A scalar Riemann boundary value problem approach to orthogonal polynomials on the circle[☆]

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Abstract

A scalar Riemann boundary value problem defining orthogonal polynomials on the unit circle and the corresponding functions of the second kind is obtained. The Riemann problem is used for the asymptotic analysis of the polynomials orthogonal with respect to an analytical real-valued weight on the circle.

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1. Introduction

Methods of complex analysis based on Riemann–Hilbert boundary value problems (BVPs) have proven their capabilities and advantages for asymptotic analysis. Many fascinating results for asymptotics of polynomials orthogonal on the real line were derived from the characterization of orthogonal polynomials as the solutions of a matrix Riemann–Hilbert problem. This reformulation

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of the orthogonality relation in terms of a matrix Riemann–Hilbert problem has been discovered first by Fokas et al. [4]. Asymptotic analysis of the matrix Riemann–Hilbert problem has been considered by Deift and Zhou [3] and extensively developed by many authors. Methods of this matrix Riemann–Hilbert problem work not only for orthogonal polynomials on the real line but also for polynomials orthogonal on the unit circle (see [2]).

There is an alternative approach to the asymptotic analysis of orthogonal polynomials based on a scalar Riemann problem and singular integral equations. This approach for polynomials orthogonal with respect to a complex weight on an interval of the real axis has been suggested by Nuttall [6] (see also some developments in [1,7]).

In this note we obtain a scalar Riemann BVP problem formulation for orthogonal polynomials on the circle $\partial U := \{z \in \mathbb{C}\colon |z| = 1\}$. This BVP brings together:

- *monic orthogonal polynomial* sequence $\{\Phi_n(z)\}$, $\Phi_n(z) = z^n + \cdots$,

$$\int_0^{2\pi} \Phi_n(\xi)\overline{\xi}^m \rho(e^{i\theta})\, d\theta = 0 \ \text{ for } m = 0, \ldots, n-1, \quad \xi = e^{i\theta}, \tag{1}$$

  with respect to a positive integrable *weight function* ρ;
- *functions of the second kind*

$$F_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\Phi_n(e^{i\theta})\rho(e^{i\theta})\, d\theta; \tag{2}$$

- *Szegő function*

$$\Pi(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho(e^{i\theta})\, d\theta\right). \tag{3}$$

We prove

Theorem 1. *The piecewise analytic function*

$$f(z) = \begin{cases} \dfrac{F_n(z)}{\Pi(z)z^n} & \text{if } z \in U := \{z \in \mathbb{C} : |z| < 1\}, \\ \dfrac{2\Phi_n(z)}{\Pi(z)z^n} & \text{if } z \in \overline{\mathbb{C}} \setminus \overline{U}. \end{cases} \tag{4}$$

is the unique solution of the following BVP

$$\begin{cases} (1)\ \ f \in H(\overline{\mathbb{C}} \setminus \partial U), \\ (2)\ \ \exists f_\pm \in L^1(\partial U) \quad \text{and} \quad f_+ - f_- = j_n \ \text{ on } \partial U, \\ (3)\ \ f(\infty) = \dfrac{2}{\Pi(\infty)}, \end{cases} \tag{5}$$

where the jump function j_n *is taken as*

$$j_n(\xi) := \frac{(F_n)_-(\xi)}{\rho(\xi)\Pi_-(\xi)\xi^n} \quad \text{if} \quad \xi \in \partial U. \tag{6}$$

Here we denote by f_+ the boundary values of the function $f(z)$ on ∂U as $z \to \xi$ from the inside of the unit circle U (respectively, f_- stands for the outside boundary values).

Remark 1. The weight $\rho(\xi)$, $\xi \in \partial U$, in (1) can be taken as a complex-valued integrable function. In this case the existence of a monic orthogonal polynomial sequence is not guaranteed. However, if $\{\Phi_n\}$ exists, then Theorem 1 remains valid.

In the next section we consider several BVPs related to the weight function ρ on the circle and present a proof of Theorem 1 in the case when ρ is complex valued and $\{\Phi_n\}$ exists. Then in the last section we demonstrate how to apply the BVP (5)–(6) for the asymptotic analysis of orthogonal polynomials on the circle. As an example we consider an integrable weight ρ which is positive symmetric ($\rho(e^{i\theta}) = \rho(e^{-i\theta})$), representing the boundary values on ∂U of an analytic function in $\overline{\mathbb{C}} \setminus \overline{U}$. In this case we derive the Szegő asymptotics (see Theorem 2). It is worth mentioning that under these conditions the convergence of the Szegő asymptotics has the rate of a geometric progression.

2. BVPs related to the circle. Proof of Theorem 1

We start with a characterization of the Carathéodory function by means of BVP. The *Carathéodory function* F is defined by

$$F(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \rho(e^{i\theta})\, d\theta\,. \tag{7}$$

Proposition 1. *There exists a unique function F satisfying the following boundary value problem*:

$$\begin{cases} (1)\ F \in H(\overline{\mathbb{C}} \setminus \partial U), \\ (2)\ \exists F_\pm \in L^1(\partial U) \quad \text{and} \quad F_+ - F_- = 2\rho \ \text{ on } \ \partial U, \\ (3)\ F(0) + F(\infty) = 0, \end{cases} \tag{8}$$

and F is given by (7).

Proof (*Existence***).** The function F given by (7) can be written in the following way:

$$F(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\xi + z}{\xi - z} \rho(\xi) \frac{d\xi}{\xi} = \frac{2}{2\pi i} \int_{\partial U} \frac{\rho(\xi)}{\xi - z}\, d\xi - \frac{1}{2\pi i} \int_{\partial U} \frac{\rho(\xi)}{\xi}\, d\xi.$$

Therefore, F has the following properties:

(1) By the properties of the Cauchy-type integral we have that $F \in H(\overline{\mathbb{C}} \setminus \partial U)$.
(2) Moreover, the Sokhotskii–Plemelj formulas assert that $F_+(\xi) - F_-(\xi) = 2\rho(\xi)$.
(3) Since $F(0) = \frac{1}{2\pi} \int_0^{2\pi} \rho(e^{i\theta})\, d\theta$ and $F(\infty) = \lim_{z\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \rho(e^{i\theta})\, d\theta = -F(0)$, the result follows.

Uniqueness: Let F_1 and F_2 be two solutions of problem (8). Then $G = F_1 - F_2 \in H(\overline{\mathbb{C}} \setminus \partial U)$. Since $(F_1)_+ - (F_1)_- = 2\rho = (F_2)_+ - (F_2)_-$, one has $(F_1 - F_2)_+ = (F_1 - F_2)_-$ and therefore $G \in H(\overline{\mathbb{C}})$. By Liouville's theorem $G(z)$ is a constant function. Since $G(0) + G(\infty) = 0$, this constant is 0. Hence $F_1 = F_2$. □

It follows from Proposition 1 that

$$\rho \in H(\overline{U}) \quad \Rightarrow \quad F = \begin{cases} 2\rho - \rho(0) & \text{in } U, \\ -\rho(0) & \text{in } \overline{\mathbb{C}} \setminus U. \end{cases}$$

The next proposition describes boundary value properties for the Szegő function (3).

Proposition 2. *There exists a unique function* Π *satisfying the BVP*

$$\begin{cases} (1)\ \Pi \in H(\overline{\mathbb{C}} \setminus \partial U), \quad \Pi(z) \neq 0, \ \ \forall z \in \overline{\mathbb{C}} \setminus \partial U, \\ (2)\ \exists \Pi_{\pm} \ \textit{with} \ \log \Pi_{\pm} \in L^1(\partial U) \quad \textit{and} \quad \Pi_+ = \rho \Pi_- \ \textit{on} \ \partial U, \\ (3)\ \Pi(0)\Pi(\infty) = 1, \end{cases} \tag{9}$$

and Π *is given by* (3).

Proof (*Existence***).** If a function Π satisfies (9), then $\log \Pi$ is such that

$$\log \Pi(\infty) = -\frac{1}{4\pi} \int_0^{2\pi} \log \rho(e^{i\theta})\, d\theta.$$

In this case $\log \Pi$ satisfies problem (8). Indeed $\log \Pi \in H(\overline{\mathbb{C}} \setminus \partial U)$, $(\log \Pi)_+, (\log \Pi)_- \in L^1(\partial U)$, $\log \Pi_+ - \log \Pi_- = \frac{2\log \rho}{2}$ and $\log \Pi(0) + \log \Pi(\infty) = 0$.

Thus, by Proposition 1,

$$\log \Pi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \frac{\log \rho(e^{i\theta})}{2}\, d\theta = \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \rho(e^{i\theta})\, d\theta$$

and therefore Π is in fact (3).

Uniqueness: The uniqueness follows from the uniqueness of the solution of BVP (8). □

The following interpolation property of the function of the second kind is well-known for positive weights ρ (see [5]).

$$F_n(z) = O(z^n) \quad \text{when } z \to 0. \tag{10}$$

It can easily be checked that this property is also true for the complex-valued weights.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We have to check that properties (1)–(3) in (5) hold.

(1) Under the assumption that Φ_n exists and from (10) we see that property (1) in (5) holds.

(2) Applying Proposition 1 to the function of the second kind F_n (see (2)), we have

$$(F_n)_+ - (F_n)_- = 2\rho\Phi_n \quad \text{on } \partial U.$$

If we divide the above equation by ξ^n and factorize the weight function by the boundary condition (2) in (9) (i.e., $\Pi_+ = \rho\Pi_-$), then we see that

$$\frac{(F_n)_+}{\xi^n} - \frac{2\Pi_+\Phi_n}{\Pi_-\xi^n} = \frac{(F_n)_-}{\xi^n}$$

which implies

$$\frac{(F_n)_+}{\xi^n\Pi_+} - \frac{2\Phi_n}{\Pi_-\xi^n} = \frac{(F_n)_-}{\xi^n\Pi_+}.$$

Then

$$\left(\frac{F_n}{\xi^n\Pi}\right)_+ - \left(\frac{2\Phi_n}{\Pi\xi^n}\right)_- = \left(\frac{F_n}{\xi^n\rho\Pi}\right)_- \tag{11}$$

on ∂U, which proves (2).

(3) By definition (4) we get (3). Indeed, $\lim_{z\to\infty} \dfrac{2\Phi_n(z)}{\Pi(z)z^n} = \frac{2}{\Pi(\infty)}$.

The theorem is proved. $\square$

Remark 2. The BVP (5) always has a unique solution which is given by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{j_n(\xi)}{\xi - z} d\xi + f(\infty). \tag{12}$$

Therefore, if ρ is complex and the orthogonal polynomial Φ_n does not exist, then BVP (5) defines via (6), (4) a function

$$\Phi_n(z) = z^n + \sum_{k=-\infty}^{n-1} \alpha_k z^k, \quad \Phi_n \in H(\mathbb{C} \setminus U), \tag{13}$$

which is orthogonal in the sense of (1).

Proof. For a given complex ρ we construct Π by (3). Then the solution of the problem (5),(6) defines by means of (4)

$$F_n(z) := f(z)\Pi(z)z^n, \ \ |z| < 1, \quad \Phi_n(z) := \frac{f(z)\Pi(z)z^n}{2}, \ \ |z| > 1. \tag{14}$$

We have from the second relation in (14) that $\Phi_n(z)$ is of the form (13). Now we check the orthogonality. The jump condition (2) in (5) gives (11) which with the help of the jump condition for the Szegő function (see condition (2) in (9)) gives

$$2\Phi_n(\xi)\rho(\xi) = F_{n+}(\xi) - F_{n-}(\xi).$$

The last relation (considered as Sokhotskii–Plemelj formula) leads to (2). From there, together with $F_n(z) = O(z^n), \ z \to 0$, we arrive at the orthogonality of the function Φ_n. $\square$

3. Application to asymptotics of Φ_n and F_n

In this section we apply Theorem 1 to obtain the Szegő asymptotics for polynomials orthogonal on the unit circle with respect to a real positive symmetric weight. In what follows we assume that the integrable weight function $\rho(z)$ has analytic non-vanishing continuation to an outer neighborhood of ∂U, i.e.,

$$\exists \delta > 0 : \ \ \rho(z) \in H(A_\delta), \ \rho \neq 0 \text{ in } A_\delta, \ \ A_\delta := \{z \in \mathbb{C} : 1 < |z| < 1 + \delta\}. \tag{15}$$

Therefore, the Fourier coefficients of the weight function

$$\rho(z) = \sum_{k=-\infty}^{\infty} c_k z^k,$$

must satisfy

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|c_k|} \leqslant \frac{1}{1+\delta} \quad \text{and} \quad \overline{\lim_{k\to\infty}} \sqrt[k]{|c_{-k}|} \leqslant 1. \tag{16}$$

We also assume that ρ is real on ∂U, i.e., we have

$$c_k = \overline{c_{-k}}, \quad \Im c_0 = 0, \tag{17}$$

and we impose the symmetry condition

$$\rho(e^{i\theta}) = \rho(e^{-i\theta}). \tag{18}$$

Theorem 2. *Let* $\{\Phi_n(z)\}$ *be defined by* (1), *where* ρ *is a real positive symmetric weight function satisfying conditions* (15)–(18), *and let* $\{F_n(z)\}$ *be the corresponding functions of the second kind* (2). *Then the following uniform asymptotic formulas hold*:

(1)

$$\frac{\Phi_n(z)}{z^n} = \frac{\Pi(z)}{\Pi(\infty)} + O\Big(\frac{1}{(1+\tilde{\delta}_K)^n}\Big), \quad n \to \infty,\ z \in K \subset \overline{\mathbb{C}} \setminus U,\ \ K \text{ compact},$$

with $\tilde{\delta}_K = \min\{\delta, dist(\partial U, K)\}$.

(2)

$$\frac{F_n(z)}{z^n} = \frac{2\Pi(z)}{\Pi(\infty)} + O\Big(\frac{1}{(1+\delta)^n}\Big), \quad n \to \infty, \quad |z| \leqslant 1.$$

Remark 3. The asymptotic formulas for Φ_n and F_n presented in this theorem are well-known (see [5] and [8]). They are valid under more general conditions. However, we emphasize that the analyticity condition (15) provides convergence of the asymptotic formulas with a fast (geometric progression) rate.

In order to prove Theorem 2 we need several propositions regarding the functions of the second kind (2).

Proposition 3. *Let* $\mathcal{F}_n \in H(\overline{\mathbb{C}} \setminus \partial U)$ *be defined as follows*:

$$\mathcal{F}_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\Phi_n(e^{i\theta})\Phi_n(e^{-i\theta})\rho(e^{i\theta})\,d\theta, \quad z \in \overline{\mathbb{C}} \setminus \partial U. \tag{19}$$

Then for the function of the second kind there is the representation

$$F_n(z) = \frac{\mathcal{F}_n(z) + \mathcal{F}_n(0)}{\Phi_n(z^{-1})}, \quad z \in \overline{\mathbb{C}} \setminus \partial U. \tag{20}$$

Proof. Transforming the kernel

$$\frac{e^{i\theta}+z}{e^{i\theta}-z} = \frac{-z^{-1}+e^{-i\theta}+2z^{-1}}{z^{-1}-e^{-i\theta}}$$

we have by orthogonality

$$F_n(z) = \frac{1}{2\pi}2z^{-1}\int_0^{2\pi} \frac{\Phi_n(e^{i\theta})\rho(e^{i\theta})\,d\theta}{z^{-1}-e^{-i\theta}}.$$

Then multiplying and dividing the right-hand side by $\Phi_n(z^{-1})$ and then adding and subtracting $\Phi_n(e^{-i\theta})$ in the numerator we arrive (using orthogonality) at

$$F_n(z) = \frac{2z^{-1}}{2\pi\Phi_n(z^{-1})}\int_0^{2\pi}\frac{\Phi_n(e^{i\theta})\Phi_n(e^{-i\theta})\rho(e^{i\theta})\,d\theta}{z^{-1}-e^{-i\theta}}.$$

Finally, adding and subtracting $z^{-1}+e^{-i\theta}$ in the numerator we obtain (20). □

The symmetry of the weight (18) gives $\Phi_n(e^{-i\theta}) = \overline{\Phi_n(e^{i\theta})}$, which transforms the expression (19) to

$$\mathcal{F}_n(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}|\Phi_n(e^{i\theta})|^2\rho(e^{i\theta})\,d\theta. \tag{21}$$

If in addition to the symmetry (18) we have that the weight function ρ is real valued, then $F_{n+} = -\overline{F_{n-}}$, and we have the boundary conditions for $\mathcal{F}_n$

$$\mathcal{F}_{n+} = -\overline{\mathcal{F}_{n-}} \quad \text{on } \partial U. \tag{22}$$

Proposition 4. *Let ρ be a positive symmetric* (*see* (18)) *weight function. Then*

$$|F_{n+}| \geqslant |F_{n-}| \quad \textit{on } \partial U. \tag{23}$$

Proof. We have from (20)

$$F_{n\pm} = \frac{\mathcal{F}_{n\pm}+\mathcal{F}_n(0)}{\overline{\Phi_n}} \quad \text{on } \partial U.$$

Property (22) implies

$$\begin{cases}\Re\mathcal{F}_{n+} = -\Re\mathcal{F}_{n-}\ ,\\ \Im\mathcal{F}_{n+} = \Im\mathcal{F}_{n-}\ .\end{cases}$$

Moreover we have from (21)

$$\Re\mathcal{F}_{n+} = |\Phi_n|^2\rho > 0$$

and $\mathcal{F}_n(0) > 0$. Thus

$$|F_{n+}|^2 = \frac{(|\Phi_n|^2\rho+\mathcal{F}_n(0))^2+(\Im\mathcal{F}_n)^2}{|\Phi_n|^2} \geqslant \frac{(-|\Phi_n|^2\rho+\mathcal{F}_n(0))^2+(\Im\mathcal{F}_n)^2}{|\Phi_n|^2} = |F_{n-}|^2. \qquad □$$

Proof of Theorem 2. The idea of the proof is to use the representation of the solution (12) of the BVP (5) with jump function (6) as an integral equation with respect to the function $F_n(\xi)$. The integral equation when $z \to \partial U$ becomes singular. However, we can use the fact that in the integral (12) along ∂U, the function j_n (see (6)) is the boundary value of an analytic function from $\mathbb{C}\setminus U$. Thus (due to Cauchy's theorem), we can deform the contour of integration ∂U to a contour γ which lies in the domain $A_\delta \subseteq \mathbb{C}\setminus U$. If we denote by $\tilde{A}_\gamma$ a ring domain bounded by ∂U and γ, then for the solution (12) of the BVP in the domain $\overline{\mathbb{C}}\setminus\tilde{A}_\gamma$ we have

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{F_n(t)\,dt}{\rho(t)\Pi(t)t^n(t-z)} + \frac{2}{\Pi(\infty)}\ , \quad z\in\overline{\mathbb{C}}\setminus\tilde{A}_\gamma. \tag{24}$$

We observe that the integral above now becomes regular when $z \to \partial U$, and we can use it for useful estimates. We set $M_n := \sup_{\xi\in\partial U} |F_{n+}(\xi)|$. First, we prove that M_n is bounded with respect to n. Let $\zeta \in \partial U$ be such that $|F_{n+}(\zeta)| = M_n$, and consider (24) when $z \to \zeta$. Taking modulus, we arrive at

$$\frac{M_n}{|\Pi(\zeta)|} = \left| \frac{1}{2\pi i} \int_\gamma \frac{F_n(t)\, dt}{\rho(t)\Pi(t)t^n(t-\zeta)} + \frac{2}{\Pi(\infty)} \right|.$$

From (23) and by maximum modulus principle we have $M_n \geqslant \sup_{\xi\in\partial U} |F_{n-}(\xi)| \geqslant |F_n(t)|$, for all $t \in \gamma$. Denoting $C_1 = \max_{\xi\in\partial U} |\Pi(\xi)|,\ C_2 = \min_{t\in\gamma} |\rho(t)\Pi(t)|$, we obtain

$$M_n \leqslant \frac{C_1 M_n}{C_2(1+\delta)^{n-1}\delta} + \frac{2C_1}{|\Pi(\infty)|}.$$

Thus

$$M_n \leqslant \frac{2C_1}{\Pi(\infty)} \left(\frac{1}{1 - \frac{C_1}{C_2(1+\delta)^{n-1}\delta}} \right) := C_3 + O\left(\frac{1}{(1+\delta)^n} \right).$$

Now we are ready to obtain the asymptotic formulas from the statement of the theorem. Let z tend to $\xi \in \partial U$ from the inside of U. Then we have

$$\left| \frac{F_{n+}(\xi)}{\Pi(\xi)\xi^n} - \frac{2}{\Pi(\infty)} \right| = \left| \frac{1}{2\pi i} \int_\gamma \frac{F_{n-}(t)\, dt}{\rho(t)\Pi(t)t^n(t-\xi)} \right| \leqslant \frac{C_3 + O(\frac{1}{(1+\delta)^n})}{C_2\delta(1+\delta)^{n-1}} = O\left(\frac{1}{(1+\delta)^n} \right).$$

This gives us statement (2) of the theorem.

For the asymptotics of $\Phi_n(z)$ we fix a compact $K \subset \overline{\mathbb{C}} \setminus U$ and fix a contour γ_K in the domain $A_\delta \setminus K$. Then for $z \in K$ we have the formula for statement (1) of the theorem.

$$\left| \frac{2\Phi_n(z)}{\Pi(z)z^n} - \frac{2}{\Pi(\infty)} \right| = \left| \frac{1}{2\pi} \int_{\gamma_K} \frac{F_{n-}(t)\, dt}{\rho(t)\Pi(t)t^n(t-z)} \right| \leqslant \frac{C_3 + O(\frac{1}{(1+\delta)^n})}{C_2\tilde{\delta_K}(1+\tilde{\delta_K})^{n-1}}$$
$$= O\left(\frac{1}{(1+\tilde{\delta_K})^n} \right). \qquad \square$$

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