambda - s\mu)t + \gamma_{1,0}(i) - s$ if $\rho > 1$.

6.2. *Representations and bounds*

Let $f_{ij}(t) = P(X(t) \leqslant j \mid X(0) = i)$. From (5) and (A.3)

$$f_{ij}(t) = \sum_{l=0}^{j} P_{il}(t) = \int_0^{\infty} Q_i(x) Q_j^*(x)\, \mathrm{e}^{-xt}\, \mathrm{d}\psi(x).$$

For a system $M/M/1$ it is known (see [1, theorem 8.1] or [5, p. 178])

$$m_i'(t) = \lambda - \mu + \mu P_{i0}(t).$$

From the representation (18) and (29), the following generalization of this result is obtained for a system $M/M/s$

$$m_i'(t) = \lambda - s\mu + \mu \sum_{j=0}^{s-1} f_{ij}(t), \quad i \geqslant 0,\ t \geqslant 0. \tag{54}$$

Differentiating the above expression we obtain $m_i''(t) = \mu \sum_{l=0}^{s-1} f_{il}'(t)$. Van Doorn [17] deduced and used this equality to study the increase of the mean.

Since $b_i(t) = s - \sum_{n=0}^{s-1} f_{in}(t)$, the above representation could be also written as

$$m_i'(t) = \lambda - \mu b_i(t). \tag{55}$$

Differentiating both sides of $b_i(t) = s - \int_0^\infty Q_i(x)\,\mathrm{e}^{-xt} \sum_{j=0}^{s-1} Q_j^*(x)\,\mathrm{d}\psi(x)$, and using (A.6) we obtain $b_i'(t) = \lambda f_{is-1}(t) - s\mu f_{is}(t) + \mu \sum_{j=0}^{s-1} f_{ij}(t)$ and then

$$c_i'(t) = m_i'(t) - b_i'(t) = \lambda - s\mu - \lambda f_{is-1}(t) + s\mu f_{is}(t).$$

Since $0 \leqslant b_i(t) \leqslant s$ and $m_i'(t) = \lambda - \mu b_i(t)$,

$$(\lambda - s\mu)t + i \leqslant m_i(t) \leqslant \lambda t + i, \quad t \geqslant 0,\ i \geqslant 0.$$

From the representation (54), we get that if $\rho > 1$ then the derivative of $m_i(t) - [(\lambda - s\mu)t + \gamma_{1,0}(i)]$ is positive. Moreover, from (30), $\lim_{t\to\infty} m_i(t) - [(\lambda - s\mu)t + \gamma_{1,0}(i)] = 0$. Therefore, if $\rho > 1$ then

$$(\lambda - s\mu)t + i \leqslant m_i(t) \leqslant (\lambda - s\mu)t + \gamma_{1,0}(i), \quad t \geqslant 0,\ i \geqslant 0. \tag{56}$$

So, the mean is between its asymptote and the parallel which crosses $(0, i)$ (i.e., the mean of $Y_i(t)$). When i increases, the distance between them decreases, since $\gamma_{1,0} = i + \mathrm{o}(1/\rho^{1/2})$ (see (32)).

A similar bound can be obtained for the variance in the case $M/M/1$. As a matter of fact, we can observe in (45) that $-xq_2(x) = 2(\lambda - \mu)xq_1(x) - 2\mu x^2$, and then from (18) and (5) it follows that for every $\rho > 0$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} M_2^{(i)}(t) = 2(\lambda - \mu)\frac{\mathrm{d}}{\mathrm{d}t} m_i(t) - 2\mu \frac{\mathrm{d}}{\mathrm{d}t} P_{i0}(t)$$

from where it is easily obtained

$$V_i'(t) = \lambda + \mu - \mu\big(1 + 2m_i(t)\big)P_{i0}(t), \quad i \geqslant 0,\ t \geqslant 0.$$

It follows that $((\lambda + \mu)t - V_i(t) + C)' \geqslant 0$, $\forall C \in \mathbb{R}$, and then

$$V_i(t) \leqslant (\lambda + \mu)t, \quad t \geqslant 0.$$

From (31), if $\rho > 1$ then $\lim_{t\to\infty} (\lambda + \mu)t + v_i - V_i(t) = 0$, where $v_i = 1 - 4Ai - 3\rho - \rho^{-i}/A^2\rho^i$, and thus

$$(\lambda + \mu)t + v_i \leqslant V_i(t) \leqslant (\lambda + \mu)t, \quad i \geqslant 0,\ t \geqslant 0.$$

So $V_i(t)$ is greater than its asymptote and less than the parallel which crosses the origin.

From (55) and the information about the sign of $m_i'(t)$ given in [17, p. 85] we have that if $\rho < 1$ then

(i) If $\lambda \geqslant i\mu$ then $b_i(t) \leqslant B$, $t \geqslant 0$.

(ii) If $Q_i(a) < 0$ then $b_i(t) > B$, $t \geqslant 0$.

(iii) If $\lambda < i\mu$ and $Q_i(a) > 0$ then $b_i(t)$ crosses once its asymptote B.

6.3. The distance of the mean to its asymptote in the $L^1(\mathbb{R}^+)$ norm

In [14], it is calculated in the ergodic case, $\rho < 1$, and when $i = 0$ the distance of the mean to its asymptote in the $L^1(\mathbb{R}^+)$ norm

$$\int_0^\infty \left| m - m_i(t) \right| \mathrm{d}t. \tag{57}$$

This quantity is considered as a measure of the convergence speed. In [7] it is generalized, providing a method to obtain (57) for an ergodic birth and death process. It might be interesting to calculate (57), for other values i, as a measure of the influence of the initial state i upon this convergence speed.

For the $M/M/s$ case, if $Q_i(a) < 0$ (see [17, p. 85]) then $m_i(t) \geqslant m$. If $i < m$ then $m > m_i(t)$. For these values of i, (57) could be easily calculate from the representation of the mean (53), and the formulae which are given in [10, appendix B] (to calculate integrals of the kind $\int_a^\infty Q_i(x)/x^l \,\mathrm{d}\psi(x)$ when $\rho < 1$).

For instance, for a $M/M/1$ system with $\rho < 1$, we obtain

$$\int_0^\infty \left| m - m_i(t) \right| \mathrm{d}t = \left| \frac{\lambda}{(1-\rho)^3} - \frac{\lambda i(i+1)}{2\rho(1-\rho)} \right|, \quad i < m \text{ or } i > \sqrt{\rho}\left(1-\sqrt{\rho}\right)^{-1}.$$

In this case, see [17, p. 63], $Q_i(a) < 0$ when $i > \sqrt{\rho}(1-\sqrt{\rho})^{-1}$.

In the case $\rho > 1$, the mean is always less than its asymptote (see (56)), and the distance of the mean to its asymptote in the $L^1(\mathbb{R}^+)$ norm, can be easily calculated from (53) and (26). For instance, for a $M/M/1$ system

$$\int_0^\infty (\lambda-\mu)t + \gamma_{1,0}(i) - m_i(t)\,\mathrm{d}t = \frac{\mu}{(\lambda-\mu)^2\rho^i}\left(i + \frac{\rho}{\rho-1}\right), \quad i \geqslant 0.$$

In particular for $i = 0$ is

$$\int_0^\infty (\lambda-\mu)t + \frac{1}{\rho-1} - m_0(t)\,\mathrm{d}t = \frac{\lambda\mu}{(\lambda-\mu)^3}.$$

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Appendix

From the recurrence relations (7) for $\{Q_n^*(x)\}$ we can prove that $\{\pi_n^{-1}(Q_n^*(x)-Q_{n-1}^*(x))\}$ satisfy the recurrence relations (4) and then

$$\pi_n Q_n(x) = Q_n^*(x) - Q_{n-1}^*(x), \quad n \geqslant 0. \tag{A.1}$$

Similarly we can prove

$$-x Q_n^*(x) = \lambda\pi_n\big(Q_{n+1}(x) - Q_n(x)\big), \quad n \geqslant 0. \tag{A.2}$$

Hence

$$Q_n^*(x) = \sum_{j=0}^{n} \pi_j Q_j(x), \qquad n \geqslant 0, \tag{A.3}$$

$$Q_n(x) = 1 - x\sum_{j=0}^{n-1} \frac{Q_j^*(x)}{\lambda\pi_j}, \quad n \geqslant 1. \tag{A.4}$$

Then

$$Q_n(0) = 1, \qquad Q_n^*(0) = \sum_{j=0}^{n} \pi_j, \qquad Q_n'(0) = -\sum_{l=0}^{n-1} \frac{1}{\lambda\pi_l}\sum_{i=0}^{l} \pi_i, \quad n \geqslant 1. \tag{A.5}$$

The addition in both hand sides of the recurrence relations (7) for $\{Q_n^*(x)\}$ from $n = 0$ to N yields

$$x\sum_{n=0}^{N} Q_n^*(x) = \mu\sum_{n=0}^{s-1} Q_n^*(x) + \lambda Q_N^*(x) - s\mu Q_{N+1}^*(x), \quad N \geqslant s-2. \tag{A.6}$$

Taking $N = s - 1$ we obtain

$$\sum_{j=0}^{s-1} Q_j^*(x) = \frac{\lambda Q_{s-1}^*(x) - s\mu Q_s^*(x)}{x-\mu}. \tag{A.7}$$

From (A.2) we obtain

$$\begin{aligned} Q_s^2(x) - Q_{s-1}(x)Q_{s+1}(x) &= \big[Q_s(x) - Q_{s+1}(x)\big]Q_s(x) + \big[Q_s(x) - Q_{s-1}(x)\big]Q_{s+1}(x) \\ &= \frac{x}{\lambda\pi_s}\big(Q_s^*(x)Q_s(x) - Q_{s-1}^*(x)Q_{s+1}(x)\big). \end{aligned} \tag{A.8}$$

From (A.1) we obtain

$$\sum_{n=0}^{N} n_{(k)}\pi_n Q_n(x) = \sum_{n=0}^{N} n_{(k)}\big(Q_n^*(x) - Q_{n-1}^*(x)\big)$$

$$= (N+1)_{(k)} Q_N^*(x) - k\sum_{n=0}^{N} n_{(k-1)} Q_n^*(x). \qquad \text{(A.9)}$$

On the other hand, from the recurrence relations (7) for $Q_n^*(x)$, a simple computation yields $\sum_{n=s-1}^{N} z^n Q_n^*(x) = (l_{N+1} - l_{s-1})/v$ where $l_n = \lambda z^{n+1} Q_{n-1}^*(x) - s\mu z^n Q_n^*(x)$ and $v = \lambda z^2 + (x - \lambda - s\mu)z + s\mu$. Thus, differentiating $k-1$ times respect to z and taking $z = 1$ we have

$$\sum_{n=s-1}^{N} n_{(k-1)} Q_n^*(x) = \left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \frac{l_{N+1} - l_{s-1}}{v}\right)_{z=1} = \eta_{k,N+1} - \eta_{k,s-1}, \qquad \text{(A.10)}$$

where $\eta_{k,n}$ is defined in (13). From (A.9) and (A.10) it follows

$$x^k \sum_{n=s-1}^{N} n_{(k)}\pi_n Q_n(x) = \Phi_{k,N+1}(x) - \Phi_{k,s-1}(x), \quad N > s-1, \qquad \text{(A.11)}$$

where $\Phi_{k,n}$ is defined in (14). A simple calculation from (14) gives

$$\Phi_{k,n}(x) = x^k n_{(k)} Q_{n-1}^*(x) + \sum_{l=0}^{k-1}\big(s\mu Q_n^*(x) n_{(k-1-l)} - \lambda Q_{n-1}^*(x)(n+1)_{(k-1-l)}\big) u_{k,l}(x), \qquad \text{(A.12)}$$

where

$$u_{k,l}(x) = k_{(l+1)} \sum_{j=0}^{[l/2]} \binom{l-j}{j} (-1)^{l-j} x^{k-1-l+j} (x + \lambda - s\mu)^{l-2j} \lambda^j.$$

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