Darboux transformation and perturbation of linear functionals

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Abstract

Let $\mathbf{L}$ be a quasi-definite linear functional defined on the linear space of polynomials with real coefficients. In the literature, three canonical transformations of this functional are studied: $\mathbf{xL}$, $\mathbf{L}+\mathbf{C}\delta(\mathbf{x})$ and $\frac{1}{\mathbf{x}}\mathbf{L}+\mathbf{C}\delta(\mathbf{x})$ where $\delta(x)$ denotes the linear functional $(\delta(x))(x^k)=\delta_{k,0}$, and $\delta_{k,0}$ is the Kronecker symbol. Let us consider the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$. This sequence satisfies a three-term recurrence relation whose coefficients are the entries of the so-called *monic Jacobi matrix*. In this paper we show how to find the monic Jacobi matrix associated with the three canonical perturbations in terms of the monic Jacobi matrix associated with $\mathbf{L}$. The main tools are Darboux transformations. In the case that the LU factorization of the monic Jacobi matrix associated with $\mathbf{xL}$ does not exist and Darboux transformation does not work, we show how to obtain the monic Jacobi matrix associated with $\mathbf{x}^2\mathbf{L}$ as a limit case. We also study perturbations of the functional $\mathbf{L}$ that are obtained by combining the canonical cases. Finally, we present explicit algebraic relations between the polynomials orthogonal with respect to $\mathbf{L}$ and orthogonal with respect to the perturbed functionals.

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1. Introduction

Let $\mathbf{L}$ be a linear functional defined on the linear space $\mathbb{P}$ of polynomials with real coefficients. We call $l_k = \mathbf{L}(x^k)$, $k = 0, 1, 2, \ldots$ the kth *moment* associated with $\mathbf{L}$. Moreover, the matrix $M = (l_{i+j})_{i,j=0}^{\infty}$ is said to be the *matrix of moments* associated with $\mathbf{L}$. The functional $\mathbf{L}$ is said to be *quasi-definite* if the principal submatrices M_n of the matrix of moments M are nonsingular for every n.

If $\mathbf{L}$ is quasi-definite, then there exists a sequence of monic polynomials $\{P_n\}$ [4] such that

(1) $\deg(P_n) = n$,
(2) $\mathbf{L}(P_n(x)P_m(x)) = K_n\delta_{n,m}$, with $K_n \neq 0$, where $\delta_{n,m}$ is the "Kronecker delta" defined by

$$\delta_{n,m} = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

Several examples of perturbations of a quasi-definite linear functional have been studied in the literature [4,5,16,18,20,22]. In particular, three canonical cases are considered.

(1) The original functional $\mathbf{L}$ is transformed into $\tilde{\mathbf{L}}_1 = \mathbf{xL}$, where

$$(\mathbf{xL})(p) := \mathbf{L}(xp),$$

for any polynomial p.
(2) The original functional is transformed into $\tilde{\mathbf{L}}_2 = \mathbf{L} + \mathbf{C}\delta(\mathbf{x})$, where

$$(\mathbf{L} + \mathbf{C}\delta(\mathbf{x}))(p) := \mathbf{L}(p) + \mathbf{C}p(0).$$

(3) The original functional is transformed into $\tilde{\mathbf{L}}_3 = \frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$, where

$$\left(\frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}\delta(\mathbf{x})\right)(p) := \mathbf{L}\left(\frac{p(x) - p(0)}{x}\right) + \mathbf{C}p(0).$$

Then, it seems to be natural to give an answer to the following questions: (1) Is the perturbed linear functional a quasi-definite functional? (2) If it is the case, does exist a relation between the sequence $\{\tilde{P}_n\}$ of monic polynomials orthogonal with respect to the perturbed functional and the sequence $\{P_n\}$?

Necessary and sufficient conditions for the quasi-definiteness of $\tilde{\mathbf{L}}_1$ are given in [4,18]. There, the representation of $\{\tilde{P}_n\}$ in terms of $\{P_n\}$ is also obtained. In [20,22] this kind of perturbation is said to be a basic *Christoffel transformation*.

Necessary and sufficient conditions for the quasi-definiteness of $\tilde{\mathbf{L}}_2$ are given in [5,16,18], where the representation of $\{\tilde{P}_n\}$ in terms of $\{P_n\}$ is obtained. Notice that $\mathbf{xL} = \tilde{\mathbf{L}}_1 = \mathbf{x}\tilde{\mathbf{L}}_2$. In [20,22] this kind of perturbation is said to be a *Uvarov transformation*.

A natural perturbation appears when we try to solve the equation $x\tilde{\mathbf{L}}_3 = \mathbf{L}$. This means that $\tilde{\mathbf{L}}_3(x^k) = \mathbf{L}(x^{k-1})$, $k = 1, 2, \ldots$, as well as $\tilde{\mathbf{L}}_3(1) = \mathbf{C}$. In [17], necessary and sufficient conditions for $\tilde{\mathbf{L}}_3$ to be a quasi-definite linear functional are studied. Moreover, a representation of $\{\tilde{P}_n\}$ in terms of $\{P_n\}$ is given. In [20,22] this kind of perturbation is said to be a *Geronimus transformation*.

The aim of our work is to analyze these kind of perturbations from a different point of view.

It is well known that the sequence of monic polynomials $\{P_n\}$ orthogonal with respect to a quasi-definite linear functional $\mathbf{L}$ satisfies a three-term recurrence relation [4]

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n = 1, 2, \ldots, \tag{1.1}$$

with initial conditions $P_0(x) = 1$, $P_1(x) = x - \beta_0$, and $\gamma_n \neq 0$, $n = 1, 2, \ldots$ Thus, there exists a semi-infinite tridiagonal matrix J given by

$$J = \begin{bmatrix} \beta_0 & 1 & 0 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & 0 & \cdots \\ 0 & \gamma_2 & \beta_2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.2}$$

such that $xp = Jp$, with $p = [P_0(x), P_1(x), P_2(x), \ldots]^{\mathrm{t}}$. The matrix J is said to be the *monic Jacobi matrix* associated with the functional $\mathbf{L}$.

Our approach is based in the relation between the matrices $\tilde{J}_i$, $i = 1, 2, 3$ associated with the linear functionals $\tilde{\mathbf{L}}_i$, $i = 1, 2, 3$, and the matrix J.

The process to compute the monic Jacobi matrix associated with $\tilde{\mathbf{L}}_1$ in terms of the monic Jacobi matrix associated with $\mathbf{L}$ is based on one step of the LR-algorithm or Darboux transformation without parameter. There are several contributions in such a direction; for instance, the works by Galant [7,8], Golub and Kautsky [10,15], and Gautschi [9]. Suppose that a given linear functional $\mathbf{L}$ is defined in terms of a weight function ω in the following way:

$$\mathbf{L}(p) = \int_{\mathbb{R}} p(x)\omega(x)\,\mathrm{d}x.$$

The work by Galant, Golub, and Kautsky is focused on the computation of the coefficients of the three-term recurrence relation satisfied by the sequence of monic polynomials orthogonal with respect to a linear perturbation of ω, i.e., $\tilde{\omega} = k(x - \alpha)\omega$, assuming that $\tilde{\omega}$ is again a weight function, which means that $\tilde{\omega}$ is a positive and L^1-integrable function in the support of ω. Gautschi extended this result for quasi-definite measures, i.e., measures μ such that the associated linear functional $\mathbf{L}$ defined by $\mathbf{L}(p(x)) = \int_{\mathbb{R}} p(x)\,\mathrm{d}\mu$ is quasi-definite. In matrix context, the Darboux transformation without parameter applied to the monic Jacobi matrix associated with a linear functional $\mathbf{L}$ generates the monic Jacobi matrix associated with the functional $\mathbf{xL}$. The main application of the above results is the evaluation of Gaussian knots in the presence of fixed knots [10]. Our main contribution to the previous issues is the following: given a symmetric linear functional **U**, i.e., a linear functional such that $U(x^{2k+1}) = 0$ for $k \geqslant 0$, the corresponding monic Jacobi matrix has no LU factorization. In such a case, it is not possible to find the monic Jacobi matrix associated with the functional **xL** applying the Darboux transformation without parameter. However, we prove that it is possible to find the monic Jacobi matrix associated with $\mathbf{x}^2\mathbf{L}$ (another symmetric functional) by the application of two Darboux transformations without parameter and with shift, as a limit case.

Grünbaum and Haine considered the monic Jacobi matrix associated with $\tilde{\mathbf{L}}_2 = \mathbf{L} + \mathbf{C}\delta(\mathbf{x})$ [12,13] in the context of the spectral analysis of fourth-order linear differential equations with polynomial coefficients. They were interested in the polynomial solutions of such differential equations, the so-called Krall orthogonal polynomials (see also [1]). They obtained those polynomials from some instances of the classical orthogonal polynomials by a combination of two processes called *Darboux transformation* and *Darboux transformation without parameter*. One of our goals is to extend their results for Krall polynomials to the general case when **L** is any quasi-definite linear functional. The Darboux transformation was introduced in the classical monograph [6], although only the case of a (continuous) Sturm–Liouville operator is discussed there. Discrete versions were introduced later and their systematic study was undertaken by Matveev and Salle [19], who are responsible for the name "Darboux transformation".

Finally, the perturbation $\tilde{\mathbf{L}}_3$ is related with an extension of the classical Christoffel formula. In the case of positive measures, there are many contributions (see [7–9,21]) when rational perturbations $\tilde{\omega}(x) = \frac{p(x)}{q(x)}\omega(x)$ of a weight function are considered. Grünbaum and Haine introduced a modification of such algorithms which allows to find the sequence of monic polynomials orthogonal with respect to $\tilde{\mathbf{L}}_3$ in an alternative way to the approach by Maroni [17]. They use such an algorithm for some particular cases of linear functionals (Laguerre and Jacobi, classical linear functionals). In our work, we consider the general problem and we introduce in a natural way the so-called *Darboux transformation* in order to find $\tilde{J}_3$ from J. The Darboux transformation is related with bispectral problems [11,12,14], as well as with differential evolution equations [19]. More recently, the analysis of rational spectral transformations, self-similarity, and orthogonal polynomials has been considered in [20,22].

The structure of the paper is the following: In Section 2 we introduce the LU and UL factorizations of a tridiagonal matrix J, as well as the Darboux transformation and the Darboux transformation without parameter. In Section 3 we show how to find the tridiagonal matrix $\tilde{J}_1$ associated with the linear functional $\tilde{\mathbf{L}}_1 = \mathbf{xL}$ in terms of the matrix J by the application of the Darboux transformation without parameter. The main result is Theorem 3.4. The case of symmetric linear functionals is also studied as a limit case. In Section 4, the tridiagonal matrix $\tilde{J}_2$ associated with the linear functional $\tilde{\mathbf{L}}_2 = \mathbf{L} + \mathbf{C}\delta(\mathbf{x})$ is obtained from J by the application of a Darboux transformation without parameter combined with a Darboux transformation. In Section 5, the Darboux transformation appears in a natural way in order to compute the tridiagonal matrix $\tilde{J}_3$ associated with the linear functional $\tilde{\mathbf{L}}_3 = \frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$. We also consider a combination of Darboux transformations and Darboux transformations without parameter that generates the monic Jacobi matrices associated with the linear functionals $\tilde{\mathbf{L}}_4 = \frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}_1\delta(\mathbf{x}) + \mathbf{C}_2\delta'(\mathbf{x})$ and $\tilde{\mathbf{L}}_5 = \mathbf{L} + \mathbf{C}_1\delta(\mathbf{x}) + \mathbf{C}_2\delta'(\mathbf{x})$, where $\delta'(x)$ denotes the first derivative of the linear functional $\delta(x)$. Finally, if $\{\tilde{P}_n\}$ denotes the sequence of monic polynomials orthogonal with respect to $\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$ or $\frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$, then we give an alternative approach to that used by other authors as Maroni, to obtain $\{\tilde{P}_n\}$ in terms of $\{P_n\}$ taking into account the algorithm that computes the Darboux transformation.

2. Darboux transformation and functionals perturbation

In this section, we define the Darboux transformation as well as the Darboux transformation without parameter, and give some auxiliary lemmas that will be very useful in next sections.

Consider a quasi-definite linear functional $\mathbf{L}$ as well as the sequence of monic polynomials $\{P_n\}$ orthogonal with respect to such a functional. This sequence of polynomials satisfies the three-term recurrence relation given in (1.1). Let J be the corresponding monic Jacobi matrix as in (1.2). We introduce the following transformation on J,

$$J = LU, \quad J^{(p)} := UL, \tag{2.1}$$

where $J = LU$ denotes the LU factorization without pivoting of J. Notice that J has an unique LU factorization if and only if the leading principal submatrices of J are nonsingular. Furthermore,

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ l_1 & 1 & 0 & \cdots \\ 0 & l_2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad U = \begin{bmatrix} u_1 & 1 & 0 & \cdots \\ 0 & u_2 & 1 & \cdots \\ 0 & 0 & u_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.2}$$

The matrix $J^{(p)}$ is semi-infinite, tridiagonal and the entries in the positions $(i, i+1)$ are ones. The transformation given in (2.1) is the so-called *Darboux transformation without parameter*. Observe that the application of a Darboux transformation without parameter to a monic Jacobi matrix generates a new monic Jacobi matrix (recall Favard's Theorem). The matrix $J^{(p)}$ is said to be the *Darboux transform without parameter* of J.

In the same way, if we express J as the product of U times L, where U and L are given in (2.2), and consider the process

$$J = UL, \quad J^{(d)} := LU,$$

the matrix $J^{(d)}$ is again a monic Jacobi matrix. The previous process is known as *Darboux transformation* and $J^{(d)}$ is said to be the *Darboux transform* of J. Notice that, while the LU factorization of a monic Jacobi matrix is unique, it is not the case with the UL factorization. It depends on a free parameter as we will prove later on.

In next lemma, we express the elements u_k in the main diagonal of the upper triangular matrix U obtained from the LU factorization of J, given in (2.2), in terms of the elements l_k in the subdiagonal of L and the entries β_k in the main diagonal of J. Moreover, we give an alternative expression of the elements u_k in terms of the values $P_n(0)$, and obtain a recursive formula for the computation of the elements l_k.

Lemma 2.1. *Let $\{P_n\}$ be the sequence of monic polynomials defined by the monic Jacobi matrix J given in* (1.2). *Assume that $P_n(0) \neq 0$, $n \geqslant 1$. If $J = LU$ denotes the LU factorization of J, then*

$$u_n = -\frac{P_n(0)}{P_{n-1}(0)}. \tag{2.3}$$

Moreover,

$$\begin{cases} u_1 = \beta_0, \\ u_n = \beta_{n-1} - l_{n-1}, \quad n \geqslant 2, \end{cases} \tag{2.4}$$

where the elements l_n can be computed in a recursive way as follows:

$$l_1 = \frac{\gamma_1}{\beta_0}, \quad l_n = \frac{\gamma_n}{\beta_{n-1} - l_{n-1}}, \quad n \geqslant 2. \tag{2.5}$$

Proof. The product of L times U gives

$$LU = \begin{bmatrix} u_1 & 1 & 0 & \cdots \\ u_1 l_1 & u_2 + l_1 & 1 & \cdots \\ 0 & u_2 l_2 & u_3 + l_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.6}$$

Comparing the elements in the same positions of matrices J and LU, the result in (2.4) is obtained, as well as

$$\gamma_k = l_k u_k \quad \text{for all } k \geqslant 1. \tag{2.7}$$

The result (2.3) is deduced by induction on k, taking into account the three-term recurrence relation (1.1) that $\{P_m\}$ satisfies. Since $P_1(0) = -\beta_0$ and $P_0(0) = 1$,

$$u_1 = \beta_0 = -\frac{P_1(0)}{P_0(0)}.$$

Assume that $u_k = -\frac{P_k(0)}{P_{k-1}(0)}$ for $k \leqslant n$. Then, taking into account (1.1),

$$P_{n+1}(0) = -\beta_n P_n(0) - \gamma_n P_{n-1}(0).$$

Dividing the previous expression by $P_n(0)$, and applying the induction hypothesis as well as (2.4) and (2.7), we get

$$-\frac{P_{n+1}(0)}{P_n(0)} = \beta_n - \frac{\gamma_n}{u_n} = \beta_n - l_n = u_{n+1}. \tag{2.8}$$

Finally, considering again (2.4) and taking into account (2.7), the result in (2.5) is obtained. □

Remark 2.2. From Lemma 2.1, the LU factorization without pivoting of a monic Jacobi matrix J exists if and only if $P_n(0) \neq 0$, $n \geqslant 1$. Moreover,

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ l_1 & 1 & 0 & \cdots \\ 0 & l_2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad U = \begin{bmatrix} \beta_0 & 1 & 0 & \cdots \\ 0 & \beta_1 - l_1 & 1 & \cdots \\ 0 & 0 & \beta_2 - l_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{2.9}$$

where

$$l_1 = \frac{\gamma_1}{\beta_0}, \quad l_n = \frac{\gamma_n}{\beta_{n-1} - l_{n-1}}, \quad n \geqslant 2.$$

It should also be noticed that, since the elements in the main diagonal of L are equal to 1, the LU factorization of the monic Jacobi matrix J is unique.

Now we express the monic Jacobi matrix (1.2) as a product of an upper triangular matrix times a lower triangular matrix, i.e., $J = UL$, where the factors L and U are as in (2.2). In this case, the factorization is not unique and depends on a free parameter, S_0. We will call such a kind of factorization, *UL factorization*, although it is not the classical one.

Definition 2.3. Let $\{P_n\}$ be a sequence of polynomials orthogonal with respect to a linear functional $\mathbf{L}$ and assume that $\{P_n\}$ satisfies a recurrence relation as in (1.1). Let $S_0 \in \mathbb{C}$. Then, $\{\hat{P}_n\}$ is said to be the sequence of *co-recursive polynomials with parameter* S_0 *associated with the linear functional* $\mathbf{L}$ if this sequence satisfies the recurrence relation given by

$$\hat{P}_{n+1}(x) = (x - \hat{\beta}_n)\hat{P}_n(x) - \hat{\gamma}_n \hat{P}_{n-1}(x), \tag{2.10}$$

where

$$\hat{\beta}_0 = \beta_0 - S_0, \quad \hat{\beta}_n = \beta_n, \quad n \geqslant 1,$$
$$\hat{\gamma}_n = \gamma_n \quad \text{for all } n.$$

Proposition 2.4. *Let J be the monic Jacobi matrix associated with a quasi-definite linear functional $\mathbf{L}$ and let $\{P_n\}$ be the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$. Assume that $J = UL$ denotes the UL factorization of J and S_i denotes the entry in the position $(i+1, i+1)$ of U, i.e., $S_i := u_{i+1}$ for $i \geqslant 0$, where the element S_0 is a free parameter generated in the factorization (since it is not unique). If $\{\hat{P}_n\}$ denotes the sequence of co-recursive polynomials with parameter S_0 associated with $\mathbf{L}$, and $\hat{P}_n(0) \neq 0$ for all n, then*

$$l_n = -\frac{\hat{P}_n(0)}{\hat{P}_{n-1}(0)}. \tag{2.11}$$

Furthermore,

$$l_n = \beta_{n-1} - S_{n-1}, \quad n \geqslant 1, \tag{2.12}$$

and S_n can be calculated in a recursive way

$$S_n = \frac{\gamma_n}{\beta_{n-1} - S_{n-1}}, \quad n \geqslant 1. \tag{2.13}$$

Proof. If U and L are matrices as in (2.2), then

$$UL = \begin{bmatrix} u_1 + l_1 & 1 & 0 & 0 & \cdots \\ u_2 l_1 & u_2 + l_2 & 1 & 0 & \cdots \\ 0 & u_3 l_2 & u_3 + l_3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.14}$$

Comparing the elements of the matrices UL and J, we get

$$l_1 = \beta_0 - u_1.$$

Let us consider u_1 a free parameter, which we denote by S_0. Notice that S_0 can take any complex value as long as $\hat{P}_n(0) \neq 0$ for all n. Then, l_1 can be expressed in terms of S_0 in the obvious way. Assume that, for some k, $l_k = \beta_{k-1} - S_{k-1} \neq 0$ and $S_{k-1} := u_k$. Then, since $u_{k+1} l_k = \gamma_k$,

$$S_k := u_{k+1} = \frac{\gamma_k}{\beta_{k-1} - S_{k-1}}.$$

On the other hand, $u_{k+1} + l_{k+1} = \beta_k$ and we get

$$l_{k+1} = \beta_k - S_k.$$

The result in (2.11) is obtained by induction and taking into account the recurrence relation that the co-recursive polynomials satisfy (2.10). □

Remark 2.5. From Proposition 2.4, we deduce that the UL factorization of a monic Jacobi matrix J exists if and only if the free parameter S_0 takes a value such that the corresponding sequence of co-recursive polynomials satisfies $\hat{P}_n(0) \neq 0$ for all n. Moreover,

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \beta_0 - S_0 & 1 & 0 & \cdots \\ 0 & \beta_1 - S_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad U = \begin{bmatrix} S_0 & 1 & 0 & \cdots \\ 0 & S_1 & 1 & \cdots \\ 0 & 0 & S_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{2.15}$$

where

$$S_n = \frac{\gamma_n}{\beta_{n-1} - S_{n-1}}, \quad n \geqslant 1.$$

3. Transformation of the functional L into xL

In the sequel, we will prove that the application of a Darboux transformation without parameter to the monic Jacobi matrix associated with the linear functional **L**, transforms this matrix into the monic Jacobi matrix associated with the functional **xL**. This result will be extended in a straightforward way to obtain the monic Jacobi matrix associated with $(\mathbf{x}-\alpha)\mathbf{L}$, $\alpha \in \mathbb{C}$.

Next lemma gives a finite version of the Darboux transformation without parameter. We use the following notation: for any square matrix A, $(A)_n$ denotes the principal submatrix of order n of A.

Lemma 3.1. *Suppose that $\tilde{L}\tilde{U}$ and LU are the LU factorizations without pivoting of $(J)_n$ and J, respectively. Then,*

$$(L)_n = \tilde{L}, \quad (U)_n = \tilde{U}.$$

Furthermore, if $J^{(p)}$ is the Darboux transform without parameter of J, then

$$(J^{(p)})_n = (U)_n(L)_n + l_n e_n e_n^{\mathrm{t}},$$

where $e_n = [0, \ldots, \overset{(n)}{1}]^{\mathrm{t}}$. We say that $(J^{(p)})_n$ is the Darboux transform without parameter of $(J)_n$.

Proof. It suffices to take into account (2.2) and the corresponding principal submatrices of order n of L and U to obtain the result in a straightforward way. □

Proposition 3.2. *Let $(J)_n$ be the principal submatrix of order n of the monic Jacobi matrix associated with a quasi-definite linear functional* **L**. *If we apply a Darboux transformation without parameter to $(J)_n$, i.e.,*

$$(J)_n = (L)_n(U)_n, \quad (J^{(p)})_n = (U)_n(L)_n + l_n e_n e_n^{\mathrm{t}},$$

then, the matrix $(J^{(p)})_n$ is the principal submatrix of order n of the monic Jacobi matrix associated with the functional **xL**.

Proof. Taking into account (2.9), we get

$$(J^{(p)})_n = \begin{bmatrix} \beta_0 + l_1 & 1 & \cdots & 0 & 0 \\ l_1(\beta_1 - l_1) & \beta_1 + l_2 - l_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & l_{n-1}(\beta_{n-1} - l_{n-1}) & \beta_{n-1} + l_n - l_{n-1} \end{bmatrix}.$$

Assume that $\{Q_n\}$ is the sequence of monic polynomials orthogonal with respect to the functional **xL** and assume that it satisfies the three-term recurrence relation given by

$$Q_{n+1}(x) = (x - \delta_n)Q_n(x) - k_n Q_{n-1}(x), \quad n \geqslant 0. \tag{3.1}$$

Assuming that $l_0 = 0$, we will prove that

$$\begin{cases} \delta_n = \beta_n + l_{n+1} - l_n, & n \geqslant 0, \\ k_n = l_n(\beta_n - l_n), & n \geqslant 1. \end{cases} \tag{3.2}$$

It is well known that $\{Q_n\}$ is the sequence of kernel polynomials associated with $\{P_n\}$ [4]. Moreover, the following explicit algebraic relation holds

$$Q_n(x) = \frac{1}{x}\left[P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)}P_n(x)\right]. \tag{3.3}$$

Replacing (3.3) in (3.1), and taking into account (2.3),

$$\begin{aligned} P_{n+2}(x) = {} & [x - \delta_n - u_{n+2}]P_{n+1}(x) + [u_{n+1}(x - \delta_n) - k_n]P_n(x) \\ & - k_n u_n P_{n-1}(x). \end{aligned}$$

Since $\{P_n\}$ satisfies the recurrence relation given in (1.1), we get

$$\begin{aligned} [\delta_n + u_{n+2} - \beta_{n+1}]P_{n+1}(x) = {} & [\gamma_{n+1} + u_{n+1}(x - \delta_n) - k_n]P_n(x) \\ & - k_n u_n P_{n-1}(x). \end{aligned} \tag{3.4}$$

Comparing the previous expression with (1.1), the following relation is obtained:

$$\delta_n = \beta_{n+1} - u_{n+2} + u_{n+1}.$$

And taking into account (2.4), we get

$$\delta_n = \beta_n + l_{n+1} - l_n.$$

On the other hand, from (1.1) and (3.4), the following relation also follows:

$$k_n = \gamma_n \frac{u_{n+1}}{u_n}.$$

Notice that we get $\gamma_n = l_n(\beta_{n-1} - l_{n-1})$ from (2.5). Then, taking into account (2.4)

$$k_n = l_n(\beta_n - l_n). \qquad \square$$

Remark 3.3. Observe from the previous proposition that the linear functional **xL** is quasi-definite if and only if the LU factorization of J exists which is equivalent to say that $P_n(0) \neq 0$ for all n.

From the previous proposition, it is immediate to obtain the corresponding result in the infinite case.

Theorem 3.4. *Let J be the monic Jacobi matrix associated with a quasi-definite linear functional* **L**. *If $\{P_n\}$ is the sequence of monic polynomials orthogonal with respect to* **L**, *and we assume that $P_n(0) \neq 0$, for $n \geqslant 1$, then the Darboux transform without parameter of J, $J^{(p)}$, is the monic Jacobi matrix associated with the functional* **xL**.

Proof. Since L and U are bidiagonal matrices, in both LU and UL products, the elements are computed from finite sums. Therefore, the proof of Proposition 3.2 can be extended to the infinite case without any problem of convergence. □

As we mentioned at the beginning of this section, the result given in Theorem 3.4 can be easily extended to the more general case $(\mathbf{x} - \alpha)\mathbf{L}$, with $\alpha \in \mathbb{C}$. The following lemma is the key for this extension.

Lemma 3.5. *Let J be the monic Jacobi matrix associated with the quasi-definite linear functional $\mathbf{L}$. Then, the matrix $J - \alpha I$, where I denotes the identity matrix, is the monic Jacobi matrix associated with the linear functional $\tilde{\mathbf{L}}$ given by*

$$\tilde{\mathbf{L}}[p(x)] = \mathbf{L}[p(x - \alpha)], \tag{3.5}$$

where $p(x)$ denotes any polynomial.

Proof. Let $\{P_n\}$ be the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$. Then, it satisfies a three-term recurrence relation as in (1.1). If we introduce the change of variable $x \to x + \alpha$, then we get

$$P_{n+1}(x + \alpha) = [x - (\beta_n - \alpha)]P_n(x + \alpha) - \gamma_n P_{n-1}(x + \alpha), \quad n \geqslant 0. \tag{3.6}$$

If $\tilde{P}_n(x) := P_n(x + \alpha)$, then $\{\tilde{P}_n\}$ is the sequence of monic orthogonal polynomials associated with the monic Jacobi matrix $J - \alpha I$. Moreover, such a sequence is orthogonal with respect to the linear functional $\tilde{\mathbf{L}}$ defined in the statement of the proposition, i.e.,

$$\begin{aligned}\tilde{\mathbf{L}}[\tilde{P}_n(x)\tilde{P}_m(x)] &= \mathbf{L}[\tilde{P}_n(x - \alpha)\tilde{P}_m(x - \alpha)] = \mathbf{L}[P_n(x)P_m(x)] \\ &= K_n \delta_{n,m}. \qquad \square\end{aligned}$$

Proposition 3.6. *Let J be the monic Jacobi matrix associated with $\mathbf{L}$, $\{P_n\}$ the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$, and $\alpha \in \mathbb{C}$ such that $P_n(\alpha) \neq 0$, for $n \geqslant 1$. If we apply the following transformation:*

$$J - \alpha I = LU, \quad \tilde{J} := UL + \alpha I,$$

then, $\tilde{J}$ is the monic Jacobi matrix associated with the functional $(\mathbf{x} - \alpha)\mathbf{L}$. The previous transformation is said to be a Darboux transformation without parameter and with shift α.

Proof. From Lemma 3.5, $J - \alpha I$ is the monic Jacobi matrix associated with the functional $\mathbf{L}_1$ defined by

$$\mathbf{L}_1[p(x)] = \mathbf{L}[p(x - \alpha)]. \tag{3.7}$$

The application of a Darboux transformation without parameter to $J - \alpha I$ generates a new monic Jacobi matrix T. From Theorem 3.4, T is the monic Jacobi matrix associated with the functional $\mathbf{xL}_1$. Then, again from Lemma 3.5, $T + \alpha I$ is the monic Jacobi matrix associated with the functional $\mathbf{L}_2$ given by

$$\mathbf{L}_2[p(x)] = (\mathbf{xL}_1)[p(x+\alpha)].$$

Hence,

$$\begin{aligned}\mathbf{L}_2[p(x)] &= (\mathbf{xL}_1)[p(x+\alpha)] = \mathbf{L}_1[xp(x+\alpha)] = \mathbf{L}[(x-\alpha)p(x)] \\ &= (\mathbf{x}-\alpha)\mathbf{L}[p(x)]. \qquad \square\end{aligned}$$

Iterating the previous result, we get the following corollary.

Corollary 3.7. *Let J be the monic Jacobi matrix associated with the quasi-definite linear functional $\mathbf{L}$, and let $\{P_n\}$ be the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$. Consider $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{C}$. We apply the following transformations to J:*

$$\begin{aligned}
T_1 &:= J - \alpha_1 I = L_1 U_1, & \tilde{T}_1 &:= U_1 L_1 + \alpha_1 I, \\
T_2 &:= \tilde{T}_1 - \alpha_2 I = L_2 U_2, & \tilde{T}_2 &:= U_2 L_2 + \alpha_2 I, \\
&\vdots \\
T_r &:= \tilde{T}_{r-1} - \alpha_r I = L_r U_r, & \tilde{T}_r &:= U_r L_r + \alpha_r I.
\end{aligned}$$

If $\{P_n^{(i)}\}$ is the sequence of monic orthogonal polynomials associated with the matrix $\tilde{T}_i$, $i \in \{1, 2, \ldots, r-1\}$, and we assume that

$$P_n(\alpha_1) \neq 0, \quad P_n(\alpha_{i+1}; i) \neq 0, \qquad n \geqslant 1, \quad i \in \{1, 2, \ldots, r-1\},$$

then $\tilde{T}_r$ is the monic Jacobi matrix associated with the functional

$$(\mathbf{x} - \alpha_r) \cdots (\mathbf{x} - \alpha_2)(\mathbf{x} - \alpha_1)\mathbf{L}.$$

This is the so-called Christoffel transform of $\mathbf{L}$.

3.1. Darboux transformation without parameter for symmetric linear functionals

Consider a positive definite linear functional $\mathbf{U}$. It is said to be *symmetric* if the odd moments associated with $\mathbf{U}$ vanish, i.e.,

$$\mathbf{U}(x^{2k+1}) = 0, \qquad k \geqslant 0.$$

In such a case, the monic Jacobi matrix associated with $\mathbf{U}$, J_0, has zeros as entries in the main diagonal and the LU factorization without pivoting of J_0 does not exist which means that there is no Darboux transform without parameter of J_0 and the functional $\mathbf{xU}$ is not quasi-definite.

In this subsection, we are interested in perturbations of a symmetric positive definite linear functional that generate new symmetric positive definite linear functionals. Notice that, if **U** is a symmetric linear functional, **xU** is not a symmetric functional anymore. However, it happens to be that $\mathbf{x}^2\mathbf{U}$ is a new symmetric linear functional. Based on previous results, the most obvious way to obtain the monic Jacobi matrix associated with $\mathbf{x}^2\mathbf{U}$ should be the application of two consecutive Darboux transformations without parameter to J_0. Nevertheless, if

$$J_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ \xi_1 & 0 & 1 & \cdots \\ 0 & \xi_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is the monic Jacobi matrix associated with a symmetric linear functional **U**, since $\beta_0 = 0$, then the LU factorization of J_0 does not exist and we cannot consider the Darboux transform without parameter of J_0. In the rest of this subsection, we will prove that the monic Jacobi matrix associated with $\mathbf{x}^2\mathbf{U}$ can be obtained from the monic Jacobi matrix associated with **U** applying two consecutive Darboux transformations without parameter with shifts $-\epsilon$ and ϵ, and taking limits when ϵ tends to zero. This problem was considered by Buhmann and Iserles [3] in a more general context. They considered a positive definite linear functional **L** and the symmetric Jacobi matrix associated with the orthonormal sequence of polynomials associated with **L**, and proved that one step of the QR method applied to the Jacobi matrix corresponds to finding the Jacobi matrix of the orthonormal polynomial system associated with $\mathbf{x}^2\mathbf{L}$.

Let us apply a Darboux transformation without parameter with shift $-\epsilon$ to the monic Jacobi matrix J_0, where ϵ is any positive number. If $J_0 + \epsilon I = L_1 U_1$ denotes the LU factorization of $J_0 + \epsilon I$ then, from (2.9),

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ l_1(\epsilon) & 1 & 0 & \cdots \\ 0 & l_2(\epsilon) & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad U_1 = \begin{bmatrix} \epsilon & 1 & 0 & \cdots \\ 0 & \epsilon - l_1(\epsilon) & 1 & \cdots \\ 0 & 0 & \epsilon - l_2(\epsilon) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.8}$$

with $l_1(\epsilon) = \frac{\xi_1}{\epsilon}$, $l_n(\epsilon) = \frac{\xi_n}{\epsilon - l_{n-1}(\epsilon)}$.

Now we compute $J_1 := U_1 L_1 - \epsilon I$. Thus we get

$$J_1 = \begin{bmatrix} l_1(\epsilon) & 1 & 0 & \cdots \\ l_1(\epsilon)(\epsilon - l_1(\epsilon)) & l_2(\epsilon) - l_1(\epsilon) & 1 & \cdots \\ 0 & l_2(\epsilon)(\epsilon - l_2(\epsilon)) & l_3(\epsilon) - l_2(\epsilon) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, from Proposition 3.6, J_1 is the monic Jacobi matrix associated with the functional $(\mathbf{x} + \epsilon)\mathbf{L}$. Now, apply a Darboux transformation without parameter with shift ϵ to J_1, i.e.,

$$J_1 - \epsilon I = L_2 U_2, \quad J_2 := U_2 L_2 + \epsilon I.$$

If the hypotheses of Corollary 3.7 are satisfied, then J_2 is the monic Jacobi matrix associated with the functional $(\mathbf{x}^2 - \epsilon^2)\mathbf{L}$. Moreover,

$$J_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ l_1(\epsilon)(\epsilon - l_2(\epsilon)) & 0 & 1 & \cdots \\ 0 & l_2(\epsilon)(\epsilon - l_3(\epsilon)) & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Notice that J_2 is the monic Jacobi matrix associated with a symmetric linear functional since the entries in the main diagonal are all zeros. If ϵ tends to zero, then we get

Proposition 3.8

$$\lim_{\epsilon \to 0} |l_n(\epsilon)[\epsilon - l_{n+1}(\epsilon)]| < \infty.$$

Therefore, if we denote by $T := \lim_{\epsilon \to 0} J_2$, then T is the monic Jacobi matrix associated with the symmetric linear functional $\mathbf{x}^2\mathbf{L}$.

In order to prove Proposition 3.8, we introduce the following lemmas:

Lemma 3.9. *If $\{R_n\}$ is the sequence of monic polynomials associated with $J_0 + \epsilon I$, then*

$$l_n(\epsilon) = -\xi_n \frac{R_{n-1}(0)}{R_n(0)}, \quad n \geqslant 1.$$

Proof. Assume that $J_0 + \epsilon I = LU$ is the LU factorization without pivoting of $J_0 + \epsilon I$. Taking into account (2.7), $\xi_n = l_n(\epsilon)u_n(\epsilon)$, where $u_n(\epsilon) = \epsilon - l_{n-1}(\epsilon)$ and $l_0(\epsilon) = 0$. Considering (2.3), the result follows in a straightforward way. □

Lemma 3.10. *Taking into account that the polynomials $\{R_n\}$ are functions of ϵ, the following statements hold.*

- *The polynomials of even degree satisfy*

$$\lim_{\epsilon \to 0} R_{2n}(0) \neq 0 \quad \text{for } n \geqslant 1.$$

- *The polynomials of odd degree satisfy*

$$R_{2n+1}(0) = \epsilon K_{2n+1}(\epsilon),$$

for a certain polynomial in ϵ, $K_{2n+1}(\epsilon)$, such that $\lim_{\epsilon \to 0} K_{2n+1}(\epsilon) \neq 0$ for $n \geqslant 0$.

Proof. The matrix $J_0 + \epsilon I$ contains the parameters of the recurrence relation that the polynomials R_n satisfy, then

$$R_{n+1}(x) = (x-\epsilon)R_n(x) - \xi_n R_{n-1}(x). \tag{3.9}$$

Since the linear functional **U** is positive definite, $\xi_n > 0$ for $n \geqslant 1$. Moreover, $R_1(0) = -\epsilon$ and $R_2(0) = \epsilon^2 - \xi_1$. Notice that $\lim_{\epsilon\to 0} R_2(0) \neq 0$ since $\xi_1 \neq 0$. Assume that $\lim_{\epsilon\to 0} R_{2n-2}(0) \neq 0$. Then, from (3.9),

$$\lim_{\epsilon\to 0} R_{2n}(0) = \lim_{\epsilon\to 0}[-\epsilon R_{2n-1}(0) - \xi_{2n-1}R_{2n-2}(0)] \neq 0.$$

On the other hand,

$$R_1(0) = -\epsilon = \epsilon K_1(\epsilon), \quad \text{where } K_1(\epsilon) := -1.$$

Assume that $R_{2n-1}(0) = \epsilon K_{2n-1}(\epsilon)$. Then, from (3.9),

$$R_{2n+1}(0) = -\epsilon R_{2n}(0) - \xi_{2n}R_{2n-1}(0) = -\epsilon R_{2n}(0) - \epsilon\xi_{2n}K_{2n-1}(\epsilon).$$

Hence,

$$R_{2n+1}(0) = \epsilon K_{2n+1}(\epsilon), \quad \text{for } n \geqslant 1, \tag{3.10}$$

where

$$K_{2n+1}(\epsilon) = -R_{2n}(0) - \xi_{2n}K_{2n-1}(\epsilon). \tag{3.11}$$

We must prove that $\lim_{\epsilon\to 0} K_{2n+1}(\epsilon) \neq 0$. First of all, observe that

$$\lim_{\epsilon\to 0} R_2(0) < 0, \quad \lim_{\epsilon\to 0} R_4(0) = \lim_{\epsilon\to 0}(-\epsilon^2 K_3(\epsilon) - \xi_3 R_2(0)) > 0. \tag{3.12}$$

We will prove by induction that the limits, when ϵ tends to zero, of two consecutive polynomials of even degree evaluated in zero, have opposite signs. In fact,

$$\lim_{\epsilon\to 0} R_{2n}(0) = -\xi_{2n-1}\lim_{\epsilon\to 0} R_{2n-2}(0), \tag{3.13}$$

and the previous assertion follows in a straightforward way. On the other hand, taking into account that $K_1(\epsilon) = -1 < 0$, from (3.11) we get

$$\lim_{\epsilon\to 0} K_3(\epsilon) = \lim_{\epsilon\to 0}[-R_2(0) - \xi_2 K_1(\epsilon)] > 0,$$

since $\lim_{\epsilon\to 0} R_2(0)$ and $\lim_{\epsilon\to 0} K_1(\epsilon)$ have the same sign and $\xi_2 > 0$. Assume that $\lim_{\epsilon\to 0} R_{2n-2}(0)$ and $\lim_{\epsilon\to 0} K_{2n-3}(\epsilon)$ have the same sign. Then, $\lim_{\epsilon\to 0} K_{2n-1}(\epsilon) = \lim_{\epsilon\to 0}[-R_{2n-2}(0) - \xi_{2n-2}K_{2n-3}(\epsilon)]$ have the opposite sign that $\lim_{\epsilon\to 0} R_{2n-2}(0)$ which, from (3.13), implies that $\lim_{\epsilon\to 0} K_{2n-1}$ and $\lim_{\epsilon\to 0} R_{2n}(0)$ have the same sign, and finally, from (3.11), we conclude that $\lim_{\epsilon\to 0} K_{2n+1}(\epsilon) \neq 0$. □

Next we prove Proposition 3.8 taking into account Lemmas 3.9 and 3.10.

Proof of Proposition 3.8. From Lemma 3.9 we get

$$l_{2n}(\epsilon)(\epsilon - l_{2n+1}(\epsilon)) = -\frac{\xi_{2n}R_{2n-1}(0)}{R_{2n}(0)}\left[\epsilon + \frac{\xi_{2n+1}R_{2n}(0)}{R_{2n+1}(0)}\right],$$

and, from Lemma 3.10,

$$l_{2n}(\epsilon)(\epsilon - l_{2n+1}(\epsilon)) = -\frac{\xi_{2n}\epsilon K_{2n-1}(\epsilon)}{R_{2n}(0)}\left[\epsilon + \frac{\xi_{2n+1}R_{2n}(0)}{\epsilon K_{2n+1}(\epsilon)}\right]$$
$$= -\frac{\xi_{2n}K_{2n-1}(\epsilon)}{R_{2n}(0)}\left[\epsilon^2 + \frac{\xi_{2n+1}R_{2n}(0)}{K_{2n+1}(\epsilon)}\right].$$

From Lemma 3.10 again, $\lim_{\epsilon\to 0} R_{2n}(0) \neq 0$ and $\lim_{\epsilon\to 0} K_{2n+1}(\epsilon) \neq 0$. Therefore,

$$\lim_{\epsilon\to 0} |l_{2n}(\epsilon)(\epsilon - l_{2n+1}(\epsilon))| < \infty.$$

Equivalently,

$$l_{2n+1}(\epsilon)(\epsilon - l_{2n+2}(\epsilon)) = -\frac{\xi_{2n+1}R_{2n}(0)}{\epsilon K_{2n+1}(\epsilon)}\left[\epsilon + \frac{\xi_{2n+2}\epsilon K_{2n+1}(\epsilon)}{R_{2n+2}(0)}\right]$$
$$= -\frac{\xi_{2n+1}R_{2n}(0)}{K_{2n+1}(\epsilon)}\left[1 + \frac{\xi_{2n+2}K_{2n+1}(\epsilon)}{R_{2n+2}(0)}\right].$$

Therefore, since $\lim_{\epsilon\to 0} R_{2n+2}(0) \neq 0$ and $\lim_{\epsilon\to 0} K_{2n+1}(\epsilon) \neq 0$, it follows that

$$\lim_{\epsilon\to 0} |l_{2n+1}(\epsilon)(\epsilon - l_{2n+2}(\epsilon))| < \infty. \qquad \square$$

4. Transformation of the functional L into $\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$

In this section, we will prove that the application of a Darboux transformation without parameter followed by a Darboux transformation to the monic Jacobi matrix associated with a linear functional **L**, yields the monic Jacobi matrix associated with the functional $\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$. This result can be extended in a simple way to obtain the monic Jacobi matrix associated with

$$\mathbf{L} + \sum_{i=1}^{k} \mathbf{C}_i \delta(\mathbf{x} - \mathbf{a}_i), \quad a_i \in \mathbb{C}.$$

Next, we define the concept of *symmetrizable functional*. This definition will be useful to prove the main result in this section.

Definition 4.1. Let **L** be a quasi-definite linear functional. The functional **L** is said to be *symmetrizable* if the linear functional **U** defined by

$$\mathbf{U}(x^{2n}) = \mathbf{L}(x^n), \quad \mathbf{U}(x^{2n+1}) = 0, \quad n \geqslant 0,$$

is also quasi-definite.

Lemma 4.2. *A linear functional* **L** *is symmetrizable if and only if* $P_n(0) \neq 0$ *for all* $n \geqslant 1$, *where* $\{P_n\}$ *is the sequence of monic polynomials orthogonal with respect to* **L**.

The following proposition is the main result in this section.

Proposition 4.3. *Let J_0 be the monic Jacobi matrix associated with the quasi-definite linear functional $\mathbf{L}$. Assume that $\{P_n\}$ is the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$, and $P_n(0) \neq 0$ for all $n \geqslant 1$. If we apply the following transformations to J_0:*

$$\begin{aligned} J_0 &= L_1U_1, \quad J_1 := U_1L_1, \\ J_1 &= U_2L_2, \quad J_2 := L_2U_2, \end{aligned}$$

then J_2 is the monic Jacobi matrix associated with the functional $\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$, where

$$C = \frac{\mu_0(\beta_0 - s)}{s}.$$

Here $\mu_0 = \mathbf{L}(1)$, $\beta_0 = \frac{\mathbf{L}(x)}{\mathbf{L}(1)}$ and s denotes the free parameter associated with the UL factorization of J_1.

Proof. Considering (2.9), the matrix J_1 can be expressed in the following way:

$$J_1 = \begin{bmatrix} \beta_0 + l_1 & 1 & 0 & \cdots \\ l_1(\beta_1 - l_1) & \beta_1 + l_2 - l_1 & 1 & \cdots \\ 0 & l_2(\beta_2 - l_2) & \beta_2 + l_3 - l_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Consider now the *UL* factorization of J_1 (which depends on a free parameter s). From Proposition 2.4,

$$J_1 = \begin{bmatrix} s & 1 & 0 & \cdots \\ 0 & S_1 & 1 & \cdots \\ 0 & 0 & S_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \beta_0 + l_1 - s & 1 & 0 & \cdots \\ 0 & \beta_1 + l_2 - l_1 - S_1 & 1 & \cdots \\ 0 & 0 & \beta_2 + l_3 - l_2 - S_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $S_1 = \frac{l_1(\beta_1 - l_1)}{\beta_0 + l_1 - s}$, $S_n = \frac{l_n(\beta_n - l_n)}{\beta_{n-1} + l_n - l_{n-1} - S_{n-1}}$, for all $n \geqslant 2$. Then, the matrix $J_2 := L_2U_2$ is given by

$$J_2 = \begin{bmatrix} s & 1 & 0 & \cdots \\ s(\beta_0 + l_1 - s) & \beta_0 + l_1 - s + S_1 & 1 & \cdots \\ 0 & S_1(\beta_1 + l_2 - l_1 - S_1) & \beta_1 + l_2 - l_1 + S_2 - S_1 & \cdots \\ 0 & 0 & S_2(\beta_2 + l_3 - l_2 - S_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We must prove that J_2 is the monic Jacobi matrix associated with the linear functional

$$\tilde{\mathbf{L}} = \mathbf{L} + \mathbf{C}\delta(\mathbf{x}),$$

where $C = \frac{\mu_0(\beta_0 - s)}{s}$, with $\mu_0 = \mathbf{L}(1)$.

Notice that $\mathbf{L}$ is a symmetrizable functional since $P_n(0) \neq 0$ (recall Definition 4.1 and Lemma 4.2). Let $\mathbf{U}$ be the symmetric linear functional associated with $\mathbf{L}$. Let $\{Q_n\}$ be the sequence of monic polynomials orthogonal with respect to $\mathbf{U}$ and $\{\xi_n\}$ the sequence of parameters given by the three-term recurrence relation that $\{Q_n\}$ satisfies, i.e.,

$$Q_{n+1}(x) = xQ_n(x) - \xi_n Q_{n-1}(x), \quad n \geqslant 0.$$

It is well known [2] that

$$\begin{cases} \beta_0 = \xi_1, \\ \beta_n = \xi_{2n} + \xi_{2n+1}, \quad n \geqslant 1, \\ \gamma_n = \xi_{2n-1}\xi_{2n}, \quad n \geqslant 1. \end{cases} \tag{4.1}$$

Assuming that $l_0 = 0$, we prove by induction that

$$\xi_{2n} = l_n, \;\; \xi_{2n+1} = \beta_n - l_n, \quad n \geqslant 1. \tag{4.2}$$

From (2.5) and (4.1),

$$\xi_1 = \beta_0 - l_0, \quad \xi_2 = \frac{\gamma_1}{\beta_0} = l_1.$$

Assume that $\xi_{2n-1} = \beta_{n-1} - l_{n-1}$. Then, from (2.5) and (4.1), we get

$$\xi_{2n} = \frac{\gamma_n}{\beta_{n-1} - l_{n-1}} = l_n.$$

On the other hand, from (4.1),

$$\xi_{2n+1} = \beta_n - l_n.$$

Suppose now that $\{\tilde{\beta}_n\}$ and $\{\tilde{\gamma}_n\}$ are the sequences of parameters associated with the recurrence relation that the polynomials orthogonal with respect to $\tilde{\mathbf{L}}$ satisfy. It is easy to prove that $\tilde{\mathbf{L}}$ is also a symmetrizable functional. Let $\tilde{\mathbf{U}}$ be the symmetric linear functional associated with $\tilde{\mathbf{L}}$ and let $\{\tilde{Q}_n\}$ be the sequence of monic polynomials orthogonal with respect to $\tilde{\mathbf{U}}$. If $\{\tilde{\xi}_n\}$ denotes the sequence of parameters given by the three-term recurrence relation that $\{\tilde{Q}_n\}$ satisfies, then

$$\begin{cases} \tilde{\beta}_0 = \tilde{\xi}_1, \\ \tilde{\beta}_n = \tilde{\xi}_{2n} + \tilde{\xi}_{2n+1}, \quad n \geqslant 1, \\ \tilde{\gamma}_n = \tilde{\xi}_{2n-1}\tilde{\xi}_{2n}, \quad n \geqslant 1. \end{cases} \tag{4.3}$$

Moreover [2,4]

$$\begin{cases} \tilde{\xi}_1 = \frac{\xi_1}{1 + \frac{C}{\mu_0}}, \\ \tilde{\xi}_{2m} = \xi_{2m-1} + \xi_{2m} - \tilde{\xi}_{2m-1}, \quad m \geqslant 1, \\ \tilde{\xi}_{2m+1} = \frac{\xi_{2m}\xi_{2m+1}}{\tilde{\xi}_{2m}}, \quad m \geqslant 1. \end{cases} \tag{4.4}$$

In the sequel, we state the relation between $\tilde{\beta}_n$ and β_n as well as the relation between $\tilde{\gamma}_n$ and γ_n. From (4.1), (4.3), and (4.4)

$$\tilde{\beta}_0 = \tilde{\xi}_1 = \frac{\xi_1\mu_0}{\mu_0 + C} = \frac{\beta_0\mu_0}{\mu_0 + C} = s,$$
$$\tilde{\gamma}_1 = \tilde{\xi}_2\tilde{\xi}_1 = (\xi_1 + \xi_2 - \tilde{\xi}_1)\tilde{\beta}_0 = \tilde{\beta}_0(\beta_0 + \xi_2 - \tilde{\beta}_0).$$

Taking into account (4.2), the previous result can be read as

$$\tilde{\gamma}_1 = s(\beta_0 + l_1 - s).$$

Taking into account that $l_0 = 0$ and denoting $S_0 := s$, we will prove that

$$\tilde{\xi}_{2n} = \beta_{n-1} + l_n - l_{n-1} - S_{n-1}, \quad \tilde{\xi}_{2n-1} = S_{n-1}, \qquad n \geqslant 1.$$

We have already proven that $\tilde{\xi}_1 = s = S_0$. From (4.4), $\tilde{\xi}_2 = \xi_1 + \xi_2 - \tilde{\xi}_1$, and we get $\tilde{\xi}_2 = \beta_0 - l_0 + l_1 - S_0$.

Assume that $\tilde{\xi}_{2n-1} = S_{n-1}$, then

$$\tilde{\xi}_{2n} = \xi_{2n-1} + \xi_{2n} - \tilde{\xi}_{2n-1} = \beta_{n-1} - l_{n-1} + l_n - S_{n-1}.$$

On the other hand, from (4.2) and (4.4)

$$\tilde{\xi}_{2n+1} = \frac{\xi_{2n}\xi_{2n+1}}{\tilde{\xi}_{2n}} = \frac{l_n(\beta_n - l_n)}{\beta_{n-1} + l_n - l_{n-1} - S_{n-1}} = S_n.$$

Hence,

$$\tilde{\beta}_k = \tilde{\xi}_{2k} + \tilde{\xi}_{2k+1} = \beta_{k-1} - l_{k-1} + l_k - S_{k-1} + S_k,$$
$$\tilde{\gamma}_k = \tilde{\xi}_{2k}\tilde{\xi}_{2k-1} = S_{k-1}(\beta_{k-1} - l_{k-1} + l_k - S_{k-1}). \qquad \square$$

Next we give the shifted version of Proposition 4.3.

Corollary 4.4. *Let J_0 be the monic Jacobi matrix associated with a quasi-definite linear functional* **L**. *Consider the following transformations on the matrix J_0:*

$$J_0 - \alpha I = L_1U_1, \quad J_1 := U_1L_1,$$
$$J_1 = U_2L_2, \quad J_2 := L_2U_2 + \alpha I.$$

Then, J_2 is the monic Jacobi matrix associated with the functional $\mathbf{L} + \mathbf{C}\delta(\mathbf{x} - \alpha)$, where

$$\mathbf{C} = \frac{\mu_0(\beta_0 - \alpha - s)}{s},$$

with $\mu_0 = \mathbf{L}(1)$, $\beta_0 = \frac{\mathbf{L}(x)}{\mathbf{L}(1)}$, and s is the parameter associated with the UL factorization of J_1.

Proof. Denote by $T_0 := J_0 - \alpha I$. From Lemma 3.5, T_0 is the monic Jacobi matrix associated with the linear functional $\mathbf{L}_1$ given by

$$\mathbf{L}_1[p(x)] = \mathbf{L}[p(x-\alpha)].$$

From Proposition 4.3, the matrix $T_1 = L_2U_2$ is the monic Jacobi matrix associated with the linear functional $\mathbf{L}_2 = \mathbf{L}_1 + \mathbf{C}\delta(\mathbf{x})$, where

$$\mathbf{C} = \frac{\tilde{\mu}_0(\tilde{\beta}_0 - s)}{s} = \frac{\mu_0(\beta_0 - \alpha - s)}{s}, \quad \text{with } \tilde{\mu}_0 = \mathbf{L}_1(1), \quad \tilde{\beta}_0 = \frac{\mathbf{L}_1(x)}{\mathbf{L}_1(1)}.$$

Finally, $J_2 = T_1 + \alpha I$ is the monic Jacobi matrix associated with the linear functional $\mathbf{L}_3$ given by

$$\mathbf{L}_3[p(x)] = \mathbf{L}_2[p(x+\alpha)]$$

and, hence

$$\mathbf{L}_3[p(x)] = \mathbf{L}_2[p(x+\alpha)] = \mathbf{L}_1[p(x+\alpha)] + \mathbf{C}p(\alpha) = \mathbf{L}[p(x)] + \mathbf{C}p(\alpha),$$

or, equivalently,

$$\mathbf{L}_3 = \mathbf{L} + \mathbf{C}\delta(\mathbf{x} - \alpha). \qquad \square$$

Next corollary shows how to include a mass point in two points symmetric with respect to the origin.

Corollary 4.5. *Consider the monic Jacobi matrix J_0 associated with the quasi-definite linear functional $\mathbf{L}$. Let us apply the following transformations to J_0*

$$\begin{aligned} J_0 - \alpha I &= L_1U_1, & J_1 &:= U_1L_1, \\ J_1 &= U_2L_2, & J_2 &:= L_2U_2 + \alpha I, \\ J_2 + \alpha I &= L_3U_3, & J_3 &:= U_3L_3, \\ J_3 &= U_4L_4, & J_4 &:= L_4U_4 - \alpha I. \end{aligned}$$

If the necessary conditions for the existence of the LU factorizations of $J_0 - \alpha I$ and $J_2 + \alpha I$ hold, then J_4 is the monic Jacobi matrix associated with the functional

$$\mathbf{L}_2 = \mathbf{L} + \mathbf{C}_1\delta(\mathbf{x} - \alpha) + \mathbf{C}_2\delta(\mathbf{x} + \alpha),$$

where

$$\mathbf{C}_1 = \frac{\mu_0(\beta_0 - \alpha - s_1)}{s_1}, \quad \mathbf{C}_2 = \frac{\mu_0(\beta_0 + \alpha - s_2) + C_1(2\alpha - s_2)}{s_2},$$

with $\mu_0 = \mathbf{L}(1)$, $\beta_0 = \frac{\mathbf{L}(x)}{\mathbf{L}(1)}$, and s_1, s_2 are the free parameters associated with the UL factorization of J_1 and J_3, respectively.

Proof. From Corollary 4.4, the matrix J_2 is associated with the linear functional

$$\mathbf{L}_1 = \mathbf{L} + \mathbf{C}_1\delta(\mathbf{x} - \alpha),$$

where

$$\mathbf{C}_1 = \frac{\mu_0(\beta_0 - \alpha - s_1)}{s_1}.$$

From Corollary 4.4 again, the matrix J_4 is associated with the linear functional

$$\mathbf{L}_2 = \mathbf{L}_1 + \mathbf{C}_2\delta(\mathbf{x}+\alpha),$$

where

$$\mathbf{C}_2 = \frac{\tilde{\mu}_0(\tilde{\beta}_0 + \alpha - s_2)}{s_2},$$

Notice that

$$\tilde{\mu}_0 = \mathbf{L}_1[1] = \mu_0 + C_1, \quad \tilde{\beta}_0 = \frac{\mathbf{L}_1(x)}{\mathbf{L}_1(1)} = \frac{\mu_1 + C_1\alpha}{\mu_0 + C_1} = \frac{\beta_0\mu_0 + C_1\alpha}{\mu_0 + C_1}.$$

If p denotes any polynomial, then

$$\mathbf{L}_2[p(x)] = \mathbf{L}_1[p(x)]\mathbf{C}_2 p(-\alpha) = \mathbf{L}[p(x)] + \mathbf{C}_1 p(\alpha) + \mathbf{C}_2 p(-\alpha),$$

or, equivalently,

$$\mathbf{L}_2 = \mathbf{L} + \mathbf{C}_1\delta(\mathbf{x}-\alpha) + \mathbf{C}_2\delta(\mathbf{x}+\alpha).$$

Moreover,

$$\begin{aligned}\mathbf{C}_2 &= \frac{(\mu_0 + \mathbf{C}_1)\left(\frac{\beta_0\mu_0 + \mathbf{C}_1\alpha}{\mu_0 + \mathbf{C}_1} + \alpha - s_2\right)}{s_2} \\ &= \frac{\mu_0(\beta_0 + \alpha - s_2) + \mathbf{C}_1(2\alpha - s_2)}{s_2}. \qquad \square\end{aligned}$$

5. Transformation of the functional L into $\frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$

Finally, we analyze the case when the initial linear functional $\mathbf{L}$ is transformed into $\frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$ by the application of a Darboux transformation to the monic Jacobi matrix associated with $\mathbf{L}$. Moreover, we consider appropriate combinations of Darboux transformations and Darboux transformations without parameter in order to obtain the monic Jacobi matrix associated with the linear functionals $\frac{1}{\mathbf{x}}\left(\frac{1}{\mathbf{x}}\mathbf{L}\right) + \mathbf{C}_1\delta(\mathbf{x}) + \mathbf{C}_2\delta'(\mathbf{x})$ and $\mathbf{L} + \mathbf{C}_1\delta(\mathbf{x}) + \mathbf{C}_2\delta'(\mathbf{x})$.

Proposition 5.1. *Let J_1 be the monic Jacobi matrix associated with the linear functional $\tilde{\mathbf{L}}$. Suppose that there exists a linear functional $\mathbf{L}$ such that $\tilde{\mathbf{L}} = \mathbf{x}\mathbf{L}$. Consider a Darboux transformation applied to J_1,*

$$J_1 = U_1 L_1, \quad J_2 := L_1 U_1.$$

Then, J_2 is the monic Jacobi matrix associated with the linear functional $\mathbf{H}$, *where* $\mathbf{H} = \frac{1}{\mathbf{x}}\tilde{\mathbf{L}} + \mathbf{C}\delta(\mathbf{x})$ *and* $\mathbf{C} = \frac{\tilde{\mathbf{L}}(1)}{s}$. *In other words,* $\mathbf{H}$ *is the Geronimus transform of* $\tilde{\mathbf{L}}$.

Proof. From Proposition 4.3, the application of a Darboux transformation without parameter followed by a Darboux transformation to the monic Jacobi matrix J_0 associated with the linear functional $\mathbf{L}$, yields the matrix J_2, which is the monic Jacobi matrix associated with the linear functional $\mathbf{L} + \mathbf{C}_1\delta(\mathbf{x})$, where $\mathbf{C}_1 = \frac{\mu_0(\beta_0 - s)}{s}$ with $\mu_0 = \mathbf{L}(1)$, $\beta_0 = \frac{\mathbf{L}(x)}{\mathbf{L}(1)}$, and s is the free parameter of the corresponding *UL* factorization. On the other hand, if J_1 is the monic Jacobi matrix obtained after the application of a Darboux transformation without parameter to the matrix J_0, J_1 is associated with the linear functional $\tilde{\mathbf{L}} = \mathbf{x}\mathbf{L}$. Then, let us prove that applying a Darboux transformation to J_1, the matrix J_2 is the monic Jacobi matrix associated with $\frac{1}{\mathbf{x}}\tilde{\mathbf{L}} + \mathbf{C}\delta(\mathbf{x})$, i.e.,

$$\left[\frac{1}{\mathbf{x}}\tilde{\mathbf{L}} + \mathbf{C}\delta(\mathbf{x})\right](p) = [\mathbf{L} + \mathbf{C}_1\delta(\mathbf{x})](p).$$

Taking into account the definition of the linear functionals involved in the left hand side of the above expression, we get

$$\begin{aligned}\left(\frac{1}{x}\tilde{\mathbf{L}}\right)(p) + Cp(0) &= \tilde{\mathbf{L}}\left(\frac{p(x) - p(0)}{x}\right) + \mathbf{C}p(0)\\ &= \mathbf{L}(p(x) - p(0)) + \mathbf{C}p(0)\\ &= \mathbf{L}(p(x)) + p(0)(\mathbf{C} - \mu_0).\end{aligned}$$

Then,

$$\mathbf{C} - \mathbf{L}(1) = \frac{\tilde{\mathbf{L}}(1)}{s} - \mathbf{L}(1) = \frac{\mathbf{L}(x)}{s} - \mathbf{L}(1) = \frac{\mu_0(\beta_0 - s)}{s} = \mathbf{C}_1.$$

Notice that $\mathbf{C}$ must be different from zero for any choice of the parameter s. □

An alternative proof of the previous proposition can be found in [13]. The finite version of Proposition 5.1 is presented below.

Proposition 5.2. *Let $(J_1)_n$ be the principal submatrix of order n of the monic Jacobi matrix associated with a quasi-definite linear functional* $\tilde{\mathbf{L}}$. *Assume that there exists a linear functional* $\mathbf{L}$ *that satisfies* $\tilde{\mathbf{L}} = \mathbf{x}\mathbf{L}$. *Apply the following transformations to* $(J_1)_n$:

$$(J_1)_n = (U)_n(L)_n, \quad (J_2)_n := (L)_n(U)_n.$$

Then, $(J_2)_n$ is the principal submatrix of order n of the monic Jacobi matrix associated with the functional $\frac{1}{\mathbf{x}}\tilde{\mathbf{L}} + \mathbf{C}\delta(\mathbf{x})$, *where* $\mathbf{C} = \frac{\tilde{\mu}_0}{s}$ *with* $\tilde{\mu}_0 = \tilde{L}(1)$, *and s is the free parameter associated with the corresponding UL factorization.*

Next we give the shifted version of Proposition 5.1.

Corollary 5.3. *Let* J_1 *be the monic Jacobi matrix associated with the functional* $\tilde{\mathbf{L}}$. *Suppose that there exists a linear functional* $\mathbf{L}$ *such that* $\tilde{\mathbf{L}} = (\mathbf{x} - \alpha)\mathbf{L}$. *Apply the following transformations to* J_1:

$$J_1 - \alpha I = U_1 L_1, \quad J_2 := L_1 U_1 + \alpha I.$$

Then, J_2 *is the monic Jacobi matrix associated with the linear functional* $\mathbf{L}_3 = \frac{\mathbf{1}}{\mathbf{x}-\alpha}\tilde{\mathbf{L}} + \mathbf{C}\delta(\mathbf{x} - \alpha)$, *where* $\mathbf{C} = \frac{\tilde{\mathbf{L}}[1]}{s}$, *and* s *is the free parameter associated with the corresponding UL factorization.*

Proof. The proof is similar to the proof of Corollary 4.4. $\square$

The application of two consecutive Darboux transformations to a monic Jacobi matrix include a new term in the transformed functional. In fact, this term is the derivative of a Dirac's delta function.

Corollary 5.4. *Let* J_1 *be the monic Jacobi matrix associated with the linear functional* $\mathbf{L}_1$. *If we apply two consecutive Darboux transformations to the matrix* J_1, *i.e.*,

$$J_1 = U_1 L_1, \quad J_2 := L_1 U_1,$$
$$J_2 = U_2 L_2, \quad J_3 := L_2 U_2,$$

then, J_3 *is the monic Jacobi matrix associated with the linear functional*

$$\mathbf{L}_3 = \frac{1}{\mathbf{x}}\left[\frac{1}{\mathbf{x}}\mathbf{L}_1\right] + \mathbf{C}_1\delta(\mathbf{x}) + \mathbf{C}_2\delta'(\mathbf{x}),$$

where

$$\mathbf{C}_1 = \frac{\tilde{\mu}_0}{s_2}, \quad \mathbf{C}_2 = \frac{\mu_0}{s_1},$$

with $\mu_0 = \mathbf{L}_1(1)$, $\tilde{\mu}_0 = \mathbf{L}_2(1)$, *and* s_1, s_2 *are the free parameters associated with the UL factorization of* J_1 *and* J_2, *respectively.*

Proof. From Proposition 5.1, J_2 is the monic Jacobi matrix associated with the linear functional

$$\mathbf{L}_2 = \frac{1}{\mathbf{x}}\mathbf{L}_1 + \mathbf{C}_2\delta(\mathbf{x}),$$

where $C_2 = \frac{\mathbf{L}_1(1)}{s_1}$. Considering again Proposition 5.1, J_3 is the monic Jacobi matrix associated with the linear functional

$$\mathbf{L}_3 = \frac{1}{\mathbf{x}}\mathbf{L}_2 + \mathbf{C}_1\delta(\mathbf{x}),$$

and $\mathbf{C}_1 = \frac{\mathbf{L}_2[1]}{s_2} = \frac{\mu_0}{s_1 s_2}$.

For any $p \in \mathbb{P}$, we get

$$\mathbf{L}_3(p) = \left(\frac{1}{\mathbf{x}}\mathbf{L}_2\right)(p) + \mathbf{C}_1 p(0) = \frac{1}{\mathbf{x}}\left[\frac{1}{\mathbf{x}}\mathbf{L}_1 + \mathbf{C}_2\delta(\mathbf{x})\right](p) + \mathbf{C}_1 p(0)$$
$$= \frac{1}{\mathbf{x}}\left[\frac{1}{\mathbf{x}}\mathbf{L}_1\right](p) - \mathbf{C}_2 p'(0) + \mathbf{C}_1 p(0). \qquad \square$$

The situation discussed in the previous corollary, with moment functionals involving not only Dirac's delta functions but also any of its derivatives, was first considered in [13]. Finally, we obtain the monic Jacobi matrix associated with the functional $\mathbf{L} + \mathbf{C}_1\delta(\mathbf{x}) + \mathbf{C}_2\delta'(\mathbf{x})$ in terms of the monic Jacobi matrix associated with $\mathbf{L}$.

Corollary 5.5. *Let J_0 be the monic Jacobi matrix associated with the quasi-definite linear functional $\mathbf{L}$. Assume that $\{P_n\}$ is the sequence of monic polynomials orthogonal with respect to $\mathbf{L}$, and $P_n(0) \neq 0$ for all $n \geqslant 1$. Apply the following transformations*

$$\begin{aligned} J_0 &= L_1U_1, & J_1 &:= U_1L_1,\\ J_1 &= L_2U_2, & J_2 &:= U_2L_2,\\ J_2 &= U_3L_3, & J_3 &:= L_3U_3,\\ J_3 &= U_4L_4, & J_4 &:= L_4U_4, \end{aligned}$$

whenever the LU factorization of J_1 exists. Then, J_4 is the monic Jacobi matrix associated with the linear functional

$$\mathbf{L} + \mathbf{C}_1\delta(\mathbf{x}) + \mathbf{C}_2\delta'(\mathbf{x}),$$

where $\mathbf{C}_1 = \frac{\boldsymbol{L}(x^2)}{s_1s_2} - \boldsymbol{L}(1)$ and $\mathbf{C}_2 = -\frac{\boldsymbol{L}(x^2)}{s_1} + \boldsymbol{L}(x)$. Here s_1 and s_2 are the free parameters associated with the UL factorization of J_2 and J_3, respectively.

Proof. Notice that

$$\frac{1}{\mathbf{x}}[\mathbf{x}\mathbf{L}] = \mathbf{L} - \mathbf{L}(1)\delta(\mathbf{x}) \quad \text{and} \quad \frac{1}{\mathbf{x}}\delta(\mathbf{x}) = -\delta'(\mathbf{x}). \tag{5.1}$$

Then, from Theorem 3.4 and Proposition 5.1, we get

- J_1 is the monic Jacobi matrix associated with the linear functional $\mathbf{xL}$.
- J_2 is the monic Jacobi matrix associated with the linear functional $\mathbf{x}^2\mathbf{L}$.
- J_3 is the monic Jacobi matrix associated with the linear functional $\mathbf{xL} + \mathbf{C}\delta(\mathbf{x})$, where $\mathbf{C} = \frac{\mathbf{L}(x^2)}{s_1} - \mathbf{L}(x)$. Here s_1 is the free parameter associated with the *UL* factorization of J_2.
- J_4 is the monic Jacobi matrix associated with the linear functional $\mathbf{L} + \mathbf{C}_1\delta(\mathbf{x}) + \mathbf{C}_2\delta'(\mathbf{x})$.

From Proposition 5.1, the linear functional associated with J_4 is

$$\frac{1}{\mathbf{x}}[\mathbf{xL} + \mathbf{C}\delta(\mathbf{x})] + \mathbf{D}\delta(\mathbf{x}),$$

with $D = \frac{\mathbf{L}(x^2)}{s_1 s_2}$. Taking into account (5.1), our statement follows. □

6. Explicit algebraic relation between the polynomials orthogonal with respect to a linear functional L and the polynomials orthogonal with respect to the linear functional $\frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$

Let $\{P_n\}$ be the sequence of monic polynomials orthogonal with respect to the quasi-definite linear functional $\mathbf{L}$. Assuming that the linear functional $\frac{1}{\mathbf{x}}\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$ is quasi-definite, let $\{Q_n\}$ be the corresponding sequence of monic orthogonal polynomials. If J_1 denotes the monic Jacobi matrix associated with $\{P_n\}$, and J_2 denotes the monic Jacobi matrix associated with $\{Q_n\}$, then, from Proposition 5.1, we get

$$J_2 L = L J_1, \tag{6.1}$$

where L denotes the lower triangular matrix corresponding to the *UL* factorization of J_1.

Moreover, if p and q denote, respectively, the column vectors whose components are the polynomials $\{P_n\}$ and $\{Q_n\}$, i.e., $p = [P_0(x), P_1(x), \ldots]^t$ and $q = [Q_0(x), Q_1(x), \ldots]^t$, taking into account that both sequences constitute a monic polynomial basis, then there exists a unique lower triangular matrix $\tilde{L}$, with ones in the main diagonal, such that

$$q = \tilde{L} p. \tag{6.2}$$

We will prove that $\tilde{L} = L$. From the definition of monic Jacobi matrix,

$$tp = J_1 p, \tag{6.3}$$

as well as

$$tq = J_2 q. \tag{6.4}$$

Replace (6.2) in (6.4). Next, multiply (6.3) by $\tilde{L}$ on the left and subtract the resulting equations. Then we get $J_2 \tilde{L} = \tilde{L} J_1$. Comparing this result with (6.1), we obtain $\tilde{L} = L$ because, if L and $\tilde{L}$ were different, then $J_2(\tilde{L} - L) = (\tilde{L} - L)J_1$. Since $\tilde{L} - L$ is a strictly lower triangular matrix, it is straightforward to prove that $\tilde{L} - L = 0$ or, in other words, $\tilde{L} = L$.

From Propositions 2.4 and 5.1, taking into account that $q = Lp$, it follows that

$$Q_n(x) = P_n(x) + (\beta_{n-1} - S_{n-1})P_{n-1}(x), \tag{6.5}$$

where $S_0 = \frac{\mathbf{L}[1]}{C}$. Recall that $l_k = \beta_{k-1} - S_{k-1}$ denotes the entry in the position $(k + 1, k)$ of the lower triangular matrix L, obtained from the *UL* factorization of the monic Jacobi matrix J_1. Let us prove that

$$l_k = \beta_{k-1} - S_{k-1} = -\frac{\hat{P}_k(0)}{\hat{P}_{k-1}(0)}, \tag{6.6}$$

where the new polynomials $\hat{P}_k$ are obtained changing the parameters in the three-term recurrence relation that $\{P_n\}$ satisfies invariant. These polynomials are said to be the *co-recursive polynomials with parameter* S_0 associated with the linear functional **L** (see [4]). Since $\hat{P}_1(x) = x - \beta_0 + S_0$ and $\hat{P}_0(x) = 1$,

$$l_1 = \beta_0 - S_0 = -\frac{\hat{P}_1(0)}{\hat{P}_0(0)}.$$

Assume that $l_k = -\frac{\hat{P}_k(0)}{\hat{P}_{k-1}(0)}$. From Proposition 2.4, since $\hat{P}_{k+1}(x) = (x - \beta_k) \times \hat{P}_k(x) - \gamma_k \hat{P}_{k-1}(x)$ for $k \geqslant 1$, we get

$$l_{k+1} = \beta_k - S_k = \beta_k - \frac{\gamma_k}{l_k} = \frac{\beta_k \hat{P}_k(0) + \gamma_k \hat{P}_{k-1}(0)}{\hat{P}_k(0)} = -\frac{\hat{P}_{k+1}(0)}{\hat{P}_k(0)}.$$

Finally, replacing (6.6) into (6.5), we obtain

$$Q_n(x) = P_n(x) - \frac{\hat{P}_n(0)}{\hat{P}_{n-1}(0)} P_{n-1}(x). \tag{6.7}$$

This expression has also been obtained in [17] using a different approach.

7. Explicit algebraic relation between the polynomials orthogonal with respect to a linear functional L and the polynomials orthogonal with respect to the linear functional $\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$

Let $\{P_n\}$ and $\{Q_n\}$ be the sequences of monic polynomials orthogonal with respect to the linear functionals **L** and $\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$, respectively. Let J_0 be the monic Jacobi matrix associated with **L**. Then, from Proposition 4.3, the matrix J_2 such that

$$\begin{aligned} J_0 &= L_1 U_1, \quad J_1 := U_1 L_1, \\ J_1 &= U_2 L_2, \quad J_2 := L_2 U_2, \end{aligned}$$

is the monic Jacobi matrix associated with $\mathbf{L} + \mathbf{C}\delta(\mathbf{x})$, where **C** can be expressed in terms of the free parameter s associated with the *UL* factorization of J_1. From Theorem 3.4, we also know that J_1 is the monic Jacobi matrix associated with the linear functional **xL**. If $\{P_n^*\}$ denotes the sequence of kernel polynomials associated with $\{P_n\}$ (recall that $\{P_n^*\}$ is also the sequence of monic orthogonal polynomials associated with J_1) then, taking into account the results in the previous section and (6.7), in particular,

$$Q_n(x) = P_n^*(x) - \frac{\hat{P}_n^*(0)}{\hat{P}_{n-1}^*(0)} P_{n-1}^*(x), \quad n \geqslant 1.$$

Considering the definition of kernel polynomials, we get

$$x\,Q_n(x) = P_{n+1}(x) - \left[\frac{P_{n+1}(0)}{P_n(0)} + \frac{\hat{P}_n^*(0)}{\hat{P}_{n-1}^*(0)} \right] P_n(x) + \frac{P_n(0)\,\hat{P}_n^*(0)}{P_{n-1}(0)\,\hat{P}_{n-1}^*(0)}\,P_{n-1}(x),$$

which is equivalent to the expression obtained in [1].

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