



On an extension of symmetric coherent pairs of orthogonal polynomials

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Abstract

Given two symmetric and positive definite linear functionals, \mathcal{W} and \mathcal{V} , we study the coefficients in the recurrence relation for the system of monic polynomials orthogonal with respect to the second linear functional assuming that the first one is classical and that there exists an algebraic–differential relation between these two families of polynomials. Moreover, we determine this companion linear functional as a rational modification of the classical one.

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1. Introduction

Let \mathcal{U} be a linear functional in the linear space \mathbb{P} of the polynomials with real coefficients. Let \mathbb{P}' be the algebraic dual space of \mathbb{P} , i.e., the linear space of the linear functionals defined on \mathbb{P} . We will denote $\langle \mathcal{U}, p \rangle$ the action of a linear functional \mathcal{U} over a polynomial p .

A sequence of monic polynomials $\{P_n\}$ is said to be orthogonal with respect to \mathcal{U} if

- (i) $\deg P_n = n$,
- (ii) $\langle \mathcal{U}, P_n P_m \rangle = k_n \delta_{nm}$, $k_n \neq 0$, $n, m \in \mathbb{N}$.

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In such a case, the linear functional \mathcal{U} is said to be quasi-definite. If $k_n \in \mathbb{R}^+$, then $\langle \mathcal{U}, p^2 \rangle > 0$ for every polynomial p . In such a situation the linear functional \mathcal{U} is said to be positive definite.

An important characterization of sequences of monic polynomials orthogonal with respect to a quasi-definite linear functional is given in terms of a three-term recurrence relation that such polynomials satisfy

$$xP_n = P_{n+1} + \beta_n P_n + \gamma_n P_{n-1}, \quad (1.1)$$

$n \geq 1$, with $\gamma_n \neq 0$. This result is known as Favard's theorem (see [2]). Moreover, in the positive definite case, $\gamma_n > 0$.

Taking into account that $\{P_n\}$ is a basis in \mathbb{P} , we can define the corresponding dual basis in \mathbb{P}' as follows.

Let $\alpha_n = \frac{P_n \mathcal{U}}{\langle \mathcal{U}, P_n^2 \rangle}$ be the linear functional such that $\langle \alpha_n, p \rangle = \langle \mathcal{U}, \frac{p P_n}{\langle \mathcal{U}, P_n^2 \rangle} \rangle$. Then it is straightforward to prove that $\langle \alpha_n, P_m \rangle = \delta_{nm}$.

An important family of linear functionals is constituted by the symmetric linear functionals, i.e., $\langle \mathcal{U}, x^{2n+1} \rangle = 0$ for every $n \in \mathbb{N}$.

Notice that if we assume \mathcal{U} is a quasi-definite linear functional then in such a situation the recurrence relation (1.1) becomes

$$xP_n = P_{n+1} + \gamma_n P_{n-1}, \quad n \geq 1,$$

i.e., we get only one sequence of parameters in order to generate our sequence of monic orthogonal polynomials.

In this contribution, we will analyze sequences of polynomials orthogonal with respect to a Sobolev inner product

$$\varphi(p, q) = \langle \mathcal{W}, pq \rangle + \lambda \langle \mathcal{V}, p'q' \rangle, \quad (1.2)$$

where $p, q \in \mathbb{P}$, $\lambda \in \mathbb{R}^+$, and \mathcal{W}, \mathcal{V} are positive definite symmetric linear functionals. In such a case, there exists a sequence of monic polynomials $\{Q_n^\lambda\}$ such that

- (i) $\deg Q_n^\lambda = n$,
- (ii) $\varphi(Q_n^\lambda, Q_m^\lambda) = \tilde{k}_n \delta_{nm}$, $\tilde{k}_n \neq 0$, $n, m \in \mathbb{N}$.

The three term recurrence relation (1.1) does not hold for the sequence $\{Q_n^\lambda\}$ and thus we need other tools in order to analyze the behavior of such polynomials.

A first approach was given by Iserles et al. [4] when they introduced the concept of symmetric coherent pairs of measures. Given two linear functionals \mathcal{W}, \mathcal{V} , they are said to be a symmetric coherent pair when the corresponding sequences of monic polynomials $\{P_n\}$ and $\{T_n\}$ orthogonal with respect to \mathcal{W} and \mathcal{V} , respectively, satisfy

$$(n+1)T_n = P'_{n+1} + \sigma_n P'_{n-1}, \quad n \geq 1,$$

where (σ_n) is a sequence of non-zero real numbers. In the paper by Iserles [4] some examples of symmetrically coherent pairs of measures are shown. In 1995, Meijer [6] described the set of symmetrically coherent pairs of measures. In such a case, there is a very simple connection between the sequences $\{P_n\}$

and $\{Q_n^\lambda\}$. Indeed

$$Q_{n+1}^\lambda + d_n^\lambda Q_{n-1}^\lambda = P_{n+1} + \sigma_n P_{n-1}, \quad n \geq 1. \quad (1.3)$$

In [3] the authors show the interest of this concept in order to provide an efficient algorithm to find the coefficients of the Fourier–Sobolev expansion. In particular, it reveals the role of symmetric coherence in the approximation of a function by its projection into polynomials and simultaneously to approximate its derivative by the derivative of the polynomial approximant. The standard projection is poor near the end points of the support of the measure \mathcal{W} whereas the Sobolev projection displays a reasonably good behaviour in the whole support.

The aim of our contribution is to analyze an inverse problem [5]. Given two symmetric and positive linear functionals \mathcal{W} and \mathcal{V} as well as a Sobolev inner product (1.2), such that (1.3) holds, then to find the relation between the measures \mathcal{W} and \mathcal{V} . We solve this problem when \mathcal{W} is a classical linear functional.

The structure of the paper is as follows. In Section 2, we find the relation between the sequences $\{P_n\}$ and $\{T_n\}$, under assumption that (1.3) is satisfied. In Section 3 we assume the linear functional \mathcal{W} is a classical one, i.e., Hermite or Gegenbauer. We find the coefficients of the three-term recurrence relation that $\{T_n\}$ satisfies, as well as the explicit expression of the linear functional \mathcal{V} in terms of \mathcal{W} . Finally, in Section 4 we show that the examples analyzed in [1] are the only cases available for our problem.

2. Sobolev inner products

Let \mathcal{W} and \mathcal{V} be two positive definite symmetric linear functionals in the space \mathbb{P} of polynomials with real coefficients and assume that they are normalized by $\langle \mathcal{W}, 1 \rangle = 1$ and $\langle \mathcal{V}, 1 \rangle = 1$. Associated with this pair of linear functionals, we define the bilinear form φ in \mathbb{P} as usual,

$$\varphi(p, q) = \langle \mathcal{W}, pq \rangle + \lambda \langle \mathcal{V}, p'q' \rangle, \quad p, q \in \mathbb{P} \quad (2.1)$$

with $\lambda \in \mathbb{R}^+$. In such a situation, the Gram matrix associated with the bilinear form φ is positive definite and thus there exists a sequence of monic polynomials orthogonal with respect to φ which we denote by $\{Q_n^\lambda\}_{n \geq 0}$.

Definition 2.1. Let \mathcal{W} and \mathcal{V} be two positive definite symmetric linear functionals and denote by $\{P_n\}_{n \geq 0}$ and $\{T_n\}_{n \geq 0}$ the sequences of monic polynomials orthogonal with respect to \mathcal{W} and \mathcal{V} , respectively. $(\mathcal{W}, \mathcal{V})$ is said to be a symmetric coherent pair of linear functionals if there exists a sequence of non-zero real numbers $(\sigma_n)_{n \geq 1}$ such that the monic orthogonal polynomials are related by

$$(n+1)T_n = P_{n+1}' + \sigma_n P_{n-1}', \quad n \geq 1.$$

Symmetric coherent pairs have been introduced by Iserles et al. in 1991 (see [4]), and described in the paper by Meijer [6].

Let us denote by p_n, t_n , and q_n^λ the squared of the norms of the polynomials P_n, T_n , and Q_n^λ with respect to \mathcal{W}, \mathcal{V} , and φ , respectively. This means $\langle \mathcal{W}, P_n^2 \rangle = p_n$, $\langle \mathcal{V}, T_n^2 \rangle = t_n$, and $\varphi(Q_n^\lambda, Q_n^\lambda) = q_n^\lambda$.

Theorem 2.2. *If $(\mathcal{W}, \mathcal{V})$ is a symmetric coherent pair of linear functionals then there exists a non-zero sequence $(d_n^\lambda)_{n \geq 1}$ such that the Sobolev polynomials $\{Q_n^\lambda\}_{n \geq 0}$ are related with $\{P_n\}_{n \geq 0}$ by*

$$Q_{n+1}^\lambda + d_n^\lambda Q_{n-1}^\lambda = P_{n+1} + \sigma_n P_{n-1}, \quad n \geq 1, \quad (2.2)$$

where $d_n^\lambda = \sigma_n p_{n-1} / q_{n-1}^\lambda$ for $n \geq 1$.

The proof of this result can be found in [4].

Now, in a bit more general situation, we consider the case when relation (2.2) holds, but with the only restriction that the coefficients which appear in the relation are non-zero.

Theorem 2.3. *Let \mathcal{W}, \mathcal{V} be two positive definite symmetric linear functionals and define the bilinear form φ by (2.1). We assume that the polynomials $\{P_n\}_{n \geq 0}$ orthogonal with respect to the linear functional \mathcal{W} and the Sobolev polynomials $\{Q_n^\lambda\}_{n \geq 0}$ orthogonal with respect to φ are related by (2.2), with $(\sigma_n)_{n \geq 1}$ and $(d_n^\lambda)_{n \geq 1}$ non-zero sequences of real numbers. Then, there exists a sequence $(c_n)_{n \geq 2}$ such that*

$$P'_{n+1} + \sigma_n P'_{n-1} = (n+1)T_n + c_n T_{n-2}, \quad n \geq 2,$$

where $\{T_n\}_{n \geq 0}$ denotes the system of monic polynomials orthogonal with respect to the linear functional \mathcal{V} .

Proof. Taking into account the definition of the bilinear form φ as well as (2.2), for $0 \leq k \leq n-2$ we get

$$0 = \varphi(Q_{n+1}^\lambda + d_n^\lambda Q_{n-1}^\lambda, x^k) = \lambda k \langle \mathcal{V}, (P'_{n+1} + \sigma_n P'_{n-1})x^{k-1} \rangle.$$

Thus $\langle \mathcal{V}, (P'_{n+1} + \sigma_n P'_{n-1})x^k \rangle = 0$ for any $0 \leq k \leq n-3$. Then, we can write $P'_{n+1} + \sigma_n P'_{n-1}$ as a linear combination of T_k for $k = n-2, n-1, n$, i.e.,

$$P'_{n+1} + \sigma_n P'_{n-1} = c_{n,n} T_n + c_{n,n-1} T_{n-1} + c_{n,n-2} T_{n-2}, \quad n \geq 2.$$

Because of the symmetry of the polynomials, the coefficient of T_{n-1} must be $c_{n,n-1} = 0$, and since we are considering monic polynomials, $c_{n,n} = n+1$. Then,

$$P'_{n+1} + \sigma_n P'_{n-1} = (n+1)T_n + c_{n,n-2} T_{n-2}, \quad n \geq 2.$$

Now we denote $c_{n,n-2} = c_n$ and we compute an explicit expression for it. First, using the orthogonality of the polynomials Q_n^λ with respect to φ we have $\varphi(Q_{n+1}^\lambda + d_n^\lambda Q_{n-1}^\lambda, x^{n-1}) = d_n^\lambda q_{n-1}^\lambda$. On the other hand, using (2.2) we get

$$\varphi(Q_{n+1}^\lambda + d_n^\lambda Q_{n-1}^\lambda, x^{n-1}) = \sigma_n p_{n-1} + \lambda(n-1)c_n t_{n-2}, \quad n \geq 2,$$

from where we deduce the explicit expression for the coefficients,

$$c_n = \frac{d_n^\lambda q_{n-1}^\lambda - \sigma_n p_{n-1}}{\lambda(n-1)t_{n-2}}, \quad n \geq 2. \quad \square$$

Furthermore, we see from the proof of the theorem that $(\mathcal{W}, \mathcal{V})$ is a symmetric coherent pair if and only if $c_n = 0$ for all $n \geq 2$, which is equivalent to

$$d_n^\lambda = \frac{\sigma_n p_{n-1}}{q_{n-1}^\lambda}, \quad n \geq 2.$$

We observe that it is the same value for the coefficient d_n^{λ} when we deduce (2.2) from the symmetric coherent condition.

3. The classical case

Let \mathcal{W} , \mathcal{V} be two symmetric linear functionals and denote by $\{P_n\}_{n \geq 0}$ and $\{T_n\}_{n \geq 0}$ the corresponding sequences of monic orthogonal polynomials. We assume that there exist sequences of non-zero real numbers $(\sigma_n)_{n \geq 2}$ and $(c_n)_{n \geq 2}$ such that

$$P'_{n+1} + \sigma_n P'_{n-1} = (n+1)T_n + c_n T_{n-2}, \quad n \geq 2. \quad (3.1)$$

In this section, we will study the case when the first functional, \mathcal{W} , is a classical linear functional, that is, the classical Hermite or the classical Gegenbauer functional.

We assume that $\{P_n\}_{n \geq 0}$ is a classical family of polynomials. Then, it is well known (see [2]) that the corresponding monic derivatives of these polynomials, which we denote by

$$R_n = \frac{1}{n+1} P'_{n+1}, \quad n \geq 0,$$

constitute again a classical system of polynomials orthogonal with respect to a symmetric linear functional \mathcal{U} . In this case, relation (3.1) becomes

$$R_n + u_{n-2} R_{n-2} = T_n + s_{n-2} T_{n-2}, \quad n \geq 2$$

with $u_{n-2} = (n-1)\sigma_n/(n+1)$ and $s_{n-2} = c_n/(n+1)$, $n \geq 2$.

Thus the study of the algebraic-differential relation (3.1) when the first linear functional is classical can be reduced to the study of a relation

$$R_n + u_{n-2} R_{n-2} = T_n + s_{n-2} T_{n-2}, \quad n \geq 2$$

with $\{R_n\}_{n \geq 0}$ a classical family of symmetric orthogonal polynomials.

Furthermore, in order to avoid trivial situations, we can assume that $u_0 \neq s_0$ and $u_1 \neq s_1$. This is an equivalent condition to $R_n \neq T_n$ for all $n \geq 2$, as we show in next result.

Lemma 3.1. *Let $\{R_n\}_{n \geq 0}$ and $\{T_n\}_{n \geq 0}$ be two sequences of monic polynomials orthogonal with respect to symmetric linear functionals \mathcal{U} and \mathcal{V} , respectively. Assume that there exist two sequences of non-zero real numbers $(u_n)_{n \geq 0}$ and $(s_n)_{n \geq 0}$ such that*

$$R_n + u_{n-2} R_{n-2} = T_n + s_{n-2} T_{n-2}, \quad n \geq 2. \quad (3.2)$$

Then, one of the following situations holds:

- (i) if $u_0 = s_0$ then, $R_n = T_n$ for all $n \geq 0$. Moreover $u_n = s_n$ for all $n \geq 0$,
- (ii) if $u_0 \neq s_0$ and $u_1 \neq s_1$, then $R_n \neq T_n$ for every $n \geq 2$,
- (iii) if $u_0 \neq s_0$ and $u_1 = s_1$, then $R_{2n} \neq T_{2n}$ for every $n \geq 1$, $R_3 = T_3$, and if there exists $N > 1$ such that $R_{2N+1} \neq T_{2N+1}$ then $R_{2n+1} \neq T_{2n+1}$ for all $n > N$.

Proof. If we assume that the linear functionals \mathcal{U} and \mathcal{V} are normalized by $\langle \mathcal{U}, 1 \rangle = 1$ and $\langle \mathcal{V}, 1 \rangle = 1$, from (3.2) we get

$$\begin{aligned} \langle \mathcal{U}, T_n \rangle &= -s_{n-2} \langle \mathcal{U}, T_{n-2} \rangle, \quad n \geq 3, & \langle \mathcal{U}, T_2 \rangle &= u_0 - s_0, & \langle \mathcal{U}, T_1 \rangle &= 0, \\ \langle \mathcal{V}, R_n \rangle &= -u_{n-2} \langle \mathcal{V}, R_{n-2} \rangle, \quad n \geq 3, & \langle \mathcal{V}, R_2 \rangle &= s_0 - u_0, & \langle \mathcal{V}, R_1 \rangle &= 0. \end{aligned}$$

(i) If $u_0 = s_0$, from the previous relations we get $\langle \mathcal{U}, T_n \rangle = 0$ and $\langle \mathcal{V}, R_n \rangle = 0$ for $n \geq 1$. Thus $R_n = T_n$ for all $n \geq 0$. Furthermore it is straightforward to see that in this situation, $u_n = s_n$ for all $n \geq 0$.

(ii) Now we assume that both $u_0 \neq s_0$ and $u_1 \neq s_1$. First, we will prove that even degree polynomials are different, that is $T_{2n} \neq R_{2n}$ for $n \geq 1$. If there exists some $n_0 \in \mathbb{N}$ such that $R_{2n_0} = T_{2n_0}$ then, from (3.2) we deduce that $u_{2n_0-2} R_{2n_0-2} = s_{2n_0-2} T_{2n_0-2}$. Since the sequences $(u_n)_{n \geq 0}$ and $(s_n)_{n \geq 0}$ have non-zero elements, it must be $u_{2n_0-2} = s_{2n_0-2}$ and $R_{2n_0-2} = T_{2n_0-2}$. Applying again the same process to $2n_0 - 2$ and so on, we get $u_0 = s_0$, which it is not possible. Thus $T_{2n} \neq R_{2n}$ for all $n \geq 1$.

In an analogous way, for odd degree polynomials we see that if there exists $n_0 \in \mathbb{N}$ such that $R_{2n_0+1} = T_{2n_0+1}$, then we get $u_1 = s_1$ a contradiction. Therefore $T_{2n+1} \neq R_{2n+1}$ for $n \geq 1$.

(iii) If $u_0 \neq s_0$, then the same process that we have done to prove assertion (ii) allows us to prove that $T_{2n} \neq R_{2n}$ for $n \geq 1$. For odd polynomials, since $u_1 = s_1$ from (3.2) we get that $R_3 = T_3$. Then, if we assume that there exists $N > 2$ such that $R_{2N-1} = T_{2N-1}$, then $R_{2N+1} = T_{2N+1}$ since if it is not true, i.e. $R_{2N+1} \neq T_{2N+1}$, from (3.2) we have $u_{2N-1} R_{2N-1} = s_{2N-1} T_{2N-1}$ a contradiction. \square

3.1. On the generation of recurrence coefficients for $\{T_n\}$

Let \mathcal{U}, \mathcal{V} be positive definite symmetric linear functionals. Consider $\{R_n\}_{n \geq 0}$ and $\{T_n\}_{n \geq 0}$ the corresponding sequences of monic orthogonal polynomials. We assume that (3.2) holds with $(u_n)_{n \geq 0}$ and $(s_n)_{n \geq 0}$ sequences of non-zero real numbers with $u_0 \neq s_0$ and $u_1 \neq s_1$. Denote by γ_n and $\tilde{\gamma}_n$ the coefficients of the three term recurrence relation for $\{R_n\}_{n \geq 0}$ and $\{T_n\}_{n \geq 0}$, respectively. This means

$$R_{n+1} = xR_n - \gamma_n R_{n-1}, \quad n \geq 1, \quad R_0 = 1, \quad R_1 = x, \quad (3.3)$$

$$T_{n+1} = xT_n - \tilde{\gamma}_n T_{n-1}, \quad n \geq 1, \quad T_0 = 1, \quad T_1 = x \quad (3.4)$$

with $\gamma_n > 0$ and $\tilde{\gamma}_n > 0$ for $n \geq 1$.

In the background of the problem we assume \mathcal{U} is a classical symmetric linear functional, that is, the family $\{R_n\}_{n \geq 0}$ is either the Hermite polynomial sequence or the Gegenbauer polynomial sequence. Thus, the recurrence coefficients for these polynomials, γ_n , are known. In this section, we present a study of the coefficients $\tilde{\gamma}_n$ in the recurrence relation for $\{T_n\}_{n \geq 0}$. More precisely, we give a way to compute these coefficients in terms of γ_n , $(u_n)_{n \geq 0}$, and $(s_n)_{n \geq 0}$.

Combining relations (3.2)–(3.4), and simplifying in an appropriate way, we get

$$\begin{aligned} &(\gamma_n + u_{n-2} - u_{n-1})R_{n-1} + u_{n-2}\gamma_{n-2}R_{n-3} \\ &= (\tilde{\gamma}_n + s_{n-2} - s_{n-1})T_{n-1} + s_{n-2}\tilde{\gamma}_{n-2}T_{n-3}, \quad n \geq 3. \end{aligned} \quad (3.5)$$

Thus, taking into account that the polynomials are monic, we have

$$\gamma_n + u_{n-2} - u_{n-1} = \tilde{\gamma}_n + s_{n-2} - s_{n-1}, \quad n \geq 3.$$

From Lemma 3.1, these expressions must be different from zero for any $n \geq 5$. Otherwise, if $\gamma_{n_0} + u_{n_0-2} - u_{n_0-1} = 0$ for some $n_0 \geq 5$, then $R_{n_0-3} = T_{n_0-3}$ which is not possible.

In such a situation, (3.5) becomes

$$R_{n-1} + \frac{u_{n-2}\gamma_{n-2}}{\gamma_n + u_{n-2} - u_{n-1}} R_{n-3} = T_{n-1} + \frac{s_{n-2}\tilde{\gamma}_{n-2}}{\tilde{\gamma}_n + s_{n-2} - s_{n-1}} T_{n-3}, \quad n \geq 5.$$

Using again (3.2) we substitute R_{n-1} in order to get

$$\left(\frac{u_{n-2}\gamma_{n-2}}{\gamma_n + u_{n-2} - u_{n-1}} - u_{n-3} \right) R_{n-3} = \left(\frac{s_{n-2}\tilde{\gamma}_{n-2}}{\tilde{\gamma}_n + s_{n-2} - s_{n-1}} - s_{n-3} \right) T_{n-3}, \quad n \geq 5,$$

from where we deduce

$$\frac{u_{n-2}\gamma_{n-2}}{\gamma_n + u_{n-2} - u_{n-1}} - u_{n-3} = \frac{s_{n-2}\tilde{\gamma}_{n-2}}{\tilde{\gamma}_n + s_{n-2} - s_{n-1}} - s_{n-3}, \quad n \geq 5,$$

which are equal to zero for $n \geq 5$ taking into account Lemma 3.1.

As a conclusion, we have shown that u_n and s_n are related with the recurrence coefficients γ_n and $\tilde{\gamma}_n$ by

$$\gamma_n + u_{n-2} - u_{n-1} = \tilde{\gamma}_n + s_{n-2} - s_{n-1}, \quad n \geq 2, \quad (3.6)$$

$$u_{n-2}\gamma_{n-2} = u_{n-3}(\gamma_n + u_{n-2} - u_{n-1}), \quad n \geq 5, \quad (3.7)$$

$$s_{n-2}\tilde{\gamma}_{n-2} = s_{n-3}(\tilde{\gamma}_n + s_{n-2} - s_{n-1}), \quad n \geq 5. \quad (3.8)$$

Notice that (3.7) and (3.8) hold in a trivial way for $n = 2$ if we set $u_{-1} = s_{-1} = 0$ and $\gamma_0 = \tilde{\gamma}_0 = 0$. Moreover, if (3.6) is different from zero for $n = 3, 4$ then, (3.7) and (3.8) hold for $n \geq 2$.

Now, if we divide (3.7) by (3.8), we obtain

$$\tilde{\gamma}_n = \frac{u_n}{u_{n-1}} \frac{s_{n-1}}{s_n} \gamma_n, \quad n \geq 3, \quad (3.9)$$

where we have used (3.6) to simplify our computations.

On the other hand, it is easy to check that $\gamma_1 - u_0 = \tilde{\gamma}_1 - s_0$. Then, using again (3.6) we can compute

$$\sum_{k=1}^{n+1} \gamma_k - u_{n-1} = \sum_{k=1}^{n+1} \tilde{\gamma}_k - s_{n-1}$$

for $n \geq 0$.

Once we have expressed the recurrence coefficients $\tilde{\gamma}_n$ in terms of u_n, s_n , and γ_n , we only need to study the properties of the sequences $(u_n)_{n \geq 0}$ and $(s_n)_{n \geq 0}$.

From (3.7) we can identify the sequence $(u_n)_{n \geq 0}$ as the solution of a non-linear difference equation. Then, we will study the properties of the solution for this difference equation.

Proposition 3.2. *If $(u_n)_{n \geq 0}$ is a solution of the difference equation (3.7), then it satisfies a quadratic difference equation*

$$u_{n+1} + \frac{\gamma_n \gamma_{n+1}}{u_{n-1}} = \gamma_{n+1} + \gamma_{n+2} + A, \quad n \geq 3, \quad (3.10)$$

where $A = (u_3 - \gamma_4)(1 - \gamma_3/u_2)$.

Proof. First, from (3.7) we put $u_{n+1} - u_n$ in terms of u_n and u_{n-1} for $n \geq 3$. We sum from $k = 3$ to $k = n$ and we get

$$u_{n+1} - u_3 = \sum_{k=3}^n \gamma_{k+2} - \sum_{k=3}^n \frac{u_k}{u_{k-1}} \gamma_k, \quad n \geq 3. \quad (3.11)$$

On the other hand, from (3.7) we express the ratio u_k/u_{k-1} in the form

$$\frac{u_k}{u_{k-1}} = \frac{\gamma_{k+1}}{u_{k-1}} - \frac{\gamma_{k-1}}{u_{k-2}} + 1, \quad n \geq 4.$$

Then, we can compute (3.11) for $n \geq 4$:

$$\begin{aligned} u_{n+1} - u_3 &= \sum_{k=3}^n \gamma_{k+2} - \frac{u_3 \gamma_3}{u_2} - \sum_{k=4}^n \left[\frac{\gamma_k \gamma_{k+1}}{u_{k-1}} - \frac{\gamma_{k-1} \gamma_k}{u_{k-2}} + \gamma_k \right] \\ &= \gamma_{n+1} + \gamma_{n+2} - \gamma_4 - \frac{\gamma_n \gamma_{n+1}}{u_{n-1}} + \frac{\gamma_3 \gamma_4}{u_2} - \frac{\gamma_3 u_3}{u_2}, \quad n \geq 4. \end{aligned}$$

Moreover, from (3.7) this identity holds for $n = 3$. If we denote by A the term which does not depend on n in the previous relation, then

$$A = u_3 - \frac{\gamma_3 u_3}{u_2} + \frac{\gamma_3 \gamma_4}{u_2} - \gamma_4 = (u_3 - \gamma_4)(1 - \gamma_3/u_2),$$

and as a consequence we obtain (3.10). \square

This new relation between u_{n+1} and u_{n-1} allows us to obtain interesting properties of the sequence of coefficients $(u_n)_{n \geq 0}$. More precisely, in order to transform (3.10) into a linear difference equation, we will express every u_n as a rational function in the variable A . We will prove this in the following:

Theorem 3.3. *If $(u_n)_{n \geq 0}$ is the solution sequence of (3.10) then there exist two sequences $\{r_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$, which are solutions of the linear difference equations*

$$r_n = [A + \gamma_{2n+1} + \gamma_{2n+2}] r_{n-1} - \gamma_{2n} \gamma_{2n+1} r_{n-2}, \quad n \geq 2, \quad (3.12)$$

$$q_{n+1} = [A + \gamma_{2n+2} + \gamma_{2n+3}] q_n - \gamma_{2n+1} \gamma_{2n+2} q_{n-1}, \quad n \geq 1, \quad (3.13)$$

such that the coefficients u_n can be expressed in the form

$$u_{2n+1} = \frac{r_n}{r_{n-1}}, \quad u_{2n} = \frac{q_n}{q_{n-1}}, \quad n \geq 1.$$

Furthermore, $\{r_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$ are both systems of orthogonal polynomials in the variable A , with $\deg r_n = n$ and $\deg q_n = n - 1$.

Proof. We define the sequences $\{r_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$ recursively by

$$r_0 = 1, \quad u_{2n+1} = \frac{r_n}{r_{n-1}}, \quad n \geq 1,$$

$$q_0 = 1, \quad u_{2n} = \frac{q_n}{q_{n-1}}, \quad n \geq 1.$$

Then, considering relation (3.10) for both odd and even indices, we can see that $\{r_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$ verify the announced relations. Moreover, we check the degree of the polynomials from the initial conditions for each system.

For the sequence $\{r_n\}_{n \geq 0}$ we have $r_0 = 1$ and $r_1 = [u_2/u_2 - \gamma_3]A + \gamma_4 - (\gamma_3\gamma_4/u_2)$, and, as a consequence, $\deg r_n = n$.

For $\{q_n\}_{n \geq 0}$ the initial conditions are $q_0 = 1$ and $q_1 = u_2$, thus $\deg q_n = n - 1$.

Finally, since $\gamma_{2n}\gamma_{2n+1} > 0$ in the three term recurrence relation for r_n , and $\gamma_{2n+1}\gamma_{2n+2} > 0$ in the three term recurrence relation for q_n , from the Favard's theorem (see [2]) we deduce that $\{r_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$ are sequences of orthogonal polynomials with respect to a positive definite linear functional. \square

As a consequence, we see that the sequences $\{r_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$ are connected with certain systems of associated polynomials.

Corollary 3.4. *Let $\{R_n\}_{n \geq 0}$ be a sequence of polynomials orthogonal with respect to a positive definite symmetric linear functional. Then, we consider $\{S_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ sequences of polynomials orthogonal with respect to positive definite linear functionals, such that*

$$R_{2n}(x) = S_n(x^2), \quad R_{2n+1}(x) = x Q_n(x^2), \quad n \geq 0. \quad (3.14)$$

Thus, the sequences $\{q_n\}_{n \geq 0}$ and $\{r_n\}_{n \geq 0}$ considered in the previous theorem are the co-recursive of associated polynomials of first kind of $\{S_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$, respectively.

Proof. First, notice that such sequences $\{S_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ exist because the polynomials $\{R_n\}_{n \geq 0}$ are orthogonal with respect to a symmetric linear functional (see [2]). Then, we consider the three-term recurrence relations for these polynomials,

$$x S_n = S_{n+1} + a_n S_n + b_n S_{n-1}, \quad n \geq 1, \quad (3.15)$$

$$x Q_n = Q_{n+1} + \tilde{a}_n Q_n + \tilde{b}_n Q_{n-1}, \quad n \geq 1. \quad (3.16)$$

Changing x by x^2 in (3.15) we substitute (3.14) and taking into account that $\{R_n\}_{n \geq 0}$ satisfy (3.3), we obtain for even degree polynomials

$$R_{2n+2} + [\gamma_{2n} + \gamma_{2n+1}]R_{2n} + \gamma_{2n-1}\gamma_{2n}R_{2n-2} = R_{2n+2} + a_n R_{2n} + b_n R_{2n-2},$$

for $n \geq 1$. Then we deduce that the coefficients of the recurrence relation for $\{S_n\}_{n \geq 0}$ can be written as a sum or a product of the recurrence coefficients for $\{R_n\}_{n \geq 0}$,

$$a_n = \gamma_{2n} + \gamma_{2n+1}, \quad b_n = \gamma_{2n-1}\gamma_{2n}, \quad n \geq 1.$$

Thus, by comparison with (3.13) and taking into account the initial conditions for these polynomials, we deduce that $\{q_n\}_{n \geq 0}$ are the co-recursive of associated polynomials of first kind of $\{S_n\}_{n \geq 0}$.

In a similar way, changing x by x^2 in (3.16) and multiplying by x , we get

$$R_{2n+3} + [\gamma_{2n+1} + \gamma_{2n+2}]R_{2n+1} + \gamma_{2n}\gamma_{2n+1}R_{2n-1} = R_{2n+3} + \tilde{a}_n R_{2n+1} + \tilde{b}_n R_{2n-1},$$

for $n \geq 1$. Then, the recurrence coefficients \tilde{a}_n and \tilde{b}_n are

$$\tilde{a}_n = \gamma_{2n+1} + \gamma_{2n+2}, \quad \tilde{b}_n = \gamma_{2n}\gamma_{2n+1}, \quad n \geq 1.$$

Finally, we compare with (3.12) and deduce that $\{r_n\}_{n \geq 0}$ are the co-recursive of associated polynomials of first kind of $\{Q_n\}_{n \geq 0}$. \square

In Section 4 we will see that in some particular cases we can identify explicitly these systems of orthogonal polynomials.

In a similar way, we can obtain analogous properties for the coefficients s_n . From (3.6), (3.7) and (3.9) we can easily obtain that the sequence (s_n) is the solution of the second-order difference equation,

$$\eta_{n+1}s_n = s_{n+1}(\eta_{n-1} - s_{n-1} + s_n), \quad n \geq 2,$$

where $\eta_n = \gamma_{n+2} + u_n - u_{n+1}$, for $n \geq 1$. This difference equation is the analogous to (3.7) for the coefficients u_n , and it can be treated in the same way.

3.2. The companion linear functional \mathcal{V}

In this section we study the companion linear functional \mathcal{V} , and we will show that it is a rational modification of the classical linear functional \mathcal{U} .

Theorem 3.5. *Let \mathcal{U} and \mathcal{V} be symmetric and positive definite linear functionals such that the corresponding systems of monic orthogonal polynomials $\{R_n\}_{n \geq 0}$ and $\{T_n\}_{n \geq 0}$ respectively, are related by (3.2), with $u_n s_n \neq 0$, $n \in \mathbb{N}$. Then there exist real numbers a , b , and μ such that*

$$(x^2 + a)\mathcal{U} = \mu(x^2 + b)\mathcal{V}. \quad (3.17)$$

Proof. Let $a \in \mathbb{R}$ be an arbitrary real number. Let us analyze the action of the polynomial modification $(x^2 + a)\mathcal{U}$ over the polynomials $\{T_n\}_{n \geq 0}$. From relation (3.2) we get

$$\langle (x^2 + a)\mathcal{U}, T_n \rangle = -s_n \langle (x^2 + a)\mathcal{U}, T_{n-2} \rangle$$

for $n \geq 5$. Because of the symmetry of the linear functional, $(x^2 + a)\mathcal{U}$ vanishes over odd polynomials, and it is easy to check the values over even polynomials of degree $n < 5$:

$$\begin{aligned} \langle (x^2 + a)\mathcal{U}, T_0 \rangle &= \gamma_1 + a, \\ \langle (x^2 + a)\mathcal{U}, T_2 \rangle &= \gamma_1 \gamma_2 + (u_0 - s_0)(\gamma_1 + a), \\ \langle (x^2 + a)\mathcal{U}, T_4 \rangle &= (u_2 - s_2)\gamma_1 \gamma_2 - s_2(u_0 - s_0)(\gamma_1 + a). \end{aligned}$$

Since $u_0 \neq s_0$ and $s_2 \neq 0$, we can choose $a \in \mathbb{R}$ such that $\langle (x^2 + a)\mathcal{U}, T_4 \rangle = 0$, and then we have $\langle (x^2 + a)\mathcal{U}, T_n \rangle = 0$ for every $n \geq 3$.

If we consider the expansion of the functional $(x^2 + a)\mathcal{U}$ in terms of the dual basis of the system of polynomials $\{T_n\}_{n \geq 0}$, then we get

$$(x^2 + a)\mathcal{U} = \sum_{j=0}^2 \lambda_j \frac{T_j \mathcal{V}}{t_j}, \quad (3.18)$$

where $\lambda_1 = 0$ and $\lambda_j = \langle (x^2 + a)\mathcal{U}, T_j \rangle$ for $j = 0, 2$. Since we have already computed these values, we substitute the appropriate value for a and we finally deduce relation (3.17) with

$$a = \frac{u_2 - s_2}{s_2(u_0 - s_0)} \gamma_1 \gamma_2 - \gamma_1, \quad b = \frac{u_2 - s_2}{u_2(u_0 - s_0)} \tilde{\gamma}_1 \tilde{\gamma}_2 - \tilde{\gamma}_1, \quad \mu = \frac{u_2 \gamma_1 \gamma_2}{s_2 \tilde{\gamma}_1 \tilde{\gamma}_2}. \quad \square$$

4. Hermite and Gegenbauer cases

In this section we analyze in particular each possible case for the classical functional \mathcal{U} , that is, when it is the classical Hermite functional defined as

$$\langle \mathcal{U}_H, p \rangle = \int_{-\infty}^{+\infty} p(x) e^{-x^2} dx, \quad p \in \mathbb{P}, \quad (4.1)$$

and when it is the classical Gegenbauer functional given by

$$\langle \mathcal{U}_G, p \rangle = \int_{-1}^1 p(x) (1 - x^2)^{\lambda-1/2} dx, \quad p \in \mathbb{P} \quad (4.2)$$

with $\lambda > -1/2$.

4.1. On the generation of recurrence coefficients for the companion polynomials

First we deal with the Hermite functional \mathcal{U}_H defined by (4.1) and the corresponding Hermite polynomials $\{H_n\}_{n \geq 0}$. Let \mathcal{V}_H a symmetric linear functional and denote by $\{T_n\}_{n \geq 0}$ the system of monic polynomials orthogonal with respect to \mathcal{V}_H . Assume that these polynomials are related to Hermite polynomials by

$$H_n + u_{n-2} H_{n-2} = T_n + s_{n-2} T_{n-2}, \quad n \geq 2, \quad (4.3)$$

where u_n, s_n are non-zero constants for all $n \geq 0$, with $u_0 \neq s_0$ and $u_1 \neq s_1$.

It is well known that Hermite polynomials verify the following three-term recurrence relation

$$H_{n+1} = x H_n - \frac{n}{2} H_{n-1}, \quad n \geq 1, \quad H_0 = 1, \quad H_1 = x.$$

As in the previous section, we denote by $\tilde{\gamma}_n$ the coefficients of the three term recurrence relation for the polynomials $\{T_n\}_{n \geq 0}$,

$$T_{n+1} = x T_n - \tilde{\gamma}_n T_{n-1}, \quad n \geq 0, \quad T_0 = 1, \quad T_1 = x$$

with $\tilde{\gamma}_n > 0$ for $n \geq 1$. Then, as in (3.9), we have

$$\tilde{\gamma}_n = \frac{n}{2} \frac{u_n}{u_{n-1}} \frac{s_{n-1}}{s_n}, \quad n \geq 3.$$

Now, by Theorem 3.3 we can write the coefficients $(u_n)_{n \geq 0}$ as a rational function. We will rewrite the statement of that theorem in this particular case and moreover we identify such a family of polynomials.

Theorem 4.1. *The coefficients $(u_n)_{n \geq 0}$ in relation (4.3) can be expressed as a rational function in the form*

$$u_{2n+1} = \frac{r_n}{r_{n-1}}, \quad u_{2n} = \frac{q_n}{q_{n-1}}, \quad n \geq 1,$$

where r_n and q_n are polynomials in the variable $A = (u_3 - 2)(1 - \frac{3}{2u_2})$, with $\deg r_n = n$ and $\deg q_n = n - 1$. Moreover, they satisfy the following three term recurrence relations:

$$r_n = [A + 2n + 3/2]r_{n-1} - n(n + 1/2)r_{n-2}, \quad n \geq 2,$$

and

$$q_{n+1} = [A + 2n + 5/2]q_n - (n + 1)(n + 1/2)q_{n-1}, \quad n \geq 1,$$

with initial conditions

$$\begin{aligned} r_0 &= 1, & r_1 &= \frac{2u_2}{2u_2 - 3}A - \frac{2u_2 - 3}{u_2}, \\ q_0 &= 1, & q_1 &= u_2. \end{aligned}$$

Then, we can identify these families of polynomials, up to a linear change of variable, as the co-recursive polynomials of associated Laguerre polynomials of first kind, with parameter $\alpha = 1/2$ for the polynomials $r_n(A)$ and $\alpha = -1/2$ for the polynomials $q_n(A)$.

On the other hand, this result agrees with Corollary 3.4. It is known, see for instance [2], that the sequences of orthogonal polynomials corresponding to Hermite polynomials of even and odd degree are the Laguerre polynomials with parameter $\alpha = -1/2$ and $1/2$, respectively, i.e.

$$H_{2n}(x) = L_n^{(-1/2)}(x^2), \quad H_{2n+1}(x) = xL_n^{(1/2)}(x^2), \quad n \geq 0.$$

Thus, for the Hermite linear functional we can compute the values of the coefficients u_n for $n \geq 4$, once we know the values of u_2 and u_3 , which are determined in terms of u_0 and u_1 according to (3.6) and (3.7). In fact, we have few possible cases depending on the vanishing of (3.6) for $n = 0, 1, 2, 3$, as we show in the table below:

	A	B
0	$u_0 = 1/2$	$u_0 \neq 1/2$
1	$u_1 = 1 + u_0$	$u_1 \neq 1 + u_0$
2	$u_2 = 3/2 + u_1$	$u_2 = 3/2 + u_1 - u_1/2u_0$
3	$u_3 = 2 + u_2$	$u_3 = 2 + u_2 - u_2/u_1$

For Gegenbauer polynomials, $\{C_n^{(\lambda)}\}_{n \geq 0}$, orthogonal with respect to the linear functional (4.2), the three term recurrence relation is

$$C_{n+1}^{(\lambda)} = xC_n^{(\lambda)} - \frac{n(n + 2\lambda - 1)}{4(n + \lambda - 1)(n + \lambda)}C_{n-1}^{(\lambda)}, \quad n \geq 1.$$

Let $\{T_n\}_{n \geq 0}$ be a sequence of monic polynomials orthogonal with respect to a symmetric linear functional \mathcal{V}_G , such that

$$C_n^{(\lambda)} + u_{n-2}C_{n-2}^{(\lambda)} = T_n + s_{n-2}T_{n-2}, \quad n \geq 2.$$

If we denote by $\tilde{\gamma}_n$ the recurrence coefficients for these polynomials, then from (3.9) we get

$$\tilde{\gamma}_n = \frac{n(n+2\lambda-1)}{4(n+\lambda-1)(n+\lambda)} \frac{u_n}{u_{n-1}} \frac{s_{n-1}}{s_n}, \quad n \geq 3.$$

Moreover, according to Theorem 3.3 we can identify the parameters $(u_n)_{n \geq 0}$:

Theorem 4.2. *The sequences of polynomials $\{r_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$ defined recursively by $r_0 = 1$, $q_0 = 1$, and*

$$u_{2n+1} = \frac{r_n}{r_{n-1}}, \quad u_{2n} = \frac{q_n}{q_{n-1}}, \quad n \geq 1,$$

verify the following three term recurrence relations:

$$r_n = \left[A + \frac{(2n+1)(n+\lambda)}{2(2n+\lambda)(2n+\lambda+1)} + \frac{(n+1)(2n+2\lambda+1)}{2(2n+\lambda+1)(2n+\lambda+2)} \right] r_{n-1} - \frac{n(2n+2\lambda-1)(2n+1)(n+\lambda)}{4(2n+\lambda-1)(2n+\lambda)^2(2n+\lambda+1)} r_{n-2}, \quad n \geq 2$$

and

$$q_{n+1} = \left[A + \frac{(n+1)(2n+2\lambda+1)}{2(2n+\lambda+1)(2n+\lambda+2)} + \frac{(2n+3)(n+\lambda+1)}{2(2n+\lambda+2)(2n+\lambda+3)} \right] q_n - \frac{(2n+1)(n+\lambda)(n+1)(2n+2\lambda+1)}{4(2n+\lambda)(2n+\lambda+1)^2(2n+\lambda+2)} q_{n-1}, \quad n \geq 1.$$

In particular, when $\lambda = 0$ we have the monic Chebyshev polynomials of the first kind, $\{C_n\}_{n \geq 0}$, orthogonal with respect to the functional

$$\langle \mathcal{U}_C, p \rangle = \int_{-1}^1 p(x) \frac{dx}{\sqrt{1-x^2}}, \quad p \in \mathbb{P}.$$

For these polynomials (see [2]) the coefficients in the recurrence relation are $\gamma_2 = 1/2$ and $\gamma_n = 1/4$ for $n \geq 2$, that is

$$C_{n+1} = xC_n - \frac{1}{4}C_{n-1}, \quad n \geq 2, \\ C_0 = 1, \quad C_1 = x, \quad C_2 = x^2 - 1/2.$$

Then, by Corollary 3.4 the sequences of polynomials $\{r_n\}_{n \geq 0}$ and $\{q_n\}_{n \geq 0}$ can be identified, up to a linear change of variable, as the co-recursive of associated Jacobi polynomials of first kind with parameters $(\alpha, \beta) = (-1/2, 1/2)$, and as the co-recursive of Chebyshev polynomials of the first kind, respectively.

4.2. The companion linear functional \mathcal{V}

Finally, in this last section we give a classification of all the companion linear functionals in the Hermite and Gegenbauer cases.

Remind that the Hermite functional \mathcal{U}_H is defined by (4.1). Then from (3.17), we see that the companion functional \mathcal{V}_H is given by

$$\langle \mathcal{V}_H, p \rangle = \int_{-\infty}^{+\infty} p(x) \frac{(x^2 + a)}{(x^2 + b)} e^{-x^2} dx, \quad p \in \mathbb{P},$$

where a and b are positive real numbers.

In the Gegenbauer case, we have two different possibilities for the companion linear functional, depending on the values of the constants a and b defined in (3.17):

$$\langle \mathcal{V}_G, p \rangle = \int_{-1}^1 p(x) \frac{(x^2 + a)}{(x^2 + b)} (1 - x^2)^{\lambda-1/2} dx, \quad p \in \mathbb{P}$$

with $a, b > 0$, and

$$\langle \mathcal{V}_G, p \rangle = \int_{-1}^1 p(x) \frac{(x^2 - a)}{(x^2 - b)} (1 - x^2)^{\lambda-1/2} dx + M[p(\sqrt{b}) + p(-\sqrt{b})], \quad p \in \mathbb{P}$$

with $a, b > 1$.

We finish with a remark about an example studied by in [1]. In this work they prove that Hermite (resp. Gegenbauer) polynomials are related to the Sobolev polynomials associated with the pair of functionals $(\mathcal{U}_H, \mathcal{V}_H)$ (resp. $(\mathcal{U}_G, \mathcal{V}_G)$) by an expression of the type (2.2). Then, by Theorem 2.3 we know that relation (3.1) holds between Hermite (resp. Gegenbauer) polynomials and the companion sequence of polynomials orthogonal with respect to the modified linear functional \mathcal{V}_H (resp. \mathcal{V}_G).

Thus, our result states that these examples are not by chance, but this is the only possible modification of the Hermite functional in order to get (2.2).

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