Second structure relation for semiclassical orthogonal polynomials

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Abstract

Classical orthogonal polynomials are characterized from their orthogonality and by a first or second structure relation. For the semiclassical orthogonal polynomials (a generalization of the classical ones), we find only the first structure relation in the literature. In this paper, we establish a second structure relation. In particular, we deduce it by means of a general finite-type relation between a semiclassical polynomial sequence and the sequence of its monic derivatives.

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1. Introduction

Classical orthogonal polynomial sequences (Hermite, Laguerre, Bessel, and Jacobi) are characterized by the property that the sequence of its monic derivatives is again orthogonal (Hahn's property, see [2,1,7–10,12,13]). Taking into account the key role of such families of polynomials in the study of hypergeometric differential equations resulting from Mathematical Physics, we find a vast literature with different approaches to the subject. For instance, the functional equation (Pearson equation) satisfied by classical linear functionals and, more generally, semiclassical linear functionals allows an efficient study of some properties of both classical and semiclassical orthogonal polynomials [3,12,14]. However, the sequences of classical orthogonal polynomials $\{\mathrm{C}_n\}_{n\geqslant 0}$ can be characterized taking into account its orthogonality as well as one of the two following differential-difference equations, the so-called structure relations.

- *First structure relation* [2,1,10,12,13]

$$E(x)\mathrm{C}_n^{[1]}(x)=\sum_{v=n}^{n+t}\lambda_{n,v}\mathrm{C}_v(x),\quad n\geqslant 0,\qquad \lambda_{n,n}\neq 0,\quad n\geqslant 0,$$

where $\mathrm{C}_n^{[1]}(x)=(n+1)^{-1}\mathrm{C}'_{n+1}(x)$, $n\geqslant 0$, and E is a monic polynomial such that $\deg E=t\leqslant 2$.

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Table 1

(A$_1$) Hermite

$$\hat{H}_n(x)=\hat{H}_n^{[1]}(x),\quad n\geqslant 0,$$

(A$_2$) Laguerre, $\hat{L}_n^{(\alpha)^{[1]}}(x)=\hat{L}_n^{(\alpha+1)}(x), n\geqslant 0,$

$$x\hat{L}_n^{(\alpha+1)}(x)=\hat{L}_{n+1}^{(\alpha)}(x)+(n+\alpha+1)\hat{L}_n^{(\alpha)}(x),\quad n\geqslant 0,$$

$$\hat{L}_n^{(\alpha)}(x)=\hat{L}_n^{(\alpha+1)}(x)+n\hat{L}_{n-1}^{(\alpha+1)}(x),\quad n\geqslant 1,$$

with $\alpha\neq -n,\quad n\geqslant 1.$

(A$_3$) Bessel, $\hat{B}_n^{(\alpha)^{[1]}}(x)=\hat{B}_n^{(\alpha+1)}(x), n\geqslant 0,$

$$x^2\hat{B}_n^{(\alpha+1)}(x)=\hat{B}_{n+2}^{(\alpha)}(x)-\frac{n+2\alpha}{(n+\alpha)(n+\alpha+1)}\hat{B}_{n+1}^{(\alpha)}(x)-\frac{(n+2\alpha)\eta_{n+1}}{n+1}\hat{B}_n^{(\alpha)}(x),\quad n\geqslant 0,$$

$$\hat{B}_n^{(\alpha)}(x)=\hat{B}_n^{(\alpha+1)}(x)+\frac{n}{(n+\alpha-1)(n+\alpha)}\hat{B}_{n-1}^{(\alpha+1)}(x)-\frac{(n-1)\eta_n}{(n+2\alpha-2)}\hat{B}_{n-2}^{(\alpha+1)}(x),\quad n\geqslant 2,$$

with $\eta_n=-\dfrac{n(n+2\alpha-2)}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)}, n\geqslant 1,\quad$ and $\quad\alpha\neq-\dfrac{n}{2},\quad n\geqslant 0.$

(A$_4$) Jacobi, $\hat{P}_n^{(\alpha,\beta)^{[1]}}(x)=\hat{P}_n^{(\alpha+1,\beta+1)}(x), n\geqslant 0,$

$$(x^2-1)\hat{P}_n^{(\alpha+1,\beta+1)}(x)=\hat{P}_{n+2}^{(\alpha,\beta)}(x)+\frac{2(\alpha-\beta)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}\hat{P}_{n+1}^{(\alpha,\beta)}(x)-\frac{(n+\alpha+\beta+2)\tau_{n+1}}{n+1}\hat{P}_n^{(\alpha,\beta)}(x),\quad n\geqslant 0,$$

$$\hat{P}_n^{(\alpha,\beta)}(x)=\hat{P}_n^{(\alpha+1,\beta+1)}(x)-\frac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\hat{P}_{n-1}^{(\alpha+1,\beta+1)}(x)-\frac{(n-1)\tau_n}{n+\alpha+\beta}\hat{P}_{n-2}^{(\alpha+1,\beta+1)}(x),\quad n\geqslant 2,$$

with $\tau_n=\dfrac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},\quad n\geqslant 1,$

and $\alpha\neq -n,\quad \beta\neq -n,\quad \alpha+\beta\neq -n-1,\ n\geqslant 1.$

- *Second structure relation* [10,12]

$$\mathrm{C}_n(x)=\sum_{\nu=0}^{n}\theta_{n,\nu}\mathrm{C}_\nu^{[1]}(x),\quad n\geqslant 0, \tag{$*$}$$

with $\theta_{n,\nu}=0$, for $\nu<n-t$, $n\geqslant t$, $0\leqslant t\leqslant 2$.

The explicit expression for the parameters involved in the first and second structure relations is given in Table 1 (for more details, see [10,12,13]).

A generalization of this family leads to semiclassical orthogonal polynomials [3–5,12,14,17]. In fact, classical orthogonal polynomials are semiclassical of class zero. It is well-known that a first structure relation for the semiclassical orthogonal polynomials was established (see [3–5,12]) and it reads as follows.

A sequence $\{B_n\}_{n\geqslant 0}$ of semiclassical orthogonal polynomials satisfies

$$E(x)B_n^{[1]}(x)=\sum_{\nu=n-\sigma}^{n+t}\lambda_{n,\nu}B_\nu(x),\quad n\geqslant\sigma,\qquad \lambda_{n,n-\sigma}\neq 0,\quad n\geqslant\sigma+1,$$

where E is a monic polynomial, $\deg E=t$, and σ is a positive integer, such that $t\leqslant\sigma+2$.

The formulation of a second structure relation for semiclassical orthogonal polynomial sequence has been the aim of several contributions. In [6], the author tried to characterize semiclassical orthogonal polynomials $\{B_n\}_{n\geqslant 0}$ by its orthogonality and the following finite-type relation:

$$B_n(x)=\sum_{v=0}^{n}\theta_{n,v}B_v^{[1]}(x),\quad n\geqslant 0,$$

with $\theta_{n,v}=0$, for $v<n-t$, $n\geqslant t$, $t\geqslant 0$.

Unfortunately, no successful result has been achieved, except the fact that no orthogonal polynomial sequences satisfy the previous relation for $t\geqslant 3$.

Recently, Maroni and Sfaxi [15] introduced the concept of diagonal sequence. It reads as follows.

Let $\{B_n\}_{n\geqslant 0}$ be a sequence of monic orthogonal polynomials and ϕ a monic polynomial with $\deg\phi=t$. If there exists an integer $s\geqslant 0$ such that

$$\phi(x)B_n(x)=\sum_{v=n-s}^{n+t}\theta_{n,v}B_v^{[1]}(x),\quad n\geqslant s,\quad \theta_{n,n-s}\neq 0,\quad n\geqslant s,$$

then the sequence $\{B_n\}_{n\geqslant 0}$ is said to be diagonal associated with ϕ and index s. The authors prove that diagonal sequences are semiclassical. Obviously, the above finite-type relation, that we will call diagonal relation, is nothing else that the second structure relation characterizing such a family. But, some semiclassical orthogonal polynomials are not diagonal. As an example, we can mention the case of a semiclassical polynomial sequence $\{Q_n\}_{n\geqslant 0}$, orthogonal with respect to the linear functional v, satisfying the functional equation: $v'+\psi v=0$, where $\psi(x)=-\mathrm{i}x^2+2x-\mathrm{i}(\alpha-1)$. In fact, the sequence $\{Q_n\}_{n\geqslant 0}$ is characterized by its orthogonality and the following relation [11],

$$(x+v_{n,0})Q_n(x)=Q_{n+1}^{[1]}(x)+\varrho_nQ_n^{[1]}(x),\quad n\geqslant 0,$$

where

$$v_{n,0}=\frac{\mathrm{i}(n+1)}{\gamma_{n+1}}-\frac{\mathrm{i}\gamma_{n+1}\gamma_{n+2}}{n+2}-\beta_n,\quad n\geqslant 1,\quad v_{0,0}=-\frac{\mathrm{i}\gamma_1\gamma_2}{2}-\beta_0,$$

$$\varrho_n=\frac{\mathrm{i}(n+1)}{\gamma_{n+1}},\quad n\geqslant 1,\quad \varrho_0=0.$$

Here, γ_{n+1} and β_n are the recurrence coefficients of the orthogonal polynomial sequence $\{Q_n\}_{n\geqslant 0}$. In fact, $\{Q_n\}_{n\geqslant 0}$ is not diagonal and the constants $v_{n,0}$ depend on the integer n. This family will be analyzed more carefully in Section 4.1.

The aim of our contribution is to give, under certain conditions, a second structure relation characterizing a sequence of semiclassical orthogonal polynomials in terms of a relation between the orthogonal polynomial sequence $\{B_n\}_{n\geqslant 0}$ and the sequence of monic derivatives $\{B_n^{[1]}\}_{n\geqslant 0}$, where $B_n^{[1]}(x)=(n+1)^{-1}B'_{n+1}(x)$, $n\geqslant 0$, as follows:

$$\sum_{v=n-\sigma}^{n+\sigma}\xi_{n,v}B_v(x)=\sum_{v=n-t}^{n+\sigma}\varsigma_{n,v}B_v^{[1]}(x),\quad n\geqslant\max(t+1,\sigma),$$

where

$$\xi_{n,n+\sigma}=\varsigma_{n,n+\sigma}=1,\quad n\geqslant\max(t+1,\sigma),$$

$$\exists\ r\geqslant\sigma+t+1,\quad \xi_{r,r-\sigma}\varsigma_{r,r-t}\neq 0.$$

Notice that for $\sigma=0$ the second structure relation $(*)$ follows.

The structure of the paper is the following. In Section 2, a basic background about linear functionals, orthogonal polynomials, and semiclassical linear functionals is given in order to allow the reader to be familiar with such concepts. In Section 3 we present two new characterizations for sequences of semiclassical orthogonal polynomials. Theorems 3.2 and 3.5 are the extensions of the second structure relations satisfied by classical orthogonal polynomials. Finally, in Section 4 we find such relations for two examples of semiclassical orthogonal polynomials of class 1.

2. Background

Let u be a linear functional in the linear space $\mathbb{P}$ of polynomials with complex coefficients and let $\mathbb{P}'$ be its algebraic dual space, i.e., the linear space of the linear functionals defined on $\mathbb{P}$. We will denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ and by $(u)_n := \langle u, x^n\rangle, n \geqslant 0$, the moments of u with respect to the sequence $\{x^n\}_{n\geqslant 0}$. Let us define the following operations in $\mathbb{P}'$. For any polynomial h and any $c \in \mathbb{C}$, let $Du = u'$, hu, and $(x-c)^{-1}u$ be the linear functionals defined on $\mathbb{P}$ by

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad f \in \mathbb{P},$$

$$\langle hu, f\rangle := \langle u, hf\rangle, \quad f, h \in \mathbb{P},$$

$$\langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c(f)\rangle, \quad f \in \mathbb{P}, \ \ c \in \mathbb{C},$$

where

$$\theta_c(f)(x) = \frac{f(x) - f(c)}{x - c}.$$

Let $\{B_n\}_{n\geqslant 0}$ be a sequence of monic polynomials (SMP), $\deg B_n = n, n \geqslant 0$, and $\{u_n\}_{n\geqslant 0}$ its dual sequence, $u_n \in \mathbb{P}', n \geqslant 0$, defined by $\langle u_n, B_m\rangle := \delta_{n,m}, n, m \geqslant 0$, where $\delta_{n,m}$ is the Kronecker symbol.

Let recall the following results [7,12].

Lemma 2.1. *For any $u \in \mathbb{P}'$ and any integer $m \geqslant 1$, the following statements are equivalent.*

(i) $\langle u, B_{m-1}\rangle \neq 0, \langle u, B_n\rangle = 0, n \geqslant m.$
(ii) *There exist $\lambda_\nu \in \mathbb{C}$, $0 \leqslant \nu \leqslant m-1$, $\lambda_{m-1} \neq 0$, such that $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.*

On the other hand, it is straightforward to prove

Lemma 2.2. *For any $(\widehat{t}, \widehat{\sigma}, \widehat{r}) \in \mathbb{N}^3$, $\widehat{r} \geqslant \widehat{\sigma} + \widehat{t} + 1$ and any sequence of monic polynomials $\{\Omega_n\}_{n\geqslant 0}$, $\deg \Omega_n = n, n \geqslant 0$, with dual sequence $\{w_n\}_{n\geqslant 0}$, such that*

$$\Omega_n(x) = \sum_{\nu=n-\widehat{t}}^{n} \lambda_{n,\nu} B_\nu(x), \quad n \geqslant \widehat{t} + \widehat{\sigma} + 1, \qquad \lambda_{\widehat{r},\widehat{r}-\widehat{t}} \neq 0,$$

$$\Omega_n(x) = B_n(x), \quad 0 \leqslant n \leqslant \widehat{t} + \widehat{\sigma},$$

we have $w_k = u_k$, $0 \leqslant k \leqslant \widehat{\sigma}$.

The linear functional u is said to be quasi-definite if the principal submatrices $H_n = ((u)_{i+j})_{i,j=0}^n$ are non-singular, for all integer $n \geqslant 0$.

Assuming u quasi-definite, there exists a sequence of monic polynomials $\{B_n\}_{n\geqslant 0}$ such that (see [7])

(i) $\deg B_n = n, n \geqslant 0$
(ii) $\langle u, B_n B_m\rangle = r_n \delta_{n,m}$, with $r_n = \langle u, B_n^2\rangle \neq 0, n \geqslant 0$.

The sequence $\{B_n\}_{n\geqslant 0}$ is said to be the sequence of monic orthogonal polynomials, in short SMOP, with respect to the linear functional u.

If $\{B_n\}_{n\geqslant 0}$ is a SMOP, with respect to the quasi-definite linear functional u, then it is well known (see [10,12]) that its corresponding dual sequence $\{u_n\}_{n\geqslant 0}$, is

$$u_n = r_n^{-1} B_n u, \quad n \geqslant 0. \tag{2.1}$$

Or, equivalently, (see [7]), the sequence $\{B_n\}_{n\geqslant 0}$ satisfies a three-term recurrence relation

$$B_{n+1}(x) = (x - \beta_n)B_n(x) - \gamma_n B_{n-1}(x), \quad n \geqslant 0, \tag{2.2}$$

with $\gamma_n \neq 0, n \geqslant 1$ and $B_{-1}(x) = 0$, $B_0(x) = 1$.

Conversely, given a SMOP $\{B_n\}_{n\geqslant 0}$ generated by a recurrence relation as above, with $\gamma_n \neq 0, n \geqslant 1$, there exists an unique quasi-definite linear functional u such that the family is the corresponding SMOP. This result is known as the Favard Theorem (see [7]).

An important family of linear functionals is constituted by the semiclassical linear functionals, i.e., when u is quasi-definite and satisfies [3,12,17]

$$(Eu)' + Fu = 0. \tag{2.3}$$

Here (E, F) is an admissible pair of polynomials, i.e., the polynomial E is monic, $\deg E = t$, $\deg F = p \geqslant 1$, and if $p = t - 1$, then $(1/p!)F^{(p)}(0) \notin \mathbb{N}^*$.

The pair (E, F) is not unique (see [12]). Eq. (2.3) can be simplified if there exists a zero ξ of E such that

$$\begin{cases} E'(\xi) + F(\xi) = 0, \\ \langle u, \theta_\xi^2(E) + \theta_\xi(F)\rangle = 0. \end{cases} \tag{2.4}$$

After simplification by $x - \xi$, the linear functional u fulfils

$$(\theta_\xi(E)u)' + (\theta_\xi^2(E) + \theta_\xi(F))u = 0. \tag{2.5}$$

We define the class of u as the minimum value of $\max(\deg(E) - 2, \deg(F) - 1)$, for all admissible pairs (E, F). The pair (E, F) giving the class σ ($\sigma \geqslant 0$ because $\deg(F) \geqslant 1$) is unique [12].

When u is semiclassical of class σ, its corresponding SMOP is said to be semi-classical of class σ.

When $\sigma = 0$, i.e., $\deg E \leqslant 2$ and $\deg F = 1$, then u is classical (Hermite, Laguerre, Bessel, and Jacobi). For more details see [2,1,10,12,13].

3. Main results

For a semiclassical linear functional u of class σ, let (E, F) be the unique admissible pair of polynomials, with E monic, $\deg E = t$, $\deg F = p \geqslant 1$, such that $\sigma := \max(t - 2, p - 1)$.

Now, given $\{B_n\}_{n\geqslant 0}$ a SMOP with respect to u, we get

$$(EB_n)'(x) = \sum_{v=0}^{n+t-1} \lambda_{n,v} B_v(x), \quad n \geqslant \max(t - 1, 0), \tag{3.1}$$

where

$$\begin{aligned} \lambda_{n,n+t-1} &= n + t, \\ \lambda_{n,v} &= \langle u, B_v^2\rangle^{-1} \langle u, (EB_n)' B_v\rangle = \langle u, B_v^2\rangle^{-1} \langle u, (EB_nB_v)' - EB_nB_v'\rangle, \\ &= -\langle u, B_v^2\rangle^{-1} \langle EB_v'u + EB_vu', B_n\rangle, \quad 0 \leqslant v \leqslant n + t - 1. \end{aligned}$$

On the other hand, if $\deg(EB_n' - (E' + F)B_n) = n + \sigma(n) + 1, n \geqslant 0$, then we easily deduce

$$E(x)B_n'(x) - (E' + F)(x)B_n(x) = \sum_{v=0}^{n+\sigma(n)+1} \tilde{\lambda}_{n,v} B_v(x), \quad n \geqslant 0, \tag{3.2}$$

where

$$\tilde{\lambda}_{n,v} = \langle u, B_v^2\rangle^{-1} \langle u, (EB_n' - (E' + F)B_n)B_v\rangle, \quad 0 \leqslant v \leqslant n + \sigma(n) + 1. \tag{3.3}$$

Notice that the coefficient of $x^{n+\sigma(n)+1}$ in $E(x)B'_n(x)-(E'+F)(x)B_n(x)$, is

$$\tilde{\lambda}_{n,n+\sigma(n)+1}=\begin{cases}-(1/p!)F^{(p)}(0) & \text{if } p>t-1\ \ (\sigma=p-1),\\ n-t-(1/p!)F^{(p)}(0) & \text{if } p=t-1\ \ (\sigma=t-2=p-1),\\ n-t & \text{if } p<t-1\ \ (\sigma=t-2).\end{cases} \tag{3.4}$$

As a consequence, the admissibility condition of the pair (E,F) yields

$$\tilde{\lambda}_{n,n+\sigma(n)+1}\neq 0 \quad \text{either } n\geqslant t+1 \text{ or } n=t \text{ and } p\geqslant t-1.$$

Moreover, for each integer $n\geqslant t$,

$$\sigma(n)=\begin{cases}\sigma & \text{either } n\geqslant t+1 \text{ or } n=t \text{ and } p\geqslant t-1,\\ <\sigma & \text{if } n=t \text{ and } p<t-1.\end{cases} \tag{3.5}$$

Lemma 3.1. *For any monic polynomial E, $\deg E=t$, and any* SMOP $\{B_n\}_{n\geqslant 0}$ *with respect to u the following statements are equivalent.*

(i) *There exists a non-negative integer σ such that the polynomials B_n satisfy*

$$(EB_n)'(x)=\sum_{\nu=n-\sigma-1}^{n+t-1}\lambda_{n,\nu}B_\nu(x), \quad n\geqslant\sigma+1, \tag{3.6}$$

$$\lambda_{n,n-\sigma-1}\neq 0, \quad n\geqslant t+\sigma+2. \tag{3.7}$$

(ii) *There exists a polynomial F,* $\deg F=p\geqslant 1$, *such that*

$$(Eu)'+Fu=0, \tag{3.8}$$

where the pair (E,F) is admissible.

(iii) *There exist a non-negative integer σ and a polynomial F,* $\deg F=p\geqslant 1$, *such that*

$$E(x)B'_n(x)-(E'+F)(x)B_n(x)=\sum_{\nu=n-t+1}^{n+\sigma(n)+1}\tilde{\lambda}_{n,\nu}B_\nu(x), \quad n\geqslant \max(t-1,0), \tag{3.9}$$

$$\tilde{\lambda}_{n,n-t+1}\neq 0, \quad n\geqslant \max(t-1,0), \tag{3.10}$$

$$\tilde{\lambda}_{n,0}=-(u)_0^{-1}\langle u,B_n^2\rangle\lambda_{0,n}, \quad 0\leqslant n\leqslant \max(t-1,0), \tag{3.11}$$

where $\sigma=\max(t-2,p-1)$ *and the pair (E,F) is admissible.*

We can write

$$\tilde{\lambda}_{n,\nu}=-\frac{\langle u,B_n^2\rangle}{\langle u,B_\nu^2\rangle}\lambda_{\nu,n}, \quad 0\leqslant\nu\leqslant n+\sigma(n)+1, \quad n\geqslant 0. \tag{3.12}$$

Proof. Assume (i) holds. For each integer $n\geqslant 0$, we have

$$\langle Eu'_n,B_m\rangle=-\langle u_n,(EB_m)'\rangle=\sum_{\nu=m-\sigma-1}^{m+t-1}\lambda_{m,\nu}\delta_{n,\nu}, \quad m\geqslant 0.$$

Therefore,

$$\langle Eu'_n,B_m\rangle=\begin{cases}0 & \text{either } m\leqslant n-t \text{ or } m\geqslant n+\sigma+2,\\ -\lambda_{m,n} & n-t+1\leqslant m\leqslant n+\sigma+1.\end{cases}$$

From Lemma 2.1, $Eu_n' = -\sum_{\nu=n-t+1}^{n+\sigma+1} \lambda_{\nu,n} u_\nu$, $n \geqslant 0$ holds. Using (2.1), we get

$$E(B_n' u + B_n u') = -\Pi_{n+\sigma+1} u, \quad n \geqslant 0, \tag{3.13}$$

where

$$\Pi_{n+\sigma+1}(x) = \sum_{\nu=n-t+1}^{n+\sigma+1} \lambda_{\nu,n} \frac{\langle u, B_n^2 \rangle}{\langle u, B_\nu^2 \rangle} B_\nu(x), \quad n \geqslant 0, \tag{3.14}$$

with $\deg \Pi_{n+\sigma+1} \leqslant n+\sigma+1$, $n \geqslant 0$, and if $n \geqslant t+1$ then $\deg \Pi_{n+\sigma+1} = n+\sigma+1$.

Taking $n=0$ in (3.13), we obtain

$$(Eu)' + Fu = 0, \tag{3.15}$$

where $F(x) = -E'(x) + \Pi_{\sigma+1}(x)$.

Putting $p = \deg F$, and $q = \deg \Pi_{\sigma+1}$, then

$$p \leqslant \max(t-1, \sigma+1), \quad q \leqslant \min(\max(t-1, p), \sigma+1).$$

Necessarily, $p \geqslant 1$, otherwise $F(x) = (u)_0^{-1} \langle u, F \rangle = (u)_0^{-1} \langle (Eu)', 1 \rangle = 0$. This implies $(Eu)' = 0$. Then, $Eu = 0$. This yields $u = 0$ because $E \neq 0$ which contradicts the quasi-definiteness of u (see [12,16]).

Inserting (3.15) into (3.13) and using the quasi-definiteness of u, we get

$$E(x) B_n'(x) - \Pi_{\sigma+1}(x) B_n(x) = -\Pi_{n+\sigma+1}(x), \quad n \geqslant 0.$$

In particular, for $n \geqslant t+1$, the analysis of the degrees of both sides in the last relation yields

when $t-1 > q$, then $t-2 = p-1 = \sigma$ (which implies $t \geqslant 2$).
when $t-1 \leqslant q$, then $q = \sigma+1$, $p \leqslant \sigma+1$, hence $\max(t-2, p-1) \leqslant \sigma$, but $q \leqslant \max(t-1, p)$, therefore $\sigma = \max(t-2, p-1)$.

In all cases, we have $\sigma = \max(t-2, p-1)$.

Obviously, the pair (E, F) is admissible, indeed if $p = t-1$, then we have

$$n - t - (1/p!) F^{(p)}(0) \neq 0, \quad n \geqslant t+1,$$

i.e., $(1/p!) F^{(p)}(0) \notin \mathbb{N}^*$.

So, relations (3.9)–(3.11) are valid. Thus, we have proved that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (iii). We have $E(x) B_n'(x) - (E' + F)(x) B_n(x) = \sum_{\nu=0}^{n+\sigma(n)+1} \tilde{\lambda}_{n,\nu} B_\nu(x)$, $n \geqslant 0$. According to (3.8), we get

$$\begin{aligned} \langle u, (EB_n' - (E'+F)B_n) B_\nu \rangle &= \langle u, E(B_n B_\nu)' - EB_n B_\nu' - (E'+F) B_n B_\nu \rangle, \\ &= \langle -(Eu)', B_n B_\nu \rangle - \langle u, EB_n B_\nu' + (E'+F) B_n B_\nu \rangle, \\ &= -\langle u, (EB_\nu)' B_n \rangle, \end{aligned}$$

and from (3.3), $-\langle u, (EB_\nu)' B_n \rangle = \tilde{\lambda}_{n,\nu} \langle u, B_\nu^2 \rangle$, $0 \leqslant \nu \leqslant n+\sigma(n)+1$, $n \geqslant 0$. Thus, $\tilde{\lambda}_{n,\nu} = 0$, $0 \leqslant \nu \leqslant n-t$, $n \geqslant t$. Moreover, for $\nu = n+t-1$ and $n \geqslant \max(t-1, 0)$,

$$-\langle u, (EB_{n-t+1})' B_n \rangle = -(n+1) \langle u, B_n^2 \rangle = \tilde{\lambda}_{n,n-t+1} \langle u, B_{n-t+1}^2 \rangle,$$

for $\nu = 0$ and $0 \leqslant n \leqslant t-1$, $(t \geqslant 1)$,

$$\tilde{\lambda}_{n,0} = -(u)_0^{-1} \langle u, E' B_n \rangle = -(u)_0^{-1} \sum_{\nu=0}^{t-1} \lambda_{0,\nu} \langle u, B_\nu B_n \rangle = -(u)_0^{-1} \langle u, B_n^2 \rangle \lambda_{0,n}.$$

Then, $E(x)B_n'(x) - (E' + F)(x)B_n(x) = \sum_{\nu=n-t+1}^{n+\sigma(n)+1} \tilde{\lambda}_{n,\nu} B_\nu(x),\ \ n \geqslant \max(t-1, 0)$, where

$$\tilde{\lambda}_{n,n-t+1} \neq 0, \quad n \geqslant \max(t-1, 0),$$
$$\tilde{\lambda}_{n,0} = -(u)_0^{-1} \langle u, B_n^2 \rangle \lambda_{0,n}, \quad 0 \leqslant n \leqslant \max(t-1, 0).$$

(iii) $\Rightarrow$ (i). From (3.9) we get $\langle u_n, EB_m' - (E' + F)B_m \rangle = \sum_{\nu=0}^{m+\sigma(m)+1} \tilde{\lambda}_{m,\nu}\delta_{n,\nu}$, $n, m \geqslant 0$, i.e.,

$$\langle (Eu_n)' + (E' + F)u_n, B_m \rangle = -\sum_{\nu=0}^{m+\sigma(m)+1} \tilde{\lambda}_{m,\nu}\delta_{n,\nu}, \quad n, m \geqslant 0.$$

Taking $n = 0$, since $u = (u)_0 u_0$, we get

$$\langle (Eu)' + (E' + F)u, B_m \rangle = \begin{cases} 0 & m \geqslant t, \\ (u)_0 \tilde{\lambda}_{m,0} & 0 \leqslant m \leqslant t-1, \quad (t \geqslant 1). \end{cases}$$

From Lemma 2.1, (2.1) and (3.11), we get

$$\begin{aligned}(Eu)' + (E' + F)u &= -(u)_0 \sum_{\nu=0}^{t-1} \tilde{\lambda}_{\nu,0} u_\nu \\ &= -(u)_0 \sum_{\nu=0}^{t-1} \frac{\tilde{\lambda}_{\nu,0}}{\langle u, B_\nu^2 \rangle} B_\nu u = \sum_{\nu=0}^{t-1} \lambda_{0,\nu} B_\nu u = E'u.\end{aligned}$$

Thus, $(Eu)' + Fu = 0$.

On the other hand,

$$\langle (Eu_n)' + (E' + F)u_n, B_m \rangle = 0, \quad m \geqslant n + t,$$
$$\langle (Eu_n)' + (E' + F)u_n, B_m \rangle = -\tilde{\lambda}_{m,n}, \quad 0 \leqslant m \leqslant n + t - 1.$$

From Lemma 2.1, $(Eu_n)' + (E' + F)u_n = -\sum_{\nu=0}^{n+t-1} \tilde{\lambda}_{\nu,n} u_\nu$, $n \geqslant 0$. Using (2.1), $(EB_n u)' + (E' + F)B_n u = -\sum_{\nu=0}^{n+t-1} \tilde{\lambda}_{\nu,n} (\langle u, B_n^2 \rangle / \langle u, B_\nu^2 \rangle) B_\nu u$, $n \geqslant 0$. Since, $(Eu)' + Fu = 0$, then $((EB_n)' + \sum_{\nu=0}^{n+t-1} \tilde{\lambda}_{\nu,n} (\langle u, B_n^2 \rangle / \langle u, B_\nu^2 \rangle) B_\nu) u = 0$, $n \geqslant 0$. According to the quasi-definiteness of u, $(EB_n)'(x) = -\sum_{\nu=0}^{n+t-1} \tilde{\lambda}_{\nu,n} (\langle u, B_n^2 \rangle / \langle u, B_\nu^2 \rangle) B_\nu(x)$, $n \geqslant 0$.

From (3.9) to (3.11), we finally obtain $\tilde{\lambda}_{\nu,n} = 0$, for $0 \leqslant \nu \leqslant n - \sigma - 2$, $n \geqslant \sigma + 2$, and $\tilde{\lambda}_{n-\sigma-1,n} \neq 0$, for $\nu = n - \sigma - 1$, $n \geqslant t + \sigma + 2$. Hence, (i) follows taking into account (3.12). □

Notice that the equivalence of parts (i) and (ii) in the above Lemma are essentially contained in Proposition 7.10 of [12].

3.1. First characterization of semiclassical polynomials

Theorem 3.2. *For any monic polynomial E, with* $\deg E = t$, *and any* SMOP $\{B_n\}_{n \geqslant 0}$ *with respect to u, the following statements are equivalent.*

(i) *There exist a non-negative integer σ, an integer $p \geqslant 1$, and an integer $r \geqslant \sigma + t + 1$, with $\sigma = \max(t-2, p-1)$, such that*

$$\sum_{\nu=n-\sigma}^{n+t} \alpha_{n,\nu} B_\nu(x) = \sum_{\nu=n-t}^{n+t} \upsilon_{n,\nu} B_\nu^{[1]}(x), \quad n \geqslant \max(\sigma, t), \tag{3.16}$$

where

$$\alpha_{n,n+t}=\upsilon_{n,n+t}=1,\quad n\geqslant \max(\sigma,t),\qquad \alpha_{r,r-\sigma}\upsilon_{r,r-t}\neq 0,$$

$$\langle (Eu)', B_n\rangle = 0,\qquad p+1\leqslant n\leqslant \sigma+2t+1,\qquad \langle (Eu)', B_p\rangle \neq 0,$$

and if $p=t-1$, *then* $\langle u, B_p^2\rangle^{-1}\langle u, EB'_p\rangle \notin \mathbb{N}^*$, (*admissibility condition*).

(ii) *There exists a polynomial* F, $\deg F = p\geqslant 1$, *such that*

$$(Eu)' + Fu = 0, \tag{3.17}$$

where the pair (E, F) *is admissible*.

Proof. (i) $\Rightarrow$ (ii). Consider the SMP $\{\Omega_n\}_{n\geqslant 0}$ defined by

$$\Omega_{n+t+1}(x) = \sum_{\nu=n-t}^{n+t} \frac{n+t+1}{\nu+1}\upsilon_{n,\nu}B_{\nu+1}(x),\qquad n\geqslant \sigma+t+1, \tag{3.18}$$

$$\Omega_n(x) = B_n(x),\qquad 0\leqslant n\leqslant \sigma+2t+1. \tag{3.19}$$

From (3.16),

$$\Omega'_{n+t+1}(x) = (n+t+1)\sum_{\nu=n-\sigma}^{n+t}\alpha_{n,\nu}B_\nu(x),\qquad n\geqslant \sigma+t+1. \tag{3.20}$$

Taking into account u is quasi-definite, then

$$\begin{aligned}\langle (Eu)', \Omega_{n+t+1}\rangle &= -\langle u, E\Omega'_{n+t+1}\rangle\\ &= -(n+t+1)\sum_{\nu=n-\sigma}^{n+t}\alpha_{n,\nu}\langle u, EB_\nu\rangle = 0,\qquad n\geqslant \sigma+t+1.\end{aligned}$$

So, from the assumption, $\langle (Eu)', \Omega_n\rangle = 0$, $n\geqslant p+1$, and $\langle (Eu)', B_p\rangle \neq 0$. If we denote $\{w_n\}_{n\geqslant 0}$ the dual sequence of $\{\Omega_n\}_{n\geqslant 0}$, and we apply Lemma 2.1, then we get

$$(Eu)' = \sum_{\nu=1}^{p}\langle (Eu)', B_\nu\rangle w_\nu. \tag{3.21}$$

On the other hand, if we take $\widehat{t}=2t$, $\widehat{\sigma}=\sigma+1$, and $\widehat{r}=r+t+1$, the expressions (3.18), and (3.19) can be written as follows

$$\Omega_n(x) = \sum_{\nu=n-\widehat{t}}^{n}\tilde{\upsilon}_{n,\nu}B_\nu(x),\qquad n\geqslant \widehat{\sigma}+\widehat{t}+1,$$

$$\Omega_n(x) = B_n(x),\qquad 0\leqslant n\leqslant \widehat{\sigma}+\widehat{t},$$

where

$$\tilde{\upsilon}_{n,\nu} = \frac{n}{\nu}\upsilon_{n-t-1,\nu-1},\qquad n-\widehat{t}\leqslant \nu\leqslant n,\quad n\geqslant \widehat{\sigma}+\widehat{t}+1,$$

$$\tilde{\upsilon}_{\widehat{r},\widehat{r}-\widehat{t}} = \frac{r+t+1}{r-t+1}\upsilon_{r,r-t}\neq 0\qquad (\widehat{r}\geqslant \sigma+2t+2=\widehat{\sigma}+\widehat{t}+1).$$

From Lemma 2.2, and (2.1), it follows $w_k = u_k = (B_k/\langle u, B_k^2\rangle)u$, $0\leqslant k\leqslant \widehat{\sigma}=\sigma+1$. So, relation (3.21) becomes

$$(Eu)' + Fu = 0, \tag{3.22}$$

where $F(x) = \sum_{\nu=1}^{p}(\langle u, EB'_\nu\rangle/\langle u, B_\nu^2\rangle)B_\nu(x)$, with $\deg F = p$, as we have $\langle u, EB'_p\rangle \neq 0$.

From the assumption, $(1/p!)F^{(p)}(0)=\langle u_0, B_p^2\rangle^{-1}\langle u, EB'_p\rangle \notin \mathbb{N}^*$, for $p=t-1$. Hence, the pair (E, F) is admissible with associated integer σ.

(ii) $\Rightarrow$ (i). From Lemma 3.1(i), and making $n \to n+1$ we have

$$(EB_{n+1})'(x) = \sum_{v=n-\sigma}^{n+t} \lambda_{n+1,v} B_v(x), \quad n \geqslant \sigma, \tag{3.23}$$

where

$$\lambda_{n+1,n+t} = n+t+1, \quad n \geqslant \sigma, \tag{3.24}$$

$$\lambda_{n+1,n-\sigma} \neq 0, \quad n \geqslant t+\sigma+1. \tag{3.25}$$

On the other hand, the orthogonality of $\{B_n\}_{n\geqslant 0}$ allows to write

$$E(x)B_{n+1}(x) = \sum_{v=n-t}^{n+t} \frac{\langle u, EB_{n+1}B_{v+1}\rangle}{\langle u, B_{v+1}^2\rangle} B_{v+1}(x), \quad n \geqslant t-1.$$

Hence,

$$(EB_{n+1})'(x) = \sum_{v=n-t}^{n+t} \frac{(v+1)\langle u, EB_{n+1}B_{v+1}\rangle}{\langle u, B_{v+1}^2\rangle} B_v^{[1]}(x), \quad n \geqslant t. \tag{3.26}$$

From (3.23) and (3.26), we obtain (3.16) with

$$\alpha_{n,v} = \frac{\lambda_{n+1,v}}{n+t+1}, \quad n-\sigma \leqslant v \leqslant n+t, \tag{3.27}$$

$$\upsilon_{n,v} = \frac{(v+1)\langle u, EB_{n+1}B_{v+1}\rangle}{(n+t+1)\langle u, B_{v+1}^2\rangle}, \quad n-t \leqslant v \leqslant n+t, \tag{3.28}$$

$$\alpha_{n,n-\sigma}\upsilon_{n,n-t} = \frac{(n-t+1)}{(n+t+1)^2}\frac{\langle u, B_{n+1}^2\rangle}{\langle u, B_{n-t+1}^2\rangle}\lambda_{n+1,n-\sigma} \neq 0, \quad n \geqslant \sigma+t+1. \tag{3.29}$$

Then,

$$\langle (Eu)', B_n\rangle = -\langle u, FB'_n\rangle = \begin{cases} 0, & p+1 \leqslant n \leqslant \sigma+2t+1, \\ -(1/p!)F^{(p)}(0)\langle u, B_p^2\rangle \neq 0, & n = p = \deg F, \end{cases}$$

and the admissibility of (E, F) yields $\langle u, B_p^2\rangle^{-1}\langle u, EB'_p\rangle \notin \mathbb{N}^*$, if $p = t-1$. □

In the case of classical linear functionals, we get the following result.

Corollary 3.3. *Let* $\{B_n\}_{n\geqslant 0}$ *be a* SMOP *with respect to* u, *and a monic polynomial* E, $\deg E = t \leqslant 2$, *such that* $\langle u, E\rangle \neq 0$, *the following statements are equivalent.*

(i) *The linear functional u is classical, i.e. there exists a polynomial* F, *with* $\deg F = 1$, *such that* $(Eu)' + Fu = 0$.
(ii) $\sum_{v=n}^{n+t} \alpha_{n,v}B_v(x) = \sum_{v=n-t}^{n+t} \upsilon_{n,v}B_v^{[1]}(x), n \geqslant t$, *where* $\alpha_{n,n+t} = \upsilon_{n,n+t} = 1, n \geqslant t$. *Furthermore, there exists an integer* $r \geqslant t+1$ *such that* $\alpha_{r,r}\upsilon_{r,r-t} \neq 0$, *and if* $t=2$ *then* $\langle u, B_1^2\rangle^{-1}\langle u, E\rangle \notin \mathbb{N}^*$ (*admissibility condition*).

3.2. Second characterization of semiclassical polynomials

From the previous characterization, we do not deduce the second structure relation of classical orthogonal polynomials $(*)$. Our goal is to establish the characterization that allows us to recover such a case.

First, we have the following result.

Proposition 3.4. *For any monic polynomial E, $\deg E = t$, and any* SMOP $\{B_n\}_{n\geqslant 0}$ *with respect to u, the following statements are equivalent.*

(i) *There exists a polynomial F, $\deg F = p \geqslant 1$, such that*

$$(Eu)' + Fu = 0, \tag{3.30}$$

where the pair (E, F) is admissible.

(ii) *There exist a non-negative integer σ and a polynomial F, $\deg F = p \geqslant 1$, such that*

$$E(x)B_n''(x) - F(x)B_n'(x) - (E'' + F')(x)B_n(x) = \sum_{\nu=n-\sigma}^{n+\sigma(n)} \vartheta_{n,\nu} B_\nu(x), \quad n \geqslant \sigma, \tag{3.31}$$

where

$$\vartheta_{n,n-\sigma} \neq 0 \quad \textit{either } n \geqslant \sigma + t + 1 \ \textit{ or } \ n = \sigma + t \ \textit{ and } \ p \geqslant t - 1, \tag{3.32}$$

$$\vartheta_{n,0} = (u)_0^{-1} \langle u, B_n^2 \rangle \vartheta_{0,n}, \quad 0 \leqslant n \leqslant \sigma, \tag{3.33}$$

where $\sigma = \max(t-2, p-1)$ and the pair (E, F) is admissible. We can write

$$\vartheta_{n,\nu} = \frac{\langle u, B_n^2 \rangle}{\langle u, B_\nu^2 \rangle} \vartheta_{\nu,n}, \quad 0 \leqslant \nu \leqslant n + \sigma(n), \ \ n \geqslant 0. \tag{3.34}$$

Proof. We have

$$E(x)B_n''(x) - F(x)B_n'(x) - (E'' + F')(x)B_n(x) = \sum_{\nu=0}^{n+\sigma(n)} \vartheta_{n,\nu} B_\nu(x), \quad n \geqslant 0, \tag{3.35}$$

where for all integers $0 \leqslant \nu \leqslant n + \sigma(n)$, and $n \geqslant 0$,

$$\begin{aligned}
\langle u, B_\nu^2 \rangle \vartheta_{n,\nu} &= \langle u, (EB_n'' - FB_n' - (E'' + F')B_n)B_\nu \rangle, \\
&= \langle B_\nu u, EB_n'' - FB_n' - (E'' + F')B_n \rangle, \\
&= \langle (EB_\nu u)'' + (FB_\nu u)' - (E'' + F')B_\nu u, B_n \rangle.
\end{aligned}$$

From (3.30),

$$\begin{aligned}
&(EB_\nu u)'' + (FB_\nu u)' - (E'' + F')B_\nu u \\
&\quad = B_\nu'' Eu + 2B_\nu'(Eu)' + B_\nu(Eu)'' + B_\nu(Fu)' + B_\nu' Fu - (E'' + F')B_\nu u, \\
&\quad = (EB_\nu'' - FB_\nu' - (E'' + F')B_\nu)u.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle u, B_\nu^2 \rangle \vartheta_{n,\nu} &= \langle u, (EB_\nu'' - FB_\nu' - (E'' + F')B_\nu)B_n \rangle, \quad \text{and inserting (3.35)} \\
&= \sum_{i=0}^{\nu+\sigma(\nu)} \vartheta_{\nu,i} \langle u, B_n^2 \rangle \delta_{i,\nu}, \\
&= \vartheta_{\nu,n} \langle u, B_n^2 \rangle.
\end{aligned}$$

In particular, for $0 \leqslant \nu \leqslant n - \sigma - 1$, then $n \geqslant \nu + \sigma + 1 \geqslant \nu + \sigma(\nu) + 1$. Thus, we deduce $\vartheta_{\nu,n} = 0$. Hence, $\vartheta_{n,\nu} = 0$, $0 \leqslant \nu \leqslant n - \sigma - 1$.

For $v = n - \sigma$, and $n \geqslant \sigma + t$, we obtain

$$\begin{aligned}
\langle u, B_{n-\sigma}^2\rangle \vartheta_{n,n-\sigma} &= \langle u, (EB'_{n-\sigma} - (E' + F)B_{n-\sigma})' B_n\rangle, \\
&= \sum_{v=0}^{n+1} \tilde{\lambda}_{n-\sigma,v} \langle u, B'_v B_n\rangle, \\
&= (n+1)\tilde{\lambda}_{n-\sigma,n+1}\langle u, B_n^2\rangle \quad \text{taking into account (3.2).}
\end{aligned}$$

But, from (3.4) to (3.5), we get $\vartheta_{n,n-\sigma} \neq 0$, either $n \geqslant \sigma + t + 1$, or $n = \sigma + t$ and $p \geqslant t - 1$.

As a consequence,

$$E(x)B_n''(x) - F(x)B_n'(x) - (E'' + F')(x)B_n(x) = \sum_{v=n-\sigma}^{n+\sigma(n)} \vartheta_{n,v} B_v(x), \quad n \geqslant \sigma,$$

where $\vartheta_{n,n-\sigma} \neq 0$, either $n \geqslant \sigma + t + 1$, or $n = \sigma + t$ and $p \geqslant t - 1$.

Thus, we have proved (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). From (3.31) we have

$$\begin{aligned}
&\langle (Eu)'' + (Fu)' - (E'' + F')u, B_n\rangle = 0, \quad n \geqslant \sigma + 1, \\
&\langle (Eu)'' + (Fu)' - (E'' + F')u, B_n\rangle = (u)_0 \vartheta_{n,0} = \langle u, B_n^2\rangle \vartheta_{0,n}, \quad 0 \leqslant n \leqslant \sigma.
\end{aligned}$$

According to Lemma 2.1,

$$\begin{aligned}
(Eu)'' + (Fu)' - (E'' + F')u &= \sum_{n=0}^{\sigma} \frac{(u)_0 \vartheta_{n,0}}{\langle u, B_n^2\rangle} B_n u, \\
&= \sum_{n=0}^{\sigma(0)} \vartheta_{0,n} B_n u = -(E'' + F')u.
\end{aligned}$$

Then $((Eu)' + Fu)' = 0$ or, equivalently, $(Eu)' + Fu = 0$.

Finally, since $\sigma(n) = \sigma$ and $\vartheta_{n,n+\sigma} = (n + \sigma + 1)\tilde{\lambda}_{n,n+\sigma+1} \neq 0$, for $n \geqslant t + 1$, then $\tilde{\lambda}_{n,n+\sigma+1} \neq 0$, $n \geqslant t + 1$. This yields the admissibility of the pair (E, F) taking into account (3.4). $\square$

Notice that this result appears as Proposition 7.16 in [12] without proof.

Our main result is the next one.

Theorem 3.5. *For any monic polynomial E, with* $\deg E = t$, *and any* SMOP $\{B_n\}_{n\geqslant 0}$ *with respect to u, the following statements are equivalent.*

(i) *There exist a non-negative integer σ, an integer $p \geqslant 1$, and an integer $r \geqslant \sigma + t + 1$, with $\sigma = \max(t - 2, p - 1)$, such that*

$$\sum_{v=n-\sigma}^{n+\sigma} \xi_{n,v} B_v(x) = \sum_{v=n-t}^{n+\sigma} \varsigma_{n,v} B_v^{[1]}(x), \tag{3.36}$$

where

$$\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1, \quad n \geqslant \max(\sigma, t + 1), \quad \xi_{r,r-\sigma}\varsigma_{r,r-t} \neq 0,$$

$$\langle (Eu)', B_n\rangle = 0, \quad p + 1 \leqslant n \leqslant 2\sigma + t + 1, \quad \langle (Eu)', B_p\rangle \neq 0,$$

and if $p = t - 1$, then $\langle u, B_p^2\rangle^{-1}\langle u, EB'_p\rangle \notin \mathbb{N}^*$ (admissibility condition).

(ii) *There exists a polynomial* F, $\deg F = p \geqslant 1$, *such that*

$$(Eu)' + Fu = 0, \tag{3.37}$$

where the pair (E, F) *is admissible*.

Proof. (i) $\Rightarrow$ (ii). Consider the SMP $\{\Pi_n\}_{n\geqslant 0}$ given by

$$\Pi_{n+\sigma+1}(x) = \sum_{v=n-t}^{n+\sigma} \frac{n+\sigma+1}{v+1} \varsigma_{n,v} B_{v+1}(x), \quad n \geqslant \sigma + t + 1, \tag{3.38}$$

$$\Pi_n(x) = B_n(x), \quad 0 \leqslant n \leqslant 2\sigma + t + 1. \tag{3.39}$$

We can easily deduce

$$\Pi'_{n+\sigma+1}(x) = (n+\sigma+1) \sum_{v=n-\sigma}^{n+\sigma} \xi_{n,v} B_v(x), \quad n \geqslant \sigma + t + 1. \tag{3.40}$$

Taking into account the linear functional u is quasi-definite, we get

$$\begin{aligned} \langle (Eu)', \Pi_{n+\sigma+1} \rangle &= -\langle u, E\Pi'_{n+\sigma+1} \rangle, \\ &= -(n+\sigma+1) \sum_{v=n-\sigma}^{n+\sigma} \xi_{n,v} \langle u, EB_v \rangle = 0, \quad n \geqslant \sigma + t + 1. \end{aligned}$$

From the assumption and Lemma 2.1, we get $\langle (Eu)', \Pi_n \rangle = 0$, $n \geqslant p+1$, and $\langle (Eu)', B_p \rangle \neq 0$. If we denote $\{w_n\}_{n\geqslant 0}$ the dual sequence of $\{\Pi_n\}_{n\geqslant 0}$, then we get

$$(Eu)' = \sum_{v=1}^{p} \langle (Eu)', B_v \rangle w_v. \tag{3.41}$$

Taking $\widehat{t} = \sigma + t$, $\widehat{\sigma} = \sigma + 1$, and $\widehat{r} = r + \sigma + 1$, (3.38), and (3.39) can be written as follows:

$$\Pi_n(x) = \sum_{v=n-\widehat{t}}^{n} \tilde{\varsigma}_{n,v} B_v(x), \quad n \geqslant \widehat{\sigma} + \widehat{t} + 1,$$
$$\Pi_n(x) = Bn(x), \quad 0 \leqslant n \leqslant \widehat{\sigma} + \widehat{t},$$

where

$$\tilde{\varsigma}_{n,v} = \frac{n}{v} \varsigma_{n-\sigma-1,v-1}, \quad n - \widehat{t} \leqslant v \leqslant n, \ \ n \geqslant \widehat{\sigma} + \widehat{t} + 1,$$

$$\tilde{\varsigma}_{\widehat{r},\widehat{r}-\widehat{t}} = \frac{r+\sigma+1}{r-\sigma+1} \varsigma_{r,r-t} \neq 0 \quad (\widehat{r} \geqslant 2\sigma + t + 2 \geqslant \widehat{\sigma} + \widehat{t} + 1).$$

From Lemma 2.2, and (2.1), it follows $w_k = u_k = (B_k / \langle u, B_k^2 \rangle) u$, $0 \leqslant k \leqslant \widehat{\sigma} = \sigma + 1$. So, (3.41) becomes

$$(Eu)' + Fu = 0, \tag{3.42}$$

where $F(x) = \sum_{v=1}^{p} (\langle u, EB'_v \rangle / \langle u, B_v^2 \rangle) B_v(x)$. Since, $\langle u, EB'_p \rangle \neq 0$, then $\deg F = p$.

From the assumption, if $p = t - 1$, then $(1/p!) F^{(p)}(0) = \langle u, B_p^2 \rangle^{-1} \langle u, EB'_p \rangle \notin \mathbb{N}^*$. Hence, the pair (E, F) is admissible with associated integer σ.

(ii) $\Rightarrow$ (i). From Lemma 3.1(iii), there exists a polynomial F, $\deg F = p \geqslant 1$, such that

$$E(x) B'_n(x) - (E' + F)(x) B_n(x) = \sum_{v=n-t+1}^{n+\sigma(n)+1} \tilde{\lambda}_{n,v} B_v(x), \quad n \geqslant \max(t-1, 0), \tag{3.43}$$

where

$$\tilde{\lambda}_{n,n-t+1} \neq 0, \quad n \geqslant \max(t-1, 0), \tag{3.44}$$

$$\tilde{\lambda}_{n,0} = -(u)_0^{-1}\langle u, B_n^2\rangle \lambda_{0,n}, \quad 0 \leqslant n \leqslant \max(t-1, 0), \tag{3.45}$$

where $\sigma = \max(t-2, p-1)$ and the pair (E, F) is admissible.

Differentiating both sides of (3.43), we get

$$E(x)B_n''(x) - F(x)B_n'(x) - (E'' + F')(x)B_n(x) = \sum_{v=n-t}^{n+\sigma(n)} \zeta_{n,v} B_v^{[1]}(x), \quad n \geqslant t, \tag{3.46}$$

where $\zeta_{n,v} = (v+1)\tilde{\lambda}_{n,v+1}$, $0 \leqslant v \leqslant n + \sigma(n)$, $n \geqslant t$.

From (3.31) and (3.46), we obtain (3.36) where

$$\xi_{n,v} = \frac{\vartheta_{n,v}}{\vartheta_{n,n+\sigma}}, \quad n - \sigma \leqslant v \leqslant n + \sigma, \tag{3.47}$$

$$\varsigma_{n,v} = \frac{(v+1)\tilde{\lambda}_{n,v+1}}{\vartheta_{n,n+\sigma}}, \quad n - t \leqslant v \leqslant n + t, \tag{3.48}$$

$$\xi_{n,n-\sigma}\varsigma_{n,n-t} = \frac{(n-t+1)}{\vartheta_{n,n+\sigma}^2}\vartheta_{n,n-\sigma}\tilde{\lambda}_{n,n-t+1} \neq 0, \quad n \geqslant \sigma + t + 1. \tag{3.49}$$

Finally,

$$\langle (Eu)', B_n\rangle = -\langle u, FB_n'\rangle = \begin{cases} 0 & p+1 \leqslant n \leqslant 2\sigma + t + 1, \\ -(1/p!)F^{(p)}(0)\langle u, B_p^2\rangle \neq 0, & n = p = \deg F. \end{cases}$$

From the admissibility of the pair (E, F), if $p = t-1$, then $\langle u, EB'_p\rangle / \langle u, B_p^2\rangle \notin \mathbb{N}^*$. $\square$

4. Examples

In order to illustrate the result of Theorem 3.5, we study two examples.

4.1. First example

Let $\{Q_n\}_{n\geqslant 0}$ be a SMOP with respect to the linear functional v solution of the functional equation [11]

$$v' + \psi v = 0, \tag{4.1}$$

where $\psi(x) = -\mathrm{i}x^2 + 2x - \mathrm{i}(\alpha - 1)$, with quasi-definiteness conditions $\alpha \notin \bigcup_{n\geqslant 0} E_n$, where $E_0 = \{\alpha \in \mathbb{C} : F(\alpha) = 0\}$, $F(\alpha) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}x^3/3 - x^2 + \mathrm{i}(\alpha-1)x}\,\mathrm{d}x$, and for each integer $n \geqslant 1$, $E_n = \{\alpha \in \mathbb{C} : \Delta_n(\alpha) = 0\}$. Here, $\Delta_n(\alpha)$ is the Hankel determinant of dimension n associated with v. Notice that v is semiclassical of class one.

The recurrence coefficients β_n and γ_{n+1}, $n \geqslant 0$, for the sequence $\{Q_n\}_{n\geqslant 0}$ are determined by the system

$$\begin{cases} \dfrac{n+1}{\gamma_{n+1}} = 2 - \mathrm{i}(\beta_n + \beta_{n+1}), & n \geqslant 0, \\ \mathrm{i}(\gamma_{n+2} + \gamma_{n+1}) = \psi(\beta_{n+1}), & n \geqslant 0, \\ \gamma_1 = -\mathrm{i}\psi(\beta_0), & \beta_0 = -\mathrm{i}\dfrac{F'(\alpha)}{F(\alpha)}. \end{cases} \tag{4.2}$$

The sequence $\{Q_n\}_{n\geqslant 0}$ is characterized by one of the two following relations (see [15])

- *First structure relation*

$$Q_n^{[1]}(x) = Q_n(x) + \lambda_{n,n-1} Q_{n-1}(x), \quad n \geqslant 1, \tag{4.3}$$

  where $\lambda_{n,n-1} = -\mathrm{i}\gamma_n\gamma_{n+1}/(n+1)$, $n\geqslant 1$.
- *Second structure relation*

$$(x + v_{n,0}) Q_n(x) = Q_{n+1}^{[1]}(x) + \varrho_n Q_n^{[1]}(x), \quad n \geqslant 0, \tag{4.4}$$

  where

$$v_{n,0} = \frac{\mathrm{i}(n+1)}{\gamma_{n+1}} - \frac{\mathrm{i}\gamma_{n+1}\gamma_{n+2}}{n+2} - \beta_n, \;\; n\geqslant 1, \quad v_{0,0} = -\frac{\mathrm{i}\gamma_1\gamma_2}{2} - \beta_0,$$

$$\varrho_n = \frac{\mathrm{i}(n+1)}{\gamma_{n+1}}, \quad n \geqslant 1, \;\; \varrho_0 = 0.$$

Lemma 4.1. *Let* $\{Q_n\}_{n\geqslant 0}$ *be the* SMOP *with respect to the linear functional* v *satisfying* (4.1), *then*

(i) *The sequence* $\{Q_n\}_{n\geqslant 0}$ *is not diagonal.*
(ii) *The coefficients* $v_{n,0}$ *depend on* n.

Proof. Assume $\{Q_n\}_{n\geqslant 0}$ is diagonal with respect to ϕ, $\deg \phi = t$, and index s. Then (see [14,15]), $(t/2) \leqslant s \leqslant t+2$ and we have the following diagonal relation:

$$\phi(x) Q_n(x) = \sum_{\nu=n-s}^{n+t} \theta_{n,\nu} Q_\nu^{[1]}(x), \;\; n \geqslant s, \quad \theta_{n,n-s} \neq 0, \;\; n\geqslant s.$$

If we denote by $\{v_n\}_{n\geqslant 0}$ and $\{v_n^{[1]}\}_{n\geqslant 0}$ the dual sequences of $\{Q_n\}_{n\geqslant 0}$ and $\{Q_n^{[1]}\}_{n\geqslant 0}$, respectively, then $v = \lambda v_0$, with $\lambda = (v)_0 \neq 0$, and the last relation is equivalent (see [15]) to the following relation that we will also call diagonal relation,

$$\phi v_n^{[1]} = \; k_n \Omega_{n+s} v, \quad n \geqslant 0, \tag{4.5}$$

where

$$k_n = \frac{\theta_{n+s,n}}{\langle v, Q_{n+s}^2\rangle},$$

and

$$\Omega_{n+s}(x) = \sum_{\nu=0}^{n+s} \frac{\theta_{\nu,n}}{\theta_{n+s,n}} \frac{\langle v, Q_{n+s}^2\rangle}{\langle v, Q_\nu^2\rangle} Q_\nu(x), \quad n \geqslant 0.$$

It is clear that v satisfies an infinite number of relations as (4.5). Indeed, by multiplying both hand sides of (4.5) by a monic polynomial, we get another diagonal relation.

For this reason, we will assume $t = \deg \phi$. It is the minimum integer number such that v satisfies diagonal relations as (4.5). Likely, (4.5) cannot be simplified.

Notice that $t \geqslant 1$. Otherwise if we suppose that $t = 0$, then $0 \leqslant s \leqslant 2$ and we recover the first structure relation characterizing classical sequences. This contradicts the fact that the sequence $\{Q_n\}_{n\geqslant 0}$ is semiclassical of class one.

Consequently, since $t \geqslant 1$ and $s \geqslant (t/2)$ then $s \geqslant 1$.

Now, differentiating both hand sides of (4.5) and using (2.1), from the functional (4.1) and $(v_n^{[1]})' = -(n+1)v_{n+1} = -(n+1)\langle v, Q_{n+1}^2\rangle^{-1} Q_{n+1} v$, $n \geqslant 0$, we obtain

$$\tilde{\phi} v_n^{[1]} = k_n \Pi_{n+s-1} v, \tag{4.6}$$

where

$$\tilde{\phi}(x) = t^{-1}\phi'(x), \quad \Pi_{n+s-1}(x) = t^{-1}(\Omega'_{n+s}(x) - \Omega_{n+s}(x)\psi(x) + d_n\phi(x)Q_{n+1}(x)),$$

$$d_n = (n+1)(\langle v, Q^2_{n+1}\rangle k_n)^{-1}, \quad n \geqslant 0.$$

Notice that the polynomials Π_{n+s-1} and $\tilde{\phi}$ are monic, with deg $\tilde{\phi} = t - 1 \geqslant 0$.

In order to determine the degree of Π_{n+s-1}, we proceed in two steps. First, we multiply (4.5) by ϕ, and using (4.6) as well as the quasi-definiteness of u, we obtain $\tilde{\phi}(x)\Omega_{n+s}(x) = \phi(x)\Pi_{n+s-1}(x)$. Second, analyzing the highest degree of the last relation, we get deg $\Pi_{n+s-1} = n+s-1$, $n \geqslant 0$. But, this contradicts the fact that $t = \deg \phi$. It is the minimum integer number such that v satisfies diagonal relations as (4.5).

Hence, (i) holds.

The assertion (ii) is a straightforward consequence of (i) and (4.4). □

From the orthogonality of $\{Q_n\}_{n\geqslant 0}$, relation (4.4) can be rewritten as follows:

$$Q_{n+1}(x) + (\beta_n + v_{n,0})Q_n(x) + \gamma_n Q_{n-1}(x) = Q^{[1]}_{n+1}(x) + \varrho_n Q^{[1]}_n(x), \quad n \geqslant 0, \tag{4.7}$$

according to Theorem 3.5, taking $E(x) = 1$ and $F(x) = \psi(x)$.

4.2. *Second example (Bachène [3])*

Consider the Hermite generalized orthogonal polynomial sequence $\{H_n(x;\mu)\}_{n\geqslant 0}$. We remind that it satisfies the following recurrence relation:

$$\begin{cases} H_{n+1}(x;\mu) = xH_n(x;\mu) - \zeta_n H_{n-1}(x;\mu), & n \geqslant 0, \\ H_0(x;\mu) = 1, \end{cases} \tag{4.8}$$

where $\zeta_0 = 0$, and $\zeta_n = \frac{1}{2}(n + \mu(1 - (-1)^n))$, $n \geqslant 1$.

The quasi-definiteness condition is $\mu \neq -n - (1/2)$, $n \geqslant 0$. Further, we are interested in the family of class one, i.e., $\mu \neq 0$. Notice that the Hermite sequence is obtained for $\mu = 0$.

The sequence $\{H_n(x;\mu)\}_{n\geqslant 0}$ is orthogonal with respect to the linear functional $\mathscr{H}(\mu)$ satisfying

$$(x\mathscr{H}(\mu))' + (2x^2 - (2\mu+1))\mathscr{H}(\mu) = 0. \tag{4.9}$$

It is well-known that $\{H_n(x;\mu)\}_{n\geqslant 0}$ is characterized by its orthogonality as well as by the following first structure relation:

$$xH^{[1]}_n(x;\mu) = H_{n+1}(x;\mu) + \omega_n H_{n-1}(x;\mu), \quad n \geqslant 0, \tag{4.10}$$

where $\omega_0 = 0$, $\omega_n = 2(n+1)^{-1}\zeta_n\zeta_{n+1}$, $n \geqslant 1$.

From Theorem 3.5, (4.9), and the symmetry of the polynomial sequence, i.e., $H_n(-x;\mu) = (-1)^n H_n(x;\mu)$, $n \geqslant 0$, the Hermite generalized polynomial sequence $\{H_n(x;\mu)\}_{n\geqslant 0}$ satisfies

$$H_{n+1}(x;\mu) + \xi_{n,n-1}H_{n-1}(x;\mu) = H^{[1]}_{n+1}(x;\mu) + \varsigma_{n,n-1}H^{[1]}_{n-1}(x;\mu), \quad n \geqslant 1, \tag{4.11}$$

where

$$\xi_{2n,2n-1} = \varsigma_{2n,2n-1} = 0, \quad n \geqslant 1,$$

$$\xi_{2n+1,2n} = \frac{(n+1)(2n+2\mu+1)}{2n+3}, \quad n \geqslant 1, \quad \varsigma_{2n+1,2n} = \frac{(n+1)(2n+1)}{2n+3}, \quad n \geqslant 1.$$

Notice that the Hermite generalized sequence can not satisfy a second structure relation

$$(x + a_n)H_n(x;\mu) = H^{[1]}_{n+1}(x;\mu) + b_n H^{[1]}_n(x;\mu) + c_n H^{[1]}_{n-1}(x;\mu), \quad n \geqslant 0, \tag{4.12}$$

where a_n, b_n, and c_n, $n \geqslant 0$, are complex numbers, with $c_0 = 0$.

This is the aim of the following Lemma.

Lemma 4.2. *For $\mu \in \mathbb{C}^*$ and $\mu \neq -n - (\frac{1}{2})$, $n \geqslant 0$, the Hermite generalized sequence $\{H_n(x;\mu)\}_{n\geqslant 0}$ does not satisfy a relation as* (4.12).

Proof. Since the polynomial sequence $\{H_n(x;\mu)\}_{n\geqslant 0}$ is symmetric, we can assume that $a_n = b_n = 0$, $n \geqslant 0$. Indeed, from (4.12), we get

$$(-x + a_n)H_n(-x;\mu) = H_{n+1}^{[1]}(-x;\mu) + b_n H_n^{[1]}(-x;\mu) + c_n H_{n-1}^{[1]}(-x;\mu), \quad n \geqslant 0.$$

The symmetry of the sequences $\{H_n(x;\mu)\}_{n\geqslant 0}$ and $\{H_n^{[1]}(x;\mu)\}_{n\geqslant 0}$ give

$$(x - a_n)H_n(x;\mu) = H_{n+1}^{[1]}(x;\mu) - b_n H_n^{[1]}(x;\mu) + c_n H_{n-1}^{[1]}(x;\mu), \quad n \geqslant 0. \tag{4.13}$$

From (4.12) and (4.13), we easily deduce

$$xH_n(x;\mu) = H_{n+1}^{[1]}(x;\mu) + c_n H_{n-1}^{[1]}(x;\mu), \quad n \geqslant 0. \tag{4.14}$$

So, this justify our assumption.

Further, multiplying both sides of (4.14) by x and taking into account (4.8) and (4.10), we get

$$\begin{cases} \omega_{n+1} + c_n = \zeta_n + \zeta_{n+1}, & n \geqslant 0, \\ c_n \omega_{n-1} = \zeta_n \zeta_{n-1}, & n \geqslant 2. \end{cases}$$

For $n = 3$, $c_3\omega_2 = \zeta_3\zeta_2$, and $\omega_4 + c_3 = \zeta_3 + \zeta_4$. Then, $\omega_4 + \omega_2^{-1}\zeta_3\zeta_2 = \zeta_3 + \zeta_4$. But, from (4.8) and (4.10), we get $\mu = 0$. This yields a contradiction. □

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