Kernels and best approximations related to the system of ultraspherical polynomials

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Abstract

We study the uniformly bounded orthonormal system $\mathcal{U}_\lambda$ of functions

$$u_n^{(\lambda)}(x) = \varphi_n^{(\lambda)}(\cos x)(\sin x)^\lambda, \quad x \in [0, \pi],$$

where $\{\varphi_n^{(\lambda)}\}_{n=0}^\infty$ $(\lambda > 0)$ is the normalized system of ultraspherical polynomials. We investigate some approximation properties of the system $\mathcal{U}_\lambda$ and we show that these properties are similar to one's of the trigonometric system. First, we obtain estimates of L^p-norms of the kernels of the system $\mathcal{U}_\lambda$. These estimates enable us to prove Nikol'skiĭ-type inequalities for $\mathcal{U}_\lambda$-polynomials. Next, we prove directly that $\mathcal{U}_\lambda$ is a basis in each L_w^p, $1 < p < \infty$, where w is an arbitrary A_p-weight function. Finally, we apply these results to get sharp inequalities for the best $\mathcal{U}_\lambda$-approximations in L^q in terms of the best $\mathcal{U}_\lambda$-approximations in L^p $(1 \leqslant p < q < \infty)$. For the trigonometric system such inequalities have been already known.

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1. Introduction

Let $0 \leqslant \lambda < \infty$ and $m_\lambda(x) = (1 - x^2)^{\lambda-1/2}$, $x \in (-1, 1)$. Denote by $\{\varphi_n^{(\lambda)}\}_{n=0}^{\infty}$ the orthonormal basis of $L^2([-1, 1], m_\lambda)$ obtained from $\{x^n\}_{n=0}^{\infty}$ by the Gram–Schmidt process. Set

$$u_n^{(\lambda)}(x) = \varphi_n^{(\lambda)}(\cos x)(\sin x)^\lambda, \quad x \in [0, \pi].$$

Then the system $\mathcal{U}_\lambda \equiv \{u_n^{(\lambda)}\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2[0, \pi]$. Moreover, this system is uniformly bounded in $[0, \pi]$ (see [13, (7.33.6)]),

$$|u_n^{(\lambda)}(x)| \leqslant M_\lambda, \quad x \in [0, \pi], \quad n = 0, 1, \ldots . \tag{1.1}$$

For $\lambda = 0$ this is the cosine system

$$u_0^{(0)}(x) = \frac{1}{\sqrt{\pi}}, \quad u_n^{(0)}(x) = \sqrt{\frac{2}{\pi}} \cos nx, \quad n \in \mathbb{N}$$

and for $\lambda = 1$ we get the sine system

$$u_n^{(1)}(x) = \sqrt{\frac{2}{\pi}} \sin(n + 1)x, \quad n = 0, 1, \ldots .$$

Askey and Wainger [2] proved the following transplantation theorem:

Theorem A. *Let $0 < \lambda < \infty$, $1 < p < \infty$, and let $\{a_n\}$ be a sequence of real numbers. Then the series $\sum_{n=0}^{\infty} a_n u_n^{(\lambda)}(x)$ is the Fourier series of some function $f \in L^p[0, \pi]$ if and only if the series $\sum_{n=0}^{\infty} a_n \cos nx$ is the Fourier series of some function $\varphi \in L^p[0, \pi]$. Moreover, $c'_p \|\varphi\|_p \leqslant \|f\|_p \leqslant c_p \|\varphi\|_p$, $c'_p > 0$.*

It follows immediately that the system $\mathcal{U}_\lambda$ is a basis in each $L^p[0, \pi]$, $1 < p < \infty$ (a direct proof of this result will be given below). Therefore the analysis of general approximation properties of this system is a natural and relevant problem. Of course, a lot of results in this direction can be derived by transplantation from the theory of trigonometric series. Nevertheless, a more extended study of the system $\mathcal{U}_\lambda$ requires an independent development of basic tools of approximation theory for this special case.

Let $0 \leqslant \lambda < \infty$. For any integer $n \geqslant 0$ denote by $\mathcal{U}_\lambda^{(n)}$ the linear span of $\{u_k^{(\lambda)}\}_{k=0}^{n}$, i.e., the set of all functions

$$U_n(x) = \sum_{k=0}^{n} a_k u_k^{(\lambda)}(x), \quad a_k \in \mathbb{R}. \tag{1.2}$$

These functions are said to be $\mathcal{U}_\lambda$-polynomials. For every k we have

$$\cos kx = \sum_{j=0}^{k} \gamma_j^{(k)} \varphi_j^{(\lambda)}(\cos x).$$

Thus, $\mathcal{U}_\lambda^{(n)}$ coincides with the set of all functions $U_n(x) = T_n(x)(\sin x)^\lambda$, where

$$T_n(x) = \sum_{k=0}^{n} \alpha_k \cos kx \quad (\alpha_k \in \mathbb{R})$$

are even trigonometric polynomials of a degree at most n.

Let $f \in L^p[0, \pi]$ $(1 \leqslant p \leqslant \infty)$. Denote by $E_n^{(\lambda)}(f)_p$ the best approximation of f by polynomials $U_n \in \mathcal{U}_\lambda^{(n)}$,

$$E_n^{(\lambda)}(f)_p = \inf_{U_n \in \mathcal{U}_\lambda^{(n)}} \|f - U_n\|_p.$$

One of the important questions in the Embedding and Approximation theories is to determine how certain smoothness or constructive properties of a function $f \in L^p$ are reflected on its corresponding properties in a more strong L^q-norm $(q > p)$. Notice that the first results in this direction concerning the embedding of Lipschitz classes were obtained by Hardy and Littlewood [6]. Afterwards, sharp different norm inequalities for moduli of continuity were found by Ul'yanov [15]. In the case of constructive characteristics (best approximations) the question can be formulated as follows: given $1 \leqslant p < q \leqslant \infty$, find sharp relations between best approximations in L^p and L^q.

For the trigonometric system this problem was posed by Ul'yanov [15] and Stechkin. Its complete solution for $1 < p < q < \infty$ was obtained in [8]. Let $E_n(f)_r$ be a best trigonometric approximation of a function f in L^r. It was proved in [8,9] that for $1 < p < q < \infty$

$$E_n(f)_q \leqslant c \left(\sum_{k=n}^{\infty} (k - n + 1)^{q/p - 2} (E_k(f)_p)^q \right)^{1/q} \tag{1.3}$$

and this inequality is sharp for *any rate* of decay of the best approximations $E_n(f)_p$. The same results are also true in the case $p = 1$; in particular, inequality (1.3) for $p = 1$ can be deduced from the case $p > 1$.

Initially, this work started from the similar question for the best approximations by $\mathcal{U}_\lambda$-polynomials. Of course, it was clear in view of Theorem A that in the case $p > 1$ the same results hold for all $\lambda > 0$. Nevertheless, we were interested in the case $p = 1$ as well as in the direct proof for $p > 1$. This led us to the study of such problems as estimates of the kernels of the $\mathcal{U}_\lambda$-system, relations between different norms of $\mathcal{U}_\lambda$-polynomials (Nikol'skiĭ-type inequalities), special $\mathcal{U}_\lambda$-polynomials with some extremal properties.

The main results of this paper are the following. In Section 2 we obtain estimates of L^p-norms of the kernels of the system $\mathcal{U}_\lambda$. These estimates enable us to prove Nikol'skiĭ-type inequalities for $\mathcal{U}_\lambda$-polynomials (Section 3). Next, in the Section 3 we construct $\mathcal{U}_\lambda$-polynomials of the form

$$U_{\nu,\mu}(x) = \sum_{k=\mu}^{\nu} a_k u_k^{(\lambda)}(x), \quad 0 \leqslant \mu < \nu \text{ are integers},$$

which have optimal order of growth of the L^p-norm for all $p \geqslant p_0 > 0$. In Section 4 we give a direct proof of the basis property of the system $\mathcal{U}_\lambda$ in $L^p_w[0, \pi]$, $1 < p < \infty$, where w is an arbitrary A_p-weight function. In particular, this gives a short proof of the Pollard's mean convergence theorem for ultraspherical polynomials. Finally, in Section 5 we apply these results to get an analogue of inequality (1.3) for the best $\mathcal{U}_\lambda$-approximations and to prove its sharpness. In this section we follow the scheme of the works [8,9].

2. Kernels of the system $\mathcal{U}_\lambda$

In this section we will prove estimates of the kernels of the system $\mathcal{U}_\lambda$. Assume that $\lambda > 0$. Let $P_n^{(\lambda)}$ be the sequence of ultraspherical polynomials defined in [13, 4.7]. Then we have

$$\varphi_n^{(\lambda)}(x) = \alpha_n^{(\lambda)} P_n^{(\lambda)}(x), \tag{2.1}$$

where

$$\alpha_n^{(\lambda)} = 2^{\lambda-1/2}\pi^{-1/2}\Gamma(\lambda)\left(\frac{(n+\lambda)\Gamma(n+1)}{\Gamma(n+2\lambda)}\right)^{1/2}.$$

In what follows we use c_λ and C_λ to denote constants (in every appearance, in principle different) depending only on the parameter λ.

Lemma 1. *Let $0 < \lambda < \infty$. Then for every $x \in [0, \pi]$ and $n \in \mathbb{N}$*

$$u_n^{(\lambda)}(x) - u_{n+1}^{(\lambda)}(x) = b_\lambda u_{n-1}^{(\lambda+1)}(x)\sin x + (1-\cos x)u_n^{(\lambda)}(x) + \frac{\beta_n(x)}{n}, \tag{2.2}$$

where b_λ is a positive constant and

$$|\beta_n(x)| \leqslant C_\lambda, \quad x \in [0, \pi], \quad n = 0, 1, \ldots .$$

Proof. We shall use the following identity [13, (4.7.27)]:

$$(n+2\lambda)tP_n^{(\lambda)}(t) - (n+1)P_{n+1}^{(\lambda)}(t) = 2\lambda(1-t^2)P_{n-1}^{(\lambda+1)}(t).$$

Taking into account (2.1), we get

$$\begin{aligned}\varphi_n^{(\lambda)}(t) - \varphi_{n+1}^{(\lambda)}(t) = {}& \left(\alpha_n^{(\lambda)}\frac{n+1}{n+2\lambda} - \alpha_{n+1}^{(\lambda)}\right)P_{n+1}^{(\lambda)}(t) \\ & + (1-t)\varphi_n^{(\lambda)}(t) + \frac{2\lambda}{n+2\lambda}\alpha_n^{(\lambda)}(1-t^2)P_{n-1}^{(\lambda+1)}(t).\end{aligned}$$

Observe that

$$\alpha_n^{(\lambda)}\frac{n+1}{n+2\lambda} = \alpha_{n+1}^{(\lambda)}\left(1 + O\left(\frac{1}{n}\right)\right)$$

and

$$\frac{2\lambda}{n+2\lambda}\alpha_n^{(\lambda)} = \alpha_{n-1}^{(\lambda+1)}\left(b_\lambda + O\left(\frac{1}{n}\right)\right), \quad b_\lambda > 0.$$

Thus, we have

$$\begin{aligned}\varphi_n^{(\lambda)}(t) - \varphi_{n+1}^{(\lambda)}(t) = b_\lambda(1-t^2)\varphi_{n-1}^{(\lambda+1)}(t) + (1-t)\varphi_n^{(\lambda)}(t)\\ + O\left(\frac{1}{n}\right)\left[(1-t^2)\varphi_{n-1}^{(\lambda+1)}(t) + \varphi_{n+1}^{(\lambda)}(t)\right]\end{aligned}$$

and, as a consequence,

$$\begin{aligned}u_n^{(\lambda)}(x) - u_{n+1}^{(\lambda)}(x) = b_\lambda u_{n-1}^{(\lambda+1)}(x)\sin x + (1-\cos x)u_n^{(\lambda)}(x)\\ + O\left(\frac{1}{n}\right)\left[u_{n+1}^{(\lambda)}(x) + u_{n-1}^{(\lambda+1)}(x)\sin x\right].\end{aligned}$$

By virtue of (1.1), this yields (2.2). The lemma is proved. □

Denote

$$K_n^{(\lambda)}(x,t) = \sum_{k=0}^{n} u_k^{(\lambda)}(x)u_k^{(\lambda)}(t).$$

From the Christoffel–Darboux formula [13, 3.2],

$$\begin{aligned}K_n^{(\lambda)}(x,t) &= \frac{\gamma_n}{\cos x - \cos t}\left[u_{n+1}^{(\lambda)}(x)u_n^{(\lambda)}(t) - u_n^{(\lambda)}(x)u_{n+1}^{(\lambda)}(t)\right]\\ &= \frac{\gamma_n}{\cos x - \cos t}\left[u_n^{(\lambda)}(t)\left(u_{n+1}^{(\lambda)}(x) - u_n^{(\lambda)}(x)\right)\right.\\ &\quad \left. + u_n^{(\lambda)}(x)\left(u_n^{(\lambda)}(t) - u_{n+1}^{(\lambda)}(t)\right)\right], \qquad (2.3)\end{aligned}$$

where

$$c'_\lambda \leqslant \gamma_n \leqslant c''_\lambda \quad (n\in\mathbb{N};\ c'_\lambda, c''_\lambda > 0).$$

Notice also that (see [13, (4.1.3)])

$$u_n^{(\lambda)}(\pi - x) = (-1)^n u_n^{(\lambda)}(x), \quad x\in[0,\pi]. \qquad (2.4)$$

Lemma 2. *Let* $0<\lambda<\infty$. *Then for any* $n\in\mathbb{N}$ *and* $x,t\in[0,\pi]$

$$|K_n^{(\lambda)}(x,t)| \leqslant c_\lambda \min(n, |x-t|^{-1}). \qquad (2.5)$$

Proof. By (1.1),

$$|K_n^{(\lambda)}(x,t)| \leqslant M_\lambda^2(n+1). \qquad (2.6)$$

We shall prove that

$$|K_n^{(\lambda)}(x,t)| \leqslant c_\lambda |x-t|^{-1}. \qquad (2.7)$$

First suppose that $x \in [0, \pi/2]$. For any $t \in [0, \pi]$ we have

$$|\cos x - \cos t| = 2 \sin \frac{|x-t|}{2} \sin \frac{x+t}{2} \geqslant \frac{1}{2\pi^2} |x-t|(x+t). \tag{2.8}$$

Denote

$$\Delta_n(x,t) \equiv |u_{n+1}^{(\lambda)}(x) u_n^{(\lambda)}(t) - u_n^{(\lambda)}(x) u_{n+1}^{(\lambda)}(t)|. \tag{2.9}$$

By (1.1),

$$\Delta_n(x,t) \leqslant M_\lambda (|u_n^{(\lambda)}(x) - u_{n+1}^{(\lambda)}(x)| + |u_n^{(\lambda)}(t) - u_{n+1}^{(\lambda)}(t)|).$$

It follows from (2.2) that for any $y \in [0, \pi]$

$$|u_n^{(\lambda)}(y) - u_{n+1}^{(\lambda)}(y)| \leqslant c_\lambda (y + 1/n).$$

If $\max(x,t) \geqslant 1/n$, then $\Delta_n(x,t) \leqslant c_\lambda (x+t)$; applying (2.8), we get (2.7). If $x, t \in [0, 1/n]$, then (2.7) follows immediately from (2.6). Thus, we have proved inequality (2.7) for $x \in [0, \pi/2]$, $t \in [0, \pi]$. If $x \in [\pi/2, \pi]$ and $t \in [0, \pi]$, then by (2.4) we have

$$K_n^{(\lambda)}(x,t) = K_n^{(\lambda)}(\pi - x, \pi - t)$$

and this case immediately reduces to the preceding one. The lemma is proved. $\square$

Let $0 \leqslant \mu < \nu$ be integer numbers. Denote

$$K_{\nu,\mu}^{(\lambda)}(x,t) = \sum_{k=\mu}^{\nu} u_k^{(\lambda)}(x) u_k^{(\lambda)}(t).$$

If $\mu \geqslant 1$, then

$$K_{\nu,\mu}^{(\lambda)}(x,t) = K_\nu^{(\lambda)}(x,t) - K_{\mu-1}^{(\lambda)}(x,t).$$

As usual, we set $p' = p/(p-1)$ for $1 \leqslant p \leqslant \infty$.

Corollary 1. *Let* $0 < \lambda < \infty$. *Then for every* $x \in [0, \pi]$

$$\|K_{\nu,\mu}^{(\lambda)}(x, \cdot)\|_p \leqslant (p')^{1/p} c_\lambda (\nu - \mu)^{1-1/p} \quad (1 < p < \infty) \tag{2.10}$$

and

$$\|K_{\nu,\mu}^{(\lambda)}(x, \cdot)\|_1 \leqslant c_\lambda \log(\nu - \mu), \tag{2.11}$$

where c_λ *is some positive constant.*

Proof. It follows from (1.1) and (2.5) that for every $x, t \in [0, \pi]$

$$|K_{\nu,\mu}^{(\lambda)}(x,t)| \leqslant c_\lambda \min(\nu - \mu, |x-t|^{-1}).$$

For a fixed $x \in [0, \pi]$ denote

$$E'_x = \{t \in [0, \pi] : |x - t| \leqslant (\nu - \mu)^{-1}\}, \quad E''_x = [0, \pi] \setminus E'_x.$$

Then for $1 \leqslant p < \infty$ we have

$$\int_0^\pi |K_{\nu,\mu}^{(\lambda)}(x, t)|^p \, dt = \left(\int_{E'_x} + \int_{E''_x} \right) |K_{\nu,\mu}^{(\lambda)}(x, t)|^p \, dt$$
$$\leqslant 2c_\lambda^p \left[(\nu - \mu)^{p-1} + \int_{(\nu-\mu)^{-1}}^\pi z^{-p} \, dz \right].$$

This implies (2.10) and (2.11). $\square$

In what follows we will use the Mehler's formula [4, p. 177]:

$$u_n^{(\lambda)}(x) = t_n(\lambda)(\sin x)^{1-\lambda} \int_0^x \frac{\cos(n + \lambda)y}{(\cos y - \cos x)^{1-\lambda}} \, dy \tag{2.12}$$

for every $x \in [0, \pi]$ and $\lambda > 0$, where

$$t_n(\lambda) = \frac{2^{2\lambda-1/2}\Gamma(\lambda + 1/2)}{\pi\Gamma(2\lambda)} \left(\frac{(n + \lambda)\Gamma(n + 2\lambda)}{\Gamma(n + 1)} \right)^{1/2} = c_\lambda n^\lambda + O(n^{\lambda-1}).$$

Denote

$$L_n^{p,\lambda} = \sup_{x \in [0,\pi]} \|K_n^{(\lambda)}(x, \cdot)\|_p \ (1 \leqslant p \leqslant \infty); \quad L_n^{1,\lambda} \equiv L_n^\lambda.$$

Theorem 1. *Let $0 < \lambda < \infty$ and $1 \leqslant p \leqslant \infty$. Then there exist positive constants c and c' depending only on p and λ such that for every $n \in \mathbb{N}$*

$$c' n^{1-1/p} \leqslant L_n^{p,\lambda} \leqslant cn^{1-1/p}, \quad \text{when } 1 < p \leqslant \infty, \tag{2.13}$$

$$c' \log(n + 1) \leqslant L_n^\lambda \leqslant c \log(n + 1), \quad \text{when } p = 1. \tag{2.14}$$

Proof. The second inequalities in (2.13) and (2.14) follow by Corollary 1.

Let $\eta_n = \pi/(8(n+\lambda))$. From (2.12) it easily follows that for any $1 \leqslant k \leqslant n$ and $0 \leqslant x \leqslant \eta_n$

$$u_k^{(\lambda)}(x) \geqslant c_\lambda k^\lambda \int_0^x (x - y)^{\lambda-1} \, dy = \frac{c_\lambda}{\lambda} (kx)^\lambda, \tag{2.15}$$

where $c_\lambda > 0$. Thus, for $0 \leqslant t \leqslant \eta_n$ we have

$$K_n^{(\lambda)}(\eta_n, t) \geqslant c_\lambda \sum_{k=[n/2]}^n u_k^{(\lambda)}(t) \geqslant c'_\lambda n^{\lambda+1} t^\lambda.$$

Hence,

$$\int_0^\pi |K_n^{(\lambda)}(\eta_n, t)|^p \, dt \geqslant \int_0^{\eta_n} |K_n^{(\lambda)}(\eta_n, t)|^p \, dt$$
$$\geqslant c_\lambda n^{p(\lambda+1)} \int_0^{\eta_n} t^{\lambda p} \, dt \geqslant c'_\lambda n^{p-1}, \quad c'_\lambda > 0.$$

This yields the left-hand side inequality in (2.13).

To prove the first inequality in (2.14) we will proceed from formula (2.3). Using notation (2.9) and applying (1.1), (2.2), and (2.15), we get for any $t \in [1/n, \pi/2]$

$$\begin{aligned}\Delta_n(\eta_n, t) \geqslant |u_{n+1}^{(\lambda)}(t) - u_n^{(\lambda)}(t)\|u_n^{(\lambda)}(\eta_n)| \\ - M_\lambda |u_{n+1}^{(\lambda)}(\eta_n) - u_n^{(\lambda)}(\eta_n)| \geqslant c_\lambda t |u_{n-1}^{(\lambda+1)}(t)| - c'_\lambda(t^2 + 1/n).\end{aligned}$$

Further, for $t \in [1/n, \pi/2]$ we have

$$0 < \cos\eta_n - \cos t = 2\sin\frac{t-\eta_n}{2}\sin\frac{t+\eta_n}{2} \leqslant t^2.$$

Using these estimates and (2.3), we get

$$\begin{aligned}\int_0^\pi |K_n^{(\lambda)}(\eta_n, t)|\,dt \geqslant \int_{1/n}^{\pi/2} |K_n^{(\lambda)}(\eta_n, t)|\,dt \\ \geqslant c_\lambda \int_{1/n}^{\pi/2} |u_{n-1}^{(\lambda+1)}(t)|\,\frac{dt}{t} - c'_\lambda\left(\frac{\pi}{2} + 1\right).\end{aligned}$$

Finally, in the last integral we will use the asymptotic formula (see [13, (8.21.18)])

$$u_n^{(\lambda)}(x) = (2/\pi)^{1/2}\cos((n+\lambda)x - \lambda\pi/2) + \tau_n(x), \tag{2.16}$$

where

$$|\tau_n(x)| \leqslant \frac{c_\lambda}{nx}, \quad x \in (0, \pi/2]. \tag{2.17}$$

We obtain

$$\begin{aligned}\int_{1/n}^{\pi/2} |u_{n-1}^{(\lambda+1)}(t)|\,\frac{dt}{t} \geqslant (2/\pi)^{1/2}\int_{1/n}^{\pi/2} |\cos((n+\lambda)t - (\lambda+1)\pi/2)|\,\frac{dt}{t} - c_\lambda \\ \geqslant c'_\lambda \log n,\end{aligned}$$

where $c'_\lambda > 0$. This implies the first inequality in (2.14). The proof is completed. $\square$

3. $\mathcal{U}_\lambda$-Polynomials

Using estimate (2.10), we get the following Nikol'skiĭ-type inequality (see [11], [3, p. 102]).

Theorem 2. *Let* $0 \leqslant \mu < \nu$ *be integer numbers*, $0 < \lambda < \infty$, *and*

$$U_{\nu,\mu}(x) \equiv U(x) = \sum_{k=\mu}^{\nu} a_k u_k^{(\lambda)}(x), \quad a_k \in \mathbb{R}.$$

Then for any $0 < p < q \leqslant \infty$

$$\|U_{\nu,\mu}\|_q \leqslant c_{p,\lambda}(\nu - \mu)^{1/p - 1/q}\|U_{\nu,\mu}\|_p. \tag{3.1}$$

Proof. First suppose that $1 \leqslant p < \infty$. We have

$$U(x) = \int_0^\pi U(t) K_{\nu,\mu}^{(\lambda)}(x,t)\, dt.$$

From here,

$$|U(x)| \leqslant \|U\|_p \|K_{\nu,\mu}^{(\lambda)}(x,\cdot)\|_{p'}$$

and by (2.10)

$$\|U\|_\infty \leqslant c_\lambda p^{1/p'} (\nu-\mu)^{1/p} \|U\|_p. \tag{3.2}$$

Let now $0 < r < 1$. Using (3.2) with $p = 1$, we have

$$\begin{aligned}\|U\|_\infty &\leqslant c_\lambda(\nu-\mu) \int_0^\pi |U(x)|\, dx \\ &\leqslant c_\lambda(\nu-\mu)\|U\|_\infty^{1-r} \int_0^\pi |U(x)|^r\, dx.\end{aligned}$$

It follows that

$$\|U\|_\infty \leqslant [c_\lambda(\nu-\mu)]^{1/r} \|U\|_r. \tag{3.3}$$

Thus, we have (3.1) for $0 < p < \infty, q = \infty$. Let now $0 < p < q < \infty$. Then by inequalities (3.2) and (3.3),

$$\begin{aligned}\int_0^\pi |U(x)|^q\, dx &\leqslant \|U\|_\infty^{q-p} \int_0^\pi |U(x)|^p\, dx \\ &\leqslant \bar{c}_{p,\lambda}^{q-p} \|U\|_p^q (\nu-\mu)^{(q-p)/p},\end{aligned}$$

where $\bar{c}_{p,\lambda} = p^{1/p'} c_\lambda$, if $p \geqslant 1$, and $\bar{c}_{p,\lambda} = c_\lambda^{1/p}$, if $0 < p < 1$. This implies (3.1). The theorem is proved. $\square$

The following lemma presents a construction of $\mathcal{U}_\lambda$-polynomials with optimal order of growth of the L^p-norm for all $p \geqslant p_0 > 0$.

Lemma 3. *Let $0 < \lambda < \infty$ and $p_0 > 0$. Then for every integer numbers $0 \leqslant \mu \leqslant \nu$ there exists a polynomial*

$$U_{\nu,\mu}(x) = \sum_{k=\mu}^{\nu} a_k u_k^{(\lambda)}(x), \quad a_k \in \mathbb{R}, \tag{3.4}$$

such that for any $p_0 \leqslant p \leqslant \infty$

$$c'(\nu-\mu+1)^{1-1/p} \leqslant \|U_{\nu,\mu}\|_p \leqslant c''(\nu-\mu+1)^{1-1/p}, \tag{3.5}$$

where c' and c'' are positive constants depending only on λ and p_0.

Proof. First notice that for any $p > 0$

$$\int_0^{\pi} |u_k^{(\lambda)}(x)|^p \, dx \geqslant c_{p,\lambda} > 0 \quad (k = 0, 1, \ldots). \tag{3.6}$$

Indeed, since $\|u_k^{(\lambda)}\|_2 = 1$, for $p \geqslant 2$ (3.6) follows by Hölder's inequality. If $0 < p < 2$, then by (1.1)

$$1 = \int_0^{\pi} (u_k^{(\lambda)}(x))^2 \, dx \leqslant M_\lambda^{2-p} \int_0^{\pi} |u_k^{(\lambda)}(x)|^p \, dx,$$

which implies (3.6).

Denote $m = [(\lambda p_0)^{-1}] + 1$. If $\nu - \mu < 2m$, then we set $U_{\nu,\mu}(x) = u_\mu^{(\lambda)}(x)$. In this case inequalities (3.5) follow from (3.6) and (1.1). Suppose that $\nu - \mu \geqslant 2m$. Clearly, we can assume that the number $s = (\nu - \mu)/(2m)$ is a positive integer. Let

$$n = \frac{\nu + \mu}{2} = \mu + ms = \nu - ms. \tag{3.7}$$

Next, denote

$$U(x) \equiv U_{\nu,\mu}(x) = s^{1-m\lambda} u_n^{(\lambda)}(x) (\varphi_s^{(\lambda)}(\cos x))^m.$$

By the Dougall's formula (see [1, p. 319]),

$$\varphi_n^{(\lambda)}(t) \varphi_s^{(\lambda)}(t) = \sum_{k=n-s}^{n+s} c_k \varphi_k^{(\lambda)}(t).$$

Applying this equality m times, we get that

$$\varphi_n^{(\lambda)}(t) (\varphi_s^{(\lambda)}(t))^m = \sum_{k=n-ms}^{n+ms} a_k \varphi_k^{(\lambda)}(t).$$

By (3.7), it follows that U is a polynomial of form (3.4).

Further, we have for $x \in (0, \pi/2]$ (see [13, (7.33.6)])

$$|\varphi_s^{(\lambda)}(\cos x)| \leqslant c \min(s^\lambda, x^{-\lambda}).$$

Using this inequality, we obtain for any $p \geqslant p_0$

$$\begin{aligned}
\int_0^{\pi} |U(x)|^p \, dx &= 2 \int_0^{\pi/2} |U(x)|^p \, dx \\
&\leqslant c s^{(1-m\lambda)p} \left(\int_0^{1/s} + \int_{1/s}^{\pi/2} \right) |\varphi_s^{(\lambda)}(\cos x)|^{mp} \, dx \\
&\leqslant c' s^{(1-m\lambda)p} \left[s^{mp\lambda - 1} + \int_{1/s}^{\infty} x^{-\lambda mp} \, dx \right] \leqslant c'' s^{p-1}
\end{aligned}$$

(note that $\lambda mp > 1$). This implies the second inequality in (3.5).

Next, we will prove the first of inequalities (3.5). Let $\eta_k = \pi/(8(k+\lambda))$. By (2.15) we have

$$|\varphi_s^{(\lambda)}(\cos x)| \geqslant c s^\lambda \quad (0 \leqslant x \leqslant \eta_s) \quad \text{and} \quad u_n^{(\lambda)}(\eta_n) \geqslant c \quad (c > 0).$$

Thus, $\|U\|_\infty \geqslant c(\nu - \mu)$. By Theorem 2, it follows that for any $p > 0$

$$\|U\|_p \geqslant c(\nu - \mu)^{-1/p} \|U\|_\infty \geqslant c'(\nu - \mu)^{1-1/p}, \quad c' > 0.$$

The proof is completed. $\square$

Remark 1. In the trigonometric case the Jackson's kernels can be used to prove Lemma 3 (see [8]). Namely, in this case the function

$$U_{\nu,\mu}(x) = \sum_{k=\mu}^{\nu} a_k \cos kx$$

satisfying condition (3.5) can be given by

$$U_{\nu,\mu}(x) = s^{1-2r} \left(\frac{\sin((s+1)x/2)}{\sin(x/2)} \right)^{2r} \cos nx,$$

where $r = [(2p_0)^{-1}] + 1$, $s = (\nu - \mu)/(2r)$, and $n = (\nu + \mu)/2$ (we assume that s is an integer).

Remark 2. In the case $\mu = 0$ we have a more simple proof of Lemma 3. Moreover, in this case non-negative polynomials can be constructed. Let $\lambda > 0$ and $p_0 > 0$. Set $r = [(\lambda + 1/p_0)/2] + 1$, $m = [\nu/r]$. Then

$$T_\nu(x) = \left(\frac{\sin((m+1)x/2)}{\sin(x/2)} \right)^{2r}$$

is an even trigonometric polynomial of degree $mr \leqslant \nu$. Thus, the function

$$U_\nu(x) = \nu^{\lambda+1-2r} T_\nu(x)(\sin x)^\lambda \tag{3.8}$$

belongs to $\mathcal{U}_\lambda^{(\nu)}$. Furthermore, for some constant $c > 0$ we have

$$\frac{1}{c} \nu^{\lambda+1} x^\lambda \leqslant U_\nu(x) \leqslant c \nu^{\lambda+1} x^\lambda, \quad x \in [0, 1/\nu]$$

and

$$U_\nu(x) \leqslant c \nu^{\lambda+1-2r} x^{\lambda-2r}, \quad x \in [1/\nu, \pi].$$

Using these inequalities, we easily get that

$$c' \nu^{1-1/p} \leqslant \|U_\nu\|_p \leqslant c'' \nu^{1-1/p} \quad (c', c'' > 0)$$

for any $p \geqslant p_0$.

Remark 3. It follows from Lemma 3 that inequality (3.1) is sharp for any $\lambda \geqslant 0$.

4. Basis property

For every polynomial (1.2) we have $U_n(0) = U_n(\pi) = 0$. Therefore, if a function $f \in C[0, \pi]$ does not vanish at the endpoints of the interval $[0, \pi]$, then the sequence of the best approximations $\{E_n^{(\lambda)}(f)_C\}$ does not tend to 0 (we set $E_n^{(\lambda)}(f)_C \equiv E_n^{(\lambda)}(f)_\infty$ for $f \in C[0, \pi]$). Denote by $C_0[0, \pi]$ the closed subspace of $C[0, \pi]$ which consists of all functions $f \in C[0, \pi]$ such that $f(0) = f(\pi) = 0$.

Proposition 1. *If $f \in C_0[0, \pi]$, then for every $0 < \lambda < \infty$*

$$\lim_{n\to\infty} E_n^{(\lambda)}(f)_C = 0.$$

Proof. Let $\varepsilon > 0$. Since $f \in C_0[0, \pi]$, then there exist a closed interval $I \subset (0, \pi)$ and a function $g \in C[0, \pi]$ such that $g(x) = 0$ for all $x \in [0, \pi] \setminus I$ and

$$\|f - g\|_C < \frac{\varepsilon}{2}.$$

The function $\varphi(x) = g(x)(\sin x)^{-\lambda}$ is uniformly continuous in $(0, \pi)$. Thus, there exists a trigonometric polynomial

$$T_n(x) = \sum_{k=0}^{n} \alpha_k \cos kx$$

such that

$$|\varphi(x) - T_n(x)| < \frac{\varepsilon}{2} \quad \text{for every } x \in (0, \pi).$$

Set $U_n(x) = T_n(x)(\sin x)^\lambda$. Then $U_n \in \mathcal{U}_\lambda^{(n)}$. Furthermore, for every $x \in (0, \pi)$ we get

$$|g(x) - U_n(x)| = |\varphi(x) - T_n(x)|(\sin x)^\lambda < \frac{\varepsilon}{2}.$$

It follows that $\|f - U_n\|_C < \varepsilon$. This completes the proof. □

Let w be a non-negative measurable function in $[0, \pi]$. Denote by $L_w^p[0, \pi]$ $(1 \leqslant p < \infty)$ the space of all measurable functions f such that

$$\|f\|_{p,w} \equiv \left(\int_0^\pi |f(x)|^p w(x)\, dx\right)^{1/p} < \infty.$$

Corollary 2. *Let $w \in L^1[0, \pi]$ be a non-negative weight function and $0 < \lambda < \infty$. Then $\mathcal{U}_\lambda$-polynomials form a dense subset in every $L_w^p[0, \pi]$, $1 \leqslant p < \infty$.*

Recall that a non-negative locally integrable function w on $\mathbb{R}$ is said to satisfy A_p-condition $(1 \leqslant p < \infty)$ if

$$\sup_I \frac{1}{|I|} \int_I w(x)\, dx \left(\frac{1}{|I|} \int_I w(x)^{-1/(p-1)}\, dx\right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I.

We have a similar definition for functions w in $[0, \pi]$ (in this case we take only intervals $I \subset [0, \pi]$). It is easy to see that if a function w satisfies A_p-condition in $[0, \pi]$ and we extend w to the whole line as an even 2π-periodic function, then the extended function also satisfies A_p-condition on $\mathbb{R}$.

Next, for any function $\varphi \in L^1[-\pi, \pi]$ denote by $\mathcal{C}^*\varphi$ the maximal conjugate function operator,

$$(\mathcal{C}^*\varphi)(x) = \sup_{0<\varepsilon\leqslant\pi} \left| \frac{1}{2\pi} \int_{\varepsilon\leqslant|x-t|\leqslant\pi} \varphi(t) \cot\frac{t-x}{2}\, dt \right|.$$

If $1 < p < \infty$ and a 2π-periodic weight function w satisfies A_p-condition, then (see [14, p. 120, Theorem 2.12]; [5, p. 255])

$$\left(\int_{-\pi}^{\pi} |(\mathcal{C}^*\varphi)(x)|^p w(x)\, dx \right)^{1/p} \leqslant c \left(\int_{-\pi}^{\pi} |\varphi(x)|^p w(x)\, dx \right)^{1/p}. \tag{4.1}$$

For $f \in L^1[0, \pi]$ and $0 < \lambda < \infty$ denote by $S_n^{(\lambda)}(f; x)$ the partial sum of the Fourier series of f,

$$S_n^{(\lambda)}(f; x) = \sum_{k=0}^{n} a_k(f) u_k^{(\lambda)}(x), \quad a_k(f) = \int_0^{\pi} f(x) u_k^{(\lambda)}(x)\, dx.$$

We have

$$S_n^{(\lambda)}(f; x) = \int_0^{\pi} f(t) K_n^{(\lambda)}(x, t)\, dt. \tag{4.2}$$

Theorem 3. *Let $0 < \lambda < \infty$, $1 < p < \infty$, and let w be a weight function satisfying the A_p-condition in $[0, \pi]$. Then for any function $f \in L_w^p[0, \pi]$*

$$\|S_n^{(\lambda)}(f)\|_{p,w} \leqslant c\|f\|_{p,w} \quad (n = 0, 1, \ldots). \tag{4.3}$$

Proof. First we suppose that $f(x) = 0$ for $x \in [\pi/2, \pi]$ and f is extended to the whole line as 2π-periodic function such that $f(x) = 0$, $x \in [-\pi, 0)$. Furthermore, as it has been noted above, we may assume that w is extended to $\mathbb{R}$ as even 2π-periodic function. For $x \in [0, \pi]$ we denote

$$A_n(x) = \{t \in [0, \pi/2] : |x - t| \geqslant 1/n\}, \quad B_n(x) = [0, \pi/2] \setminus A_n(x).$$

By (4.2),

$$\begin{aligned} S_n^{(\lambda)}(f; x) &= \int_{A_n(x)} f(t) K_n^{(\lambda)}(x, t)\, dt \\ &\quad + \int_{B_n(x)} f(t) K_n^{(\lambda)}(x, t)\, dt \equiv \sigma_n(x) + \tau_n(x). \end{aligned} \tag{4.4}$$

First, we have

$$|\tau_n(x)| \leqslant cn \int_{B_n(x)} |f(t)|\, dt \leqslant cMf(x),$$

where *Mf* is the Hardy–Littlewood maximal function. Then (see [5, p. 255])

$$\|\tau_n\|_{p,w} \leqslant c\|Mf\|_{p,w} \leqslant c'\|f\|_{p,w}. \tag{4.5}$$

Next, in order to estimate $\|\sigma_n\|_p$ we will apply formula (2.3). First, (2.3) implies that

$$|K_n^{(\lambda)}(x,t)| \leqslant C \quad \text{for } x \in [2\pi/3, \pi],\ t \in [0, \pi/2]. \tag{4.6}$$

Further, by (2.2),

$$u_n^{(\lambda)}(x) - u_{n+1}^{(\lambda)}(x) = \sin x \left[b_\lambda u_{n-1}^{(\lambda+1)}(x) + u_n^{(\lambda)}(x) \tan\frac{x}{2}\right] + \frac{\beta_n(x)}{n}, \tag{4.7}$$

where

$$|\beta_n(x)| \leqslant C_\lambda \quad (x \in [0,\pi],\ n = 0, 1, \ldots).$$

We have also

$$\frac{\sin t}{\cos x - \cos t} = \frac{1}{2}\left(\cot\frac{t-x}{2} + \cot\frac{t+x}{2}\right). \tag{4.8}$$

Now for $t \in [0, \pi]$ set

$$f_n(t) = f(t)u_n^{(\lambda)}(t), \quad g_n(t) = f(t)\left[b_\lambda u_{n-1}^{(\lambda+1)}(t) + u_n^{(\lambda)}(t)\tan\frac{t}{2}\right].$$

Since $f(t) = 0$ for $t \in [\pi/2, \pi]$, we have

$$|f_n(t)| + |g_n(t)| \leqslant c|f(t)|, \quad 0 \leqslant t \leqslant \pi.$$

Extend the functions f_n and g_n to be 0 in $(-\pi, 0)$ and then periodically with the period 2π to the whole real line. Using (2.3), (4.7), and (4.8), we easily get for $x \in [0, 2\pi/3]$

$$\begin{aligned}|\sigma_n(x)| \leqslant c\Bigg(&\left|\int_{A_n(x)} f_n(t)\left(\cot\frac{t-x}{2} - \cot\frac{t+x}{2}\right)dt\right| \\ &+ \left|\int_{A_n(x)} g_n(t)\left(\cot\frac{t-x}{2} + \cot\frac{t+x}{2}\right)dt\right| \\ &+ \frac{1}{n}\int_{A_n(x)} \frac{|f(t)|}{|\cos x - \cos t|}\,dt\Bigg).\end{aligned}$$

If $t \in A_n(x)$, then $|\cos x - \cos t| \geqslant (x+t)/(\pi^2 n)$. Thus, for $x \in [0, 2\pi/3]$ we have

$$|\sigma_n(x)| \leqslant c\left[(\mathcal{C}^* f_n)(x) + (\mathcal{C}^* g_n)(x) + \int_0^{\pi/2} \frac{|f(t)|}{x+t}\,dt\right].$$

If $x \in [0, \pi]$, then

$$\psi(x) \equiv \int_0^{\pi/2} \frac{|f(t)|}{x+t}\,dt \leqslant c(H\varphi)(x),$$

where $\varphi(x) = |f(-x)|\chi_{[-\pi/2,0]}(x)$ and $H\varphi$ is the Hilbert transform of φ. Hence (see [5, p. 255]), $\|\psi\|_{p,w} \leqslant c\|f\|_{p,w}$. Next, by (4.1)

$$\|\mathcal{C}^* f_n\|_{p,w} + \|\mathcal{C}^* g_n\|_{p,w} \leqslant c \left(\int_{-\pi}^{\pi} (|f_n(x)| + |g_n(x)|)^p w(x) dx \right)^{1/p} \leqslant c' \|f\|_{p,w}.$$

Thus, we get

$$\left(\int_0^{2\pi/3} |\sigma_n(x)|^p w(x)\, dx \right)^{1/p} \leqslant c\|f\|_{p,w}. \tag{4.9}$$

Further, applying (4.2), (4.6), Hölder inequality, and A_p-condition, we obtain

$$\begin{aligned} &\left(\int_{2\pi/3}^{\pi} |S_n^{(\lambda)}(f; x)|^p w(x)\, dx \right)^{1/p} \\ &\qquad \leqslant c \int_0^{\pi} |f(t)|\, dt \left(\int_0^{\pi} w(x)\, dx \right)^{1/p} \leqslant c' \|f\|_{p,w}. \end{aligned}$$

From this inequality, (4.4), (4.5), and (4.9), it follows (4.3).

If supp $f \subset [\pi/2, \pi]$, then we consider the function $f_1(x) = f(\pi - x)$. We have (see (2.4))

$$\begin{aligned} S_n^{(\lambda)}(f_1; \pi - x) &= \int_0^{\pi} f(\pi - t) K_n^{(\lambda)}(\pi - x, t)\, dt \\ &= \int_0^{\pi} f(t) K_n^{(\lambda)}(x, t)\, dt = S_n^{(\lambda)}(f; x). \end{aligned}$$

Therefore (4.3) follows from the preceding case. This completes the proof. □

Corollary 3. *Suppose that $0 < \lambda < \infty$, $1 < p < \infty$, and a weight function w satisfies the A_p-condition in $[0, \pi]$. Then the system $\mathcal{U}_\lambda$ is a basis in $L_w^p[0, \pi]$.*

To prove this, observe that the system $\mathcal{U}_\lambda$ is minimal in $L_w^p[0, \pi]$, that is, no $u_k^{(\lambda)}$ belongs to the closure of the linear span of $\{u_n^{(\lambda)}\}_{n \neq k}$ in $L_w^p[0, \pi]$ (see [7, p. 6]). Indeed, if Q is an element in this linear span, then by orthogonality, (1.1), and Hölder inequality, we have

$$\begin{aligned} 1 &= \int_0^{\pi} (u_k^{(\lambda)}(x))^2\, dx = \int_0^{\pi} [u_k^{(\lambda)}(x) - Q(x)] u_k^{(\lambda)}(x)\, dx \\ &\leqslant M_\lambda c_w \|u_k^{(\lambda)} - Q\|_{p,w}, \quad \text{where } c_w = \left(\int_0^{\pi} w(x)^{-1/(p-1)}\, dx \right)^{1/p'} < \infty. \end{aligned}$$

Now Corollary 3 follows immediately from the criterion of a basis property (see [7, p. 10]).

Remark 4. The system $\mathcal{U}_\lambda$ is not a basis neither in $C_0[0, \pi]$ nor in $L^1[0, \pi]$. Indeed, it is easy to see that for any $0 < \lambda < \infty$, $n \in \mathbb{N}$, $\varepsilon > 0$, and $x \in [0, \pi]$ there exists a function $f \in C_0[0, \pi]$ with $\|f\|_C = 1$ such that

$$S_n^{(\lambda)}(f; x) > \int_0^{\pi} |K_n^{(\lambda)}(x, t)|\, dt - \varepsilon.$$

Thus, we have

$$\sup\{\|S_n^{(\lambda)}(f)\|_C : f \in C_0[0,\pi],\ \|f\| = 1\} = L_n^{(\lambda)}.$$

Applying (2.14) and the uniform boundedness principle, we immediately get the following statement:

Proposition 2. *For any* $0 < \lambda < \infty$ *there exists a function* $f \in C_0[0,\pi]$ *such that the sequence* $\{\|S_n^{(\lambda)}(f)\|_C\}$ *is unbounded.*

The similar proposition is true in the case of L^1-norm.

Remark 5. Let $0 < \lambda < \infty$ and $\mathcal{L}_\lambda^p \equiv L^p([-1,1],(1-t^2)^{\lambda-1/2})$. Suppose that $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$. The Pollard's mean convergence theorem [12, Theorem 8.1] asserts that for any function $g \in \mathcal{L}_\lambda^p$ the series

$$\sum_{n=0}^{\infty} c_n(g)\varphi_n^{(\lambda)}(t), \quad c_n(g) \equiv \int_{-1}^{1} g(t)\varphi_n^{(\lambda)}(t)(1-t^2)^{\lambda-1/2}\,dt,$$

converges to g in $\mathcal{L}_\lambda^p$. Observe that this theorem can be derived from Theorem 3. Indeed, it is easy to see that the function $w(x) = (\sin x)^{(2-p)\lambda}$ satisfies A_p-condition in $[0,\pi]$. Set $f(x) = g(\cos x)(\sin x)^\lambda$. Then

$$c_n(g) = \int_0^\pi f(x)u_n^{(\lambda)}(x)\,dx$$

and

$$\int_{-1}^{1} |g(t)|^p(1-t^2)^{\lambda-1/2}\,dt = \int_0^\pi |f(x)|^p w(x)\,dx.$$

Applying Theorem 3, we easily get Pollard's theorem.

5. Different norm inequalities for best approximations

In this section we will study the following problems. First, given $1 \leqslant p < q < \infty$, find sharp conditions on the best approximations $E_n^{(\lambda)}(f)_p$ of a function $f \in L^p[0,\pi]$ which guarantee that f belongs to $L^q[0,\pi]$. Furthermore, if these conditions hold, then find a sharp estimate of $E_n^{(\lambda)}(f)_q$ in terms of $E_n^{(\lambda)}(f)_p$.

As it was mentioned in the Introduction, for the trigonometric system these problems have been already solved. In our case we can apply the same scheme with the corresponding modifications.

The crucial role is played by the following lemma [8,10]:

Lemma 4. *Let* $0 < p < \infty$ *and let* $\{h_k(x)\}$ *be a sequence of non-negative functions* $h_k \in L^\infty[a,b]$ *such that*

$$\|h_k\|_p \leqslant d_k \quad (k = 1, 2, \ldots),$$

where the sequence $\{d_k\}$ *satisfies the condition*

$$d_{k+1} \leqslant \beta d_k \quad (0 < \beta < 1;\ k = 1, 2, \ldots).$$

Then for any $q \in (p, \infty)$

$$\left\| \sum_{k=1}^{\infty} h_k \right\|_q \leqslant c \left(\sum_{k=1}^{\infty} \|h_k\|_{\infty}^{q-p} d_k^p \right)^{1/q}.$$

We will use also the following Hardy-type inequalities.

Lemma 5. *Let* $\alpha_n \geqslant 0$, $\varepsilon_n > 0$, *and for some* $\beta \in (0, 1)$

$$\varepsilon_{n+1} \leqslant \beta \varepsilon_n \quad (n = 1, 2, \ldots).$$

Then for any $r > 0$

$$\sum_{n=1}^{\infty} \varepsilon_n \left(\sum_{k=1}^{n} \alpha_k \right)^r \leqslant c \sum_{n=1}^{\infty} \varepsilon_n \alpha_n^r, \tag{5.1}$$

$$\sum_{n=1}^{\infty} \varepsilon_n^{-1} \left(\sum_{k=n}^{\infty} \alpha_k \right)^r \leqslant c \sum_{n=1}^{\infty} \varepsilon_n^{-1} \alpha_n^r. \tag{5.2}$$

Inequality (5.1) was proven in [9]; the proof of (5.2) is similar.

Theorem 4. *Let* $1 \leqslant p < q < \infty$ *and* $0 < \lambda < \infty$. *Then for any function* $f \in L^p[0, \pi]$

$$\|f\|_q \leqslant c \left[\|f\|_p + \left(\sum_{n=1}^{\infty} n^{q/p-2} (E_n^{(\lambda)}(f)_p)^q \right)^{1/q} \right], \tag{5.3}$$

where c is a constant which only depends on p, q, *and* λ.

Inequality (5.3) is a direct analogue of the Ul'yanov's inequality [15] for the best approximations by trigonometric polynomials. A generalization as well as an alternative proof of Ul'yanov's inequality was given in [10]. The proof in our case can be provided exactly as in [10, Theorem 4] and we omit it.

Next, it was proven in [8] for $\lambda = 0$ that inequality (5.3) is sharp for any rate of decay of the best trigonometric approximations $E_n(f)_p$. Following the scheme given in [8], we immediately get a similar result for all $\lambda \geqslant 0$. The only change we need is to use polynomials (3.8) instead of Fejér's kernels (see [8, Theorem 3]).

Now we will consider the main problem in this section, the relations between best approximations in different norms. First, it follows immediately from (5.3) that for $1 \leqslant p < q < \infty$

$$E_n^{(\lambda)}(f)_q \leqslant c \left(n^{1/p-1/q} E_n^{(\lambda)}(f)_p + \left(\sum_{k=n}^{\infty} k^{q/p-2} (E_k^{(\lambda)}(f)_p)^q \right)^{1/q} \right). \tag{5.4}$$

However, it is easy to see that this inequality is not sharp if the sequence $\{E_n^{(\lambda)}(f)_p\}$ tends to 0 sufficiently rapidly (for example, with a geometric rate). In the case of the trigonometric system the sharp estimate was found in [8] (see also [9]). We will obtain similar results for all $\lambda > 0$.

Since the system $\mathcal{U}_\lambda$ is a basis in $L^p[0,\pi]$ $(1 < p < \infty)$, then for every $f \in L^p[0,\pi]$ we have

$$E_n^{(\lambda)}(f)_p \leqslant \|f - S_n^{(\lambda)}(f)\|_p \leqslant c_p E_n^{(\lambda)}(f)_p. \tag{5.5}$$

Theorem 5. *Let $1 \leqslant p < q < \infty$ and $0 \leqslant \lambda < \infty$. Then for any function $f \in L^p[0,\pi]$*

$$E_n^{(\lambda)}(f)_q \leqslant c \left(\sum_{k=n}^{\infty} (k-n+1)^{q/p-2} (E_k^{(\lambda)}(f)_p)^q \right)^{1/q} \tag{5.6}$$

for every $n = 0, 1, \ldots$, where c is a constant which only depends on p, q, and λ.

Proof. Set $S_n(x) = S_n^{(\lambda)}(f;x)$. First we suppose that $p > 1$. Denote $\varepsilon_n = E_n^{(\lambda)}(f)_p$. Fix $n \in \mathbb{N}$ and set

$$\nu_1 = n, \quad \nu_{k+1} = \min\left\{ \nu \geqslant \nu_k : \varepsilon_\nu \leqslant \frac{1}{2}\varepsilon_{\nu_k} \right\}, \quad k = 1, 2, \ldots . \tag{5.7}$$

Then

$$\varepsilon_{\nu_{k+1}} \leqslant \frac{1}{2}\varepsilon_{\nu_k} \quad \text{and} \quad \varepsilon_{\nu_k} < 2\varepsilon_\nu, \quad \nu_k \leqslant \nu < \nu_{k+1}. \tag{5.8}$$

By (5.5), we have (convergence in L^p)

$$f(x) = S_n(x) + \sum_{k=1}^{\infty} [S_{\nu_{k+1}}(x) - S_{\nu_k}(x)].$$

Thus,

$$E_n^{(\lambda)}(f)_q \leqslant \|f - S_n\|_q \leqslant \left\| \sum_{k=1}^{\infty} h_k \right\|_q, \tag{5.9}$$

where $h_k(x) = |S_{\nu_{k+1}}(x) - S_{\nu_k}(x)|$. Once again applying (5.5), we get

$$\|h_k\|_p \leqslant \|f - S_{\nu_{k+1}}\|_p + \|f - S_{\nu_k}\|_p \leqslant c\varepsilon_{\nu_k}. \tag{5.10}$$

Furthermore, by Theorem 2,

$$\|h_k\|_\infty \leqslant c(\nu_{k+1} - \nu_k)^{1/p} \|h_k\|_p \leqslant c'(\nu_{k+1} - \nu_k)^{1/p} \varepsilon_{\nu_k}. \tag{5.11}$$

Now Lemma 4 and inequalities (5.9)–(5.11) yield

$$E_n^{(\lambda)}(f)_q \leqslant c \left(\sum_{k=1}^{\infty} (\nu_{k+1} - \nu_k)^{q/p-1} \varepsilon_{\nu_k}^q \right)^{1/q}.$$

Changing the order of summation, we get

$$\begin{aligned}\sum_{k=1}^{\infty}(\nu_{k+1}-\nu_k)^{q/p-1}\varepsilon_{\nu_k}^q &\leqslant c\sum_{k=1}^{\infty}\varepsilon_{\nu_k}^q\sum_{m=1}^{\nu_{k+1}-n} m^{q/p-2}\\ &\leqslant c\sum_{m=1}^{\infty} m^{q/p-2}\sum_{k=k_m}^{\infty}\varepsilon_{\nu_k}^q,\end{aligned}$$

where $k_m=\min\{k:\nu_{k+1}\geqslant m+n\}$. By virtue of (5.8),

$$\sum_{k=k_m}^{\infty}\varepsilon_{\nu_k}^q\leqslant 2^{1+q}\varepsilon_{m+n-1}^q$$

and we get (5.6).

Now assume that $p=1$. Choose some $1<r<q$. By (5.6) and (5.2), we have

$$\begin{aligned}E_n^{(\lambda)}(f)_q &\leqslant c\left(\sum_{k=1}^{\infty}k^{q/r-2}(E_{n+k-1}^{(\lambda)}(f)_r)^q\right)^{1/q}\\ &\leqslant c'\left(\sum_{m=0}^{\infty}2^{m(q/r-1)}(E_{n+2^m-1}^{(\lambda)}(f)_r)^q\right)^{1/q}\\ &\leqslant c'\left(\sum_{m=0}^{\infty}2^{m(q/r-1)}\|f-S_{n+2^m-1}\|_r^q\right)^{1/q}\\ &\leqslant c'\left(\sum_{m=0}^{\infty}2^{m(q/r-1)}\left(\sum_{\nu=m}^{\infty}\|S_{n+2^{\nu+1}-1}-S_{n+2^\nu-1}\|_r\right)^q\right)^{1/q}\\ &\leqslant c''\left(\sum_{m=0}^{\infty}2^{m(q/r-1)}\|S_{n+2^{m+1}-1}-S_{n+2^m-1}\|_r^q\right)^{1/q}.\end{aligned}$$

Further, let $U_n(x)$ be the $\mathcal{U}_\lambda$-polynomial of best approximation of degree n to f in $L^1[0,\pi]$. Using orthogonality, we have

$$\begin{aligned}S_{n+2^{m+1}-1}(x)-S_{n+2^m-1}(x) &= \int_0^\pi f(t)\sum_{k=n+2^m}^{n+2^{m+1}-1}u_k^{(\lambda)}(x)u_k^{(\lambda)}(t)\,dt\\ &= \int_0^\pi (f(t)-U_{n+2^m-1}(t))\sum_{k=n+2^m}^{n+2^{m+1}-1}u_k^{(\lambda)}(x)u_k^{(\lambda)}(t)\,dt.\end{aligned}$$

By Minkowski inequality and (2.10),

$$\begin{aligned}&\|S_{n+2^{m+1}-1}-S_{n+2^m-1}\|_r\\ &\quad\leqslant\int_0^\pi|f(t)-U_{n+2^m-1}(t)|\left(\int_0^\pi\left|\sum_{k=n+2^m}^{n+2^{m+1}-1}u_k^{(\lambda)}(x)u_k^{(\lambda)}(t)\right|^r dx\right)^{1/r}dt\\ &\quad\leqslant c2^{m(1-1/r)}\|f-U_{n+2^m-1}\|_1=c2^{m(1-1/r)}E_{n+2^m-1}^{(\lambda)}(f)_1.\end{aligned}$$

Thus, we get

$$E_n^{(\lambda)}(f)_q \leqslant c\left(\sum_{m=0}^{\infty} 2^{m(q-1)}(E_{n+2^m-1}^{(\lambda)}(f)_1)^q\right)^{1/q}$$

$$\leqslant c'\left(\sum_{k=1}^{\infty} k^{q-2}(E_{n+k-1}^{(\lambda)}(f)_r)^q\right)^{1/q}.$$

This is inequality (5.6) for $p = 1$. The proof is now complete. $\square$

It was proven in [8,9] for $\lambda = 0$ that inequality (5.6) is sharp for any rate of the decay of the best approximations $E_n^{(\lambda)}(f)_p$. Following the same scheme we obtain a similar result for all $\lambda \geqslant 0$. The main tools are Lemmas 3 and 4.

Let $\mathcal{H}$ be the set of all positive sequences $\varepsilon \equiv \{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$. Suppose that $1 \leqslant p < \infty$, $0 \leqslant \lambda < \infty$, and $\varepsilon \in \mathcal{H}$. Then $L_p^{(\lambda)}(\varepsilon)$ will denote the class of all functions $f \in L^p[0, \pi]$ such that $E_n^{(\lambda)}(f)_p \leqslant \varepsilon_n$.

Next, for $0 < \lambda < \infty$, $1 \leqslant p < q < \infty$, and $\varepsilon \in \mathcal{H}$ denote

$$\mathcal{E}_n(\varepsilon; p, q) = \sup_{f \in L_p^{(\lambda)}(\varepsilon)} E_n^{(\lambda)}(f)_q \quad (n = 0, 1, \ldots).$$

Theorem 6. *Let $0 \leqslant \lambda < \infty$ and $1 \leqslant p < q < \infty$. Then there exist positive constants c and c' (depending only on p, q, and λ) such that for every sequence $\varepsilon \in \mathcal{H}$ and every $n = 0, 1, \ldots$*

$$c'\mathcal{R}_n(\varepsilon; p, q) \leqslant \mathcal{E}_n(\varepsilon; p, q) \leqslant c\mathcal{R}_n(\varepsilon; p, q), \tag{5.12}$$

where

$$\mathcal{R}_n(\varepsilon; p, q) = \left(\sum_{k=n}^{\infty} (k - n + 1)^{q/p-2} \varepsilon_k^q\right)^{1/q}. \tag{5.13}$$

Proof. The second inequality in (5.12) follows immediately from Theorem 5. We will prove the first inequality. Fix $n \in \mathbb{N}$ and set

$$\nu_1 = n, \quad \nu_{k+1} = \min\left\{\nu \geqslant \nu_k : \varepsilon_\nu \leqslant \frac{1}{2}\varepsilon_{\nu_k}\right\}, \quad k = 1, 2, \ldots . \tag{5.14}$$

It follows that

$$\varepsilon_{\nu_{k+1}} \leqslant \frac{1}{2}\varepsilon_{\nu_k} \quad \text{and} \quad \varepsilon_{\nu_k} < 2\varepsilon_\nu, \quad \nu_k \leqslant \nu < \nu_{k+1}. \tag{5.15}$$

Set $p_0 = \min(1, p/(q-1))/2$ and apply Lemma 3 with $\nu = \nu_{k+1}$ and $\mu = \nu_k + 1$. Thus we obtain $\mathcal{U}_\lambda$-polynomials

$$\tau_k(x) = \sum_{j=\nu_k+1}^{\nu_{k+1}} a_j^{(k)} u_j^{(\lambda)}(x)$$

which satisfy the inequalities ($c' > 0$)

$$c'(\nu_{k+1}-\nu_k)^{1-1/r} \leqslant \|\tau_k\|_r \leqslant c''(\nu_{k+1}-\nu_k)^{1-1/r} \tag{5.16}$$

for each $r \in [p_0, \infty]$. Next, we consider the function

$$f(x) \equiv f_n(x) = \frac{1}{4}\sum_{k=1}^{\infty} \varepsilon_{\nu_k}\tau_k(x)/\|\tau_k\|_p \tag{5.17}$$

(it follows from (5.15) that the last series converges in L^p). Let $S_m(x)$ be the partial sums of the Fourier series of the function f with respect to the system $\mathcal{U}_\lambda$. Note that $S_m(x) = 0$ for $0 \leqslant m \leqslant n$. By (5.15), we get for any $\nu_k \leqslant m < \nu_{k+1}$

$$E_m^{(\lambda)}(f)_p \leqslant \|f - S_{\nu_k}\|_p \leqslant \frac{1}{4}\sum_{j=k}^{\infty} \varepsilon_{\nu_j} \leqslant \varepsilon_m.$$

Hence, $f \in L_p^{(\lambda)}(\varepsilon)$. If $f \notin L^q[0,\pi]$, then by Theorem 4 series (5.13) diverges and (5.12) trivially holds. Suppose that $f \in L^q[0,\pi]$. Set

$$g_N(x) = \sum_{k=1}^{N} h_k(x), \quad \text{where } h_k(x) = (\nu_{k+1}-\nu_k)^{(q-1)/p-1}\varepsilon_{\nu_k}^{q-1}\tau_k(x). \tag{5.18}$$

Taking into account the orthogonality of the system $\{\tau_k(x)\}$ as well as (5.16), we have

$$\begin{aligned}\int_0^\pi & f(x) g_N(x)\,dx \\ &\geqslant c\sum_{k=1}^{N}(\nu_{k+1}-\nu_k)^{q/p-2}\varepsilon_{\nu_k}^{q}\|\tau_k\|_2^2 \geqslant c'\sum_{k=1}^{N}(\nu_{k+1}-\nu_k)^{q/p-1}\varepsilon_{\nu_k}^{q}\end{aligned} \tag{5.19}$$

($c' > 0$). On the other hand, by the Hölder inequality and (5.5),

$$\begin{aligned}\int_0^\pi f(x) g_N(x)\,dx &\leqslant \|f\|_q\|g_N\|_{q'} \\ &= \|f - S_n\|_q\|g_N\|_{q'} \leqslant cE_n^{(\lambda)}(f)_q\|g_N\|_{q'}\end{aligned} \tag{5.20}$$

(we have used also that $S_n = 0$). Next, by (5.16) we have (see (5.18))

$$\|h_k\|_{p/(q-1)} \leqslant c\varepsilon_{\nu_k}^{q-1} \quad \text{and} \quad \|h_k\|_\infty \leqslant c(\nu_{k+1}-\nu_k)^{(q-1)/p}\varepsilon_{\nu_k}^{q-1}.$$

Applying Lemma 4, we get

$$\|g_N\|_{q'} \leqslant c\left(\sum_{k=1}^{N}(\nu_{k+1}-\nu_k)^{q/p-1}\varepsilon_{\nu_k}^{q}\right)^{1/q}. \tag{5.21}$$

It follows from (5.19)–(5.21) that

$$E_n^{(\lambda)}(f)_q \geqslant c\left(\sum_{k=1}^{\infty}(\nu_{k+1}-\nu_k)^{q/p-1}\varepsilon_{\nu_k}^{q}\right)^{1/q} \equiv cA_n,$$

where c is a positive constant that does not depend on n and ε.

The last step is similar to one carried out in the proof of Theorem 5. Namely, applying Lemma 5, changing the order of summation, and using (5.15), we get

$$A_n^q \geqslant c\sum_{k=1}^{\infty}(\nu_{k+1}-n)^{q/p-1}\varepsilon_{\nu_k}^q \geqslant c'\sum_{k=1}^{\infty}\varepsilon_{\nu_k}^q\sum_{m=1}^{\nu_{k+1}-n} m^{q/p-2}$$

$$\geqslant 2^q c'\sum_{m=1}^{\infty} m^{q/p-2}\varepsilon_{m+n-1}^q = 2^q c'\mathcal{R}_n(\varepsilon;\,p,q)^q.$$

This yields the first inequality in (5.12). The proof is now complete. $\square$

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