GENERATING FUNCTIONS AND COMPANION SYMMETRIC LINEAR FUNCTIONALS

Esther M. García-Caballero
Samuel G. Moreno*
Francisco Marcellán**

Abstract

In this contribution we analyze the generating functions for polynomials orthogonal with respect to a symmetric linear functional u, i.e., a linear application in the linear space of polynomials with complex coefficients such that $u(x^{2n+1}) = 0$.

In some cases we can deduce explicitly the expression for the generating function

$$\mathcal{P}(x,\omega) = \sum_{n=0}^{\infty} c_n P_n(x)\omega^n,$$

where $\{P_n\}_n$ is the sequence of orthogonal polynomials with respect to u.

1. Introduction

It is very well known that classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) are eigenfunctions of a second order linear differential operator L

$$L(y) = a_2(x)y'' + a_1(x)y',$$

where $(a_k)_{k=1}^2$ are polynomials with $\deg a_k \leq k$. S. Bochner [2] proved that, in fact, they are the unique solutions of such an eigenproblem up to a linear change in the variable which are also orthogonal polynomials. H. L. Krall [11], considered the following extension of such a problem:

Given a fourth order linear differential operator $\mathcal{L}$

$$\mathcal{L}(y) = \sum_{k=1}^{4} a_k(x)y^{(k)},$$

Mathematics subject classification number: 33C45, 42C05.

Key words and phrases: orthogonal polynomials, symmetric linear functionals, generating functions.

* Research supported by Junta de Andalucía, Grupo de Investigación FQM 0178.

** Research supported by Dirección general de Investigación (Ministerio de Ciencia y Tecnología) of Spain, Grant BFM 2000-0206-C04-01 and INTAS Project 2000-272.

where $(a_k)_{k=1}^4$ are polynomials with $\deg a_k \leq k$, to find the sequences of orthogonal polynomials which are eigenfuntions of such a differential operator. He proved that the solutions are the classical orthogonal polynomials (when $\mathcal{L} = L^2$) as well as three new classes of orthogonal polynomials:

(i) Laguerre-type orthogonal polynomials.
The corresponding measure of orthogonality is

$$\chi_{\mathbb{R}^+} e^{-x}\, dx + M\delta(x),$$

where $\chi_{\mathbb{R}^+}$ is the characteristic function of $\mathbb{R}^+$ and $\delta(x)$ is the Dirac measure supported on $\{0\}$.

(ii) Legendre-type orthogonal polynomials.
The corresponding measure of orthogonality is

$$\frac{M}{2}\chi_{[-1,1]}\, dx + \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1).$$

(iii) Jacobi-type orthogonal polynomials.
The corresponding measure of orthogonality is

$$\chi_{[0,1]}(1-x)^\alpha + M\delta(x), \quad \alpha > -1.$$

From the 80's two ways are considered in order to generalize the above polynomials.

The first one emphasizes the algebraic and analytic properties of polynomials orthogonal with respect to a general classical measure (Laguerre, Jacobi) when mass points are added. These polynomials are said to be classical-type orthogonal polynomials.

For the measure of orthogonality

$$\chi_{\mathbb{R}^+} x^\alpha e^{-x}\, dx + M\delta(x), \quad \alpha > -1, \tag{1.1}$$

R. Koekoek [8] deduced the recurrence relation as well as the representation as hypergeometric series of the corresponding orthogonal polynomials. In fact, this paper was motivated by a previous work by T. H. Koornwinder [10] where similar problems are considered for the measure of orthogonality

$$\chi_{[-1,1]}(1-x)^\alpha(1+x)^\beta\, dx + M\delta(x+1) + N\delta(x-1), \quad \alpha,\, \beta > -1. \tag{1.2}$$

For a general approach to perturbations of general measures by addition of Dirac masses see [12].

The second way is focused on differential properties of the classical-type orthogonal polynomials, i.e., to find linear differential operators

$$\mathcal{L}(y) = \sum_{k=1}^{\infty} a_k(x) y^{(k)},$$

where a_k are polynomials of degree at most k, such that our classical-type orthogonal polynomials are eigenfunctions.

J. Koekoek and R. Koekoek [6] deduced the differential equation for the Jacobi-type polynomials orthogonal with respect to (1.2) and proved that the order of the differential operator is infinite up to $\alpha \in \mathbb{N}$ or $\beta \in \mathbb{N}$. Moreover, the order is

$$\begin{cases} 2\beta + 4 & \text{if } M > 0,\ N = 0 \text{ and } \beta \in \mathbb{N}, \\ 2\alpha + 4 & \text{if } M = 0,\ N > 0 \text{ and } \alpha \in \mathbb{N}, \\ 2\alpha + 2\beta + 6 & \text{if } M > 0,\ N > 0 \text{ and } \alpha,\ \beta \in \mathbb{N}. \end{cases}$$

For the measure of orthogonality (1.1), J. Koekoek and R. Koekoek [7] deduced the linear differential operator $\mathcal{L}$

$$\mathcal{L}(y) = \sum_{k=1}^{\infty} a_k(x) y^{(k)},$$

such that their eigenfunctions are the Laguerre-type orthogonal polynomials. In fact, $\mathcal{L}$ is an infinite order differential operator up to $\alpha \in \mathbb{N}$. In such a case, the order of the differential operator $\mathcal{L}$ is $2\alpha + 4$.

For a general and recent survey about orthogonal polynomials as eigenfunctions of finite order differential operators, see the excellent review [5] by W. N. Everitt *et al.*

The first aim of the present contribution is to obtain a generating function for Laguerre-type orthogonal polynomials in order to complete the framework of properties of such polynomials. From the generating function and using the Darboux method (see [13]) we can deduce some relevant information about the asymptotics of such polynomials.

The second aim is to deduce a generating function for the symmetrized sequence of a given family of orthogonal polynomials whose generating function is explicitly given. As an application we obtain a generating function for the generalized Hermite-type orthogonal polynomials.

The structure of this paper is as follows.

In Section 2, we introduce the symmetrized linear functional associated with a linear functional u. We deduce its generating function assuming that there is a generating function for the sequences of orthogonal polynomials with respect to the linear functionals u and xu, respectively. As an example, we consider classical Laguerre orthogonal polynomials and their symmetrized (the generalized Hermite polynomials). As a nice application of the results we deduce its generating function.

In Section 3, we obtain a generating function for Laguerre-type orthogonal polynomials and, again, we get a generating function for the so-called generalized Hermite-type orthogonal polynomials which are related to the symmetrization process for the Laguerre-type orthogonal polynomials.

2. Symmetric linear functionals and generating functions

Let $\mathbb{P}$ be the linear space of polynomials with complex coefficients and let $\mathbb{P}_n$ be the linear subspace of polynomials of degree at most n.

If u is a linear functional on $\mathbb{P}$, then the sequence of complex numbers $\{u_n\}_n$ defined by $u_n = \langle u, x^n \rangle$, where $\langle \cdot, \cdot \rangle$ means the duality bracket, is called the sequence of moments associated with u, and u is said to be the linear functional determined by the moment sequence $\{u_n\}_n$.

Definition 2.1. A linear functional u is said to be quasi-definite if the principal submatrices of the Hankel matrix $(u_{i+j})_{i,j=0}^{\infty}$ are nonsingular.

Proposition 2.2. *A linear functional u is quasi-definite if and only if there exists a sequence of monic polynomials $\{P_n\}_n$ with $\deg P_n = n$ such that*

i) $\langle u, P_n P_m \rangle = 0$, $n \neq m$.

ii) $\langle u, P_n^2 \rangle \neq 0$, *for every* $n \in \mathbb{N}$.

Such a sequence is said to be a sequence of monic orthogonal polynomials with respect to the linear functional u if the leading coefficient of P_n is 1.

Definition 2.3. A linear functional u is said to be positive definite if $\langle u, P \rangle > 0$ for every polynomial P that is not identically zero and such that $P(x) \geq 0$ for every real number x.

Definition 2.4. A functional u is said to be symmetric if all of its moments of odd order are 0, i.e., if

$$\langle u, x^{2n+1} \rangle = 0, \quad n = 0, 1, \ldots$$

Let u be a quasi-definite linear functional and let $\{P_n\}_n$ denote the sequence of monic orthogonal polynomials with respect to the functional u. We define the functional v by

$$\langle v, x^{2n} \rangle = \langle u, x^n \rangle, \quad \langle v, x^{2n+1} \rangle = 0, \quad n \geq 0.$$

The linear functional v is said to be the symmetrized linear functional of the linear functional u. In ([4], Chapter 1, Section 8) necessary and sufficient conditions for the quasi-definite character of v are given. Under such assumptions, we will denote by $\{S_n\}_n$ the sequence of monic orthogonal polynomials with respect to the functional v.

We introduce the linear functional $\tilde{v} = v + \lambda \delta_0$ via the addition of a Dirac linear functional. Thus $\tilde{v}$ is also symmetric. A necessary and sufficient condition in order to $\tilde{v}$ be quasi-definite is (see [12])

$$1 + \lambda L_n(0,0) \neq 0, \quad \forall n \in \mathbb{N},$$

where

$$L_n(x,y) = \sum_{j=0}^{n} \frac{S_j(x)S_j(y)}{\langle v, S_j^2\rangle}, \quad n \geq 0,$$

are the kernel polynomials corresponding to $\{S_n\}_n$.

LEMMA 2.5. *If v is the symmetrized linear functional associated with the functional u, then $\tilde{v}$ is the the symmetrized linear functional associated with the functional $\tilde{u} = u + \lambda\delta_0$* (see [1]).

PROPOSITION 2.6. *If $\mathcal{P}(x,\omega) = \sum_{n=0}^{\infty} c_n P_n(x)\omega^n$ is a generating function for the polynomials $\{P_n\}_n$ and the series $\mathcal{K}(x,\omega) = \sum_{n=0}^{\infty} d_n \frac{\langle u, P_n^2\rangle}{P_n(0)} K_n(0,x)\omega^n$ (where $K_n(x,y)$ are the kernel polynomials corresponding to $\{P_n\}_n$) is convergent, then*

$$\mathcal{S}(x,\omega) = \sum_{n=0}^{\infty} a_n S_n(x)\omega^n = \mathcal{P}(x^2,\omega^2) + x\omega\mathcal{K}(x^2,\omega^2),$$

is a generating function for the polynomials $\{S_n\}_n$, where $c_n = a_{2n}$, $d_n = a_{2n+1}$.

PROOF. The sequence $\{S_n\}_n$ defined by

$$S_{2n}(x) = P_n(x^2),$$
$$S_{2n+1}(x) = xQ_n(x^2), \quad n \geq 0,$$

where $Q_n(x) = \frac{\langle u, P_n^2\rangle}{P_n(0)} K_n(0,x)$, is the monic orthogonal polynomial sequence with respect to v. Therefore

$$\begin{aligned}
\mathcal{S}(x,\omega) &= \sum_{n=0}^{\infty} a_n S_n(x)\omega^n = \sum_{n=0}^{\infty} c_n S_{2n}(x)\omega^{2n} + \sum_{n=0}^{\infty} d_n S_{2n+1}(x)\omega^{2n+1} \\
&= \sum_{n=0}^{\infty} c_n P_n(x^2)(\omega^2)^n + x\omega \sum_{n=0}^{\infty} d_n \frac{\langle u, P_n^2\rangle}{P_n(0)} K_n(0,x^2)(\omega^2)^n \\
&= \mathcal{P}(x^2,\omega^2) + x\omega\mathcal{K}(x^2,\omega^2).
\end{aligned}$$

□

2.1 Classical Laguerre polynomials

For an arbitrary real number α, Laguerre polynomials are defined by (see [14], p. 100–102)

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k, \quad n = 0,1,2,\ldots, \tag{2.1}$$

where $\binom{a}{k}$ denotes the generalized binomial coefficient

$$\binom{a}{k} = \frac{(a-k+1)_k}{k!}$$

and $(a-k+1)_k$ stands for the so-called *Pochhammer's symbol* defined by

$$(b)_0 = 1, \quad (b)_n = b(b+1)\cdots(b+n-1), \quad b \in \mathbb{R},\ n \geq 1.$$

From the above definition $L_n^{(\alpha)}$ is a polynomial of degree n with leading coefficient

$$k_n = \frac{(-1)^n}{n!}.$$

Furthermore

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{(1)_n}, \tag{2.2}$$

and

$$\sum_{k=0}^{n} L_k^{(\alpha)}(x) = L_n^{(\alpha+1)}(x). \tag{2.3}$$

When α is not a negative integer, Laguerre polynomials are orthogonal with respect to a quasi-definite linear functional $u^{(\alpha)}$. This linear functional is positive definite for $\alpha > -1$. In fact, if $\alpha > -1$ $\{L_n^{(\alpha)}\}_n$ is orthogonal with respect to the inner product

$$(f,g) = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}dx.$$

Moreover, if $\alpha > -1$, then

$$\| L_n^{(\alpha)} \|^2 = \int_0^{+\infty} \left(L_n^{(\alpha)}(x)\right)^2 x^\alpha e^{-x}dx = \frac{\Gamma(n+\alpha+1)}{n!}. \tag{2.4}$$

Denote by

$$K_n^{(\alpha)}(x,y) = \sum_{j=0}^{n} \frac{L_j^{(\alpha)}(x)L_j^{(\alpha)}(y)}{\| L_j^{(\alpha)} \|^2}, \quad n = 0,1,2,\ldots,$$

the reproducing kernel of degree n associated with the family of orthogonal polynomials $\{L_n^{(\alpha)}\}_n$.

Using (2.2) and (2.3), for $n \geq 1$, we get

$$\text{i)} \qquad K_{n-1}^{(\alpha)}(0,0) = \frac{1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n-1} = \frac{(\alpha+1)_n}{(1)_{n-1}\Gamma(\alpha+2)}, \tag{2.5}$$

$$\text{ii)} \qquad K_{n-1}^{(\alpha)}(x,0) = \frac{1}{\Gamma(\alpha+1)} L_{n-1}^{(\alpha+1)}(x). \tag{2.6}$$

A generating function for Laguerre polynomials is obtained by F. Brafman [3] (see also [9])

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\alpha+1)_n} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\gamma} \, {}_1F_1\left(\gamma;\, \alpha+1;\, \frac{x\omega}{\omega-1}\right), \tag{2.7}$$

where ${}_1F_1(a_1; b_1; z)$ denotes the confluent hypergeometric function

$${}_1F_1(a_1; b_1; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k}{(b_1)_k} \frac{z^k}{k!}.$$

Notice that for $\gamma = \alpha + 1$ we get

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n &= (1-\omega)^{-(\alpha+1)} \sum_{k=0}^{\infty} \left(\frac{x\omega}{\omega-1}\right)^k \frac{1}{k!} \\ &= (1-\omega)^{-(\alpha+1)} \exp\left(\frac{x\omega}{\omega-1}\right), \end{aligned} \tag{2.8}$$

(see also [14] p. 101).

Furthermore, taking into account (2.2) and (2.7) we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \omega^n &= \sum_{n=0}^{\infty} \frac{(1)_n}{(\alpha+1)_n} L_n^{(\alpha)}(x)\omega^n \\ &= (1-\omega)^{-1} \, {}_1F_1\left(1; \alpha+1; \frac{x\omega}{\omega-1}\right). \end{aligned} \tag{2.9}$$

2.2 Generalized Hermite polynomials

As an example of the symmetrization process, generalized Hermite polynomials are defined by

$$H_n^{(\mu)}(x) = 2^n S_n^{(\alpha)}(x), \quad \mu = \alpha + \frac{1}{2},\ \alpha > -1.$$

Here

$$\begin{aligned} S_{2n}^{(\alpha)}(x) &= (-1)^n n! L_n^{(\alpha)}(x^2), \\ S_{2n+1}^{(\alpha)}(x) &= (-1)^n n! x L_n^{(\alpha+1)}(x^2), \quad n \geq 0, \end{aligned}$$

are the monic orthogonal polynomials with respect to the symmetrized linear functional associated with the Laguerre linear functional $u^{(\alpha)}$. We denote this functional by $v^{(\alpha)}$. In particular, if $\alpha = -1/2$, the polynomials $H_n^{(0)}$ are the classical Hermite polynomials (see [4], Chapter 5, Section 2, (2.43)).

Using the proposition 2.6 we are going to obtain a generating function for the generalized Hermite polynomials.

$$\begin{aligned}
\mathcal{H}_1^{(\mu)}(x,\omega) &= \sum_{n=0}^{\infty} \frac{1}{\left[\frac{n}{2}\right]!} H_n^{(\mu)}(x)\omega^n = \sum_{n=0}^{\infty} \frac{1}{n!} H_{2n}^{(\mu)}(x)\omega^{2n} + \sum_{n=0}^{\infty} \frac{1}{n!} H_{2n+1}^{(\mu)}(x)\omega^{2n+1} \\
&= \sum_{n=0}^{\infty} 2^{2n}(-1)^n L_n^{(\mu-\frac{1}{2})}(x^2)\omega^{2n} + x\omega \sum_{n=0}^{\infty} 2^{2n+1}(-1)^n L_n^{(\mu+\frac{1}{2})}(x^2)\omega^{2n} \\
&= (1+4\omega^2)^{-\mu-\frac{1}{2}} \exp\left(\frac{4x^2\omega^2}{1+4\omega^2}\right) + 2x\omega(1+4\omega^2)^{-\mu-\frac{3}{2}} \exp\left(\frac{4x^2\omega^2}{1+4\omega^2}\right) \\
&= (1+4\omega^2+2x\omega)(1+4\omega^2)^{-(\mu+\frac{3}{2})} \exp\left(\frac{4x^2\omega^2}{1+4\omega^2}\right)
\end{aligned}$$

(see [4], Chapter 5, Section 2, (2.49)).

Another generating function for generalized Hermite polynomials can be given in the following way

$$\begin{aligned}
\mathcal{H}_2^{(\mu)}(x,\omega) &= \sum_{n=0}^{\infty} \tilde{c}_n H_n^{(\mu)}(x)\omega^n \\
&= \sum_{n=0}^{\infty} \tilde{c}_{2n} H_{2n}^{(\mu)}(x)\omega^{2n} + \sum_{n=0}^{\infty} \tilde{c}_{2n+1} H_{2n+1}^{(\mu)}(x)\omega^{2n+1} \\
&= \sum_{n=0}^{\infty} \tilde{c}_{2n} 2^{2n}(-1)^n n! L_n^{(\mu-\frac{1}{2})}(x^2)\omega^{2n} \\
&\quad + x\omega \sum_{n=0}^{\infty} \tilde{c}_{2n+1} 2^{2n+1}(-1)^n n! L_n^{(\mu+\frac{1}{2})}(x^2)\omega^{2n}.
\end{aligned}$$

Next we can choose

$$\begin{aligned}
\tilde{c}_{2n} &= \frac{(-1)^n}{2^{2n} n! L_n^{(\mu-\frac{1}{2})}(0)}, \quad n \geq 0, \\
\tilde{c}_{2n+1} &= \frac{(-1)^n}{2^{2n+1} n! (\mu+\frac{1}{2}) L_n^{(\mu+\frac{1}{2})}(0)}, \quad n \geq 0.
\end{aligned}$$

Thus we get

$$\mathcal{H}_2^{(\mu)}(x,\omega) = \sum_{n=0}^{\infty} \frac{L_n^{(\mu-\frac{1}{2})}(x^2)}{L_n^{(\mu-\frac{1}{2})}(0)} \omega^{2n} + x\omega \sum_{n=0}^{\infty} \frac{L_n^{(\mu+\frac{1}{2})}(x^2)}{(\mu+\frac{1}{2}) L_n^{(\mu+\frac{1}{2})}(0)} \omega^{2n}.$$

Taking into account (2.9) as well as

$$\frac{1}{\mu+\frac{1}{2}} \, {}_1F_1\left(1; \mu+\frac{3}{2}; \frac{x^2\omega^2}{\omega^2-1}\right) = \frac{\omega^2-1}{x^2\omega^2}\left[{}_1F_1\left(1; \mu+\frac{1}{2}; \frac{x^2\omega^2}{\omega^2-1}\right) - 1\right], \quad (2.10)$$

we get

$$\begin{aligned}\mathcal{H}_2^{(\mu)}(x,\omega) &= (1-\omega^2)^{-1}\,{}_1F_1\left(1;\mu+\frac{1}{2};\frac{x^2\omega^2}{\omega^2-1}\right)\\ &\quad + (1-\omega^2)^{-1}x\omega\left[{}_1F_1\left(1;\mu+\frac{1}{2};\frac{x^2\omega^2}{\omega^2-1}\right)-1\right]\frac{\omega^2-1}{x^2\omega^2}\\ &= (1-\omega^2)^{-1}\left(\frac{\omega^2-1+x\omega}{x\omega}\right){}_1F_1\left(1;\mu+\frac{1}{2};\frac{x^2\omega^2}{\omega^2-1}\right)+\frac{1}{x\omega},\end{aligned}$$

which is a new generating formula for generalized Hermite polynomials.

3. Laguerre-type orthogonal polynomials

Let $\left\{\tilde{L}_n^{(\alpha)}\right\}_n$ denote the sequence of orthogonal polynomials with respect to the functional $\tilde{u}^{(\alpha)} = u^{(\alpha)} + \lambda\delta_0$, where $u^{(\alpha)}$ is the Laguerre functional, $\alpha > -1$ and $\lambda \geq 0$. $\tilde{L}_n^{(\alpha)}$ is normalized by the condition that the leading coefficient of $\tilde{L}_n^{(\alpha)}(x)$ equals the leading coefficient of $L_n^{(\alpha)}(x)$.

From the orthogonality conditions we are able to obtain a representation of $\tilde{L}_n^{(\alpha)}(x)$ in terms of the $L_n^{(\alpha)}(x)$, (see [12]).

Proposition 3.1.

$$\tilde{L}_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) - \frac{\lambda}{\lambda_{n-1}}L_n^{(\alpha)}(0)K_{n-1}^{(\alpha)}(x,0), \tag{3.1}$$

where $\lambda_n = 1 + \lambda K_n^{(\alpha)}(0,0)$.

Notice that from (3.1) we get

$$\tilde{L}_n^{(\alpha)}(0) = \frac{L_n^{(\alpha)}(0)}{\lambda_{n-1}}.$$

In this way

$$\frac{\tilde{L}_n^{(\alpha)}(x)}{\tilde{L}_n^{(\alpha)}(0)} = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} + \lambda\frac{K_{n-1}^{(\alpha)}(0,0)}{L_n^{(\alpha)}(0)}L_n^{(\alpha)}(x) - \lambda K_{n-1}^{(\alpha)}(x,0).$$

If we multiply by ω^n, we deduce

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\tilde{L}_n^{(\alpha)}(x)}{\tilde{L}_n^{(\alpha)}(0)}\omega^n &= 1+\sum_{n=1}^{\infty}\frac{\tilde{L}_n^{(\alpha)}(x)}{\tilde{L}_n^{(\alpha)}(0)}\omega^n\\ &= \sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\omega^n + \frac{\lambda}{\Gamma(\alpha+2)}\sum_{n=1}^{\infty}nL_n^{(\alpha)}(x)\omega^n\end{aligned}$$

$$-\frac{\lambda}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty} L_{n-1}^{(\alpha+1)}(x)\omega^n,$$

where we used (2.2), (2.5) and (2.6).

On the other hand from (2.8)

$$\sum_{n=1}^{\infty} L_{n-1}^{(\alpha+1)}(x)\omega^n = \omega\sum_{n=0}^{\infty} L_{n}^{(\alpha+1)}(x)\omega^n = \omega(1-\omega)^{-(\alpha+2)}\exp\left(\frac{x\omega}{\omega-1}\right).$$

Taking derivatives with respect to ω in (2.8) we get

$$\begin{aligned}\sum_{n=1}^{\infty} nL_n^{(\alpha)}(x)\omega^{n-1} &= (\alpha+1)(1-\omega)^{-(\alpha+2)}\exp\left(\frac{x\omega}{\omega-1}\right)\\ &\quad + (1-\omega)^{-(\alpha+1)}\frac{x(\omega-1)-x\omega}{(\omega-1)^2}\exp\left(\frac{x\omega}{\omega-1}\right)\\ &= (1-\omega)^{-(\alpha+2)}\left(\alpha+1+\frac{x}{\omega-1}\right)\exp\left(\frac{x\omega}{\omega-1}\right),\end{aligned}$$

and thus

$$\sum_{n=1}^{\infty} nL_n^{(\alpha)}(x)\omega^{n} = \omega(1-\omega)^{-(\alpha+2)}\left(\alpha+1+\frac{x}{\omega-1}\right)\exp\left(\frac{x\omega}{\omega-1}\right).$$

As a conclusion we obtain a generating function for Laguerre-type orthogonal polynomials.

Theorem 3.2. *For* $|\omega|<1$

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\tilde{L}_n^{(\alpha)}(x)}{\tilde{L}_n^{(\alpha)}(0)}\omega^n\\ &= (1-\omega)^{-1}\left[{}_1F_1\left(1;\alpha+1;\frac{x\omega}{\omega-1}\right)-\frac{\lambda}{\Gamma(\alpha+2)}\frac{x\omega}{(1-\omega)^{\alpha+2}}\exp\left(\frac{x\omega}{\omega-1}\right)\right].\end{aligned} \tag{3.2}$$

3.1 Generalized Hermite-type orthogonal polynomials

Let $\left\{\tilde{H}_n^{(\mu)}\right\}_n$ denote the sequence of orthogonal polynomials with respect to the symmetric linear functional $\tilde{v}^{(\alpha)} = v^{(\alpha)} + \lambda\delta_0$, where $v^{(\alpha)}$ is the symmetrized linear functional associated with the Laguerre functional $u^{(\alpha)}$, $\alpha = \mu - 1/2 > -1$, $\lambda \geq 0$ and the leading coefficient of $\tilde{H}_n^{(\mu)}$ is equal to the leading coefficient of the generalized Hermite polynomials $H_n^{(\mu)}$. $\tilde{H}_n^{(\mu)}$ are called generalized Hermite-type orthogonal polynomials.

By Lemma (2.5) $\tilde{v}^{(\alpha)}$ is the symmetrized linear functional of the functional $\tilde{u}^{(\alpha)} = u^{(\alpha)} + \lambda\delta_0$. Therefore the sequence $\{\tilde{S}_n\}_n$ defined by

$$\tilde{S}_{2n}^{(\alpha)}(x) = (-1)^n n! \tilde{L}_n^{(\alpha)}(x^2),$$

$$\tilde{S}_{2n+1}^{(\alpha)}(x) = x\frac{\left\langle \tilde{u}^{(\alpha)}, \left(\tilde{L}_n^{(\alpha)}\right)^2\right\rangle}{\tilde{L}_n^{(\alpha)}(0)}\tilde{K_n}^{(\alpha)}(0,x^2) = (-1)^n n! x L_n^{(\alpha+1)}(x^2), \quad n \geq 0,$$

where $\tilde{K_n}^{(\alpha)}(0,x)$ are the kernel polynomials corresponding to the Laguerre-type polynomials $\tilde{L}_n^{(\alpha)}$ and $L_n^{(\alpha)}$ are classical Laguerre polynomials, is the monic orthogonal polynomial sequence with respect to the functional $\tilde{v}^{(\alpha)}$. Therefore, $\tilde{H}_n^{(\mu)} = 2^n \tilde{S}_n^{(\alpha)}(x)$.

In (3.2) we have deduced a generating function for Laguerre-type orthogonal polynomials. Therefore in a straightforward way we can deduce a generating function for generalized Hermite-type orthogonal polynomials.

Indeed, we denote

$$\tilde{c}_{2n} = \frac{(-1)^n}{2^{2n} n! \tilde{L}_n^{(\mu-\frac{1}{2})}(0)}, \quad n \geq 0,$$

$$\tilde{c}_{2n+1} = \frac{(-1)^n}{2^{2n+1} n!}, \quad n \geq 0.$$

Then

$$\begin{aligned}
\tilde{\mathcal{H}}_1(x,\omega) &= \sum_{n=0}^{\infty} \tilde{c}_n \tilde{H}_n^{(\mu)}(x)\omega^n \\
&= \sum_{n=0}^{\infty} \tilde{c}_{2n} \tilde{H}_{2n}^{(\mu)}(x)\omega^{2n} + \sum_{n=0}^{\infty} \tilde{c}_{2n+1} \tilde{H}_{2n+1}^{(\mu)}(x)\omega^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{\tilde{L}_n^{(\mu-\frac{1}{2})}(x^2)}{\tilde{L}_n^{(\mu-\frac{1}{2})}(0)}\omega^{2n} + x\omega \sum_{n=0}^{\infty} L_n^{(\mu+\frac{1}{2})}(x^2)\omega^{2n}.
\end{aligned}$$

Taking into account (3.2) and (2.8) we get

$$\begin{aligned}
\tilde{\mathcal{H}}_1(x,\omega) = (1-\omega^2)^{-1}\Bigg[{}_1F_1\left(1; \mu + \frac{1}{2}; \frac{x^2\omega^2}{\omega^2-1}\right) \\
- \frac{\lambda}{\Gamma(\mu+\frac{3}{2})} x^2\omega^2 (1-\omega^2)^{-(\mu+\frac{3}{2})} \exp\left(\frac{x^2\omega^2}{\omega^2-1}\right)\Bigg] \\
+ x\omega(1-\omega^2)^{-(\mu+\frac{3}{2})}\exp\left(\frac{x^2\omega^2}{\omega^2-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (1-\omega^2)^{-1}\Bigg[{}_1F_1\left(1;\mu+\frac{1}{2};\frac{x^2\omega^2}{\omega^2-1}\right)\\
&\qquad + x\omega(1-\omega^2)^{-(\mu+\frac{1}{2})}\exp\left(\frac{x^2\omega^2}{\omega^2-1}\right)\Bigg]\\
&\quad - \frac{\lambda}{\Gamma(\mu+\frac{3}{2})}x^2\omega^2(1-\omega^2)^{-(\mu+\frac{5}{2})}\exp\left(\frac{x^2\omega^2}{\omega^2-1}\right).
\end{aligned}
$$

Another choice for the generating function is

$$
\begin{aligned}
\tilde{c}_{2n} &= \frac{(-1)^n}{2^{2n}n!\tilde{L}_n^{(\mu-\frac{1}{2})}(0)}, \quad n \geq 0,\\
\tilde{c}_{2n+1} &= \frac{(-1)^n}{2^{2n+1}n!(\mu+\frac{1}{2})L_n^{(\mu+\frac{1}{2})}(0)}, \quad n \geq 0. \qquad (3.3)
\end{aligned}
$$

Then

$$
\begin{aligned}
\tilde{\mathcal{H}}_2(x,\omega) &= \sum_{n=0}^{\infty}\tilde{c}_{2n}\tilde{H}_{2n}^{(\mu)}(x)\omega^{2n} + \sum_{n=0}^{\infty}\tilde{c}_{2n+1}\tilde{H}_{2n+1}^{(\mu)}(x)\omega^{2n+1}\\
&= \sum_{n=0}^{\infty}\frac{\tilde{L}_n^{(\mu-\frac{1}{2})}(x^2)}{\tilde{L}_n^{(\mu-\frac{1}{2})}(0)}\omega^{2n} + \frac{x\omega}{\mu+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{L_n^{(\mu+\frac{1}{2})}(x^2)}{L_n^{(\mu+\frac{1}{2})}(0)}\omega^{2n}.
\end{aligned}
$$

Taking into account (3.2) and (2.9) we get

$$
\begin{aligned}
\tilde{\mathcal{H}}_2(x,\omega) &= (1-\omega^2)^{-1}\Bigg[{}_1F_1\left(1;\mu+\frac{1}{2};\frac{x^2\omega^2}{\omega^2-1}\right) \qquad (3.4)\\
&\qquad - \frac{\lambda}{\Gamma(\mu+\frac{3}{2})}x^2\omega^2(1-\omega^2)^{-(\mu+\frac{3}{2})}\exp\left(\frac{x^2\omega^2}{\omega^2-1}\right)\Bigg]\\
&\quad + x\omega\frac{(1-\omega^2)^{-1}}{\mu+\frac{1}{2}}\,{}_1F_1\left(1;\mu+\frac{3}{2};\frac{x^2\omega^2}{\omega^2-1}\right)\\
&= (1-\omega^2)^{-1}\Bigg[{}_1F_1\left(1;\mu+\frac{1}{2};\frac{x^2\omega^2}{\omega^2-1}\right) \qquad (3.5)\\
&\qquad + \frac{x\omega}{\mu+\frac{1}{2}}\,{}_1F_1\left(1;\mu+\frac{3}{2};\frac{x^2\omega^2}{\omega^2-1}\right)\Bigg]\\
&\quad - \frac{\lambda}{\Gamma(\mu+\frac{3}{2})}x^2\omega^2(1-\omega^2)^{-(\mu+\frac{5}{2})}\exp\left(\frac{x^2\omega^2}{\omega^2-1}\right). \qquad (3.6)
\end{aligned}
$$

Using (2.10), the substitution in (3.4) yields

Theorem 3.3. *For $|\omega| < 1$*

$$\tilde{\mathcal{H}}_2(x,\omega) = \sum_{n=0}^{\infty} \tilde{c}_n \tilde{H}_n^{(\mu)}(x)\omega^n$$
$$= (1-\omega^2)^{-1}\frac{\omega^2-1+x\omega}{x\omega}\,{}_1F_1\left(1;\mu+\frac{1}{2};\frac{x^2\omega^2}{\omega^2-1}\right)+\frac{1}{x\omega}$$
$$-\frac{\lambda}{\Gamma(\mu+\frac{3}{2})}x^2\omega^2(1-\omega^2)^{-(\mu+\frac{5}{2})}\exp\left(\frac{x^2\omega^2}{\omega^2-1}\right).$$

where $\tilde{c}_n$ is given in (3.3).

Acknowledgements. The authors wish express their gratitude to the referee. His/her valuable remarks and suggestions have been very useful in order to improve our manuscript. In particular he/she pointed out us the paper by Brafman [3].

REFERENCES

[1] J. Arvesú, J. Atia and F. Marcellán, On semiclassical linear functionals: the symmetric Companion *Comm. Anal. Th. Cont. Fractions* **10** (2002), 13–29.
[2] S. Bochner, Über Sturm–Liouvillesche Polynomsysteme, *Math. Z.* **89** (1929), 730–736.
[3] F. Brafman, Some generating functions for Laguerre and Hermite polynomials, *Canad. J. Math.* **9** (1957), 180–187.
[4] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
[5] W. N. Everitt, K. H. Kwon, L. L. Littlejohn and R. Wellman, Orthogonal polynomial solutions of linear ordinary differential equations, *J. Comp. Appl. Math.* **133** (2001), 85–109.
[6] J. Koekoek and R. Koekoek, Differential equations for generalized Jacobi polynomials, *J. Comp. Appl. Math.* **126** (2000), 1–31.
[7] J. Koekoek and R. Koekoek, On a differential equation for Koornwinder's generalized Laguerre polynomials, *Proc. Amer. Math. Soc.* **112** (1991), 1045–1054.
[8] R. Koekoek, Koornwinder's Laguerre polynomials, *Delft Progress Report*, **12** (1988), 393–404.
[9] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Delft University of Technology, Report no. 98–17, 1998.
[10] T. H. Koornwinder, Orthogonal polynomials with weight function $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$, *Can. Math. Bull.* **27** (1984), 205–214.
[11] H. L. Krall, *On orthogonal polynomials satisfying a certain fourth order differential equation*, The Pennsylvania State College Studies, No. 6, State College, Pa., 1940.
[12] F. Marcellán and P. Maroni, Sur l'adjonction d'une masse de Dirac à une forme régulière et semi-classique, *Ann. Mat. Pura ed Appl.* **162** (1992), 1–22.

[13] E. M. Nikishin and V. N. Sorokin, *Rational Approximations and Orthogonality*, Transl. of Math. Monographs, Vol. 92, Amer. Math. Soc. Providence, R.I., 1991.
[14] G. Szegő, *Orthogonal Polynomials* (4^{th} edn.), Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 1975.

(Received: May 7, 2002)

Esther M. García-Caballero
Departamento de Matemáticas
Universidad de Jaén
23700 Linares
Spain
E-mail: emgarcia@ujaen.es

Samuel G. Moreno
Departamento de Matemáticas
Universidad de Jaén
23700 Linares
Spain
E-mail: samuel@ujaen.es

Francisco Marcellán
Departamento de Matemáticas
Universidad Carlos III de Madrid
28911 Leganés
Spain
E-mail: pacomarc@ing.uc3m.es