ZEROS OF JACOBI-SOBOLEV ORTHOGONAL POLYNOMIALS

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Abstract. We investigate zeros of Jacobi-Sobolev orthogonal polynomials with respect to

$$\phi(f,g)=\int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx+\gamma\int_{-1}^{1} f'(x)g'(x)(1-x)^{\alpha+1}(1+x)^{\beta}dx$$

where $\alpha>-1$, $-1<\beta\le 0$ and $\gamma>0$.

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1. Introduction

Consider a Sobolev inner product on the space $\mathbb{P}$ of real polynomials given by

$$\phi(p,q):=\langle\sigma,pq\rangle+\gamma\langle\tau,p'q'\rangle \tag{1.1}$$

where σ and τ are positive-definite moment functionals and $\gamma>0$. Let $\{P_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{S_n^{(\gamma)}(x)\}_{n=0}^{\infty}$ be the sequences of monic polynomials orthogonal with respect to σ, τ, and $\phi(\cdot,\cdot)$ respectively. Set

$$\langle\sigma,P_n^2(x)\rangle=u_n,\ \langle\tau,Q_n^2(x)\rangle=v_n,\ \phi(S_n^{(\gamma)},S_n^{(\gamma)})=s_n(\gamma),\ n\ge 0.$$

Then it is well known([5]) that both $P_n(x)$ and $Q_n(x)$ have n real simple zeros. There also have been many works on zeros of Sobolev orthogonal polynomials $S_n^{(\gamma)}$ for various choices of σ and τ ([2, 3, 6, 16]). Recently, Marcellán, Pérez, and Piñar showed that $S_n^{(\gamma)}$ has n real simple zeros, which interlace with zeros of $P_n(x)$ when $\sigma=\tau$ is the Laguerre moment functional([13]) and the Gegenbauer moment functional([11]). These results not only extend the previous works by Althammer[2] and Cohen[6] but also motivate the works by M. G. de Bruin and H. G. Meijer[4, 15]. In [15], they presented an exhaustive overview about the location of zeros of $S_n^{(\gamma)}(x)$ when $\{\sigma,\tau\}$ is a coherent pair.

Here, we are interested in the location of zeros of Jacobi-Sobolev orthogonal polynomials $\{S_n^{(\gamma)}(x)\}_{n=0}^{\infty}$ when

$$\sigma=(1-x)^{\alpha}(1+x)^{\beta}dx \quad\text{and}\quad \tau=(1-x)^{\alpha+1}(1+x)^{\beta}dx$$

on $[-1,1]$. In this case, $\{\sigma,\tau\}$ is a coherent pair of type C if $-1<\beta<0$, type B if $\beta=0$, and type A and C if $\beta>0$ according to the classification in [15]. So we can deduce from [15, Theorem 4.2] that for $-1<\beta<0$, $S_n^{(\gamma)}(x)$ has n real simple zeros, all of which lie in $(-1,1)$ except possibly the smallest zero. Furthermore, they showed ([15, Theorem 5.2]) that the smallest zero must be greater than

$$\frac{\alpha-\beta}{\alpha+\beta+2}-\frac{5}{2}.$$

Note that the above lower bound for zeros of $S_n^{(\gamma)}(x)$ is always less than -1 for $\alpha>-1$ and $-1<\beta<0$.

In this work, we give more precise location for the smallest zero with respect to the point -1.

2. Main Results

Assume that $\{\sigma, \tau\}$ is a coherent pair ([4, 8, 10]), that is, there are non-zero constants a_n such that

$$P_n'(x) + a_{n-1}P_{n-1}'(x) = nQ_{n-1}(x), \quad n \geq 2.$$

Expanding $P_n(x) + a_{n-1}P_{n-1}(x)$ in terms of $\{S_k^{(\gamma)}(x)\}_{k=0}^n$, we obtain

$$P_n(x) + a_{n-1}P_{n-1}(x) = S_n^{(\gamma)}(x) + d_{n-1}(\gamma)S_{n-1}^{(\gamma)}(x), \quad n \geq 2; \tag{2.1}$$

$$d_{n-1}(\gamma) = \frac{a_{n-1}u_{n-1}}{s_{n-1}(\gamma)}, \quad n \geq 2.$$

Set $\phi_{ij} := \phi(x^i, x^j)$ and $\Delta_n(\phi) := det[\phi_{ij}]_{i,j=0}^n$. Then $s_n(\gamma) = \frac{\Delta_n(\phi)}{\Delta_{n-1}(\phi)}$ $(\Delta_{-1}(\phi) = 1)$, $n \geq 0$ so that

$$d_n(\gamma) = \frac{a_n u_n \Delta_{n-1}(\phi)}{\Delta_n(\phi)}, \ n \geq 1.$$

Since $\Delta_n(\phi)$ is a polynomial in γ of degree n, $\lim_{\gamma\to\infty} d_n(\gamma) = 0$ for $n \geq 1$.

It is easy to see from the orthogonality that $S_n^{(\infty)}(x) := \lim_{\gamma\to\infty} S_n^{(\gamma)}(x)$ exists for $n \geq 0$. Since $\lim_{\gamma\to\infty} d_n(\gamma) = 0$, by (2.1),

$$S_n^{(\infty)}(x) = P_n(x) + a_{n-1}P_{n-1}(x), \quad n \geq 2. \tag{2.2}$$

Hence, $S_n^{(\infty)}(x)$ is quasi-orthogonal of order n with respect to σ so that $S_n^{(\infty)}(x)$ $(n \geq 2)$ has n real simple zeros $\{y_{nk}(\infty)\}_{k=1}^n$ satisfying

$$y_{n1}(\infty) < x_{n1} < y_{n2}(\infty) < x_{n2} < \cdots < y_{nn}(\infty) < x_{nn}. \tag{2.3}$$

If we write $S_n^{(\gamma)}(x) = x^n + \sum_{k=0}^{n-1} C_k^{(n)}(\gamma)x^k$, $n \geq 1$, then we can easily obtain from the orthogonality of $\{S_n^{(\gamma)}(x)\}_{n=0}^\infty$

$$C_k^{(n)}(\gamma) = -\Delta_{n-1}^{(k)}(\phi)/\Delta_{n-1}(\phi), \quad 0 \leq k \leq n-1,$$

where $\Delta_{n-1}^{(k)}(\phi)$ is the determinant of $[\phi_{ij}]_{i,j=0}^{n-1}$ whose k-th column is replaced by $[\phi(x^n, x^j)]_{j=0}^{n-1}$. Note that $\Delta_{n-1}(\phi)$ and $\Delta_{n-1}^{(k)}(\phi)$ $(0 \leq k \leq n-1)$ are polynomials in γ of degree at most $n-1$. Since $\Delta_{n-1}(\phi) \neq 0$, zeros of $S_n^{(\gamma)}(x)$ $(n \geq 1)$ are continuous functions in γ for $\gamma \geq 0$.

Consider now the Jacobi differential equation for $\alpha + \beta \neq -1, -2, \cdots$

$$(1-x^2)y''(x) + \{(\beta - \alpha) - (\alpha + \beta + 2)x\}y'(x) + n(\alpha + \beta + n + 1)y(x) = 0, \tag{2.4}$$

which is admissible ([9]) so that it has for each $n \geq 0$ a unique monic polynomial solution of degree n, i.e., Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x) = \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k.$$

For $\alpha + \beta \neq -1, -2, \cdots$ (see [1])

$$P_n^{(\alpha,\beta)}(x) + a_{n-1}P_{n-1}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta-1)}(x), \quad n \geq 1 \tag{2.5}$$

where

$$a_n = \frac{2(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n \geq 0$$

and

$$P_n^{(\alpha,\beta-1)}(x)' = nP_{n-1}^{(\alpha+1,\beta)}(x), \quad n \geq 0. \tag{2.6}$$

For $\alpha, \beta > -1$ and $\gamma > 0$, let

$$\phi(p,q) := \left\langle \sigma_J^{(\alpha,\beta)}, pq \right\rangle + \gamma \left\langle \sigma_J^{(\alpha+1,\beta)}, p'q' \right\rangle$$

and $\{S_n^{(\gamma)}(x;\alpha,\beta)\}_{n=0}^{\infty}$ the monic Jacobi-Sobolev orthogonal polynomials with respect to $\phi(\cdot,\cdot)$, where $\sigma_J^{(\alpha,\beta)}$ is the positive-definite Jacobi moment functional defined by

$$\left\langle \sigma_J^{(\alpha,\beta)}, p(x) \right\rangle := \int_{-1}^{1} p(x)(1-x)^{\alpha}(1+x)^{\beta}dx, \ p \in \mathbb{P}.$$

Then by the relations (2.5) and (2.6), $\{\sigma_J^{(\alpha,\beta)}, \sigma_J^{(\alpha+1,\beta)}\}$ is a coherent pair so that

$$S_n^{(\gamma)}(x;\alpha,\beta) + d_{n-1}(\gamma)S_{n-1}^{(\gamma)}(x;\alpha,\beta) = P_n^{(\alpha,\beta)}(x) + a_{n-1}P_{n-1}^{(\alpha,\beta)}(x), \ n \geq 1 \tag{2.7}$$

for some constants $d_{n-1}(\gamma)$, which are positive since $a_n > 0$ for $\alpha, \beta > -1$. We also have

$$S_n^{(\infty)}(x;\alpha,\beta) := \lim_{\gamma\to\infty} S_n^{(\gamma)}(x;\alpha,\beta) = P_n^{(\alpha,\beta-1)}(x), \quad n \geq 0.$$

According to the classification in [15], $\{\sigma_J^{(\alpha,\beta)}, \sigma_J^{(\alpha+1,\beta)}\}$ is of type C so that by Theorem 4.1 and Theorem 5.2 in [15], $S_n^{(\gamma)}(x;\alpha,\beta)$ has n real simple zeros $y_{nk} = y_{nk}(\gamma)$ $(1 \leq k \leq n)$ such that

$$\frac{\alpha-\beta}{\alpha+\beta+2} - \frac{5}{2} < y_{n1} < y_{n2} < \cdots < y_{nn} < 1 \quad \text{and} \quad y_{n2} > -1.$$

Moreover, if $\beta > 0$, then $\{\sigma_J^{(\alpha,\beta)}, \sigma_J^{(\alpha+1,\beta)}\}$ is also of type A. Hence by Theorem 4.2 in [15], $\{y_{nk}\}_{k=1}^{n}$ interlace with the zeros $\{x_{nk}\}_{k=1}^{n}$ of $S_n^{(\infty)}(x;\alpha,\beta) = P_n^{(\alpha,\beta-1)}(x)$ as

$$x_{n1} < y_{n1} < x_{n2} < y_{n2} < \cdots < x_{nn} < y_{nn}.$$

In particular, $y_{n1} > -1$ if $\beta > 0$.

We are now concerned with the location of the smallest zero y_{n1} of $S_n^{(\gamma)}(x;\alpha,\beta)$ with respect to the point -1 for $\alpha > -1$ and $-1 < \beta \leq 0$.

Theorem 2.1. *If $\alpha > -1$ and $\gamma > 0$, then $S_n^{(\gamma)}(x;\alpha,0)$ $(n \geq 2)$ has n real simple zeros $\{y_{nk}\}_{k=1}^{n}$ with*

$$-1 < y_{n1} < x_{n1} < y_{n2} < x_{n2} < \cdots < y_{nn} < x_{nn} < 1 \tag{2.8}$$

where $\{x_{nk}\}_{k=1}^{n}$ are zeros of $P_n^{(\alpha,0)}(x)$.

We will prove Theorem 2.1 taking into account ideas used in ([11, 13]). Set

$$\mathcal{F}[\cdot] = (1-x^2)I - \gamma(1-x)[(-(\alpha+\beta+1)x - (\alpha-\beta+1))D + (1-x^2)D^2]$$

where $D = \frac{d}{dx}$. Then it is shown in [12] that $\mathcal{F}[\cdot]$ is a symmetric operator for $\phi(\cdot,\cdot)$ in the sense that $\phi(\mathcal{F}[p],q) = \phi(p,\mathcal{F}[q])$ for any $p(x)$ and $q(x)$ in $\mathbb{P}$.

Lemma 2.2. *Let $\beta = 0$. Then for any polynomial $p(x)$ of degree k (≥ 0), there exists a unique polynomial $p_1(x)$ of degree k such that*

$$\mathcal{F}[p_1(x)] = (1-x^2)p(x).$$

Proof. When $\beta = 0$, $\mathcal{F}[\cdot] = (1-x^2)[I + \gamma(\alpha+1)D - \gamma(1-x)D^2]$. Let

$$p_1(x) = \sum_{i=0}^{k} b_i(1+x)^i \text{ and } p(x) = \sum_{i=0}^{k} c_i(1+x)^i.$$

Then we obtain the following linear system of equations

$$\begin{cases} b_k = c_k, \\ b_{k-1} + \gamma k(\alpha+k)b_k = c_{k-1} \\ b_i + \gamma(i+1)(\alpha+i+1)b_{i+1} - 2\gamma(i+1)(i+2)b_{i+2} = c_i, \quad 0 \leq i \leq k-2 \end{cases} \tag{2.9}$$

from which $\{b_i\}_{i=0}^{k}$ can be obtained recursively. □

Lemma 2.3. *$sgnS_n^{(\gamma)}(-1;\alpha,0) = (-1)^n$, $n \geq 0$.*

Proof. From (2.5) and (2.7),

$$S_n^{(\gamma)}(-1;\alpha,0) + d_{n-1}(\gamma)S_{n-1}^{(\gamma)}(-1;\alpha,0) = P_n^{(\alpha,-1)}(-1) = 0, \quad n \geq 1.$$

Since $d_n(\gamma) > 0$, $n \geq 0$ and $S_0(x;\alpha,0) = 1$, $\mathrm{sgn}S_n^{(\gamma)}(-1;\alpha,0) = (-1)^n$, $n \geq 0$. □

Proof of Theorem 2.1. Fix any $n \geq 1$ and set

$$w_i(x) = \frac{P_n^{(\alpha,0)}(x)}{x - x_i},\ 1 \leq i \leq n, \text{ where } x_i = x_{ni}\ (1 \leq i \leq n) \text{ are zeros of } P_n^{(\alpha,0)}(x).$$

By Lemma 2.2, there exists a unique polynomial $p_i(x)$ of degree $n-1$ such that

$$\mathcal{F}[p_i(x)] = (1-x^2)w_i(x),\ 1 \leq i \leq n.$$

Hence,

$$\begin{aligned} &\phi(S_n^{(\gamma)}(x;\alpha,0), p_i(x)) \\ &\quad= \textstyle\int_{-1}^{1} S_n^{(\gamma)}(x;\alpha,0)p_i(x)(1-x)^\alpha dx + \gamma \int_{-1}^{1} S_n^{(\gamma)}(x;\alpha,0)'p_i'(x)(1-x)^{\alpha+1}dx \\ &\quad= \textstyle\int_{-1}^{1} S_n^{(\gamma)}(x;\alpha,0)[p_i(x) + \gamma(\alpha+1)p_i'(x) - \gamma(1-x)p_i''(x)](1-x)^\alpha dx \\ &\qquad -2^{\alpha+1}\gamma S_n^{(\gamma)}(-1;\alpha,0)p_i'(-1) \\ &\quad= \textstyle\int_{-1}^{1} S_n^{(\gamma)}(x;\alpha,0)w_i(x)(1-x)^\alpha dx - 2^{\alpha+1}\gamma S_n^{(\gamma)}(-1;\alpha,0)p_i'(-1) \\ &\quad= \lambda_i S_n^{(\gamma)}(x_i;\alpha,0)w_i(x_i) - 2^{\alpha+1}\gamma S_n^{(\gamma)}(-1;\alpha,0)p_i'(-1) \end{aligned}$$

where λ_i's are the Christoffel numbers for the Jacobi polynomial $P_n^{(\alpha,0)}(x)$.
Since $\mathrm{sgn}w_i(x_i) = \mathrm{sgn}P_n^{(\alpha,0)}(x_i)' = (-1)^{n-i}$, then

$$\mathrm{sgn}S_n^{(\gamma)}(x_i;\alpha,0) = \mathrm{sgn}(w_i(x_i)S_n^{(\gamma)}(-1;\alpha,0)p_i'(-1)) = (-1)^i\mathrm{sgn}p_i'(-1).$$

Because $c_{n-1} = 1$ and $w_i(x)$ has $n-1$ simple zeros in $(-1,1)$, we obtain by the Cardano-Vieta formula $(-1)^{n-i-1}c_i > 0$, $0 \le i \le n-1$. Then we have from (2.9)

$$\mathrm{sgn} b_i = (-1)^{n-1-i} \quad \text{and} \quad \mathrm{sgn} p_i'(-1) = \mathrm{sgn} b_1 = (-1)^n, \quad 0 \le i \le n-1.$$

Hence, $\mathrm{sgn} S_n^{(\gamma)}(x_i;\alpha,0) = (-1)^{n-i}$, $1 \le i \le n$. Since $\mathrm{sgn} S_n^{(\gamma)}(-1;\alpha,0) = (-1)^n$, $S_n^{(\gamma)}(x;\alpha,0)$ has n real simple zeros $\{y_{nk}\}_{k=1}^n$ with

$$-1 < y_{n1} < x_1 < y_{n2} < x_2 < \cdots < y_{nn} < x_n < 1. \qquad \square$$

Note that Theorem 2.1 also follows Theorem 4.1 in [15] and the subsequent remark, where different arguments are used.
For $\alpha > -1$ and $-1 < \beta < 0$, we have the following which is the Jacobi version of Theorem 5.1 in [15].

Theorem 2.4. *[cf. Theorem 5.1 in [15]] Let $y_{n1}(\gamma)$ denote the smallest zero of $S_n^{(\gamma)}(x;\alpha,\beta)$ and $y_{n,1}(\infty)$ the smallest zero of $S_n^{(\infty)}(x;\alpha,\beta)$. For $\alpha > -1$ and $-1 < \beta < 0$, we have*

(i) *$y_{n1}(\infty) < -1$ if $n \ge 2$;*
(ii) *$y_{21}(\infty)$ is a lower bound for the zeros of $S_n^{(\gamma)}(x;\alpha,\beta)$ for all $n \ge 1$ and all $\gamma > 0$;*
(iii) *if $n \ge 3$, then for γ large*

$$y_{n-1,1}(\gamma) < y_{n,1}(\gamma) < y_{n,1}(\infty)$$

and for γ small

$$y_{n,1}(\infty) < -1 < y_{n,1}(\gamma) < y_{n-1,1}(\gamma).$$

Proof. From the relation in (2.5)

$$P_n^{(\alpha,\beta-1)}(x) = P_n^{(\alpha,\beta)}(x) + \frac{2n(n+\alpha)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)} P_{n-1}^{(\alpha,\beta)}(x), \quad n \ge 1$$

and the three-term recurrence relation ([5, (2.29), page 153]) for monic Jacobi polynomials

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) =& \left(x - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)}\right) P_{n-1}^{(\alpha,\beta)}(x) \\ &- \frac{4(n-1)(n+\alpha-1)(n+\beta-1)(n+\alpha+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-3)} P_{n-2}^{(\alpha,\beta)}(x), \quad n \ge 1 \end{aligned}$$

we get the relations

$$\frac{(\alpha+\beta+2)(\alpha+\beta+3)}{(\alpha+1)(\alpha+2)} S_2^{(\infty)}(x;\alpha,\beta)(x) = (x+1)^2 + 2\left(\frac{\beta+1}{\alpha+1}\right)(x^2-1) + \frac{\beta(\beta+1)}{(\alpha+2)(\alpha+1)}(x-1)^2$$

and for $n \ge 3$

$$S_n^{(\infty)}(x;\alpha,\beta) = (x+1) P_{n-1}^{(\alpha,\beta)}(x) - \frac{2(n+\beta-1)(n+\alpha+\beta-1))}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)} S_{n-1}^{(\infty)}(x;\alpha,\beta). \tag{2.10}$$

Then by the same arguments used to prove Theorem 5.1 in [15], we have the theorem. $\square$

We see from Theorem 2.4 that if $\alpha > -1$ and $-1 < \beta < 0$, then for $n \ge 2$, the smallest zero $y_{n1}(\gamma)$ of $S_n^{(\gamma)}(x;\alpha,\beta)$ can be less than or equal to or greater than -1 depending on γ.

Theorem 2.5. *Let $n \ge 2$. Then for the zeros $\{y_{nk}(\gamma)\}_{k=1}^n$ of $S_n^{(\gamma)}(x;\alpha,\beta)$ $(\gamma > 0)$, we have*

(i) *$y_{n-1,1}(\gamma) < y_{n,1}(\gamma) < y_{n1}(\infty)$ or $y_{n-1,1}(\gamma) > y_{n,1}(\gamma) > y_{n1}(\infty)$ or $y_{n-1,1}(\gamma) = y_{n,1}(\gamma) = y_{n1}(\infty)$;*

(ii) $\{y_{n,1}(\infty)\}_{n=2}^{\infty}$ *is strictly increasing with an upper bound* -1;

(iii) *if there is an* $n \geq 3$ *such that* $y_{n-1,1}(\gamma) \leq y_{n,1}(\gamma)$ *then* $y_{m-1,1}(\gamma) < y_{m,1}(\gamma),\ m > n$, *so that* $\{y_{n,1}(\gamma)\}_{n=2}^{\infty}$ *is strictly decreasing or* $\{y_{n,1}(\gamma)\}_{n=m}^{\infty}$ *is strictly increasing for some* $m \geq 2$;

(iv) *for* $0 \leq \gamma \leq \infty$, $\lim_{n\to\infty} y_{n,1}(\gamma) = -1$.

Proof. **(i)** From (i) of Theorem 2.4 and (ii) of Theorem 4.1 in [15], we have

$$y_{n1}(\infty) < -1 < y_{n2}(\gamma) \quad \text{and} \quad y_{n1}(\gamma) < x_{n1} < y_{n2}(\gamma), \quad n \geq 3, \tag{2.11}$$

where x_{n1} is the smallest zero of $P_n^{(\alpha,\beta)}(x)$. We deduce from (2.2) that if $y_{n1}(\gamma) = y_{n-1,1}(\gamma)$, then $y_{n1}(\gamma) = y_{n-1,1}(\gamma) = y_{n1}(\infty)$. If $y_{n1}(\gamma) < y_{n-1,1}(\gamma)$, then $y_{n1}(\infty) < y_{n1}(\gamma) < y_{n-1,1}(\gamma)$ since

$$sgnS_n^{(\infty)}(y_{n1}(\gamma)) = sgnS_{n-1}^{(\gamma)}(y_{n1}(\gamma)) = (-1)^{n-1}.$$

If $y_{n1}(\gamma) > y_{n-1,1}(\gamma)$, then $y_{n1}(\infty) > y_{n1}(\gamma) > y_{n-1,1}(\gamma)$ since

$$sgnS_n^{(\infty)}(y_{n1}(\gamma)) = sgnS_{n-1}^{(\gamma)}(y_{n1}(\gamma)) = (-1)^{n}.$$

(ii) From (2.10), we obtain

$$sgn(S_n^{(\infty)}(y_{n-1,1}(\infty)) = sgn(y_{n-1,1}(\infty) + 1)sgnP_{n-1}^{(\alpha,\beta)}(y_{n-1,1}(\infty)) = (-1)^n.$$

Hence for $n \geq 2$, $y_{n,1}(\infty) < y_{n+1,1}(\infty) < -1$. Thus $\{y_{n,1}(\infty)\}_{n=2}^{\infty}$ is a strictly increasing sequence with an upper bound -1.

(iii) Assume that there exists an $n \geq 3$ such that $y_{n-1,1}(\gamma) \leq y_{n1}(\gamma)$ but $y_{n1}(\gamma) \geq y_{n+1,1}(\gamma)$. Then from (i), we have

$$y_{n-1,1}(\gamma) \leq y_{n1}(\gamma) \leq y_{n1}(\infty) \quad \text{and} \quad y_{n+1,1}(\infty) \leq y_{n+1,1}(\gamma) \leq y_{n,1}(\gamma).$$

It implies that $y_{n+1,1}(\infty) \leq y_{n1}(\infty)$, which is a contradiction to (ii). Thus $\{y_{n,1}(\gamma)\}_{n=2}^{\infty}$ is a strictly decreasing sequence or $\{y_{n,1}(\gamma)\}_{n=m}^{\infty}$ is a strictly increasing sequence for some $m \geq 2$.

(iv) For $\gamma = 0$, $y_{n1}(0) = x_{n1}$. Thus $\{y_{n1}(0)\}_{n=2}^{\infty}$ is a decreasing sequence and $\lim_{n\to\infty} y_{n,1}(0) = -1$. Let $\gamma \in (0, \infty)$. Then (iii) shows that $\{y_{n,1}(\gamma)\}_{n=2}^{\infty}$ converges :

$$\lim_{n\to\infty} y_{n,1}(\gamma) := y(\gamma).$$

On the other hand, from Theorem 2 and Corollary 1 in [7], we have that the limit $y(\gamma)$ lies in $[-1, 1]$, which shows that $y(\gamma) = -1$. For $\gamma = \infty$, (ii) implies that

$$\lim_{n\to\infty} y_{n,1}(\infty) := y(\infty) \leq -1.$$

Finally, by Theorem 2.4 (iii), there is a γ with $0 < \gamma < \infty$ and

$$y_{2,1}(\gamma) < y_{3,1}(\gamma) < y_{3,1}(\infty)$$

so that we have from (i) and (iii) that

$$y_{n-1,1}(\gamma) < y_{n,1}(\gamma) < y_{n,1}(\infty), \quad n \geq 3,$$

which shows that $y(\infty) = -1$. □

Since $S_n^{(\infty)}(x; \alpha, \beta) = P_n^{(\alpha,\beta-1)}(x),\ n \geq 0$, we have the following.

Corollary 2.6. *For* $\alpha > -1$ *and* $-2 < \beta < -1$, *the smallest zeros of* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ *converge to* -1.

The following table gives numerical computations of the smallest zeros of $S_n^{(\gamma)}(x; -\frac{1}{2}, -\frac{1}{2})$.

	$\gamma = 0$	$\gamma = 1$	$\gamma = 100$	$\gamma = 100000$	$\gamma = \infty$
$n = 2$	−0.7071067	−1.1150692	−1.3621040	−1.3660250	−1.3660254
$n = 3$	−0.8660254	−1.1202612	−1.1209032	−1.1206532	−1.1206532
$n = 4$	−0.9238795	−1.0627289	−1.0601785	−1.0601489	−1.0601489
$n = 5$	−0.9510565	−1.0367560	−1.0360529	−1.0360461	−1.0360461
$n = 6$	−0.9659258	−1.0242415	−1.0240185	−1.0240162	−1.0240162
$n = 7$	−0.9749279	−1.0172377	−1.0171494	−1.0171485	−1.0171485
$n = 8$	−0.9807852	−1.0128997	−1.0128590	−1.0128586	−1.0128586
$n = 9$	−0.9848077	−1.0100208	−1.0099999	−1.0099997	−1.0099997
$n = 10$	−0.987688363	−1.008010659	−1.007999075	−1.007998969	−1.007998963

Table of the smallest zeros of $S_n^{(\gamma)}(x; -\frac{1}{2}, -\frac{1}{2})$

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