Spectral transformations for Hermitian Toeplitz matrices

Leyla Daruis[a], Javier Hernández[b], Francisco Marcellán[b,*]

[a]*Departamento de Análisis Matemático, Universidad de La Laguna, Spain*
[b]*Departamento de Matemáticas, Universidad Carlos III de Madrid, Spain*

Abstract

In this paper we deal with some perturbations of probability measures supported on the unit circle as well as, in a more general framework, with Hermitian linear functionals. We focus our attention in the Hessenberg matrix associated with the multiplication operator in terms of an orthogonal basis in the linear space of polynomials with complex coefficients. The LU and QR factorizations of such a matrix are introduced. Then, the connection between the above-mentioned perturbations and such factorizations is presented.

1. Introduction

Given a linear functional u in the linear space $\mathscr{P}$ of polynomials with real coefficients we define an inner product

$$\langle p, q \rangle = u(pq), \quad p, q \in \mathscr{P}.$$

Notice that the Gram matrix G of this inner product with respect to the canonical basis $\{x^n\}_{n \geqslant 0}$ is a Hankel matrix, i.e., $\langle x^m, x^n \rangle = u(x^{m+n})$. This means that the antidiagonals of G have the same entries.

If the leading principal submatrices of G are nonsingular, then the linear functional u is said to be quasi-definite. In such a case, there exists a sequence $\{P_n\}_{n \geqslant 0}$ of monic polynomials with $\deg P_n = n$ such that $u(P_n P_m) = 0$ if $n \neq m$ and $u(P_n^2) \neq 0$. $\{P_n\}_{n \geqslant 0}$ is said to be orthogonal with respect to u.

It is very well known that $\{P_n\}_{n \geqslant 0}$ satisfies a three-term recurrence relation

$$x P_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \geqslant 1,$$

with $c_n \neq 0$ for every $n \geqslant 1$.

This means that the matrix representation for the multiplication operator $(hq)(x) = xq(x)$, $q \in \mathscr{P}$, with respect to the basis $\{P_n\}_{n \geqslant 0}$ is a tridiagonal matrix H_p. We will denote $xP = H_p P$, where $P = [P_0(x), P_1(x), \dots]^{\mathrm{t}}$.

---

* Corresponding author. Tel.: +34 916249442; fax: +34 916249151.
  E-mail addresses: ldaruis@ull.es (L. Daruis), jhbenite@math.uc3m.es (J. Hernández), pacomarc@ing.uc3m.es (F. Marcellán).

Several examples of perturbations $\tilde{u}$ of a quasi-definite linear functional have been considered in the literature [3,5,22–24] in the framework of some factorization process of the matrix H_p. In particular, three canonical cases appear.

(i) $\tilde{u}$ is a linear functional defined by

$$\tilde{u}(p(x)) = u(xp(x)), \quad p \in \mathscr{P}.$$

$\tilde{u}$ is said to be the Christoffel transformation of u.

(ii) $\tilde{u}$ is a linear functional defined by

$$\tilde{u}(p(x)) = u(p(x)) + mp(0), \quad p \in \mathscr{P}, \ m \in \mathbb{R}.$$

$\tilde{u}$ is said to be the Uvarov transformation of u.

(iii) $\tilde{u}$ is a linear functional defined by

$$\tilde{u}(p(x)) = u\left(\frac{p(x) - p(0)}{x}\right) + mp(0), \quad p \in \mathscr{P}, \ m \in \mathbb{R}.$$

$\tilde{u}$ is said to be the Geronimus transformation of u.

Necessary and sufficient conditions for the quasi-definite character of $\tilde{u}$ appear in [3,23]. These three canonical transformations are called basic spectral transformations (see [22,24]).

A natural question is to find the connection between the tridiagonal matrices associated with the monic polynomial bases orthogonal with respect to u and $\tilde{u}$, respectively. In such cases, the LU and UL factorization of H_p play the main role in order to obtain explicitly the tridiagonal matrix associated with the perturbed linear functional $\tilde{u}$ [3,23].

In the theory of integrable systems, these transformations are called Darboux transformations. See [4,22,23] for more details.

The aim of our work is to analyze the analog of spectral transforms in the framework of inner products associated with Hermitian Toeplitz matrices. They appear in prediction theory as covariance matrices of discrete stationary stochastic process and the corresponding sequences of orthogonal polynomials are prediction filters (see [13,15]). The analysis of such families of orthogonal polynomials has intensively attracted the interest of researchers in the last five years. The monograph by Simon [21] is a good sample of it.

From the point of view of perturbations of positive-definite Hermitian Toeplitz matrices or, equivalently, probability measures supported on the unit circle, there is a wide literature (see [2,7–9,16,18,19], among others) emphasizing the analytic properties of polynomials orthogonal with respect to the perturbed measures.

In [11,10], an operator approach based in the matrix representation of the multiplication operator is given. We get a lower Hessenberg matrix that is almost unitary. More recently, in [6], a five diagonal matrix that is unitary and such that the characteristic polynomials of their leading principal submatrices are the monic orthogonal polynomials with respect to the spectral probability measure is introduced.

The goal of our contribution is to present the connection between the infinite Hessenberg matrices associated with canonical perturbations of probability measures supported on the unit circle using LU and QR factorization of the original one. We will analyze this problem in a more general framework that will be described in the sequel. When the measure has a finite support, then the Hessenberg matrix is finite dimensional. In such a case, there is a vast literature (see [1,12,20]) about the QR and LU factorization for eigenvalue computations and for the least-squares approximation by trigonometric polynomials. There the representation in terms of a set of n parameters, sometimes referred as Schur parameters or Verblunsky parameters (see [21]), both before and after the spectral transformation is emphasized because this makes efficient computation possible.

Let $\mathscr{S}$ be a linear functional in the linear space $\mathbb{L}$ of Laurent polynomials with complex coefficients. Denote by $\mathscr{L}(p(z), q(z)) := \mathscr{S}(p(z)\bar{q}(1/z))$ a bilinear functional, with $p, q \in \mathbb{P}$, where $\mathbb{P}$ is the linear space of polynomials with complex coefficients.

Definition 1.1 (*Jones et al. [14], Marcellán and Alfaro [17]*). Consider the matrix $T = (t_{i,j})_{i,j=0}^{\infty}$, where $t_{i,j} := \mathscr{L}(z^i, z^j) = \mathscr{S}(z^{i-j})$, and let T_n be the leading principal submatrix of order n. Then

(i) If $t_{i,j} = \overline{t_{j,i}}$, then $\mathscr{L}$ is said to be Hermitian in $\mathbb{P}$.

(ii) If det $T_n \neq 0$, for every nonnegative integer n, then $\mathscr{L}$ is said to be a quasi-definite bilinear functional in $\mathbb{P}$.

(iii) If $\mathscr{L}(p, p) > 0$, for every $p \in \mathbb{P}$, $p(z) \neq 0$, or, equivalently, det $T_n > 0$, for every nonnegative integer n, then $\mathscr{L}$ is said to be a positive definite bilinear functional in $\mathbb{P}$.

If $\mathscr{L}$ is quasi-definite, then there exists a sequence $\{P_n\}_{n \geqslant 0}$ of monic polynomials orthogonal with respect to $\mathscr{L}$, i.e.,

$$\deg P_n = n \quad \text{and} \quad \mathscr{L}(P_k, P_n) = \mathbf{k}_n \delta_{k,n}, \quad \mathbf{k}_n \neq 0, \quad 0 \leqslant k \leqslant n,$$

where $\delta_{k,n}$ is the Kronecker delta.

The sequence $\{P_n\}_{n \geqslant 0}$ satisfies two recurrence relations

(i) $P_{n+1}(z) = z P_n(z) + P_{n+1}(0) P_n^*(z),$

(ii) $P_{n+1}(z) = (1 - |P_{n+1}(0)|^2) z P_n(z) + P_{n+1}(0) P_{n+1}^*(z).$

They are known in the literature [7,13,14,21] as forward and backward recurrence relations, respectively. Here $P_n^*(z) = z^n \overline{P_n(\bar{z}^{-1})}$ is the so-called *reversed polynomial* of P_n.

Notice that $\mathscr{L}$ is quasi-definite if and only if $|P_n(0)| \neq 1$ for every $n \geqslant 1$. $\mathscr{L}$ is positive-definite if and only if $|P_n(0)| < 1$ for every $n \geqslant 1$.

On the other hand, if $\mathscr{L}$ is positive-definite and $P_n(\alpha) = 0$ then $|\alpha| < 1$. For the quasi-definite case we need a deep analysis as we show in below.

If for certain n, $P_n(\alpha) = P_n^*(\alpha) = 0$ with $|\alpha| \neq 1$, then from the backward recurrence relation we get

$$P_1(\alpha) = P_1^*(\alpha) = 0.$$

But

$$P_1(z) = z - \alpha \quad \text{and} \quad P_1^*(z) = 1 - \bar{\alpha} z,$$

i.e.,

$$1 - |\alpha|^2 = 0 \quad \text{a contradiction.}$$

Thus, if $|\alpha| \neq 1$ and $P_n(\alpha) = 0$, then $P_n^*(\alpha) \neq 0$.

If $|\alpha| = 1$ and $P_n(\alpha) = 0$, then $P_n^*(\alpha) = 0$. Using the backward recurrence relation $P_1(z) = z - \alpha$, with $|P_1(0)| = 1$, a contradiction. Thus, if $|\alpha| = 1$ then $P_n(\alpha) \neq 0$ for every n.

Consider the semi-infinite lower Hessenberg matrix H_p associated with $\{P_n\}_{n \geqslant 0}$, such that $z P = H_p P$, where $P = [P_0(z), P_1(z), \ldots, P_n(z), \ldots]^t$.

We will deal with the study of polynomial perturbations of $\mathscr{L}$ of the form (see [8,16])

$$\mathscr{L}_2(p, q) := \mathscr{L}((z - \alpha)p, (z - \alpha)q), \quad p, q \in \mathbb{P}, \tag{1}$$

and how the Hessenberg matrices associated with the original and the perturbed bilinear functionals are related to each other by using certain factorizations of them. As a convention we will denote $\mathscr{L}_2$ as $|z - \alpha|^2 \mathscr{L}$.

In Section 2, in order to study the perturbation of $\mathscr{L}$ given by (1), we will first consider the following perturbation of $\mathscr{L}$ defined by

$$\mathscr{L}_1(p, q) := \mathscr{L}((z - \alpha)p, q), \quad p, q \in \mathbb{P},$$

and then we will finally consider the perturbation:

$$\mathscr{L}_2(p, q) = \mathscr{L}_1(p, (z - \alpha)q) = \mathscr{L}((z - \alpha)p, (z - \alpha)q), \quad p, q \in \mathbb{P}.$$

In this process we will see that the corresponding Hessenberg matrices are related using the LU factorization of the Hessenberg matrix associated with $\mathscr{L}$.

In Section 3, we will consider the perturbation (1) directly. Since in this case we will deal with orthonormal polynomials, we assume the bilinear functional $\mathscr{L}$ to be positive definite. In this case, we will establish how the Hessenberg matrices associated with $\mathscr{L}$ and $\mathscr{L}_2$, respectively, are related to each other by using now the QR factorization.

In Section 4, we consider another perturbation of $\mathscr{L}(p, q)$ defined by

$$\mathscr{L}_3(p, q) = \mathscr{L}(p, q) + mp(\alpha)\overline{q(\alpha)}, \quad |\alpha| = 1.$$

By using the results obtained in Section 3, we will establish a relation between the Hessenberg matrices associated with $\mathscr{L}$ and $\mathscr{L}_3$, respectively, again using QR factorization. Finally, we will analyze a new perturbation

$$\mathscr{L}_4(p, q) = \mathscr{L}(p, q) + mp(\alpha)\bar{q}(\alpha^{-1}) + mp(\bar{\alpha}^{-1})\bar{q}(\bar{\alpha}),$$

with $|\alpha| \neq 1$ and $m \in \mathbb{R}$.

2. Quasi-definite linear functionals and LU factorization

According to the notation given in the Introduction, assume that $\mathscr{S}$ is a quasi-definite linear functional in the linear space of Laurent polynomials with complex coefficients as well as the bilinear functional $\mathscr{L}$ is Hermitian. $\mathbb{P}$ will denote the linear space of polynomials with complex coefficients and $\mathbb{P}_n$ is the linear subspace of polynomials of degree at most n.

If $\{P_n\}_{n \geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to $\mathscr{L}$ and H_p is the semi-infinite lower Hessenberg matrix associated with $\{P_n\}_{n \geqslant 0}$, such that $zP = H_p P$, where $P = [P_0(z), P_1(z), \dots]^t$, then $\mathscr{L}(P, P^t) = D_p$, where D_p is a nonsingular diagonal matrix.

As we have mentioned in the Introduction, in this section we will study the polynomial perturbation (1) of $\mathscr{L}$ in two steps.

First of all consider the perturbation of $\mathscr{L}$ defined by

$$\mathscr{L}_1(p, q) := \mathscr{L}((z - \alpha)p, q).$$

In this case, $\mathscr{L}_1$ is not Hermitian. Therefore, we cannot consider the sequence of monic polynomials orthogonal with respect to $\mathscr{L}_1$. However, we can look for sequences of polynomials which are either *left* or *right* orthogonal with respect to $\mathscr{L}_1$.

Definition 2.1. Let $\mathscr{F}$ be a bilinear functional in $\mathbb{P}$, $\{L_n\}_{n \geqslant 0}$, $\{R_n\}_{n \geqslant 0}$ sequences of monic polynomials.

(i) $\{L_n\}_{n \geqslant 0}$ is said to be left orthogonal with respect to $\mathscr{F}$ if for every $n \geqslant 0$, we get
  - $\deg L_n = n$,
  - $\mathscr{F}(L_n(z), z^k) = 0, 0 \leqslant k \leqslant n - 1$,
  - $\mathscr{F}(L_n(z), z^n) \neq 0$.
(ii) $\{R_n\}_{n \geqslant 0}$ is said to be right orthogonal with respect to $\mathscr{F}$ if for every $n \geqslant 0$, we get
  - $\deg R_n = n$,
  - $\mathscr{F}(z^k, R_n(z)) = 0, 0 \leqslant k \leqslant n - 1$,
  - $\mathscr{F}(z^n, R_n(z)) \neq 0$.

Indeed, one can prove the following:

Proposition 2.1. *Suppose that* $P_n(\alpha) \neq 0$ *for every* $n \geqslant 1$. *Then*

(i) *The sequence*

$$R_n(z, \alpha) = \frac{\mathbf{k}_n}{P_n(\alpha)} K_n(z, \alpha) = P_n(z) + \frac{\mathbf{k}_n}{P_n(\alpha)} \sum_{j=0}^{n-1} \mathbf{k}_j^{-1} \overline{P_j(\alpha)} P_j(z) \tag{2}$$

is right orthogonal with respect to $\mathscr{L}_1$. K_n *is the kernel polynomial of degree n associated with the bilinear functional* $\mathscr{L}$.

(ii) *The sequence*

$$S_n(z, \alpha) = \frac{1}{z - \alpha} \left(P_{n+1}(z) - \frac{P_{n+1}(\alpha)}{P_n(\alpha)} P_n(z) \right) \tag{3}$$

is left orthogonal with respect to $\mathscr{L}_1$.

Proof. (i) From (2), $\deg R_n = \deg P_n = n$.

The kernel polynomial $K_n(z, \alpha)$ satisfies the *Reproducing property*

$$\mathscr{L}(p(z), K_n(z, \alpha)) = p(\alpha) \quad \text{for every } p \in \mathbb{P}_n,$$

as well as the Christoffel–Darboux formula (see [7,13,21])

$$K_n(z, \alpha) = \frac{1}{\mathbf{k}_{n+1}} \frac{\overline{P_{n+1}^*(\alpha)} P_{n+1}^*(z) - \overline{P_{n+1}(\alpha)} P_{n+1}(z)}{1 - \bar{\alpha}z}. \tag{4}$$

Thus, for $0 \leqslant k \leqslant n$,

$$
\begin{aligned}
\mathscr{L}_1(z^k, R_n(z, \alpha)) &= \frac{\mathbf{k}_n}{P_n(\alpha)} \mathscr{L}_1(z^k, K_n(z, \alpha)) \\
&= \frac{\mathbf{k}_n}{P_n(\alpha)} \mathscr{L}(z^k(z - \alpha), K_n(z, \alpha)) \\
&= \frac{\mathbf{k}_n}{P_n(\alpha)} \mathscr{L}(z^k(z - \alpha), K_{n+1}(z, \alpha) - \mathbf{k}_{n+1}^{-1} P_{n+1}(z) \overline{P_{n+1}(\alpha)}) \\
&= - \mathbf{k}_n \frac{P_{n+1}(\alpha)}{P_n(\alpha)} \delta_{n,k}.
\end{aligned}
$$

(ii) For $0 \leqslant k \leqslant n$,

$$
\begin{aligned}
\mathscr{L}_1(S_n(z, \alpha), z^k) &= \mathscr{L}((z - \alpha) S_n(z, \alpha), z^k) \\
&= \mathscr{L}\left(P_{n+1}(z) - \frac{P_{n+1}(\alpha)}{P_n(\alpha)} P_n(z), z^k \right) \\
&= \mathscr{L}(P_{n+1}(z), z^k) - \frac{P_{n+1}(\alpha)}{P_n(\alpha)} \mathscr{L}(P_n(z), z^k) \\
&= - \mathbf{k}_n \frac{P_{n+1}(\alpha)}{P_n(\alpha)} \delta_{n,k}. \qquad \square
\end{aligned}
$$

The sequence $\{S_n(z, \alpha)\}_{n \geqslant 0}$ was used in [4] for nonsymmetric perturbations of symmetric bilinear functionals. We will work with the family $\{R_n(z, \alpha)\}_{n \geqslant 0}$.

We will now establish a relation between the Hessenberg matrices H_p and H_r associated with $\{P_n\}_{n \geqslant 0}$ and $\{R_n(z, \alpha)\}_{n \geqslant 0}$, respectively. Notice that $\{P_n\}_{n \geqslant 0}$ and $\{R_n(z, \alpha)\}_{n \geqslant 0}$ are monic polynomial bases in the linear space $\mathbb{P}$ of polynomials with complex coefficients. Thus, there exists a lower triangular matrix L_{pr} with 1 as diagonal entries such that $P = L_{pr} R$, where $R = [R_0(z, \alpha), R_1(z, \alpha), \dots, R_n(z, \alpha), \dots]^{\mathrm{t}}$. Observe that $D_r = \mathscr{L}_1(R, R^{\mathrm{t}})$ is a nonsingular lower triangular matrix.

Lemma 2.1.

(i) $\mathscr{L}_1(P, P^{\mathrm{t}}) = (H_p - \alpha I) \mathscr{L}(P, P^{\mathrm{t}}) = (H_p - \alpha I) D_p$,
(ii) $\mathscr{L}_1((z - \alpha) P, P^{\mathrm{t}}) = (H_p - \alpha I)^2 D_p$,

where I denotes the infinite unit matrix.

Proof.

$$\mathscr{L}_1(P, P^{\mathsf{t}}) = \mathscr{L}((z - \alpha)P, P^{\mathsf{t}}) = \mathscr{L}((H_p - \alpha I)P, P^{\mathsf{t}})$$
$$= (H_p - \alpha I)\mathscr{L}(P, P^{\mathsf{t}}) = (H_p - \alpha I)D_p.$$

In a similar way, the second statement follows. $\square$

Proposition 2.2. *Let* L_{pr} *be the lower triangular matrix with* 1 *as diagonal entries such that* $P = L_{pr}R$. *Then* $H_p - \alpha I = LU$, *where*

$$L = L_{pr}D_r \qquad\qquad (5)$$

is a lower triangular matrix and

$$U = L_{pr}^* D_p^{-1} \qquad\qquad (6)$$

is an upper triangular matrix.

Proof. By the above Lemma, one can write

$$(H_p - \alpha I)D_p = \mathscr{L}_1(P, P^{\mathsf{t}}) = \mathscr{L}_1(L_{pr}R, R^{\mathsf{t}}L_{pr}^{\mathsf{t}})$$
$$= L_{pr}\mathscr{L}_1(R, R^{\mathsf{t}})L_{pr}^* = L_{pr}D_r L_{pr}^*.$$

Therefore, $H_p - \alpha I = L_{pr}D_r L_{pr}^* D_p^{-1}$. $\square$

Remark 2.1. Observe that the *LU* factorization given in the last Proposition is not unique.

Next, we give a result that shows that the matrix H_r is similar to the product UL.

Proposition 2.3. *If* $H_p - \alpha I = LU$, *where* L *and* U *are given by* (5) *and* (6), *respectively*, *then*

$$H_r = D_r(UL)D_r^{-1} + \alpha I.$$

Recall that D_r *is a nonsingular lower triangular matrix.*

Proof. From the above result, we get $H_p - \alpha I = LU$, with L and U given by (5) and (6), respectively. Therefore, taking into account the last Proposition, on one hand we have

$$\mathscr{L}_1((z - \alpha)R, R^{\mathsf{t}}) = \mathscr{L}_1((z - \alpha)L_{pr}^{-1}P, P^{\mathsf{t}}L_{pr}^{-\mathsf{t}})$$
$$= L_{pr}^{-1}\mathscr{L}_1((z - \alpha)P, P^{\mathsf{t}})(L_{pr}^*)^{-1}$$
$$= L_{pr}^{-1}(H_p - \alpha I)^2 D_p(L_{pr}^*)^{-1} = L_{pr}^{-1}(LU)^2 D_p(L_{pr}^*)^{-1}.$$

On the other hand, $\mathscr{L}_1((z - \alpha)R, R^{\mathsf{t}}) = (H_r - \alpha I)\mathscr{L}_1(R, R^{\mathsf{t}}) = (H_r - \alpha I)D_r$. Thus,

$$H_r - \alpha I = L_{pr}^{-1}L_{pr}D_r UL D_r^{-1} = D_r UL D_r^{-1}. \qquad \square$$

The next step is to define now a perturbation of $\mathscr{L}_1$ as follows: $\mathscr{L}_2(p, q) := \mathscr{L}_1(p, (z-\alpha)q)$. Notice that $\mathscr{L}_2(p, q) = \mathscr{L}((z - \alpha)p, (z - \alpha)q)$ and that the bilinear functional $\mathscr{L}_2$ is Hermitian (see [17]).

Proposition 2.4. *The bilinear functional* $\mathscr{L}_2$ *is quasi-definite if and only if* $K_n(\alpha, \alpha) \neq 0$, *for every* $n \geqslant 0$.

Proof. Assume that $\mathscr{L}_2$ is a quasi-definite bilinear functional and consider the sequence $\{Q_n\}_{n \geqslant 0}$ of monic polynomials orthogonal with respect to $\mathscr{L}_2$. Then

$$\mathbb{P}_n = (z - \alpha)\mathbb{P}_{n-1} \oplus L[K_n(z, \alpha)]$$
$$= L[(z - \alpha)Q_{n-1}(z)] \oplus (z - \alpha)\mathbb{P}_{n-2} \oplus L[K_n(z, \alpha)],$$

where $L[K_n(z, \alpha)]$ means the span of $K_n(z, \alpha)$.

On the other hand,

$$\mathbb{P}_n = \mathbb{P}_{n-1} \oplus L[P_n(z)]$$
$$= L[P_n(z)] \oplus L[K_{n-1}(z, \alpha)] \oplus (z - \alpha)\mathbb{P}_{n-2}.$$

Thus,

$$L[(z - \alpha)Q_{n-1}(z)] \oplus L[K_n(z, \alpha)] = L[P_n(z)] \oplus L[K_{n-1}(z, \alpha)].$$

Therefore,

$$P_n(z) = (z - \alpha)Q_{n-1}(z) + \beta_n K_{n-1}(z, \alpha).$$

If $K_{n_0-1}(\alpha, \alpha) = 0$ for some n_0, then, from the above expression, $P_{n_0}(\alpha) = 0$. This means that $K_n(\alpha, \alpha) = 0$, for every $n \geqslant n_0$ and thus $P_{n+1}(\alpha) = 0$, $n \geqslant n_0$.

If $|\alpha| \neq 1$, taking into account $P_n^*(\alpha) \neq 0$, $n \geqslant n_0$, as well as the Christoffel–Darboux formula (4), then for $n \geqslant n_0$ we get

$$K_n(\alpha, \alpha) = \mathbf{k}_{n+1}^{-1} \frac{|P_{n+1}^*(\alpha)|^2 - |P_{n+1}(\alpha)|^2}{1 - |\alpha|^2} = \mathbf{k}_{n+1}^{-1} \frac{|P_{n+1}^*(\alpha)|^2}{1 - |\alpha|^2} \neq 0,$$

a contradiction.

If $|\alpha| = 1$, then $P_n^*(\alpha) = 0$, thus $P_{n-1}(\alpha) = 0$ and $P_1(\alpha) = 0$, i.e., $P_1(z) = z - \alpha$, hence $|P_1(0)| = 1$, a contradiction with the quasi-definite character of $\mathscr{L}$. Hence $K_n(\alpha, \alpha) \neq 0$, for every $n \geqslant 0$.

Conversely, assume that $K_n(\alpha, \alpha) \neq 0$ for every $n \geqslant 0$ and consider the family of monic polynomials $\{Q_n\}_{n \geqslant 0}$ given by

$$(z - \alpha)Q_n(z) = P_{n+1}(z) - \frac{P_{n+1}(\alpha)}{K_n(\alpha, \alpha)} K_n(z, \alpha). \tag{7}$$

Then, for $0 \leqslant k \leqslant n$

$$\mathscr{L}_2(Q_n(z), (z - \alpha)^k) = \mathscr{L}((z - \alpha)Q_n(z), (z - \alpha)^{k+1})$$

$$= \mathscr{L}\left(P_{n+1}(z) - \frac{P_{n+1}(\alpha)}{K_n(\alpha, \alpha)} K_n(z, \alpha), (z - \alpha)^{k+1}\right)$$

$$= \mathscr{L}(P_{n+1}, (z - \alpha)^{k+1}) - \frac{P_{n+1}(\alpha)}{K_n(\alpha, \alpha)} \mathscr{L}(K_n(z, \alpha), (z - \alpha)^{k+1})$$

$$= \mathbf{k}_{n+1}\delta_{n,k} - \frac{P_{n+1}(\alpha)}{K_n(\alpha, \alpha)} \mathscr{L}\left(K_{n+1}(z, \alpha) - \frac{\overline{P_{n+1}(\alpha)}}{\mathbf{k}_{n+1}} P_{n+1}(z), (z - \alpha)^{k+1}\right)$$

$$= \mathbf{k}_{n+1}\delta_{n,k} + \frac{P_{n+1}(\alpha)\overline{P_{n+1}(\alpha)}}{K_n(\alpha, \alpha)}\delta_{n,k}$$

$$= \mathbf{k}_{n+1} \frac{K_{n+1}(\alpha, \alpha)}{K_n(\alpha, \alpha)}\delta_{n,k}.$$

Hence $\{Q_n\}_{n \geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to $\mathscr{L}_2$. $\square$

Let H_q be the lower Hessenberg matrix such that $zQ = H_q Q$, where $Q = [Q_0(z), Q_1(z), \ldots, Q_n(z), \ldots]^{\mathrm{t}}$. We will show that H_q can be obtained from the LU factorization of H_r.

First we need to prove the following:

Lemma 2.2.

(i) $\mathscr{L}_2(R, R^{\mathrm{t}}) = \mathscr{L}_1(R, R^{\mathrm{t}})(H_r - \alpha I)^* = D_r(H_r - \alpha I)^*.$
(ii) $\mathscr{L}_2(R, (z - \alpha)R^{\mathrm{t}}) = D_r((H_r - \alpha I)^*)^2.$

Proof. (i)

$$\mathcal{L}_2(R, R^{\mathrm{t}}) = \mathcal{L}_1(R, (z - \alpha)R^{\mathrm{t}}) = \mathcal{L}_1(R, R^{\mathrm{t}})(H_r - \alpha I)^* = D_r(H_r - \alpha I)^*.$$

In a similar way, we deduce (ii). $\quad\square$

Proposition 2.5. *Let L_{rq} be the lower triangular matrix with 1 as entries in the main diagonal such that $R = L_{rq}Q$. Then, $H_r - \alpha I = L_{rq}\tilde{U}$, where $\tilde{U}$ denotes a nonsingular upper triangular matrix.*

Proof. From the previous Lemma

$$\begin{aligned}
D_r(H_r - \alpha I)^* &= \mathcal{L}_2(R, R^{\mathrm{t}}) = \mathcal{L}_2(L_{rq}Q, Q^{\mathrm{t}}L_{rq}^{\mathrm{t}}) \\
&= L_{rq}\mathcal{L}_2(Q, Q^{\mathrm{t}})L_{rq}^* \\
&= L_{rq}D_q L_{rq}^*, \quad (D_q = \mathcal{L}_2(Q, Q^{\mathrm{t}}) \text{ is a diagonal matrix}).
\end{aligned}$$

Therefore, $H_r - \alpha I = L_{rq}D_q^* L_{rq}^* (D_r^*)^{-1}$ and

$$H_r - \alpha I = \tilde{L}\tilde{U} \quad \text{where} \quad \tilde{L} = L_{rq} \quad \text{and} \quad \tilde{U} = D_q^* L_{rq}^* (D_r^*)^{-1}. \quad\square$$

Proposition 2.6. *Let L_{rq} be the lower triangular matrix with 1 as entries in the main diagonal such that $R = L_{rq}Q$. If $H_r - \alpha I = \tilde{L}\tilde{U}$ denotes the LU factorization without pivoting of $H_r - \alpha I$, then*

$$H_q - \alpha I = \tilde{U}\tilde{L}.$$

Proof.

$$\begin{aligned}
\mathcal{L}_2(Q, (z - \alpha)Q^{\mathrm{t}}) &= \mathcal{L}_2(L_{rq}^{-1}R, (z - \alpha)R^{\mathrm{t}}L_{rq}^{-\mathrm{t}}) \\
&= L_{rq}^{-1}\mathcal{L}_2(R, (z - \alpha)R^{\mathrm{t}})(L_{rq}^*)^{-1}.
\end{aligned}$$

Thus,

$$\mathcal{L}_2(Q, Q^{\mathrm{t}})(H_q - \alpha I)^* = L_{rq}^{-1}D_r((H_r - \alpha I)^*)^2(L_{rq}^*)^{-1}$$

or, equivalently,

$$D_q(H_q - \alpha I)^* = L_{rq}^{-1}D_r D_r^{-1}L_{rq}D_q L_{rq}^* D_r^{-1}L_{rq}D_q L_{rq}^*(L_{rq}^*)^{-1}.$$

Finally

$$(H_q - \alpha I)^* = L_{rq}^* D_r^{-1}L_{rq}D_q \quad \text{i.e.,}$$

$$H_q - \alpha I = D_q^* L_{rq}^*(D_r^*)^{-1}L_{rq} = \tilde{U}\tilde{L}. \quad\square$$

3. Positive-definite linear functionals and QR factorization

Assume now that $\mathcal{L}$ is positive definite, and denote by $\{\varphi_n\}_{n \geqslant 0}$ the corresponding sequence of orthonormal polynomials, i.e., $\varphi_n(z) = \kappa_n P_n(z)$, where $\kappa_n = \sqrt{\mathbf{k}_n^{-1}}$.

We will deal with the study of the polynomial perturbations $\mathcal{L}_2$ of $\mathcal{L}$ given in (1). Notice that the bilinear functional $\mathcal{L}_2$ is also Hermitian and positive definite. Let $\{\psi_n\}_{n \geqslant 0}$ denote the corresponding sequence of orthonormal polynomials.

Our aim is to obtain a relation between H_φ and H_ψ by using (7). For convenience, we will rewrite such a formula in terms of the orthonormal polynomials $\{\varphi_n\}_{n \geqslant 0}$ and $\{\psi_n\}_{n \geqslant 0}$ with respect to $\mathcal{L}$ and $\mathcal{L}_2$, respectively. Thus,

$$(z - \alpha)\psi_n(z) = \sqrt{\frac{K_n(\alpha, \alpha)}{K_{n+1}(\alpha, \alpha)}}\varphi_{n+1}(z) - \sum_{j=0}^{n}\frac{\varphi_{n+1}(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{K_{n+1}(\alpha, \alpha)K_n(\alpha, \alpha)}}\varphi_j(z). \tag{8}$$

If we consider

$$\varphi = [\varphi_0(z), \varphi_1(z), \ldots, \varphi_n(z), \ldots]^{\mathrm{t}} \quad \text{and} \quad \psi = [\psi_0(z), \psi_1(z), \ldots, \psi_n(z), \ldots]^{\mathrm{t}},$$

then the matrix expression of (8) is

$$(z - \alpha)\psi = M\varphi, \tag{9}$$

where M is a lower Hessenberg matrix with entries $m_{i,j}$ given by

$$m_{i,j} = \begin{cases} -\dfrac{\varphi_{i+1}(\alpha)}{\sqrt{K_{i+1}(\alpha, \alpha) K_i(\alpha, \alpha)}} \overline{\varphi_j(\alpha)} & \text{if } j \leqslant i, \\[2ex] \sqrt{\dfrac{K_i(\alpha, \alpha)}{K_{i+1}(\alpha, \alpha)}} & \text{if } j = i + 1, \\[2ex] 0 & \text{if } j > i + 1. \end{cases} \tag{10}$$

The matrix M satisfies

Proposition 3.1.

$$MM^* = I, \tag{11}$$

where I denotes the infinite unit matrix.

Proof. From the orthogonality of $\{\varphi_n\}_{n \geqslant 0}$ and $\{\psi_n\}_{n \geqslant 0}$ with respect to $\mathscr{L}$ and $\mathscr{L}_2$ respectively, and by (9), we get

$$I = \mathscr{L}_2(\psi, \psi^{\mathrm{t}}) = \mathscr{L}((z - \alpha)\psi, (z - \alpha)\psi^{\mathrm{t}})$$
$$= \mathscr{L}(M\varphi, \varphi^{\mathrm{t}} M^{\mathrm{t}}) = M\mathscr{L}(\varphi, \varphi^{\mathrm{t}})M^* = MM^*. \qquad \square$$

Remark 3.1. Notice that $M^*M \neq I$.

To obtain the relation between the Hessenberg matrices H_φ (i.e., $H_\varphi\varphi = z\varphi$, see [11,10]) and H_ψ, we must introduce the lower triangular matrix L such that $\varphi = L\psi$. We will show that such a matrix can be expressed in terms of H_φ and M in the following way:

Proposition 3.2.

$$L = (H_\varphi - \alpha I)M^*. \tag{12}$$

Proof. Let $\varphi = L\psi$. Then $(z - \alpha)\psi = (z - \alpha)L^{-1}\varphi$. From (9), $M\varphi = L^{-1}(H_\varphi - \alpha I)\varphi$. Therefore, $LM = H_\varphi - \alpha I$ and since $MM^* = I$, the statement follows. $\square$

From this, as a straightforward consequence, we get

Proposition 3.3.

$$H_\psi - \alpha I = ML.$$

Proof. From (9), $(z - \alpha)\psi = M\varphi$. So $(H_\psi - \alpha I)\psi = ML\psi$ and the result follows. $\square$

Therefore, to compute H_ψ from H_φ, first we need to determine the lower triangular matrix L. We can explicitly do these calculations. Just take into account that the coefficients of Hessenberg matrix H_φ are

$$h_{i,j} = \begin{cases} -\dfrac{\kappa_j}{\kappa_i} P_{i+1}(0)\overline{P_j(0)} & \text{if } 0 \leqslant j \leqslant i, \\ \dfrac{\kappa_i}{\kappa_{i+1}} & \text{if } j = i+1, \\ 0 & \text{if } j > i+1, \end{cases}$$

and the expressions (10) and (12), we deduce by a simple computation.

Proposition 3.4. *The entries $l_{i,j}$ of the matrix L are*

$$l_{i,j} = \begin{cases} \dfrac{\kappa_i}{\kappa_{i+1}}\sqrt{\dfrac{K_{i+1}(\alpha,\alpha)}{K_i(\alpha,\alpha)}}, & i = j, \\[3mm] \dfrac{\overline{\varphi_i(\alpha)}((P_{i+1}(0)/\kappa_i)\kappa_{i-1}\varphi_{i-1}^*(\alpha))}{\sqrt{K_i(\alpha,\alpha)K_{i-1}(\alpha,\alpha)}} - (P_{i+1}(0)\overline{P_i(0)} + \alpha)\sqrt{\dfrac{K_{i-1}(\alpha,\alpha)}{K_i(\alpha,\alpha)}}, & j = i-1, \\[3mm] \dfrac{P_{i+1}(0)}{\kappa_i\sqrt{K_j(\alpha,\alpha)K_{j+1}(\alpha,\alpha)}}(\kappa_j\overline{\varphi_{j+1}(\alpha)}\varphi_j^*(\alpha) - \overline{\varphi_{j+1}(0)}K_j(\alpha,\alpha)), & j \leqslant i-2. \end{cases}$$

From Proposition 3.3, to obtain H_ψ we need to multiply again the matrices M and L. To do this explicitly is very complicated, so we have chosen an example to show these computations. Indeed, let

$$\mathcal{L}(p, q) = \int_{-\pi}^{\pi} p(e^{i\theta})\overline{q(e^{i\theta})}\frac{d\theta}{2\pi}.$$

It is very well known that the nth orthonormal polynomial in this case is $P_n(z) = \varphi_n(z) = z^n$, for every $n \geqslant 0$.

Consider the parameter α with $|\alpha| = 1$. Then, the reproducing kernel is

$$K_n(z, \alpha) = \sum_{j=0}^{n} \frac{z^j}{\alpha^j} = \frac{1}{\alpha^n}\frac{z^{n+1} - \alpha^{n+1}}{z - \alpha},$$

and, as a consequence, $K_n(\alpha, \alpha) = n + 1$, for every $n \geqslant 0$.

On the other hand,

$$H_\varphi - \alpha I = \begin{bmatrix} -\alpha & 1 & 0 & \cdots \\ 0 & -\alpha & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

The entries $m_{i,j}$ of M are

$$m_{i,j} = \begin{cases} -\dfrac{\alpha^{i-j+1}}{\sqrt{(i+2)(i+1)}} & \text{if } j \leqslant i, \\[3mm] \sqrt{\dfrac{i+1}{i+2}} & \text{if } j = i+1, \\[2mm] 0 & \text{if } j > i+1. \end{cases} \tag{13}$$

In this case, from Proposition 3.4, $l_{i,j} = 0$, for $j \leqslant i - 2$. Hence, the matrix L is a lower bidiagonal matrix with entries

$$l_{i,i-1} = -\alpha\sqrt{\frac{i}{i+1}} \quad \text{and} \quad l_{i,i} = \sqrt{\frac{i+2}{i+1}}. \tag{14}$$

We will analyze several cases.

For $j \geqslant i + 2$, one can easily see that $M_{(i)}L^{(j)} = 0$. (Note that the resulting matrix must be Hessenberg).
For $j = i + 1$,

$$M_{(i)}L^{(i+1)} = m_{i,i+1}l_{i+1,i+1} = \sqrt{\frac{i+1}{i+2}}\sqrt{\frac{i+3}{i+2}} = \frac{\sqrt{(i+1)(i+3)}}{i+2}.$$

For $j = i$,

$$M_{(i)}L^{(i)} = m_{i,i}l_{i,i} + m_{i,i+1}l_{i+1,i}$$

$$= -\frac{\alpha}{\sqrt{(i+2)(i+1)}}\sqrt{\frac{i+2}{i+1}} - \alpha\frac{i+1}{i+2} = -\alpha\frac{i+2+(i+1)^2}{(i+1)(i+2)}.$$

Finally, for $j < i$,

$$M_{(i)}L^{(j)} = m_{i,j}l_{j,j} + m_{i,j+1}l_{j+1,j}$$

$$= \frac{\alpha^{i-j+1}}{\sqrt{(j+2)(j+1)}}\left(\sqrt{\frac{j+1}{j+2}} - \sqrt{\frac{j+2}{j+1}}\right) = -\frac{\alpha^{i-j+1}}{\sqrt{(i+2)(i+1)(j+2)(j+1)}}.$$

Hence, the entries $\tilde{h}_{i,j}$ of the Hessenberg matrix $H_\psi - \alpha I$ are

$$\tilde{h}_{i,j} = \begin{cases} 0 & \text{if } j \geqslant i + 2, \\ \dfrac{\sqrt{(i+1)(i+3)}}{i+2} & \text{if } j = i + 1, \\ -\alpha\dfrac{i+2+(i+1)^2}{(i+1)(i+2)} & \text{if } j = i, \\ -\dfrac{\alpha^{i-j+1}}{\sqrt{(i+2)(i+1)(j+2)(j+1)}} & \text{if } j < i. \end{cases}$$

As we have seen in this example, all the calculations need a lot of work. This is the reason why we tried to relate, in a general case, the computation of the Hessenberg matrix H_ψ with certain factorization of the original Hessenberg matrix H_φ. More precisely, we will use the QR factorization of $(H_\varphi - \alpha I)^*$ to obtain $H_\psi - \alpha I$. Indeed, assume that Q^*R^* is the QR−factorization of $(H_\varphi - \alpha I)^*$, where $QQ^* = I$ and R^* is upper triangular with strictly positive diagonal entries. Then, $H_\varphi - \alpha I = RQ$ and one can prove the following:

Proposition 3.5. *If L is such that $\varphi = L\psi$, then $R = L$.*

Proof. Notice that $\mathscr{L}_2(\varphi, \varphi^t) = \mathscr{L}_2(L\psi, \psi^t L^t) = L\mathscr{L}_2(\psi, \psi^t)L^* = LL^*$.
  On the other hand,

$$\mathscr{L}_2(\varphi, \varphi^t) = \mathscr{L}((z - \alpha)\varphi, (z - \alpha)\varphi^t)$$

$$= (H_\varphi - \alpha I)\mathscr{L}(\varphi, \varphi^t)(H_\varphi - \alpha I)^*$$

$$= (H_\varphi - \alpha I)(H_\varphi - \alpha I)^*$$

$$= (RQ)(Q^*R^*) = RR^*.$$

From Proposition 3.4, one can see that the diagonal entries $l_{i,i}$ of L are all strictly positive. Therefore, LL^* represents the Cholesky decomposition of the Gram matrix of the bilinear form $\mathscr{L}_2$ with respect to the orthonormal basis $\{\varphi_n\}_{n \geqslant 0}$. Thus, $R = L$. $\quad\square$

  Now, we will prove the following:

Proposition 3.6. $H_\psi - \alpha I = QL$.

Proof.

$$H_\psi - \alpha I = (H_\psi - \alpha I)\mathscr{L}_2(\psi, \psi^{\mathrm{t}})$$
$$= \mathscr{L}_2(H_\psi - \alpha I)\psi, \psi^{\mathrm{t}})$$
$$= \mathscr{L}_2((z - \alpha)\psi, \psi^{\mathrm{t}})$$
$$= \mathscr{L}_2((z - \alpha)L^{-1}\varphi, \varphi^{\mathrm{t}}L^{-\mathrm{t}})$$
$$= L^{-1}\mathscr{L}_2((z - \alpha)\varphi, \varphi^{\mathrm{t}})(L^{-1})^*$$
$$= L^{-1}(H_\varphi - \alpha I)\mathscr{L}_2(\varphi, \varphi^{\mathrm{t}})(L^*)^{-1}$$
$$= L^{-1}(LQ)(LL^*)(L^*)^{-1}$$
$$= QL. \qquad \square$$

Remark 3.2. Notice that, according to Propositions 3.3 and 3.6, we get $Q = M$.

In order to give a finite version of the last Proposition we will prove the following:

Proposition 3.7. *Let $(H_\varphi - \alpha I)_n$ be the leading principal submatrix of order n of $H_\varphi - \alpha I$ and consider the factorization $(H_\varphi - \alpha I)_n = R_n Q_n$ where R_n is a lower triangular matrix and Q_n is a unitary matrix such that $(H_\varphi - \alpha I)_n^* = Q_n^* R_n^*$ is the QR factorization of $(H_\varphi - \alpha I)_n^*$. Then*

$$(H_\psi - \alpha I)_{n-1} = (Q_n R_n)_{n-1}.$$

Proof. Consider the factorization $H_\varphi - \alpha I = LM$ and let L_{11}, M_{11} be the leading principal submatrix of order n of L and M, respectively. Then

$$H_\varphi - \alpha I = \left[\begin{array}{c|c} L_{11} & 0 \\ \hline L_{21} & L_{22} \end{array}\right]\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]$$

$$= \left[\begin{array}{c|c} L_{11}M_{11} & L_{11}M_{12} \\ \hline L_{21}M_{11} + L_{22}M_{21} & L_{21}M_{12} + L_{22}M_{22} \end{array}\right],$$

and, as a consequence $(H_\varphi - \alpha I)_n = L_{11}M_{11}$.
   On the other hand,

$$H_\psi - \alpha I = ML$$
$$= \left[\begin{array}{c|c} M_{11}L_{11} + M_{12}L_{21} & M_{12}L_{21} \\ \hline M_{21}L_{11} + M_{22}L_{21} & M_{22}L_{21} \end{array}\right].$$

Thus,

$$(H_\psi - \alpha I)_n = M_{11}L_{11} + M_{12}L_{21},$$

but

$$M_{12}L_{21} = \begin{bmatrix} 0 & \cdots & \\ \vdots & & \\ 0 & 0 & \cdots \\ m_{n-1,n} & 0 & \cdots \end{bmatrix}\begin{bmatrix} l_{n,0} & \cdots & l_{n-1} \\ \vdots & & \vdots \\ & & \end{bmatrix} = m_{n-1,n}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ l_{n,0} & \cdots & l_{n-1} \end{bmatrix}.$$

Thus,

$$(H_\psi - \alpha I)_{n-1} = (M_{11} L_{11})_{n-1}. \tag{15}$$

Since $MM^* = I$, we get

$$M_{11} M_{11}^* = I_n - |m_{n-1,n}|^2 E_{nn},$$

where

$$E_{nn} = \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

is a matrix of order n and I_n is the unit matrix of order n.

Now, consider

$$E := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ 0 & \cdots & 0 & \dfrac{1}{\delta} \end{bmatrix},$$

where $\delta = \sqrt{1 - |m_{n-1,n}|^2} > 0$. Let $\hat{Q} = E M_{11}$ and $\hat{R} = L_{11} E^{-1}$. Then, we will check that $\hat{Q}$ is a unitary matrix of order n.

Let $\hat{q}_i$, $i = 0, 1, \ldots, n-1$ be the ith row of $\hat{Q}$. Then

$$\hat{q}_i = \begin{cases} m_i, & 0 \leqslant i \leqslant n-2, \\ \dfrac{1}{\delta} m_n, & i = n-1, \end{cases}$$

where m_i, $i = 0, 1, \ldots, n-1$ is the ith row of M_{11}.

If $0 \leqslant i, j \leqslant n-2$, then we get

$$(\hat{Q}\hat{Q}^*)_{i,j} = \hat{q}_i \hat{q}_j^* = m_i m_j^* = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

If $i = n-1$, or $j = n-1$ and $i \neq j$, then

$$\hat{q}_i \hat{q}_j^* = \frac{1}{\delta} m_i m_j^* = 0.$$

Finally,

$$\hat{q}_{n-1} \hat{q}_{n-1}^* = \frac{1}{\delta^2} m_{n-1} m_{n-1}^* = \frac{1}{\delta^2} (1 - |m_{n-1,n}|^2) = 1.$$

$\hat{R}$ is a lower triangular matrix, with positive diagonal entries.

Furthermore,

$$\hat{Q}^* \hat{R}^* = M_{11}^* E E^{-1} L_{11}^* = M_{11}^* L_{11}^* = (H_\varphi - \alpha I)_n^*.$$

Thus, by the uniqueness of the QR factorization of $(H_\varphi - \alpha I)_n^*$, we get

$$\hat{Q} = Q_n \quad \text{and} \quad \hat{R} = R_n.$$

But Q_n and M_{11} differ in the last row and R_n and L_{11} differ in the last column. Hence,

$$(Q_n R_n)_{n-1} = (M_{11} L_{11})_{n-1} = (H_\psi - \alpha I)_{n-1}. \qquad \square$$

4. Perturbation

$$\mathscr{L}_3(p, q) = \mathscr{L}(p, q) + mp(\alpha)\overline{q(\alpha)}, \quad |\alpha| = 1.$$

Let consider now the perturbation $\mathscr{L}_3$ of the original bilinear functional $\mathscr{L}$ defined by

$$\mathscr{L}_3(p, q) = \mathscr{L}(p, q) + mp(\alpha)\overline{q(\alpha)}, \quad p, q \in \mathbb{P}, \tag{16}$$

where $m \in \mathbb{R}$ and $|\alpha| = 1$. Notice that, since $m \in \mathbb{R}$, such a bilinear functional is also Hermitian.

Proposition 4.1. $\mathscr{L}_3$ *is quasi-definite if and only if* $1 + mK_{n-1}(\alpha, \alpha) \neq 0$ *for every* $n \geqslant 1$.

Proof. Let $\{U_n\}_{n \geqslant 0}$ be the sequence of monic polynomials orthogonal with respect to $\mathscr{L}_3$. We wish to obtain a relation between the nth monic polynomials U_n and P_n, where P_n denotes the nth monic orthogonal polynomial with respect to $\mathscr{L}$.

We can write

$$U_n(z) = P_n(z) + \sum_{j=0}^{n-1} \lambda_{n,j} P_j(z), \tag{17}$$

where $\{\lambda_{n,j}\}_{j=0}^{n-1}$ are the Fourier coefficients given by

$$\lambda_{n,j} = \frac{\mathscr{L}(U_n, P_j)}{\mathscr{L}(P_j, P_j)}, \quad j = 0, \ldots, n-1.$$

From (16) and the orthogonality conditions of U_n with respect to $\mathscr{L}_3$, one has

$$\lambda_{n,j} = \frac{\mathscr{L}_3(U_n, P_j) - mU_n(\alpha)\overline{P_j(\alpha)}}{\mathscr{L}(P_j, P_j)} = -\frac{mU_n(\alpha)\overline{P_j(\alpha)}}{\mathscr{L}(P_j, P_j)}, \quad j = 0, \ldots, n-1.$$

Thus,

$$U_n(z) = P_n(z) + \sum_{j=0}^{n-1} \lambda_{n,j} P_j(z) = P_n(z) - mU_n(\alpha)K_{n-1}(z, \alpha).$$

If we set $z = \alpha$ in the last formula, then one can easily obtain that

$$P_n(\alpha) = U_n(\alpha)(1 + mK_{n-1}(\alpha, \alpha)).$$

If $1 + mK_{n_0-1}(\alpha, \alpha) = 0$ for some n_0, then $P_{n_0}(\alpha) = 0$. This means $1 + mK_{n_0}(\alpha, \alpha) = 0$, i.e., $P_{n_0+1}(\alpha) = 0$. As a consequence

$$P_n(\alpha) = 0 \quad \text{for every } n \geqslant n_0.$$

On the other hand, taking into account $|\alpha| = 1$, $P_{n_0}^*(\alpha) = 0$, i.e., $P_{n_0-1}(\alpha) = 0$, from the backward recurrence relation and thus $P_1(\alpha) = 0$, hence $P_1(z) = z - \alpha$ and $|P_1(0)| = 1$, a contradiction. Hence $1 + mK_{n-1}(\alpha, \alpha) \neq 0$ for every $n \geqslant 1$.
Conversely, assume that $1 + mK_{n-1}(\alpha, \alpha) \neq 0$ for every $n \geqslant 1$, and define the polynomial

$$U_n(z) = P_n(z) - \frac{mP_n(\alpha)}{1 + mK_{n-1}(\alpha, \alpha)} K_{n-1}(z, \alpha). \tag{18}$$

Then, for $0 \leqslant k \leqslant n$,

$$\mathscr{L}_3(U_n(z), (z-\alpha)^k)) = \mathscr{L}(P_n(z), (z-\alpha)^k) - \frac{m P_n(\alpha)}{1 + m K_{n-1}(\alpha, \alpha)} \mathscr{L}(K_{n-1}(z, \alpha), (z-\alpha)^k)$$

$$= \mathbf{k}_n \delta_{n,k} - \frac{m P_n(\alpha)}{1 + m K_{n-1}(\alpha, \alpha)} \mathscr{L}\left(K_n(z, \alpha) - \frac{\overline{P_n(\alpha)}}{\mathbf{k}_n} P_n(z), (z-\alpha)^k\right)$$

$$= \mathbf{k}_n \delta_{n,k} + m \frac{P_n(\alpha)\overline{P_n(\alpha)}}{1 + m K_{n-1}(\alpha, \alpha)} \delta_{n,k}$$

$$= \mathbf{k}_n \frac{1 + m K_n(\alpha, \alpha)}{1 + m K_{n-1}(\alpha, \alpha)} \delta_{n,k}.$$

Thus, $\{U_n\}_{n \geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to $\mathscr{L}_3$. $\square$

If $\mathscr{L}$ is positive definite and $m > 0$, then $1 + m K_n(\alpha, \alpha) > 0$ for every $n \geqslant 0$. Thus, $\mathscr{L}_3$ is also positive definite.

In order to rewrite (18) in terms of the sequences $\{\varphi_n\}_{n \geqslant 0}$ and $\{\sigma_n\}_{n \geqslant 0}$ of orthonormal polynomials with respect to $\mathscr{L}$ and $\mathscr{L}_3$, respectively, we must compute the norm of U_n. Indeed,

$$\|U_n\|_{\mathscr{L}_3}^2 = \mathscr{L}_3(U_n, U_n) = \frac{1}{\kappa_n^2} \frac{1 + m K_n(\alpha, \alpha)}{1 + m K_{n-1}(\alpha, \alpha)}.$$

Therefore,

$$\sigma_n(z) = \left(\frac{1}{\kappa_n \|U_n\|_{\mathscr{L}_3}}\right) \varphi_n(z) - \sum_{j=0}^{n-1} \frac{m P_n(\alpha)}{\|U_n\|_{\mathscr{L}_3}(1 + m K_{n-1}(\alpha, \alpha))} \overline{\varphi_j(\alpha)} \varphi_j(z)$$

$$= \left(\sqrt{\frac{1 + m K_{n-1}(\alpha, \alpha)}{1 + m K_n(\alpha, \alpha)}}\right) \varphi_n(z) - \sum_{j=0}^{n-1} \frac{m \varphi_n(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{(1 + m K_n(\alpha, \alpha))(1 + m K_{n-1}(\alpha, \alpha))}} \varphi_j(z). \quad (19)$$

Hence, we have shown the following:

Proposition 4.2. *Let L_1 be the lower triangular matrix such that $\sigma = L_1 \varphi$, where $\sigma = [\sigma_0, \sigma_1, \dots]^t$ and $\varphi = [\varphi_0, \varphi_1, \dots]^t$. Then, the entries $l_{i,j}^{(1)}$ of L_1 are given by*

$$l_{i,j}^{(1)} = \begin{cases} \sqrt{\dfrac{1 + m K_{i-1}(\alpha, \alpha)}{1 + m K_i(\alpha, \alpha)}}, & i = j, \\[3mm] -\dfrac{m \varphi_i(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{(1 + m K_i(\alpha, \alpha))(1 + m K_{i-1}(\alpha, \alpha))}}, & i > j. \end{cases}$$

To compute L_1^{-1}, just set $P_n(z) = U_n(z) + \sum_{j=0}^{n-1} \gamma_{n,j} U_j(z)$ where now, for $0 \leqslant j \leqslant n-1$,

$$\gamma_{n,j} = \frac{\mathscr{L}_3(P_n, U_j)}{\mathscr{L}_3(U_j, U_j)} = \frac{\mathscr{L}(P_n, U_j) + m P_n(\alpha)\overline{U_j(\alpha)}}{\|U_j\|_{\mathscr{L}_3}^2} = \frac{m P_n(\alpha)\overline{U_j(\alpha)}}{\|U_j\|_{\mathscr{L}_3}^2}.$$

Thus, for orthonormal polynomials one has

$$\varphi_n(z) = \kappa_n \|U_n\|_{\mathscr{L}_3} \sigma_n(z) + \sum_{j=0}^{n-1} \frac{m \varphi_n(\alpha)\overline{U_j(\alpha)}}{\|U_j\|_{\mathscr{L}_3}} \sigma_j(z)$$

$$= \sqrt{\frac{1 + m K_n(\alpha, \alpha)}{1 + m K_{n-1}(\alpha, \alpha)}} \sigma_n(z) + \sum_{j=0}^{n-1} \frac{m \varphi_n(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{(1 + m K_j(\alpha, \alpha))(1 + m K_{j-1}(\alpha, \alpha))}} \sigma_j(z).$$

Thus, we get

Proposition 4.3. *The entries $\hat{l}_{i,j}^{(1)}$ of L_1^{-1} are given by*

$$\hat{l}_{i,j}^{(1)} = \begin{cases} \sqrt{\dfrac{1 + mK_i(\alpha, \alpha)}{1 + mK_{i-1}(\alpha, \alpha)}}, & i = j, \\[4mm] \dfrac{m\varphi_i(\alpha)\overline{\varphi_j(\alpha)}}{\sqrt{(1 + mK_j(\alpha, \alpha))(1 + mK_{j-1}(\alpha, \alpha))}}, & i > j. \end{cases}$$

We will see how we can use the results in Section 2 to establish a relation between the Hessenberg matrix associated with the original bilinear functional $\mathscr{L}$ and the Hessenberg matrix corresponding to $\mathscr{L}_3$.

Notice that if we apply to $\mathscr{L}_3$ the transformation defined in (1) we get

$$|z - \alpha|^2 \mathscr{L}_3 = |z - \alpha|^2 \mathscr{L} = \mathscr{L}_2. \tag{20}$$

Hence, on one hand, by Section 2 we know that the QR-factorization of $(H_\varphi - \alpha I)^*$ is M^*L^*, where M is the matrix given by (9) and L is such that $\varphi = L\psi$. By Proposition 3.6, $H_\psi - \alpha I = ML$.

Taking into account (20), we can apply the same process to the functional $\mathscr{L}_3$. Therefore, for the orthonormal polynomials $\{\sigma_n\}_{n \geqslant 0}$ and $\{\psi_n\}_{n \geqslant 0}$ with respect to $\mathscr{L}_3$ and $\mathscr{L}_2$, respectively, we have the following relations:

$$(z - \alpha)\psi = M_3\sigma \quad \text{and} \quad \sigma = L_3\psi.$$

Then, again the QR-factorization of $(H_\sigma - \alpha I)^*$ is $M_3^*L_3^*$, and, by Proposition 3.6, $H_\psi - \alpha I = M_3L_3$.

Proposition 4.4.

$$L_3 = L_1L \quad \text{and} \quad M_3 = ML_1^{-1}. \tag{21}$$

Proof. Since $\sigma = L_1\varphi$ and $\varphi = L\psi$, then $\sigma = L_1L\psi$. On the other hand, $\sigma = L_3\psi$. Thus, $L_3 = L_1L$.

Since $H_\psi - \alpha I = ML = M_3L_3$, then

$$M_3 = MLL_3^{-1} = ML(L^{-1}L_1^{-1}) = ML_1^{-1}. \qquad \square$$

Therefore, to compute $H_\sigma - \alpha I$ from $H_\varphi - \alpha I$, we just need to apply the QR factorization of $(H_\varphi - \alpha I)^*$ to obtain the matrices M and L. Then we compute M_3 and L_3 according to the formulas given in (4.4) and finally, $H_\sigma - \alpha I = L_3M_3$.

On the other hand, for the leading principal submatrices we get

Proposition 4.5. *Let $(H_\varphi - \alpha I)_n$ be the leading principal submatrix of order n of $H_\varphi - \alpha I$, and consider the factorization $(H_\varphi - \alpha I)_n = R_n Q_n$ where R_n is a lower triangular matrix and Q_n is a unitary matrix such that $(H_\varphi - \alpha I)_n^* = Q_n^* R_n^*$. Then*

$$(H_\sigma - \alpha I)_{n-1} = (\hat{L}_{11} R_n Q_n \hat{L}_{11}^{-1})_{n-1},$$

where $\hat{L}_{11}$ is the leading principal submatrix of order n of the matrix L_1, that satisfies $\sigma = L_1\varphi$.

Proof. Consider the factorization $H_\varphi - \alpha I = LM$, and let L_{11}, M_{11} be the leading principal submatrices of order n of L and M, respectively. Then

$$(M_3)_n = M_{11}\hat{L}_{11}^{-1} + M_{12}\tilde{L}_{21}$$

and

$$(L_3)_n = \hat{L}_{11}L_{11},$$

where

$$L_1^{-1} = \left[\begin{array}{c|c} \hat{L}_{11}^{-1} & 0 \\ \hline \tilde{L}_{21} & \tilde{L}_{22} \end{array} \right].$$

Hence

$$(L_3 M_3)_n = \hat{L}_{11} L_{11} (M_{11} \hat{L}_{11}^{-1} + M_{12} \tilde{L}_{21}),$$

but $\hat{L}_{11} L_{11} M_{11} \hat{L}_{11}^{-1}$ and $(L_3 M_3)_n$ differ in the last row. As a consequence,

$$(H_\sigma - \alpha I)_{n-1} = (L_3 M_3)_{n-1} = (\hat{L}_{11} L_{11} M_{11} \hat{L}_{11}^{-1})_{n-1}.$$

Thus,

$$(H_\sigma - \alpha I)_{n-1} = (\hat{L}_{11} R_n Q_n \hat{L}_{11}^{-1})_{n-1},$$

since $R_n Q_n = L_{11} E^{-1} E M_{11} = L_{11} M_{11}$. $\square$

In order to illustrate the procedure of finding $H_\sigma - \alpha I$, consider the bilinear functional $\mathscr{L}_3$ defined by

$$\mathscr{L}_3(p, q) = \int_{-\pi}^{\pi} p(e^{i\theta}) \overline{q(e^{i\theta})} \frac{d\theta}{2\pi} + p(1)\overline{q(1)}.$$

It is straightforward consequence that $1 + K_n(1, 1) = n + 2$, for every $n \geqslant 0$.

From (13) and (14) we get the entries $m_{i,j}$ of M and $l_{i,j}$ of L, respectively,

$$m_{i,j} = \begin{cases} -\dfrac{1}{\sqrt{(i+2)(i+1)}} & \text{if } j \leqslant i \\ \sqrt{\dfrac{i+1}{i+2}} & \text{if } j = i+1 \\ 0 & \text{if } j > i+1 \end{cases} \quad \text{and} \quad l_{i,j} = \begin{cases} -\sqrt{\dfrac{i}{i+1}} & \text{if } j = i-1, \\ \sqrt{\dfrac{i+2}{i+1}} & \text{if } j = i, \\ 0 & \text{if } j < i+1, \\ 0 & \text{if } j > i. \end{cases}$$

The entries $l_{i,j}^{(1)}$ and $\hat{l}_{i,j}^{(1)}$ of the lower triangular matrices L_1 and L_1^{-1}, respectively, are

$$l_{i,j}^{(1)} = \begin{cases} \sqrt{\dfrac{i+1}{i+2}} & \text{if } i = j, \\ -\dfrac{1}{\sqrt{(i+1)(i+2)}} & \text{if } i > j, \end{cases} \qquad \hat{l}_{i,j}^{(1)} = \begin{cases} \sqrt{\dfrac{i+2}{i+1}} & \text{if } i = j, \\ \dfrac{1}{\sqrt{(i+1)(i+2)}} & \text{if } i > j. \end{cases}$$

The next step is compute L_3 and M_3. These matrices are given by (21), and as consequence, the entries $(m_3)_{i,j}$ of M_3 are

$$(m_3)_{i,j} = \begin{cases} -\dfrac{1}{(i+2)(i+1)} & \text{if } j = i, \\ \dfrac{\sqrt{(i+1)(i+3)}}{i+2} & \text{if } j = i+1, \\ 0 & \text{if } j > i+1 \text{ or } j < i, \end{cases}$$

and, the entries $(l_3)_{i,j}$ of L_3 are

$$(l_3)_{i,j} = \begin{cases} -\dfrac{i+1}{\sqrt{i(i+2)}} & \text{if } j = i-1, \\ 1 & \text{if } j = i, \\ -\dfrac{1}{\sqrt{(i+1)(i+2)(j+1)(j+2)}} & \text{if } j < i-1, \\ 0 & \text{if } j > i. \end{cases}$$

Finally, to obtain $H_\sigma - I$, we need to multiply L_3 by M_3. This result is following:

$$(H_\sigma - I)_{i,j} = \begin{cases} -\dfrac{1}{(i+1)(i+2)} - 1 & \text{if } j = i, \\[2mm] \dfrac{\sqrt{(i+1)(i+3)}}{i+2} & \text{if } j = i+1, \\[2mm] 0 & \text{if } j > i+1, \\[2mm] -\dfrac{j+3}{(j+2)^{\frac{3}{2}}\sqrt{(i+1)(i+2)(j+1)}} & \text{if } j < i. \end{cases}$$

5. Perturbation

$$\mathscr{L}_4(p, q) = \mathscr{L}(p, q) + mp(\alpha)\bar{q}(\alpha^{-1}) + mp(\bar{\alpha}^{-1})\bar{q}(\bar{\alpha}), \quad |\alpha| \neq 1.$$

Let consider now the perturbation $\mathscr{L}_4$ of the original bilinear functional $\mathscr{L}$ defined by

$$\mathscr{L}_4(p, q) = \mathscr{L}(p, q) + mp(\alpha)\bar{q}(\alpha^{-1}) + mp(\bar{\alpha}^{-1})\bar{q}(\bar{\alpha}), \tag{22}$$

with $|\alpha| \neq 1$ and $m \in \mathbb{R}$.

Notice that $\mathscr{L}_4$ is a Hermitian bilinear functional.

Proposition 5.1. *The bilinear functional $\mathscr{L}_4$ is quasi-definite if and only if*

$$\Delta_n := \begin{vmatrix} 1 + mK_n(\alpha, \bar{\alpha}^{-1}) & mK_n(\alpha, \alpha) \\ mK_n(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) & 1 + mK_n(\bar{\alpha}^{-1}, \alpha) \end{vmatrix} \neq 0,$$

for every $n \geqslant 0$.

Proof. Assume that $\mathscr{L}_4$ is quasi-definite, and let $\{V_n\}_{n \geqslant 0}$ be the sequence of monic polynomials orthogonal with respect to $\mathscr{L}_4$. Then

$$V_n(z) = P_n(z) + \sum_{j=0}^{n-1} \lambda_{n,j} P_j(z),$$

where

$$\lambda_{n,j} = \frac{\mathscr{L}(V_n, P_j)}{\mathscr{L}(P_j, P_j)} = -\frac{m}{\mathbf{k}_j}(V_n(\alpha)\bar{P}_j(\alpha^{-1}) + V_n(\bar{\alpha}^{-1})\overline{P_j(\alpha)}).$$

Hence,

$$V_n(z) = P_n(z) - mV_n(\alpha)\sum_{j=0}^{n-1} \mathbf{k}_j^{-1}\overline{P_j(\bar{\alpha}^{-1})}P_j(z) - mV_n(\bar{\alpha}^{-1})\sum_{j=0}^{n-1} \mathbf{k}_j^{-1}\overline{P_j(\alpha)}P_j(z)$$

$$= P_n(z) - mV_n(\alpha)K_{n-1}(z, \bar{\alpha}^{-1}) - mV_n(\bar{\alpha}^{-1})K_{n-1}(z, \alpha).$$

If we evaluate the above expression for $z = \alpha$ and $z = \bar{\alpha}^{-1}$, respectively, then we get

$$-P_n(\alpha) + (1 + mK_{n-1}(\alpha, \bar{\alpha}^{-1}))V_n(\alpha) + mK_{n-1}(\alpha, \alpha)V_n(\bar{\alpha}^{-1}) = 0,$$

$$-P_n(\bar{\alpha}^{-1}) + mK_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1})V_n(\alpha) + (1 + mK_{n-1}(\bar{\alpha}^{-1}, \alpha))V_n(\bar{\alpha}^{-1}) = 0.$$

According to the uniqueness of the values $V_n(\alpha)$ and $V_n(\bar{\alpha}^{-1})$, the matrix of the above linear system

$$B_{n-1} := \begin{bmatrix} 1 + mK_{n-1}(\alpha, \bar{\alpha}^{-1}) & mK_{n-1}(\alpha, \alpha) \\ mK_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) & 1 + mK_{n-1}(\bar{\alpha}^{-1}, \alpha) \end{bmatrix}$$

must be nonsingular. Hence $\Delta_{n-1} = \det B_{n-1} \neq 0$ for every $n \geqslant 1$.

Conversely, we assume that $\Delta_n \neq 0$, for every $n \geqslant 0$, and consider the polynomial

$$V_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} P_n(z) & mK_{n-1}(z, \bar{\alpha}^{-1}) & mK_{n-1}(z, \alpha) \\ P_n(\alpha) & 1 + mK_{n-1}(\alpha, \bar{\alpha}^{-1}) & mK_{n-1}(\alpha, \alpha) \\ P_n(\bar{\alpha}^{-1}) & mK_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) & 1 + mK_{n-1}(\bar{\alpha}^{-1}, \alpha) \end{vmatrix}, \quad n \geqslant 0. \tag{23}$$

For $0 \leqslant k \leqslant n-1$ we get

$$\mathscr{L}_4(V_n(z), (z-\alpha)^k)$$

$$= \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mathscr{L}_4(P_n(z), (z-\alpha)^k) & m\mathscr{L}_4(K_{n-1}(z, \bar{\alpha}^{-1}), (z-\alpha)^k) & m\mathscr{L}_4(K_{n-1}(z, \alpha), (z-\alpha)^k) \\ P_n(\alpha) & 1 + mK_{n-1}(\alpha, \bar{\alpha}^{-1}) & mK_{n-1}(\alpha, \alpha) \\ P_n(\bar{\alpha}^{-1}) & mK_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) & 1 + mK_{n-1}(\bar{\alpha}^{-1}, \alpha) \end{vmatrix}.$$

On the other hand, for $0 \leqslant k \leqslant n-1$,

$$\mathscr{L}_4(P_n(z), (z-\alpha)^k) = mP_n(\alpha)(\alpha^{-1} - \bar{\alpha})^k,$$

$$\mathscr{L}_4(K_{n-1}(z, \bar{\alpha}^{-1}), (z-\alpha)^k) = (\alpha^{-1} - \bar{\alpha})^k(1 + mK_n(\alpha, \bar{\alpha}^{-1})),$$

$$\mathscr{L}_4(K_{n-1}(z, \alpha), (z-\alpha)^k) = mK_{n-1}(\alpha, \alpha)(\alpha^{-1} - \bar{\alpha})^k.$$

Thus,

$$\mathscr{L}_4(V_n(z), (z-\alpha)^k)$$

$$= \frac{m(\bar{\alpha}^{-1} - \bar{\alpha})^k}{\Delta_{n-1}} \begin{vmatrix} P_n(\alpha) & 1 + mK_{n-1}(\alpha, \bar{\alpha}^{-1}) & mK_{n-1}(\alpha, \alpha) \\ P_n(\alpha) & 1 + mK_{n-1}(\alpha, \bar{\alpha}^{-1}) & mK_{n-1}(\alpha, \alpha) \\ P_n(\bar{\alpha}^{-1}) & mK_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) & 1 + mK_{n-1}(\bar{\alpha}^{-1}, \alpha) \end{vmatrix} = 0.$$

Now,

$$\mathscr{L}_4(V_n, V_n) = \mathscr{L}_4(V_n, P_n)$$

$$= \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mathscr{L}_4(P_n, P_n) & m\mathscr{L}_4(K_{n-1}(z, \bar{\alpha}^{-1}), P_n) & m\mathscr{L}_4(K_{n-1}(z, \alpha), P_n) \\ P_n(\alpha) & 1 + mK_{n-1}(\alpha, \bar{\alpha}^{-1}) & mK_{n-1}(\alpha, \alpha) \\ P_n(\bar{\alpha}^{-1}) & mK_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) & 1 + mK_{n-1}(\bar{\alpha}^{-1}, \alpha) \end{vmatrix}.$$

But

$$\mathscr{L}_4(P_n, P_n) = \mathbf{k}_n + mP_n(\alpha)\overline{P_n(\bar{\alpha}^{-1})} + mP_n(\bar{\alpha}^{-1})\overline{P_n(\alpha)},$$

$$\mathscr{L}_4(K_{n-1}(z, \bar{\alpha}^{-1}), P_n) = m(K_{n-1}(\alpha, \bar{\alpha}^{-1})\overline{P_n(\bar{\alpha}^{-1})} + K_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1})\overline{P_n(\alpha)}),$$

$$\mathscr{L}_4(K_{n-1}(z, \alpha), P_n) = m(K_{n-1}(\alpha, \alpha)\overline{P_n(\bar{\alpha}^{-1})} + K_{n-1}(\bar{\alpha}^{-1}, \alpha)\overline{P_n(\alpha)}).$$

Thus,

$$\mathscr{L}_4(V_n, V_n) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mathbf{k}_n & -m\overline{P_n(\alpha^{-1})} & -m\overline{P_n(\alpha)} \\ P_n(\alpha) & 1 + mK_{n-1}(\alpha, \bar{\alpha}^{-1}) & mK_{n-1}(\alpha, \alpha) \\ P_n(\bar{\alpha}^{-1}) & mK_{n-1}(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) & 1 + mK_{n-1}(\bar{\alpha}^{-1}, \alpha) \end{vmatrix}$$

$$= \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mathbf{k}_n & -m\overline{P_n(\alpha^{-1})} & -m\overline{P_n(\alpha)} \\ 0 & 1 + mK_n(\alpha, \bar{\alpha}^{-1}) & mK_n(\alpha, \alpha) \\ 0 & mK_n(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) & 1 + mK_n(\bar{\alpha}^{-1}, \alpha) \end{vmatrix}$$

$$= \mathbf{k}_n \frac{\Delta_n}{\Delta_{n-1}} \neq 0. \quad \square \tag{24}$$

If $\mathscr{L}$ is a positive-definite bilinear functional, then we get

Proposition 5.2. *The bilinear functional $\mathscr{L}_4$ is positive definite if and only if $\Delta_{n+1}\Delta_n > 0$, for every $n \geqslant 0$.*

Proof. This is a straightforward consequence of (24). □

Under these conditions, let $\{\phi_n\}_{n \geqslant 0}$ be the sequence of orthonormal polynomials associated with $\mathscr{L}_4$. Then $\phi_n(z) = (1/\|V_n\|)V_n(z)$, where

$$\|V_n\| = \|P_n\| \sqrt{\frac{\Delta_n}{\Delta_{n-1}}}. \tag{25}$$

Proposition 5.3. *Let $\tilde{L}$ be the lower triangular matrix such that $\phi = \tilde{L}\varphi$, where $\phi = [\phi_0, \phi_1, \dots]^t$ and $\varphi = [\varphi_0, \varphi_1, \dots]^t$. Then, the entries $\tilde{l}_{i,j}$ of $\tilde{L}$ are given by*

$$\tilde{l}_{i,j} = \begin{cases} \sqrt{\dfrac{\Delta_{i-1}}{\Delta_i}}, & i = j, \\[2ex] -\dfrac{m}{\sqrt{\Delta_{i-1}\Delta_i}}(A_i(\alpha)\overline{\varphi_j(\bar{\alpha}^{-1})} + A_i(\bar{\alpha}^{-1})\overline{\varphi_j(\alpha)}), & i > j, \end{cases}$$

where $A_n(z) := \varphi_n(z) + m\varphi_n(z)K_{n-1}(\bar{z}^{-1}, z) - m\varphi_n(\bar{z}^{-1})K_{n-1}(z, z)$. The entries $\tilde{l}_{i,j}^{(-1)}$ of $\tilde{L}^{-1}$ are given by

$$\tilde{l}_{i,j}^{(-1)} = \begin{cases} \dfrac{1}{\sqrt{\Delta_{i-1}\Delta i}}(\Delta_{i-1} + m\varphi_i(\alpha)\overline{A_i(\bar{\alpha}^{-1})} + m\varphi_i(\bar{\alpha}^{-1})\overline{A_i(\alpha)}), & i = j, \\[2ex] \dfrac{m}{\sqrt{\Delta_{j-1}\Delta_j}}(A_i(\alpha)\overline{\varphi_j(\bar{\alpha}^{-1})} + A_i(\bar{\alpha}^{-1})\overline{\varphi_j(\alpha)}), & i > j. \end{cases}$$

Proof. We can express

$$V_i(z) = P_i(z) + \sum_{j=0}^{i-1} \gamma_{i,j} P_j(z),$$

where $\gamma_{i,j} = \mathscr{L}(V_i, P_j)/\|P_j\|^2, 0 \leqslant j \leqslant i - 1$.

Thus, from (25), for orthonormal polynomials we get

$$\phi_i(z) = \sqrt{\frac{\Delta_{i-1}}{\Delta_i}}\varphi_i(z) + \sum_{j=0}^{i-1} \gamma_{i,j}\|P_j\|\sqrt{\frac{\Delta_{i-1}}{\Delta_i}}\varphi_j(z). \tag{26}$$

By (22) and the orthogonality conditions of V_n with respect to $\mathscr{L}_4$

$$\gamma_{i,j} = -\frac{m}{\|P_j\|^2}(V_i(\alpha)\overline{P_j(\bar{\alpha}^{-1})} + V_i(\bar{\alpha}^{-1})\overline{P_j(\alpha)})$$

$$= -\frac{m}{\|P_j\|}(V_i(\alpha)\overline{\varphi_j(\bar{\alpha}^{-1})} + V_i(\bar{\alpha}^{-1})\overline{\varphi_j(\alpha)}).$$

On the other hand, from (23)

$$V_i(\alpha) = \frac{\|P_i\|}{\Delta_{i-1}}A_i(\alpha) \quad \text{and} \quad V_i(\bar{\alpha}^{-1}) = \frac{\|P_i\|}{\Delta_{i-1}}A_i(\bar{\alpha}^{-1}).$$

Thus, if $i > j$ then

$$\tilde{l}_{i,j} = \gamma_{i,j}\|P_j\|\sqrt{\frac{\Delta_{i-1}}{\Delta_i}} = -\frac{m}{\sqrt{\Delta_{i-1}\Delta_i}}(A_i(\alpha)\overline{\varphi_j(\bar{\alpha}^{-1})} + A_i(\bar{\alpha}^{-1})\overline{\varphi_j(\alpha)}).$$

From (26), we get $\tilde{l}_{i,i} = \sqrt{\Delta_{i-1}/\Delta_i}$.

Now, we can also express

$$\varphi_i(z) = \sum_{j=0}^{i} \beta_{i,j} \phi_j(z),$$

with $\beta_{i,j} = \mathscr{L}_4(\varphi_i, \phi_j)$.

Thus,

$$\beta_{i,j} = \mathscr{L}(\varphi_i, \phi_j) + m\varphi_i(\alpha)\overline{\phi_j(\bar{\alpha}^{-1})} + m\varphi_i(\bar{\alpha}^{-1})\overline{\phi_j(\alpha)}.$$

From (23) and (25), we get

$$\phi_j(\alpha) = \frac{1}{\sqrt{\Delta_{j-1}\Delta_j}}A_j(\alpha) \quad \text{and} \quad \phi_j(\bar{\alpha}^{-1}) = \frac{1}{\sqrt{\Delta_{j-1}\Delta_j}}A_j(\bar{\alpha}^{-1}).$$

Hence, if $i > j$

$$\tilde{l}_{i,j}^{(-1)} = \beta_{i,j} = \frac{m}{\sqrt{\Delta_{j-1}\Delta_j}}(\varphi_i(\alpha)\overline{A_j(\bar{\alpha}^{-1})} + \varphi_i(\bar{\alpha}^{-1})\overline{A_j(\alpha)}),$$

and, if $i = j$

$$\tilde{l}_{i,i}^{(-1)} = \beta_{i,i} = \sqrt{\frac{\Delta_{i-1}}{\Delta_i}} + \frac{m}{\sqrt{\Delta_{i-1}\Delta_i}}(\varphi_i(\alpha)\overline{A_i(\bar{\alpha}^{-1})} + \varphi_i(\bar{\alpha}^{-1})\overline{A_i(\alpha)}). \quad \square$$

Notice that if we apply to $\mathscr{L}_4$ the transformation defined in Section 2, i.e., if we set $|z - \alpha|^2 \mathscr{L}_4$, then we get

$$|z - \alpha|^2 \mathscr{L}_4 = |z - \alpha|^2 \mathscr{L} = \mathscr{L}_2. \tag{27}$$

Hence, by Section 2 we know that the QR-factorization of $(H_\varphi - \alpha I)^*$ is $M^* L^*$, where M is the matrix given by (10) and L is such that $\varphi = L\psi$. By Proposition 3.6, $H_\psi - \alpha I = ML$.

Taking into account (27), we can apply the same process to the bilinear functional $\mathscr{L}_4$. Therefore, for the orthonormal polynomials $\{\phi_n\}_{n \geqslant 0}$ and $\{\psi_n\}_{n \geqslant 0}$ with respect to $\mathscr{L}_4$ and $\mathscr{L}_2$, respectively, we get

$$(z - \alpha)\psi = M_4\sigma \quad \text{and} \quad \sigma = L_4\psi.$$

Then, again the QR-factorization of $(H_\phi - \alpha I)^*$ is $M_4^* L_4^*$ and, from Proposition 3.6, $H_\psi - \alpha I = M_4 L_4$.

Proposition 5.4.

$$L_4 = \tilde{L}L \quad \text{and} \quad M_4 = M\tilde{L}^{-1}, \tag{28}$$

where M is given by (10) and L is given in the Proposition 3.4.

The finite version of the above result is presented in the following:

Proposition 5.5. Let $(H_\varphi - \alpha I)_n$ be the leading principal submatrix of order n of $H_\varphi - \alpha I$ and consider the factorization $(H_\varphi - \alpha I)_n = R_n Q_n$ where R_n is a lower triangular matrix and Q_n is a unitary matrix such that $(H_\varphi - \alpha I)_n^* = Q_n^* R_n^*$. Then

$$(H_\phi - \alpha I)_{n-1} = (\tilde{L}_{11} R_n Q_n \tilde{L}_{11}^{-1})_{n-1},$$

where $\tilde{L}_{11}$ is the leading principal submatrix of order n of the matrix $\tilde{L}$, that satisfies $\phi = \tilde{L}\varphi$.

Acknowledgements

References

[1] G.S. Ammar, W.B. Gragg, L. Reichel, Downdating of Szegő polynomials and data fitting applications, Linear Algebra Appl. 172 (1992) 315–336.

[2] A. Branquinho, L. Golinskii, F. Marcellán, Orthogonal polynomials and rational modification of the Lebesgue measure on the unit circle. An inverse problem, Complex Variables 38 (1999) 137–154.

[3] M.I. Bueno, F. Marcellán, Darboux transformation and perturbation of linear functionals, Linear Algebra Appl. 384 (2004) 215–242.

[4] M.I. Bueno, F. Marcellán, Polynomial perturbations of bilinear functionals and Hessenberg matrices, Linear Algebra Appl. 414 (2006) 64–83.

[5] M. Buhmann, A. Iserles, On orthogonal polynomials transformed by the QR algorithm, J. Comput. Appl. Math. 43 (1992) 117–134.

[6] M.J. Cantero, L. Moral, L. Velázquez, Five-diagonal matrices and zeros of orthogonal polynomials, Linear Algebra Appl. 362 (2003) 29–56.

[7] Ya.L. Geronimus, Polynomials orthogonal on a circle and their applications, Amer. Math. Soc. Trans. 3 (1) (1962) 1–78.

[8] E. Godoy, F. Marcellán, An analogue of the Christoffel formula for polynomial modification of a measure on the unit circle, Boll. Un. Mat. Ital. A 5 (1991) 1–12.

[9] E. Godoy, F. Marcellán, Orthogonal polynomials and rational modification of measures, Canad. J. Math. 45 (1993) 930–943.

[10] L. Golinskii, P. Nevai, F. Pintér, W. Van Assche, Perturbation of orthogonal polynomials on an arc of the unit circle, II, J. Approx. Theory 96 (1999) 1–32.

[11] L. Golinskii, P. Nevai, W. Van Assche, Perturbation of orthogonal polynomials on an arc of the unit circle, J. Approx. Theory 83 (1995) 392–422.

[12] W.B. Gragg, The QR algorithm for unitary Hessenberg matrices, J. Comput. Appl. Math. 16 (1986) 1–8.

[13] U. Grenander, G. Szegő, Toeplitz forms and their Applications, second ed., University of California Press, Berkeley, 1958 Chelsea, New York, 1984.

[14] W.B. Jones, O. Njastad, W.J. Thron, Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, Bull. London Math. Soc. 21 (1989) 113–152.

[15] T. Kailath, B. Porat, State-space generators for orthogonal polynomials, in: V. Mandrekar, H. Salehi (Eds.), Prediction Theory and Harmonic Analysis, The Pesi Masani Volume, North-Holland, Amsterdam, 1983, pp. 131–163.

[16] X. Li, F. Marcellán, Representation of orthogonal polynomials for modified measures, Comm. Anal. Theory Cont. Fract. 7 (1999) 9–22.

[17] F. Marcellán, M. Alfaro, Recent trends in orthogonal polynomials on the unit circle, in: C. Brezinski et al. (Eds.), Proceedings Erice International Symposium on Orthogonal Polynomials and their Applications, IMACS Annals on Computing and Application in Mathematics J. C. Baltzer, Basel, 1991, pp. 3–14.

[18] F. Marcellán, F. Peherstorfer, R. Steinbauer, Orthogonality properties of linear combinations of orthogonal polynomials I, Adv. Comput. Math. 5 (1996) 281–295.

[19] F. Marcellán, F. Peherstorfer, R. Steinbauer, Orthogonality properties of linear combinations of orthogonal polynomials II, Adv. Comput. Math. 7 (1997) 401–428.

[20] L. Reichel, G.S. Ammar, W.B. Gragg, Discrete least squares approximation by trigonometric polynomials, Math. Comput. 57 (1991) 273–289.

[21] B. Simon, Orthogonal Polynomials on the Unit Circle, AMS Colloquium Publication Series, vol. 54, AMS Providence, RI, 2005.

[22] V. Spiridonov, A. Zhedanov, Spectral transformations self-similar reductions and orthogonal polynomials, J. Phys. A: Math. Gen. 30 (1997) 7621–7637.

[23] G.J. Yoon, Darboux transforms and orthogonal polynomials, Bull. Korean Math. Soc. 39 (2002) 359–376.

[24] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997) 67–86.