Second structure relation for q-semiclassical polynomials of the Hahn Tableau ☆

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Abstract

The q-classical orthogonal polynomials of the q-Hahn Tableau are characterized from their orthogonality condition and by a first and a second structure relation. Unfortunately, for the q-semiclassical orthogonal polynomials (a generalization of the classical ones) we find only in the literature the first structure relation. In this paper, a second structure relation is deduced. In particular, by means of a general finite-type relation between a q-semiclassical polynomial sequence and the sequence of its q-differences such a structure relation is obtained.

Keywords: Finite-type relation; Recurrence relation; q-Polynomials; q-Semiclassical polynomials

1. Introduction

The q-classical orthogonal polynomial sequences (Big q-Jacobi, q-Laguerre, Al-Salam Carlitz I, q-Charlier, etc.) are characterized by the property that the sequence of its monic q-difference polynomials is, again, orthogonal (Hahn's property, see [6]). In fact, the q-difference operator is a particular case of the Hahn operator which is defined as

$$L_{q,\omega}(f)(x) = \frac{f(qx+\omega) - f(x)}{(q-1)x+\omega}, \quad \omega \in \mathbb{C},\ q \in \mathbb{C},\ |q| \neq 1.$$

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In the sequel, we are going to work with q-semiclassical orthogonal polynomials and q-classical polynomials of the Hahn Tableau, hence we will consider the q-linear lattice $x(s)$, i.e., $x(s+1) = qx(s)+\omega$. Therefore, for the sake of convenience we will denote $\Delta^{(1)} \equiv L_{q,\omega}$. Notice that for $q=1$ we get the forward difference operator Δ. In such a case, when $w \to 0$ we recover the standard semiclassical orthogonal polynomials [13].

Taking into account the role of such families of q-polynomials in the analysis of hypergeometric q-difference equations resulting from physical problems as the q-Schrödinger equation, q-harmonic oscillators, the connection and the linearization problems among others there is an increasing interest to study them. Moreover, the connection between the representation theory of quantum algebras and the q-orthogonal polynomials is well known (see [2] and references therein).

We also find many different approaches to the subject in the literature. For instance, the functional equation (the so-called Pearson equation) satisfied by the corresponding moment functionals allows an efficient study of some properties of q-classical polynomials [3,7,8,17]. However, the q-classical sequences of orthogonal polynomials $\{C_n\}_{n\geqslant 0}$ can also be characterized taking into account its orthogonality as well as one of the two following difference equations, the so-called structure relations.

- First structure relation [1,9,18]

$$\Phi(s)C_n^{[1]}(s) = \sum_{\nu=n}^{n+t} \lambda_{n,\nu} C_\nu(s), \quad n \geqslant 0, \qquad \lambda_{n,n} \neq 0, \quad n \geqslant 0, \tag{1}$$

  where Φ is a polynomial with $\deg \Phi = t \leqslant 2$ and $C_n^{[1]}(s) := [n+1]^{-1}\Delta^{(1)}C_{n+1}(s)$, being

$$[n] := \big(q^n - 1\big)/(q-1), \quad n \geqslant 0.$$

•• Second structure relation [16,17]

$$C_n(s) = \sum_{\nu=n-t}^{n} \theta_{n,\nu} C_\nu^{[1]}(s), \quad n \geqslant t,\ 0 \leqslant t \leqslant 2, \qquad \theta_{n,n} = 1, \quad n \geqslant t. \tag{2}$$

The q-classical orthogonal polynomials were introduced by W. Hahn [6] and also analyzed in [1]. The generalization of this families leads to q-semiclassical orthogonal polynomials which were introduced by P. Maroni and extensively studied in the last decade by himself, L. Kheriji, J.C. Medem, and others (see [7,16]).

For q-classical orthogonal polynomial sequences, which are q-semiclassical of class zero, the structure relations (1) and (2) become

$$\begin{aligned}
&\phi(s)L_{q,\omega}P_n(s) = \tilde{\alpha}_n P_{n+1}(s) + \tilde{\beta}_n P_n(s) + \tilde{\gamma}_n P_{n-1}(s), \quad \tilde{\gamma}_n \neq 0,\\
&\sigma(s)L_{1/q,\omega/q}P_n(s) = \hat{\alpha}_n P_{n+1}(s) + \hat{\beta}_n P_n(s) + \hat{\gamma}_n P_{n-1}(s), \quad \hat{\gamma}_n \neq 0,\\
&P_n(s) = P_n^{[1]}(s) + \delta_n P_{n-1}^{[1]}(s) + \epsilon_n P_{n-2}^{[1]}(s).
\end{aligned}$$

In particular, in Table 1 we describe these parameters for some families of q-classical orthogonal polynomials.

The first structure relation for the q-semiclassical orthogonal polynomials was established (see [7]), and it reads as follows.

Table 1
Some families of q-polynomials of the Hahn Tableau

(A_1)	Big q-Jacobi $\widehat{P}_n(x;a,b,c;q)$, $x \equiv x(s) = q^s$,
	$P_n^{[1]}(x;a,b,c;q) = q^{-n}\widehat{P}_n(qx;aq,bq,cq;q)$,
	$\phi(x) = aq(x-1)(bx-c)$, $\sigma(x) = q^{-1}(x-aq)(x-cq)$,
	$\hat{\alpha}_n = abq[n]$, $\tilde{\alpha}_n = q^{-n}[n]$,
	$\hat{\beta}_n = -aq[n](1-abq^{n+1})\frac{c+ab^2q^{2n+1}+b(1-cq^n-cq^{n+1}-aq^n(1+q-cq^{n+1}))}{(1-abq^{2n})(1-abq^{2n+2})}$,
	$\tilde{\beta}_n = q[n](1-abq^{n+1})\frac{c+a^2bq^{2n+1}+a(1-cq^n-cq^{n+1}-bq^n(1+q-cq^{n+1}))}{(1-abq^{2n})(1-abq^{2n+2})}$,
	$\hat{\gamma}_n = aq[n]\frac{(1-aq^n)(1-bq^n)(1-abq^n)(c-abq^n)(1-cq^n)(1-abq^{n+1})}{(1-abq^{2n})^2(1-abq^{2n-1})(1-abq^{2n+1})}$,
	$\tilde{\gamma}_n = q^n\hat{\gamma}_n$, $\delta_n = -\frac{q^n(1-q)}{1-abq^{n+1}}\hat{\beta}_n$, $\epsilon_n = abq^{2n}\frac{(1-q^{n-1})(1-q)}{(1-abq^n)(1-abq^{n+1})}\hat{\gamma}_n$.
(A_2)	q-Laguerre $\widehat{L}_n^{(\alpha)}(x;q)$, $x \equiv x(s) = q^s$,
	$L_n^{[1]\,(\alpha)}(x;q) = q^{-n}\widehat{L}_n^{(\alpha+1)}(qx;q)$,
	$\phi(x) = ax(x+1)$, $\sigma(x) = q^{-1}x$,
	$\hat{\alpha}_n = a[n]$, $\hat{\beta}_n = q^{-2n-1}[n](1+q-aq^{n+1})$, $\hat{\gamma}_n = a^{-1}q^{1-4n}[n](1-aq^n)$,
	$\tilde{\alpha}_n = 0$, $\tilde{\beta}_n = q^{-n}[n]$, $\tilde{\gamma}_n = a^{-1}q^{1-3n}(1-aq^n)$,
	$\delta_n = a^{-1}(1-q)\hat{\beta}_n$, $\epsilon_n = a^{-1}(1-q^{n-1})(1-q)\hat{\gamma}_n$.
(A_3)	Al-Salam Carlitz I $\widehat{U}_n^{(a)}(x;q)$, $x \equiv x(s) = q^s$,
	$U_n^{[1](a)}(x;q) = \widehat{U}_n^{(a)}(x;q)$,
	$\phi(x) = a$, $\sigma(x) = (1-x)(a-x)$, $\tilde{\alpha}_n = q^{1-n}[n]$, $\tilde{\beta}_n = q(1+a)[n]$, $\tilde{\gamma}_n = aq^n[n]$.
(A_4)	q-Charlier $\widehat{C}_n(q^{-s};a;q)$,
	$C_n^{[1]}(q^{-s};a;q) = \widehat{C}_n(q^{-s};aq^{-1};q)$,
	$\phi(x) = x(x-1)$, $\sigma(x) = q^{-1}ax$,
	$\hat{\alpha}_n = [n]$, $\hat{\beta}_n = q^{-2n-1}[n](a+aq+q^{n+1})$, $\hat{\gamma}_n = aq^{1-4n}[n](a+q^n)$,
	$\tilde{\alpha}_n = 0$, $\tilde{\beta}_n = aq^{-n}[n]$, $\tilde{\gamma}_n = q^n\hat{\gamma}_n$, $\delta_n = (1-q)\hat{\beta}_n$, $\epsilon_n = (1-q^{n-1})(1-q)\hat{\gamma}_n$.

An orthogonal polynomial sequence, $\{B_n\}_{n\geqslant 0}$, is said to be *q-semiclassical* if

$$\Phi(s)B_n^{[1]}(s) = \sum_{\nu=n-\sigma}^{n+t}\lambda_{n,\nu}B_\nu(s), \quad n \geqslant \sigma, \qquad \lambda_{n,n-\sigma} \neq 0, \quad n \geqslant \sigma+1,$$

where Φ is a polynomial of degree t and σ is a non-negative integer such that $\sigma \geqslant \max\{t-2, 0\}$.

Recently, F. Marcellán and R. Sfaxi [12] have established a second structure relation for the standard semiclassical polynomials which reads as follows:

Theorem 1.1. *For any integer $\sigma \geqslant 0$, any monic polynomial Φ, with* $\deg\Phi = t \leqslant \sigma+2$, *and any SMOP $\{B_n\}_{n\geqslant 0}$ with respect to a linear functional u, the following statements are equivalent*:

(i) *There exist an integer* $p \geqslant 1$ *and an integer* $r \geqslant \sigma + t + 1$, *with* $\sigma = \max(t-2, p-1)$, *such that*

$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(x) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} B_\nu^{[1]}(x), \quad n \geqslant \max(\sigma, t+1), \tag{3.36}$$

where $B_n^{[1]}(x) = (n+1)^{-1} B'_{n+1}(x)$,

$$\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1, \quad n \geqslant \max(\sigma, t+1), \qquad \xi_{r,r-\sigma}\varsigma_{r,r-t} \neq 0,$$
$$\big\langle (\Phi u)', B_n \big\rangle = 0, \quad p+1 \leqslant n \leqslant 2\sigma + t + 1, \qquad \big\langle (\Phi u)', B_p \big\rangle \neq 0 \quad (\sigma \geqslant 1),$$

and if $p = t-1$ *then* $\langle u, B_p^2 \rangle^{-1} \langle u, \Phi B'_p \rangle \notin \mathbb{N}^*$.

(ii) *The linear functional* u *satisfies*

$$(\Phi u)' + \Psi u = 0,$$

where the pair (Φ, Ψ) *is admissible, i.e., the polynomial* Φ *is monic,* $\deg \Phi = t$, $\deg \Psi = p \geqslant 1$, *and if* $p = t-1$ *then* $\frac{1}{n!}\Psi^{(n)}(0) \notin -\mathbb{N}^*$, *with associated integer* σ.

Now, we are going to extend this result for the q-semiclassical polynomials of the Hahn Tableau.

Some years ago, P. Maroni and R. Sfaxi [15] introduced the concept of diagonal sequence for the standard semiclassical polynomials. The following definition extends this definition to the q-semiclassical case.

Definition 1.1. Let $\{B_n\}_{n\geqslant 0}$ be a sequence of monic orthogonal polynomials and ϕ a monic polynomial with $\deg \phi = t$. When there exists an integer $\sigma \geqslant 0$ such that

$$\phi(s) B_n(s) = \sum_{\nu=n-\sigma}^{n+t} \theta_{n,\nu} B_\nu^{[1]}(s), \quad \theta_{n,n-\sigma} \neq 0, \quad n \geqslant \sigma, \tag{3}$$

the sequence $\{B_n\}_{n\geqslant 0}$ is said to be *diagonal associated with* ϕ *and index* σ.

Obviously, the above finite-type relation, that we will call diagonal relation, is nothing else that an example of second structure relation for such a family. But, some q-semiclassical orthogonal polynomials are not diagonal. As an example, we can mention the case of a q-semiclassical polynomial sequence $\{Q_n\}_{n\geqslant 0}$ orthogonal with respect to the linear functional v, such that the functional equation $\Delta^{(1)} v = \Psi v$, with $\deg \Psi = 2$, holds. In fact, the sequence $\{Q_n\}_{n\geqslant 0}$ satisfies the following relation:

$$\big(x(s+1) + v_{n,0}\big) Q_n(s) = q\, Q_{n+1}^{[1]}(s) + \rho_n Q_n^{[1]}(s), \quad n \geqslant 0,$$

where the lattice, $x(s)$, is q-linear, i.e., $x(s+1) - q x(s) = \omega$,

$$\rho_n = \frac{q^{n+1}}{\mathfrak{C}} \frac{[n+1]}{\gamma_{n+1}}, \quad n \geqslant 1, \qquad \rho_0 = 0,$$
$$v_{n,0} = \frac{\gamma_{n+2}\gamma_{n+1}}{q^n [n+2]} \mathfrak{C} + \rho_n - q\beta_n - \omega, \quad n \geqslant 0.$$

Here $\mathfrak{C}$ is a constant, γ_n and β_n are the coefficients of the three-term recurrence relation (TTRR) that the orthogonal polynomial sequence $\{Q_n\}_{n\geqslant 0}$ satisfies. In fact, this sequence is not diagonal and it will be analyzed more carefully in Section 5.1.

The aim of our contribution is to give, under certain conditions, the second structure relation characterizing a q-semiclassical polynomial sequence by a new relation between the sequence of q-polynomials, $\{B_n\}_{n\geqslant 0}$, and the polynomial sequence of monic q-differences, $\{B_n^{[1]}\}_{n\geqslant 0}$, as follows:

$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(s) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} B_n^{[1]}(s), \quad n \geqslant \max(t+1, \sigma),$$

where $\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1$, $n \geqslant \max(t+1, \sigma)$, and there exists $r \geqslant \sigma + t + 1$ such that $\xi_{r,r-\sigma}\varsigma_{r,r-t} \neq 0$.

Notice that when $\sigma = 0$ we get the second structure relation (2).

2. Preliminaries and notation

Let u be a linear functional in the linear space $\mathbb{P}$ of polynomials with complex coefficients and let $\mathbb{P}'$ be its algebraic dual space, i.e., the linear space of the linear functionals defined on $\mathbb{P}$. We will denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$ and by $(u)_n := \langle u, x^n\rangle$, $n \geqslant 0$, the moments of u with respect to the sequence $\{x^n\}_{n\geqslant 0}$.

Let us define the following operations in $\mathbb{P}'$. For any polynomial h and any $c \in \mathbb{C}$, let $\Delta^{(1)}u$, hu, and $(x-c)^{-1}u$ be the linear functionals defined on $\mathbb{P}$ by (see [7,14])

(i) $\langle \Delta^{(1)}u, f\rangle := -\langle u, \Delta^{(1)} f\rangle$, $f \in \mathbb{P}$,
(ii) $\langle gu, f\rangle := \langle u, gf\rangle$, $f, g \in \mathbb{P}$,
(iii) $\langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c(f)\rangle$, $f \in \mathbb{P}$, $c \in \mathbb{C}$, where $\theta_c(f)(x) = \frac{f(x)-f(c)}{x-c}$.

Furthermore, for any linear functional u and any polynomial g we get

$$L_{q,\omega}(gu) := \Delta^{(1)}(gu) = g\big(q^{-1}(x-\omega)\big)\Delta^{(1)}u + \Delta^{(1)}\big(g\big(q^{-1}(x-\omega)\big)\big)u. \tag{4}$$

Let $\{B_n\}_{n\geqslant 0}$ be a sequence of monic polynomials (SMP) with $\deg B_n = n$, $n \geqslant 0$, and $\{u_n\}_{n\geqslant 0}$ its dual sequence, i.e., $u_n \in \mathbb{P}'$, $n \geqslant 0$, and $\langle u_n, B_m\rangle := \delta_{n,m}$, $n, m \geqslant 0$, where $\delta_{n,m}$ is the Kronecker symbol. The next results are very well known [7].

Lemma 2.1. *For any $u \in \mathbb{P}'$ and any integer $m \geqslant 1$, the following statements are equivalent*:

(i) $\langle u, B_{m-1}\rangle \neq 0$, $\langle u, B_n\rangle = 0$, $n \geqslant m$.
(ii) *There exist $\lambda_\nu \in \mathbb{C}$, $0 \leqslant \nu \leqslant m-1$, $\lambda_{m-1} \neq 0$, such that $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.*

On the other hand, it is straightforward to prove

Lemma 2.2. *For any $(\hat{t}, \hat{\sigma}, \hat{r}) \in \mathbb{N}^3$, $\hat{r} \geqslant \hat{\sigma} + \hat{t} + 1$, and any sequence of monic polynomials $\{\Omega_n\}_{n\geqslant 0}$, $\deg \Omega_n = n$, $n \geqslant 0$, with dual sequence $\{w_n\}_{n\geqslant 0}$ such that*

$$\Omega_n(x) = \sum_{\nu=n-\hat{t}}^{n} \lambda_{n,\nu} B_\nu(x), \quad n \geqslant \hat{t} + \hat{\sigma} + 1,\ \lambda_{\hat{r},\hat{r}-\hat{t}} \neq 0,$$

$$\Omega_n(x) = B_n(x), \quad 0 \leqslant n \leqslant \hat{t} + \hat{\sigma},$$

we have that $w_k = u_k$ for every $0 \leqslant k \leqslant \hat{\sigma}$.

The linear functional u is said to be quasi-definite if, for every non-negative integer, the leading principal Hankel submatrices $H_n = ((u)_{i+j})_{i,j=0}^{n}$ are non-singular for every $n \geqslant 0$. Assuming u is quasi-definite, there exists a sequence of monic polynomials $\{B_n\}_{n\geqslant 0}$ such that (see [4])

(i) $\deg B_n = n$, $n \geqslant 0$,
(ii) $\langle u, B_n B_m\rangle = r_n \delta_{n,m}$, with $r_n = \langle u, B_n^2\rangle \neq 0$, $n \geqslant 0$.

The sequence $\{B_n\}_{n\geqslant 0}$ is said to be the sequence of monic orthogonal polynomials, in short SMOP, with respect to the linear functional u.

If $\{B_n\}_{n\geqslant 0}$ is a SMOP, with respect to the quasi-definite linear functional u, then it is well known (see [14]) that its corresponding dual sequence $\{u_n\}_{n\geqslant 0}$, is

$$u_n = r_n^{-1} B_n u, \quad n \geqslant 0. \tag{5}$$

Remark 2.1. We assume $u_0 = u$, i.e., the linear functional u is normalized.

On the other hand, (see [4]), the sequence $\{B_n\}_{n\geqslant 0}$ satisfies a three-term recurrence relation (TTRR)

$$B_{n+1}(x) = (x - \beta_n)B_n(x) - \gamma_n B_{n-1}(x), \quad n \geqslant 0, \tag{6}$$

with $\gamma_n \neq 0$, $n \geqslant 1$, and $B_{-1}(x) = 0$, $B_0(x) = 1$.

Conversely, given a SMP, $\{B_n\}_{n\geqslant 0}$, generated by a recurrence relation (6) as above with $\gamma_n \neq 0$, $n \geqslant 1$, there exists a unique normalized quasi-definite linear functional u such that the family $\{B_n\}_{n\geqslant 0}$ is the corresponding SMOP. This result is known as Favard theorem (see [4]).

An important family of linear functionals is constituted by the q-semiclassical linear functionals, i.e., when u is quasi-definite and satisfies

$$\Delta^{(1)}(\Phi u) = \Psi u. \tag{7}$$

Here (Φ, Ψ) is an admissible pair of polynomials, i.e., the polynomial Φ is monic, $\deg \Phi = t$, $\deg \Psi = p \geqslant 1$, and if $p = t - 1$, then the following condition holds:

$$\lim_{q\uparrow 1} \frac{1}{[p]!}\left[\Delta^{(1)}\right]^p \Psi(0) := \lim_{q\uparrow 1} \frac{1}{[p]!} \overbrace{\Delta^{(1)} \cdots \Delta^{(1)}}^{p} \Psi(0) \neq -n, \quad n \in \mathbb{N}^*,$$

where $[m]! = [1][2]\cdots[m]$, $m \in \mathbb{N}^*$, is the q-analog of the usual factorial.

The pair (Φ, Ψ) is not unique. In fact, under certain conditions (7) can be simplified, so we define the class of u as the minimum value of $\max(\deg(\Phi) - 2, \deg(\Psi) - 1)$, for all admissible pairs (Φ, Ψ). The pair (Φ, Ψ) giving the class σ ($\sigma \geqslant 0$ because $\deg(\Psi) \geqslant 1$) is unique [7].

When u is q-semiclassical of class σ, the corresponding SMOP is said to be q-semiclassical of class σ.

When $\sigma = 0$, i.e., $\deg \Phi \leqslant 2$ and $\deg \Psi = 1$, then u is q-classical (Askey–Wilson, q-Racah, Big q-Jacobi, q-Charlier, etc.). For more details see [10,17,18].

3. Main results

First, we will present particular cases of *diagonal sequences*.

Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be sequences of monic polynomials, $\{v_n\}_{n\geqslant 0}$ and $\{w_n\}_{n\geqslant 0}$ their corresponding dual sequences. Let ϕ be a monic polynomial of degree t.

Definition 3.1. The sequence $\{P_n\}_{n\geqslant 0}$ is said to be compatible with ϕ if $\phi v_n \neq 0$, $n \geqslant 0$.

Lemma 3.1. [14, Proposition 2.1] *Let ϕ be as above. For any sequence $\{P_n\}_{n\geqslant 0}$ compatible with ϕ, the following statements are equivalent*:

(i) *There is an integer $\sigma \geqslant 0$ such that*

$$\phi(x)Q_n(x) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} P_\nu(x), \quad n \geqslant \sigma, \tag{8}$$

$$\exists r \geqslant \sigma: \quad \lambda_{r,r-\sigma} \neq 0. \tag{9}$$

(ii) *There are an integer $\sigma \geqslant 0$ and a mapping from $\mathbb{N}$ into $\mathbb{N}: m \mapsto \mu(m)$ satisfying*

$$\max\{0, m-t\} \leqslant \mu(m) \leqslant m+\sigma, \quad m \geqslant 0, \tag{10}$$

$$\exists m_0 \geqslant 0 \quad \text{with } \mu(m_0) = m_0 + \sigma, \tag{11}$$

such that

$$\phi v_m = \sum_{\nu=m-t}^{\mu(m)} \lambda_{\nu,m} w_\nu, \quad m \geqslant t,$$

$$\lambda_{\mu(m),m} \neq 0, \quad m \geqslant 0. \tag{12}$$

Proposition 3.1. [14, Proposition 2.2] *Assume $\{Q_n\}_{n\geqslant 0}$ is orthogonal and $\{P_n\}_{n\geqslant 0}$ is compatible with ϕ. Then the sequences $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ fulfill the finite-type relations* (8)–(9) *if and only if there are an integer $\sigma \geqslant 0$ and a mapping from $\mathbb{N}$ into $\mathbb{N}: m \mapsto \mu(m)$ satisfying* (10) *and* (11). *Moreover, there exist $\{k_m\}_{m\geqslant 0}$ and a sequence $\{\Lambda_{\mu(m)}\}_{m\geqslant 0}$ of monic polynomials with* $\deg(\Lambda_{\mu(m)}) = \mu(m)$, *$m \geqslant 0$, such that*

$$\phi v_m = k_m \Lambda_{\mu(m)} w_0, \quad m \geqslant 0. \tag{13}$$

From these two results we get

Corollary 3.1. [15, Proposition 1.6] *Let ϕ be as above. For sequences of monic orthogonal polynomials* (*SMOP*) *$\{P_n\}_{n\geqslant 0}$ and $\{B_n\}_{n\geqslant 0}$ orthogonal with respect to linear functionals v and u, respectively, the following statements are equivalent*:

(i) *There exists an integer $\sigma \geqslant 0$ such that*

$$\phi(s)P_n(s) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n,\nu} B_\nu^{[1]}(s), \quad \lambda_{n,n-\sigma} \neq 0, \quad n \geqslant \sigma.$$

(ii) *There exist a monic polynomial sequence $\{\Omega_{n+\sigma}\}_{n\geqslant 0}$, with* $\deg(\Omega_{n+\sigma}) = n+\sigma$, *$n \geqslant 0$, and non-zero constants k_n, $n \geqslant 0$, such that*

$$\phi u_n^{[1]} = k_n \Omega_{n+\sigma} v_0, \tag{14}$$

where $\{u_n^{[1]}\}_{n\geqslant 0}$ is the dual sequence of $\{B_n^{[1]}\}_{n\geqslant 0}$.

Thus we can prove

Proposition 3.2. *Any diagonal sequence, $\{B_n\}_{n\geqslant 0}$, orthogonal with respect a linear functional u is necessarily semiclassical and u satisfies*

$$\Delta^{(1)}\big(\phi(qx+\omega)\Omega_{n+\sigma}(x)u\big)=\psi_n(x)u, \quad n\geqslant 0, \tag{15}$$

where

$$\psi_n(s)=\frac{\phi(s+1)-\phi(s-1)}{\Delta x(s)}\Omega_{n+\sigma}(s)-d_n\phi(s)\phi(s-1)B_{n+1}(s), \tag{16}$$

and

$$d_n=[n+1]\frac{\langle u, B_{n+\sigma}^2\rangle}{\langle u, B_{n+1}^2\rangle\lambda_{n+\sigma,n}}, \quad n\geqslant 0. \tag{17}$$

Furthermore, the sequence $\{\Omega_{n+s}\}_{n\geqslant 0}$ satisfies

$$\begin{aligned}&\Omega_{n+\sigma}(s)\Delta^{(1)}\Omega_\sigma(s)-\Omega_\sigma(s)\Delta^{(1)}\Omega_{n+\sigma}(s)\\ &\quad=\phi(s+1)\big\{d_n\Omega_\sigma(s)B_{n+1}(s+1)-d_0\Omega_{n+\sigma}(s)B_1(s+1)\big\}.\end{aligned} \tag{18}$$

Proof. Let $\{B_n\}_{n\geqslant 0}$ be a diagonal sequence in the sense of Definition 1.1 and assume the linear functional u is normalized. Then from Lemma 3.1 there exist a sequence of monic polynomials $\{\Omega_{n+\sigma}\}_{n\geqslant 0}$ and non-zero constants $\{k_n\}_{n\geqslant 0}$ such that

$$\phi u_n^{[1]}=k_n\Omega_{n+\sigma}u.$$

Then

$$\begin{aligned}k_n\Delta^{(1)}(\Omega_{n+\sigma}u)&=\Delta^{(1)}\big(\phi\big(q^{-1}(x-\omega)\big)\big)u_n^{[1]}+\phi\big(q^{-1}(x-\omega)\big)\Delta^{(1)}u_n^{[1]}\\ &=\Delta^{(1)}\big(\phi\big(q^{-1}(x-\omega)\big)\big)u_n^{[1]}-\frac{[n+1]}{\langle u, B_{n+1}^2\rangle}\phi\big(q^{-1}(x-\omega)\big)B_{n+1}(x)u(s),\end{aligned} \tag{19}$$

as well as

$$\Delta^{(1)}\big(\phi(s)\phi(s-1)\big)=\phi(s)\frac{\phi(s+1)-\phi(s-1)}{\Delta x(s)}. \tag{20}$$

Combining (19) and (20), a straightforward calculation yields (15)–(17). Taking (15) for $n=0$ and cancelling out $\Delta^{(1)}(\phi(qx+\omega)u)$, from the quasi-definite character of u we obtain (18). □

Corollary 3.2. [15, Corollary 2.3] *If $\{B_n\}_{n\geqslant 0}$ is a diagonal sequence given by* (3)*, then we get*

$$\frac{1}{2}t\leqslant\sigma\leqslant t+2. \tag{21}$$

For a linear functional u, let (Φ,Ψ) be the minimal admissible pair of polynomials with Φ monic, $\deg\Phi=t$, and $\deg\Psi=p\geqslant 1$, defined as above. To this pair we can associate the non-negative integer $\sigma:=\max(t-2,p-1)\geqslant 0$.

Now, given $\{B_n\}_{n\geqslant 0}$, a SMOP with respect to u, we get

$$\Phi(s)B_n^{[1]}(s)=\sum_{\nu=0}^{n+t}\lambda_{n,\nu}B_\nu(s), \quad n\geqslant\max(t-1,0), \tag{22}$$

where $\lambda_{n,n+t}=1$ and

$$\lambda_{n,\nu}=r_\nu^{-1}\langle u,\Phi(s)B_n^{[1]}(s)B_\nu(s)\rangle=\frac{r_\nu^{-1}}{[n+1]}\langle B_\nu\Phi u,\Delta^{(1)}B_{n+1}\rangle$$
$$=-\frac{r_\nu^{-1}}{[n+1]}\langle B_\nu\big(q^{-1}(x-\omega)\big)\Delta^{(1)}(\Phi u)+\Delta^{(1)}\big(B_\nu\big(q^{-1}(x-\omega)\big)\big)\Phi u,B_{n+1}\rangle,$$
$$0\leqslant\nu\leqslant n+t.$$

Lemma 3.2. [7, Proposition 3.2] *For any monic polynomial Φ,* $\deg\Phi=t$*, and any SMOP $\{B_n\}_{n\geqslant0}$ with respect to u, the following statements are equivalent*:

(i) *There exists a non-negative integer σ such that*

$$\Phi(s)B_n^{[1]}(s)=\sum_{\nu=n-\sigma}^{n+t}\lambda_{n,\nu}B_\nu(s),\quad n\geqslant\sigma, \tag{23}$$

$$\lambda_{n,n-\sigma}\neq0,\quad n\geqslant\sigma+1. \tag{24}$$

(ii) *There exists a polynomial Ψ,* $\deg\Psi=p\geqslant1$*, such that*

$$\Delta^{(1)}(\Phi u)=\Psi u, \tag{25}$$

where the pair (Φ,Ψ) is admissible.

(iii) *There exist a non-negative integer σ and a polynomial Ψ, with* $\deg\Psi=p\geqslant1$*, such that*

$$\Phi(s)\Delta^{(1)}B_n(s-1)+\Psi(s)B_n(s-1)=\sum_{\nu=n-t}^{n+\sigma(n)}\tilde\lambda_{n,\nu}B_{\nu+1}(s),\quad n\geqslant t, \tag{26}$$

$$\tilde\lambda_{n,n-t}\neq0,\quad n\geqslant t, \tag{27}$$

where $\sigma=\max(p-1,t-2)$*, the pair (Φ,Ψ) is admissible, and*

$$\sigma(n)=\begin{cases}p-1, & n=0,\\ \sigma, & n\geqslant1.\end{cases} \tag{28}$$

We can write

$$\tilde\lambda_{n,\nu}=-[\nu+1]\frac{\langle u,B_n^2\rangle}{\langle u,B_{\nu+1}^2\rangle}\lambda_{\nu,n},\quad 0\leqslant\nu\leqslant n+\sigma. \tag{29}$$

Proof. (i) ⇒ (ii), (iii). Assuming (i), from Lemma 3.1 and taking $P_n=B_n$ and $Q_n=B_n^{[1]}$, we get

$$\Phi u_m=\sum_{\nu=0}^{\mu(m)}\lambda_{\nu,m}u_\nu^{[1]},\quad m\geqslant0.$$

On the other hand, (24) implies $\mu(m)=m+\sigma$, $m\geqslant1$. Taking into account that

$$\Delta^{(1)}u_m^{[1]}=-[m+1]u_{m+1},\quad m\geqslant0, \tag{30}$$

we have

$$\Delta^{(1)}(\Phi u_m) = -\sum_{\nu=0}^{\mu(m)} \lambda_{\nu,m}[\nu+1]u_{\nu+1}, \quad m \geqslant 0.$$

In accordance with the orthogonality of $\{B_n\}_{n\geqslant 0}$, we get

$$\Delta^{(1)}(\Phi B_m u) = -\Psi_{\mu(m)+1}u, \quad m \geqslant 0, \tag{31}$$

with

$$\Psi_{\mu(m)+1}(s) = \sum_{\nu=0}^{\mu(m)} \lambda_{\nu,m}[\nu+1]B_{\nu+1}(s), \quad m \geqslant 0. \tag{32}$$

Taking $m = 0$ in (31), we have

$$\Delta^{(1)}(\Phi u) = -\Psi_{\mu(0)+1}u. \tag{33}$$

Inserting (33) in (31) and because u is quasi-definite, we get

$$\Phi(s)\Delta^{(1)}B_m(s-1) - \Psi_{\mu(0)+1}(s)B_m(s-1) = -\Psi_{\mu(m)+1}(s), \quad m \geqslant 0.$$

The consideration of the degrees in both-hand sides leads to

- If $t-1 > \mu(0)+1$, which implies $t \geqslant 3$, then $t = \sigma+2$, $\mu(0) < \sigma$.
- If $t-1 \leqslant \mu(0)+1$, then $\mu(0) = \sigma$, $t \leqslant \sigma+2$.

Obviously, the pair $(\Phi, -\Psi_{\mu(0)+1})$ is admissible and putting $p = \mu(0)+1$, we have $\sigma = \max(p-1, t-2)$. So (26) and (27) are valid from (29).

Thus, we have proved that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (iii). Consider $m \geqslant 0$. Thus

$$\Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1) = \sum_{\nu=0}^{m+\sigma(m)+1} \lambda'_{m,\nu}B_\nu(s).$$

We successively derive from this

$$\big\langle u, \big(\Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1)\big)B_\mu\big\rangle = \lambda'_{m,\mu}\big\langle u, B_\mu^2\big\rangle, \quad 0 \leqslant \mu \leqslant m+\sigma+1.$$

A straightforward calculation yields

$$\big\langle u, \big(\Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1)\big)B_\mu\big\rangle = -\big\langle u, \Phi(s)B_m(s)\Delta^{(1)}B_\mu(s)\big\rangle. \tag{34}$$

Then

$$-\big\langle u, \Phi(s)B_m(s)\Delta^{(1)}B_\mu(s)\big\rangle = \lambda'_{m,\mu}\big\langle u, B_\mu^2\big\rangle.$$

Consequently, $\lambda'_{m,\mu} = 0$, $0 \leqslant \mu \leqslant m-t$, $\lambda'_{m,0} = 0$, $m \geqslant 0$. Moreover, for $\mu = m-t+1$, $m \geqslant t$,

$$-\big\langle u, \Phi(s)P_m(s)\Delta^{(1)}P_{m-t+1}(s)\big\rangle = -[m-t+1]\big\langle u, B_m^2\big\rangle = \lambda'_{m,m-t+1}\big\langle u, B_{m-t+1}^2\big\rangle.$$

Therefore, for $m \geqslant t$,

$$\Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1) = \sum_{\nu=m-t}^{m+\sigma(m)} \lambda'_{m,\nu+1}B_{\nu+1}(s), \quad \lambda'_{m,m-t+1} \neq 0.$$

(iii) ⇒ (i). From (26), we get

$$\sum_{\nu=0}^{m+\sigma(m)} \tilde{\lambda}_{m,\nu}\delta_{n,\nu+1} = \big\langle u_n, \Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1)\big\rangle$$
$$= -\big\langle \Delta^{(1)}(\Phi u_n) - \Psi u_n, B_m(s-1)\big\rangle.$$

For $n=0$, $\langle \Psi u - \Delta^{(1)}(\Phi u), B_m(s-1)\rangle = 0$, $m \geqslant 0$. Therefore

$$\Delta^{(1)}(\Phi u) = \Psi u. \tag{35}$$

Moreover, using (34) and the orthogonality of $\{B_n\}_{n\geqslant 0}$, we get

$$\big\langle u_n, \Phi(s)\Delta^{(1)}B_m(s-1) + \Psi(s)B_m(s-1)\big\rangle = -r_n^{-1}\big\langle u, \Phi(s)B_m(s)\Delta^{(1)}B_n(s)\big\rangle.$$

Furthermore, making $n \to n+1$, we obtain

$$\begin{cases} \big\langle \big(\Phi\Delta^{(1)}B_{n+1}\big)u, B_m\big\rangle = 0, & m \geqslant n+t+1,\ n \geqslant 0, \\ \big\langle \big(\Phi\Delta^{(1)}B_{n+1}\big)u, B_{n+t}\big\rangle = -r_{n+1}\tilde{\lambda}_{n+t,n} \neq 0, & n \geqslant 0. \end{cases}$$

According to Lemma 2.1,

$$\big(\Phi\Delta^{(1)}B_{n+1}\big)u = -\sum_{\nu=n-\sigma}^{n+t} r_n\tilde{\lambda}_{\nu,n}u_\nu, \quad n \geqslant \sigma.$$

The orthogonality of $\{B_n\}_{n\geqslant 0}$ leads to

$$\big(\Phi\Delta^{(1)}B_{n+1}\big)u = -\sum_{\nu=n-\sigma}^{n+t} \left(\tilde{\lambda}_{\nu,n}\frac{\langle u, B_{n+1}^2\rangle}{\langle u, B_\nu^2\rangle}B_\nu\right)u, \quad n \geqslant 0.$$

From (35) and taking into account u is quasi-definite, we finally obtain (23)–(24) in accordance with (29). □

In an analog way we can prove the following result.

Lemma 3.3. [12, Lemma 3.1] *For any monic polynomial Φ,* deg $\Phi = t$, *and any SMOP $\{B_n\}_{n\geqslant 0}$ with respect to u, the following statements are equivalent*:

(i) *There exists a non-negative integer σ such that the polynomials B_n satisfy*

$$\Delta^{(1)}\big(\Phi(s-1)B_n(s)\big) = \sum_{\nu=n-\sigma-1}^{n+t-1} \lambda_{n,\nu}B_\nu(s), \quad n \geqslant \sigma+1, \tag{36}$$
$$\lambda_{n,n-\sigma-1} \neq 0, \quad n \geqslant t+\sigma+2. \tag{37}$$

(ii) *There exists a polynomial Ψ,* deg $\Psi = p \geqslant 1$, *such that*

$$\Delta^{(1)}(\Phi u) = \Psi u, \tag{38}$$

where the pair (Φ, Ψ) is admissible.

(iii) *There exist a non-negative integer* σ *and a polynomial* Ψ, $\deg\Psi = p \geqslant 1$, *such that*

$$\begin{aligned}\Phi(s)\Delta^{(1)}B_n(s-1) + \Psi(s)B_n(s-1) - B_n(s)\Delta^{(1)}\Phi(s-1)\\ = \sum_{\nu=n-t+1}^{n+\sigma(n)+1} \tilde{\lambda}_{n,\nu}B_\nu(s), \quad n \geqslant t,\end{aligned} \tag{39}$$

$$\tilde{\lambda}_{n,n-t+1} \neq 0, \quad n \geqslant t, \tag{40}$$

where $\sigma = \max(p-1, t-2)$ *and the pair* (Φ, Ψ) *is admissible. We can write*

$$\tilde{\lambda}_{n,\nu} = -\frac{\langle u, B_m^2\rangle}{\langle u, B_\nu^2\rangle}\lambda_{\nu,n}, \quad 0 \leqslant \nu \leqslant n+\sigma(n)+1, \quad n \geqslant 0. \tag{41}$$

3.1. *First characterization of* q*-semiclassical polynomials*

Theorem 3.1. *For a monic polynomial* Φ, $\deg\Phi = t$, *and any SMOP* $\{B_n\}_{n\geqslant 0}$ *with respect to* u, *the following statements are equivalent*:

(i) *There exist a non-negative integer* σ, *an integer* $p \geqslant 1$, *and an integer* $r \geqslant \sigma + t + 1$, *with* $\sigma = \max(t-2, p-1)$, *such that*

$$\sum_{\nu=n-\sigma}^{n+t} \alpha_{n,\nu}B_\nu(s) = \sum_{\nu=n-t}^{n+t} \upsilon_{n,\nu}B_\nu^{[1]}(s), \quad n \geqslant \max(\sigma, t), \tag{42}$$

where $\alpha_{n,n+t} = \upsilon_{n,n+t} = 1$, $n \geqslant \max(\sigma, t)$, $\alpha_{r,r-\sigma}\upsilon_{r,r-t} \neq 0$,

$$\bigl\langle \Delta^{(1)}(\Phi u), B_n\bigr\rangle = 0, \quad p+1 \leqslant n \leqslant \sigma + 2t + 1, \qquad \bigl\langle \Delta^{(1)}(\Phi u), B_p\bigr\rangle \neq 0,$$

and if $p = t-1$, *then* $\lim_{q\uparrow 1}\langle u, B_p^2\rangle^{-1}\langle u, \Phi\Delta^{(1)}B_p\rangle \neq -m$, $m \in \mathbb{N}^*$.

(ii) *There exists a polynomial* Ψ, $\deg\Psi = p \geqslant 1$, *such that*

$$\Delta^{(1)}(\Phi u) = \Psi u,$$

and the pair (Φ, Ψ) *is admissible.*

Proof. (i) ⇒ (ii). Consider the SMP $\{\Omega_n\}_{n\geqslant 0}$ defined by

$$\Omega_{n+t+1}(s) = \sum_{\nu=n-t}^{n+t} \frac{[n+t+1]}{[\nu+1]}\upsilon_{n,\nu}B_{\nu+1}(s), \quad n \geqslant \sigma + t + 1,$$

$$\Omega_n(s) = B_n(s), \quad 0 \leqslant n \leqslant \sigma + 2t + 1.$$

From (42),

$$\Delta^{(1)}\bigl(\Omega_{n+t+1}(s)\bigr) = [n+t+1]\sum_{\nu=n-\sigma}^{n+t} \alpha_{n,\nu}B_\nu(s), \quad n \geqslant \sigma + t + 1. \tag{43}$$

Since u is quasi-definite, then

$$\begin{aligned}\bigl\langle \Delta^{(1)}(\Phi u), \Omega_{n+t+1}\bigr\rangle &= -\bigl\langle u, \Phi\Delta^{(1)}\Omega_{n+t+1}\bigr\rangle\\ &= -[n+t+1]\sum_{\nu=n-\sigma}^{n+t} \alpha_{n,\nu}\langle u, \Phi B_\nu\rangle = 0, \quad n \geqslant \sigma + t + 1.\end{aligned}$$

Therefore, $\langle \Delta^{(1)}(\Phi u), \Omega_n \rangle = 0$, $n \geqslant \sigma + 2t + 1$, and by hypothesis $\langle \Delta^{(1)}(\Phi u), \Omega_n \rangle = 0$, $p+1 \leqslant n \leqslant \sigma + 2t + 1$, then $\langle \Delta^{(1)}(\Phi u), \Omega_n \rangle = 0$ for $n \geqslant p+1$, and $\langle \Delta^{(1)}(\Phi u), \Omega_p \rangle \neq 0$. Hence, if we denote $\{w_n\}_{n\geqslant 0}$ the dual sequence of $\{\Omega_n\}_{n\geqslant 0}$ and apply Lemma 2.1, then

$$\Delta^{(1)}(\Phi u) = \sum_{\nu=1}^{p} \langle \Delta^{(1)}(\Phi u), B_\nu \rangle w_\nu. \tag{44}$$

On the other hand, if we take $\hat{t} = 2t$, $\hat{\sigma} = \sigma + 1$, and $\hat{r} = r + t + 1$, then

$$\Omega_n(s) = \sum_{\nu=n-\hat{t}}^{n} \tilde{v}_{n,\nu} B_\nu(s), \quad n \geqslant \hat{\sigma} + \hat{t} + 1,$$
$$\Omega_n(s) = B_n(s), \quad 0 \leqslant n \leqslant \hat{\sigma} + \hat{t},$$

where

$$\tilde{v}_{n,\nu} = \frac{[n]}{[\nu]} v_{n-t-1,\nu-1}, \quad n - \hat{t} \leqslant \nu \leqslant n, \quad n \geqslant \hat{\sigma} + \hat{t} + 1,$$
$$\tilde{v}_{\hat{r},\hat{r}-\hat{t}} = \frac{[r+t+1]}{[r-t+1]} v_{r,r-t} \neq 0, \quad \hat{r} \geqslant \sigma + 2t + 2 = \hat{\sigma} + \hat{t} + 1.$$

From Lemma 2.2 and (5), it follows that $w_k = u_k = \langle u, B_k^2 \rangle^{-1} B_k$, $0 \leqslant k \leqslant \hat{\sigma} = \sigma + 1$. So, relation (44) becomes

$$\Delta^{(1)}(\Phi u) = \Psi u,$$

where

$$\Psi(s) = -\sum_{\nu=1}^{p} \langle u, B_\nu^2 \rangle^{-1} \langle u, \Phi \Delta^{(1)} B_\nu \rangle B_\nu(s),$$

with $\deg \Psi = p$, as well as we have $\langle u, \Phi \Delta^{(1)} B_p \rangle \neq 0$ and, as a consequence, the pair (Φ, Ψ) is admissible with associated integer σ.

(ii) ⇒ (i). From Lemma 3.3(i) and making $n \to n+1$ we have

$$\Delta^{(1)}\big(\Phi(s-1)B_{n+1}(s)\big) = \sum_{\nu=n-\sigma}^{n+t} \lambda_{n+1,\nu} B_\nu(s), \quad n \geqslant \sigma, \tag{45}$$

where $\lambda_{n+1,n+t} = [n+t+1]$, $n \geqslant \sigma$, and $\lambda_{n+1,n-\sigma} \neq 0$, $n \geqslant t + \sigma + 1$.

On the other hand, the orthogonality of $\{B_n\}_{n\geqslant 0}$ yields

$$\Phi(s-1)B_{n+1}(s) = \sum_{\nu=n-t}^{n+t} \frac{\langle u, \Phi(s-1)B_{n+1}(s)B_{\nu+1}(s) \rangle}{\langle u, B_{\nu+1}^2 \rangle} B_{\nu+1}(s), \quad n \geqslant t-1.$$

Hence,

$$\Delta^{(1)}\big(\Phi(s-1)B_{n+1}(s)\big) = \sum_{\nu=n-t}^{n+t} \frac{[\nu+1]\langle u, \Phi(s-1)B_{n+1}(s)B_{\nu+1}(s) \rangle}{\langle u, B_{\nu+1}^2 \rangle} B_\nu^{[1]}(s), \quad n \geqslant t. \tag{46}$$

From (45) and (46), we obtain (42) with

$$\alpha_{n,\nu} = \frac{\lambda_{n+1,\nu}}{[n+t+1]}, \quad n-\sigma \leqslant \nu \leqslant n+t,$$
$$\upsilon_{n,\nu} = \frac{[\nu+1]\langle u, \Phi(s-1)B_{n+1}(s)B_{\nu+1}(s)\rangle}{[n+t+1]\langle u, B_{\nu+1}^2\rangle}, \quad n-t \leqslant \nu \leqslant n+t,$$
$$\alpha_{n,n-\sigma}\upsilon_{n,n-t} \neq 0, \quad n \geqslant \sigma+t+1.$$

Then,

$$\big\langle \Delta^{(1)}(\Phi u), B_n\big\rangle = -\big\langle u, \Phi\Delta^{(1)}B_n\big\rangle = \begin{cases} 0, & p+1 \leqslant n \leqslant \sigma+2t+1, \\ \frac{1}{[p]!}[\Delta^{(1)}]^p\Psi(0)\langle u, B_p^2\rangle, & n = p = \deg\Psi, \end{cases}$$

and if $p = t-1$, the q-admissibility of (Φ, Ψ) yields $\lim_{q\uparrow 1}\langle u, B_p^2\rangle^{-1}\langle u, \Phi\Delta^{(1)}B_p\rangle \neq -m$, $m \in \mathbb{N}^*$. □

In the case of q-classical linear functionals, we get the following result

Corollary 3.3. *Let $\{B_n\}_{n\geqslant 0}$ be a SMOP with respect to u and a monic polynomial Φ, with* $\deg\Phi = t \leqslant 2$, *such that $\langle u, \Phi\rangle \neq 0$, then the following statements are equivalent*:

(i) *The linear functional u is q-classical, i.e., there exists a polynomial Ψ with* $\deg\Psi = 1$ *such that $\Delta^{(1)}(\Phi u) = \Psi u$.*
(ii) $\sum_{\nu=n}^{n+t}\alpha_{n,\nu}B_\nu(s) = \sum_{\nu=n-t}^{n+t}\upsilon_{n,\nu}B_\nu^{[1]}(s)$, *$n \geqslant t$. Furthermore, there exists an integer $r \geqslant t+1$ such that $\alpha_{r,r}\upsilon_{r,r-t} \neq 0$, and if $t = 2$ then* $\lim_{q\uparrow 1}\langle u, B_1^2\rangle^{-1}\langle u, \Phi\rangle \neq -m$, *$m \in \mathbb{N}^*$.*

3.2. Second characterization of q-semiclassical polynomials

From the previous characterization, we can not recover the second structure relation of q-classical orthogonal polynomials (2). Our goal is to establish the characterization that allows us to deduce such a case.

First, we have the following result.

Proposition 3.3. *For any monic polynomial Φ, with* $\deg\Phi = t$, *and any SMOP $\{B_n\}_{n\geqslant 0}$ with respect to u, the following statements are equivalent*:

(i) *There exists a polynomial Ψ,* $\deg\Psi = p \geqslant 1$, *such that*

$$\Delta^{(1)}(\Phi u) = \Psi u, \tag{47}$$

where the pair (Φ, Ψ) is admissible.
(ii) *There exist a non-negative integer σ and a polynomial Ψ, with* $\deg\Psi = p \geqslant 1$, *such that*

$$\Phi(s)\big[\Delta^{(1)}\big]^2 B_n(s-1) + \Delta^{(1)}\big(\Psi(s)B_n(s-1)\big) - B_n(s)\big[\Delta^{(1)}\big]^2\Phi(s-1)$$
$$= \sum_{\nu=n-\sigma}^{n+\sigma(n)} \vartheta_{n,\nu}B_\nu(s), \quad n \geqslant \sigma, \tag{48}$$

where $\vartheta_{n,n-\sigma} \neq 0$ *either* $n \geqslant \sigma + t + 1$ *or* $n = \sigma + t$ *and* $p \geqslant t - 1$, $\sigma = \max(t-2, p-1)$, *and the pair* (Φ, Ψ) *is admissible. We can write*

$$\vartheta_{n,\nu} = \frac{\langle u, B_n^2\rangle}{\langle u, B_\nu^2\rangle}\vartheta_{\nu,n}, \quad 0 \leqslant \nu \leqslant n + \sigma(n),\ n \geqslant 0. \tag{49}$$

Proof. We have

$$\begin{aligned}&\Phi(s)\big[\Delta^{(1)}\big]^2 B_n(s-1) + \Delta^{(1)}\big(\Psi(s)B_n(s-1)\big) - B_n(s)\big[\Delta^{(1)}\big]^2\Phi(s-1)\\ &\quad= \sum_{\nu=0}^{n+\sigma(n)} \vartheta_{n,\nu}B_\nu(s), \quad n \geqslant 0,\end{aligned} \tag{50}$$

where for all integers $0 \leqslant \nu \leqslant n + \sigma(n)$, and $n \geqslant 0$,

$$\begin{aligned}\big\langle u, B_\nu^2\big\rangle\vartheta_{n,\nu} = \big\langle u, &\big(\Phi(s)\big[\Delta^{(1)}\big]^2 B_n(s-1) + \Delta^{(1)}\big(\Psi(s)B_n(s-1)\big)\\ &- B_n(s)\big[\Delta^{(1)}\big]^2\Phi(s-1)\big)B_\nu\big\rangle.\end{aligned}$$

Taking into account (5) and (48), a straightforward calculation leads to

$$\begin{aligned}\big\langle u, B_\nu^2\big\rangle\vartheta_{n,\nu} = \big\langle u, &\big(\Phi(s)\big[\Delta^{(1)}\big]^2 B_\nu(s-1) + \Delta^{(1)}\big(\Psi(s)B_\nu(s-1)\big)\\ &- B_\nu(s)\big[\Delta^{(1)}\big]^2\Phi(s-1)\big)B_n\big\rangle.\end{aligned}$$

Therefore, inserting (50)

$$\big\langle u, B_\nu^2\big\rangle\vartheta_{n,\nu} = \sum_{i=0}^{\nu+\sigma(\nu)} \vartheta_{\nu,i}\big\langle u, B_n^2\big\rangle\delta_{i,n} = \vartheta_{\nu,n}\big\langle u, B_n^2\big\rangle.$$

In particular, for $0 \leqslant \nu \leqslant n - \sigma - 1$, then $n \geqslant \nu + \sigma + 1 \geqslant \nu + \sigma(\nu) + 1$. Thus, we deduce $\vartheta_{\nu,n} = 0$. Hence $\vartheta_{n,\nu} = 0$, for $0 \leqslant \nu \leqslant n - \sigma - 1$.

For $\nu = n - \sigma$ and $n \geqslant \sigma + t$, we obtain

$$\begin{aligned}\big\langle u, B_{n-\sigma}^2\big\rangle\vartheta_{n,n-\sigma} &= \big\langle u, \Delta^{(1)}\big(\Phi(s)\Delta^{(1)}B_{n-\sigma}(s-1) + \Psi(s)B_{n-\sigma}(s-1)\big)\big\rangle\\ &\quad- \big\langle u, \Delta^{(1)}\big(B_{n-\sigma}(s)\Delta^{(1)}\Phi(s-1)\big)B_n\big\rangle\\ &= \sum_{nu=0}^{n+1} \tilde{\lambda}_{n-\sigma,\nu}\big\langle u, B_n\Delta^{(1)}B_\nu\big\rangle\\ &= [n+1]\tilde{\lambda}_{n-\sigma,n+1}\big\langle u, B_n^2\big\rangle.\end{aligned}$$

But, from (40), we get $\vartheta_{n,n-\sigma} \neq 0$, either $n \geqslant \sigma + t + 1$, or $n = \sigma + t$ and $p \geqslant t - 1$. As a consequence,

$$\begin{aligned}&\Phi(s)\big[\Delta^{(1)}\big]^2 B_n(s-1) + \Delta^{(1)}\big(\Psi(s)B_n(s-1)\big) - B_n(s)\big[\Delta^{(1)}\big]^2\Phi(s-1)\\ &\quad= \sum_{\nu=n-\sigma}^{n+\sigma(n)} \vartheta_{n,\nu}B_\nu(s), \quad n \geqslant \sigma.\end{aligned}$$

(ii) ⇒ (i). From (48)

$$\big\langle \Delta^{(1)}\big(\Phi(s-1)\Delta^{(1)}u\big) + \big(\big(\Delta^{(1)}\Phi(s-1)\big) - \Psi(s)\big)\Delta^{(1)}u, B_n(s-1)\big\rangle = 0, \quad n \geqslant \sigma+1,$$
$$\big\langle \Delta^{(1)}\big(\Phi(s-1)\Delta^{(1)}u\big) + \big(\big(\Delta^{(1)}\Phi(s-1)\big) - \Psi(s)\big)\Delta^{(1)}u, B_n(s-1)\big\rangle = \langle u,1\rangle\vartheta_{n,0},$$
$$n \leqslant \sigma.$$

According to Lemma 2.1

$$\Delta^{(1)}\big(\Phi(s-1)\Delta^{(1)}u\big) + \big(\big(\Delta^{(1)}\Phi(s-1)\big) - \Psi(s)\big)\Delta^{(1)}u$$
$$= \sum_{n=0}^{\sigma} \frac{\langle u,1\rangle\vartheta_{n,0}}{\langle u, B_n^2\rangle} B_n(\nabla u - u) = \sum_{n=0}^{\sigma(0)} \vartheta_{0,n} B_n(\nabla u - u).$$

Finally, a direct calculation yields

$$\Delta^{(1)}\big(\Delta^{(1)}(\Phi u) - \Psi u\big) = 0,$$

then $\Delta^{(1)}(\Phi u) - \Psi u = 0$.

Moreover, since $\sigma(n) = \sigma$ and $\vartheta_{n,n+\sigma} = [n+\sigma+1]\tilde{\lambda}_{n,n+\sigma+1} \neq 0$, for $n \geqslant t+1$, then $\tilde{\lambda}_{n,n+\sigma+1} \neq 0$, $n \geqslant t+1$. The q-admissibility of the pair (Φ, Ψ) follows taking into account the value of $\tilde{\lambda}_{n+\sigma(n)+1}$. □

Our main result is the next one.

Theorem 3.2. *For any monic polynomial Φ, $\deg\Phi = t$, and any SMOP $\{B_n\}_{n\geqslant 0}$ with respect to u, the following statements are equivalent*:

(i) *There exist a non-negative integer σ, an integer $p \geqslant 1$, and an integer $r \geqslant \sigma+t+1$, with $\sigma = \max(t-2, p-1)$, such that*

$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(s) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} B_\nu^{[1]}(s), \tag{51}$$

where $\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1$, $n \geqslant \max(\sigma, t+1)$, $\xi_{r,r-\sigma}\varsigma_{r,r-t} \neq 0$,

$$\begin{cases} \big\langle \Delta^{(1)}(\Phi u), B_m\big\rangle = 0, & p+1 \leqslant m \leqslant 2\sigma+t+1, \\ \big\langle \Delta^{(1)}(\Phi u), B_p\big\rangle \neq 0, \end{cases}$$

and if $p = t-1$, then $\lim_{q\uparrow 1}\langle u, B_p^2\rangle^{-1}\langle u, \Phi\Delta^{(1)}B_p\rangle \neq m$, $m \in \mathbb{N}^*$ (q-admissibility condition).

(ii) *There exists a polynomial Ψ, $\deg\Psi = p \geqslant 1$, such that*

$$\Delta^{(1)}(\Phi u) = \Psi u, \tag{52}$$

where the pair (Φ, Ψ) is admissible.

Proof. (i) ⇒ (ii). Let us consider the SMP $\{\Xi_n\}_{n\geqslant 0}$ given by

$$\Xi_{n+\sigma+1}(x) = \sum_{\nu=n-t}^{n+\sigma} \frac{[n+\sigma+1]}{[\nu+1]} \varsigma_{n,\nu} B_{\nu+1}(x), \quad n \geqslant \sigma+t+1,$$
$$\Xi_n(x) = B_n(x), \quad 0 \leqslant n \leqslant 2\sigma+t+1.$$

A direct calculation yields

$$\Delta^{(1)} \Xi_{n+\sigma+1}(s) = [n+\sigma+1] \sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(s), \quad n \geqslant \sigma + t + 1.$$

Taking into account the linear functional u is quasi-definite, we get

$$\begin{aligned}\left\langle \Delta^{(1)}(\Phi u), \Xi_{n+\sigma+1} \right\rangle &= -\left\langle u, \Phi \Delta^{(1)} \Xi_{n+\sigma+1}(s) \right\rangle \\ &= -[n+\sigma+1] \sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} \langle u, \Phi B_\nu \rangle = 0, \quad n \geqslant \sigma + t + 1.\end{aligned}$$

From the assumption and Lemma 2.1, if we denote $\{w_n\}_{n\geqslant 0}$ the dual sequence of $\{\Xi_n\}_{n\geqslant 0}$, then we get

$$\Delta^{(1)}(\Phi u) = \sum_{\nu=0}^{p} \left\langle \Delta^{(1)}(\Phi u), B_\nu \right\rangle w_k. \tag{53}$$

Taking $\hat{t} = \sigma + t$, $\hat{\sigma} = \sigma + 1$, and $\hat{r} = r + \sigma + 1$, the polynomials Ξ_n can be rewritten as

$$\begin{aligned}&\Xi_n(x) = \sum_{\nu=n-\hat{t}}^{n} \tilde{\varsigma}_{n,\nu} B_\nu(x), \quad n \geqslant \hat{\sigma} + \hat{t} + 1, \\ &\Xi_n(x) = B_n(x), \quad 0 \leqslant n \leqslant \hat{\sigma} + \hat{t},\end{aligned}$$

where

$$\begin{aligned}&\tilde{\varsigma}_{n,\nu} = \frac{[n]}{[\nu]} \varsigma_{n-\sigma-1,\nu-1}, \quad n - \hat{t} \leqslant \nu \leqslant n,\ n \geqslant \sigma + \hat{t} + 1, \\ &\tilde{\varsigma}_{\hat{r},\hat{r}-\hat{t}} = \frac{[r+\sigma+1]}{[r-t+1]} \varsigma_{r,r-t} \neq 0, \quad \hat{r} \geqslant 2\sigma + t + 2 \geqslant \hat{\sigma} + \hat{t} + 1.\end{aligned}$$

From Lemma 2.2, $w_k = u_k = \langle u, B_k^2 \rangle^{-1} B_k u$, $0 \leqslant k \leqslant \hat{\sigma} = \sigma + 1$. So, (53) becomes

$$\Delta^{(1)}(\Phi u) = \sum_{\nu=1}^{p} \left(\frac{\langle \Delta^{(1)}(\Phi u), B_\nu \rangle}{\langle u, B_\nu^2 \rangle} B_\nu \right) u = \Psi u.$$

Since $\langle \Delta^{(1)}(\Phi u), B_p \rangle \neq 0$, then $\deg \Psi = p$.

From the assumption, if $p = t - 1$, then

$$\lim_{q\uparrow 1} \frac{1}{[p]!} \left[\Delta^{(1)}\right]^p \Psi(0) = \lim_{q\uparrow 1} \frac{\langle \Delta^{(1)}(\Phi u), B_p \rangle}{\langle u, B_p^2 \rangle} = -\lim_{q\uparrow 1} \frac{\langle u, \Phi \Delta^{(1)} B_p \rangle}{\langle u, B_p^2 \rangle} \neq -m, \quad m \in \mathbb{N}^*.$$

Hence, the pair (Φ, Ψ) is admissible with associated integer σ.

(ii) $\Rightarrow$ (i). From Lemma 3.2(iii), there exists a polynomial Ψ, $\deg \Psi = p \geqslant 1$, such that

$$\begin{aligned}&\Phi(s-1)\Delta^{(1)} B_n(s-1) + \Psi(s) B_n(s-1) - B_n(s) \Delta^{(1)} \Phi(s-1) \\ &\quad = \sum_{\nu=n-t+1}^{n+\sigma(n)+1} \tilde{\lambda}_{n,\nu} B_\nu(s), \quad n \geqslant t,\end{aligned} \tag{54}$$

where $\tilde{\lambda}_{n,n-t+1} \neq 0$, $n \geqslant t$, $\sigma = \max(t-2, p-1)$, and the pair (Φ, Ψ) is admissible.

Taking q-differences in both-hand sides of (54), we get

$$\Phi(s)\big[\Delta^{(1)}\big]^2 B_n(s-1)+\Delta^{(1)}\big(\Psi(s)B_n(s-1)\big)-B_n(s)\big[\Delta^{(1)}\big]^2\Phi(s-1)$$
$$=\sum_{\nu=n-t}^{n+\sigma(n)}\zeta_{n,\nu}B_\nu^{[1]}(s),\quad n\geqslant t,\tag{55}$$

where $\zeta_{n,\nu}=[\nu+1]\tilde{\lambda}_{n,\nu+1}$, $0\leqslant\nu\leqslant n+\sigma(n)$, $n\geqslant t$.

From (48) and (55), we obtain (51) where

$$\xi_{n,\nu}=\frac{\vartheta_{n,\nu}}{\vartheta_{n,n+\sigma}},\quad n-\sigma\leqslant\nu\leqslant n+\sigma,$$
$$\varsigma_{n,\nu}=\frac{[\nu+1]\tilde{\lambda}_{n,\nu+1}}{\vartheta_{n,n+\sigma}},\quad n-t\leqslant\nu\leqslant n+t,$$
$$\xi_{n,n-\sigma}\,\varsigma_{n,n-t}=\frac{[n-t+1]}{\vartheta^2_{n,n+\sigma}}\,\vartheta_{n,n-\sigma}\,\tilde{\lambda}_{n,n-t+1}\neq 0,\quad n\geqslant\sigma+t+1.$$

Finally,

$$\big\langle\Delta^{(1)}(\Phi u),B_n\big\rangle=\langle u,\Psi B_n\rangle=\begin{cases}0, & p+1\leqslant n\leqslant 2\sigma+t+1,\\ \frac{\langle u,B_p^2\rangle}{[p]!}[\Delta^{(1)}]^p\Psi(0)\neq 0, & n=p=\deg\Psi.\end{cases}$$

From the admissibility of the pair (Φ,Ψ), if $p=t-1$, then $\langle u,B_p^2\rangle^{-1}\langle u,\Phi\Delta^{(1)}B_p\rangle\neq m$, $m\in\mathbb{N}^*$. □

4. The uniform lattice $x(s)=s$

As a direct consequence from the operator $L_{q,\omega}$ and the q-linear lattice $x(s)$, we can recover the uniform lattice setting $x(s)=(q^s-1)/(q-1)$ and taking limit $q\to 1$. For instance, for Δ-classical orthogonal polynomials the structure relations (1) and (2) have been studied in [5].

Theorem 4.1 *(First characterization of discrete semiclassical polynomials). For a monic polynomial Φ,* $\deg\Phi=t$*, and any SMOP $\{B_n\}_{n\geqslant 0}$ with respect to u, the following statements are equivalent*:

(i) *There exist a non-negative integer σ, an integer $p\geqslant 1$, and an integer $r\geqslant\sigma+t+1$, with* $\sigma=\max(t-2,p-1)$*, such that*

$$\sum_{\nu=n-\sigma}^{n+t}\alpha_{n,\nu}B_\nu(s)=\sum_{\nu=n-t}^{n+t}\upsilon_{n,\nu}B_\nu^{[1]}(s),\quad n\geqslant\max(\sigma,t),\tag{56}$$

where $B_n^{[1]}(s):=(n+1)^{-1}\Delta B_{n+1}(s)$, $\alpha_{n,n+t}=\upsilon_{n,n+t}=1$, $n\geqslant\max(\sigma,t)$, $\alpha_{r,r-\sigma}\upsilon_{r,r-t}\neq 0$,

$$\big\langle\Delta(\Phi u),B_n\big\rangle=0,\quad p+1\leqslant n\leqslant\sigma+2t+1,\qquad\big\langle\Delta(\Phi u),B_p\big\rangle\neq 0,$$

and if $p=t-1$, then $\langle u,B_p^2\rangle^{-1}\langle u,\Phi\Delta B_p\rangle\neq -m$, $m\in\mathbb{N}^*$.

(ii) *There exists a polynomial Ψ,* $\deg\Psi=p\geqslant 1$*, such that*

$$\Delta(\Phi u)=\Psi u,$$

and the pair (Φ,Ψ) is admissible.

Theorem 4.2 *(Second characterization of discrete semiclassical polynomials). For any monic polynomial Φ, $\deg\Phi = t$, and any SMOP $\{B_n\}_{n\geqslant 0}$ with respect to u, the following statements are equivalent*:

(i) *There exist a non-negative integer σ, an integer $p \geqslant 1$, and an integer $r \geqslant \sigma + t + 1$, with $\sigma = \max(t-2, p-1)$, such that*

$$\sum_{\nu=n-\sigma}^{n+\sigma} \xi_{n,\nu} B_\nu(s) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} B_\nu^{[1]}(s), \tag{57}$$

where $\xi_{n,n+\sigma} = \varsigma_{n,n+\sigma} = 1$, $n \geqslant \max(\sigma, t+1)$, $\xi_{r,r-\sigma}\varsigma_{r,r-t} \neq 0$,

$$\begin{cases} \big\langle \Delta(\Phi u), B_m \big\rangle = 0, & p+1 \leqslant m \leqslant 2\sigma + t + 1, \\ \big\langle \Delta(\Phi u), B_p \big\rangle \neq 0, \end{cases}$$

and if $p = t-1$, then $\langle u, B_p^2\rangle^{-1}\langle u, \Phi\Delta B_p\rangle \neq m$, $m \in \mathbb{N}^*$ (admissibility condition).

(ii) *There exists a polynomial Ψ, $\deg\Psi = p \geqslant 1$, such that*

$$\Delta(\Phi u) = \Psi u, \tag{58}$$

where the pair (Φ, Ψ) is admissible.

The proofs are analogous to the original ones setting $\omega = 1$, and taking limit $q \uparrow 1$. Therefore $L_{q,1} \equiv \Delta^{(1)}$ becomes Δ and $[n]$ becomes n.

Remark 4.1. Δ-semiclassical linear functionals have been studied in [11].

5. Examples

5.1. First example

Let $\{Q_n\}_{n\geqslant 0}$ be a SMOP that satisfies the following relation

$$\big(x(s+1) + v_{n,0}\big) Q_n(s) = q\, Q_{n+1}^{[1]}(s) + \rho_n(s) Q_n^{[1]}(s), \tag{59}$$

where the lattice, $x(s)$, is q-linear, i.e., $x(s+1) - q x(s) = \omega$,

$$\rho_n = \frac{q^{n+1}}{\mathfrak{C}} \frac{[n+1]}{\gamma_{n+1}}, \quad n \geqslant 1, \qquad \rho_0 = 0,$$

$$v_{n,0} = \frac{\gamma_{n+2}\gamma_{n+1}}{q^n[n+2]}\mathfrak{C} + \rho_n - q\beta_n - \omega, \quad n \geqslant 0,$$

and $\mathfrak{C}$ is a constant, being $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_n\}_{n\geqslant 0}$ the coefficients of the TTRR

$$x Q_n = Q_{n+1} + \beta_n Q_n + \gamma_n Q_{n-1}, \quad n \geqslant 1.$$

Then, from the above TTRR and Theorem 3.1, we get $\{Q_n\}_{n\geqslant 0}$ is a sequence of q-semiclassical orthogonal polynomials with respect to the linear functional v, solution of the Pearson equation

$$\Delta^{(1)} v = \Psi v, \tag{60}$$

of class $\sigma = 1$, with $\Phi(x) = 1$ and $\deg\Psi = 2$.

Then, it also satisfies the following relation

$$Q_n^{[1]}(s) = Q_n(s) + \lambda_{n,n-1} Q_{n-1}(s), \tag{61}$$

where $\lambda_{n,n-1} = \frac{\gamma_{n+1}\gamma_n}{q^n[n+1]}\mathfrak{C}$.

In fact, a straightforward calculation gives $\Psi(x) = -\frac{\mathfrak{C}}{q} Q_2(x) - \frac{1}{\gamma_1} Q_1(x)$.

Lemma 5.1. *Let $\{Q_n\}_{n\geqslant 0}$ be a SMOP with respect to the linear functional v satisfying* (60)*. Then the sequence $\{Q_n\}_{n\geqslant 0}$ is not diagonal.*

Proof. Assume $\{Q_n\}_{n\geqslant 0}$ is diagonal with respect to ϕ, with $\deg\phi = t$, and index σ. Then from Corollary 3.2, $t/2 \leqslant \sigma \leqslant t+2$ and we have the following diagonal relation:

$$\phi(s) Q_n(s) = \sum_{\nu=n-\sigma}^{n+t} \theta_{n,\nu} Q_\nu^{[1]}(s), \quad \theta_{n,n-\sigma} \neq 0, \ n \geqslant \sigma.$$

If we denote by $\{v_n\}_{n\geqslant 0}$ and $\{v_n^{[1]}\}_{n\geqslant 0}$ the dual sequences of $\{Q_n\}_{n\geqslant 0}$ and $\{Q_n^{[1]}\}_{n\geqslant 0}$, respectively, then by Proposition 3.1 the last relation is equivalent to

$$\phi v_n^{[1]} = k_n \Omega_{n+\sigma} v, \quad n \geqslant 0, \tag{62}$$

where $k_n = \langle v, Q_{n+\sigma}^2\rangle^{-1}\theta_{n+\sigma,n}$ and

$$\Omega_{n+\sigma}(x) = \sum_{\nu=0}^{n+\sigma} \frac{\theta_{\nu,n}}{\theta_{n+\sigma,n}} \frac{\langle v, Q_{n+\sigma}^2\rangle}{\langle v, Q_\nu^2\rangle} Q_\nu(x), \quad n \geqslant 0.$$

It is clear that v satisfies an infinite number of relations as (62). Indeed, by multiplying both-hand sides of (62) by a monic polynomial, we get another diagonal relation.

For this reason, we will assume $t = \deg\phi$ is the minimum non-negative integer such that v satisfies diagonal relations as (62), i.e., Eq. (62) cannot be simplified.

Notice that $t \geqslant 1$. Indeed, if we suppose that $t = 0$, then $0 \leqslant \sigma \leqslant 2$ and we recover the first structure relation characterizing q-classical sequences. This contradicts the fact that the sequence $\{Q_n\}_{n\geqslant 0}$ is q-semiclassical of class one.

Consequently, since $t \geqslant 1$ then $\sigma \geqslant 1$. Taking q-differences in both-hand sides of (62) and using (5), from (60) and $\Delta^{(1)} v_n^{[1]} = -[n+1]v_{n+1}$, we obtain

$$\tilde{\phi} v_n^{[1]} = k_n \psi_n v, \quad n \geqslant 0, \tag{63}$$

where

$$\begin{aligned}
&\tilde{\phi}(s) = [t]^{-1}\Delta^{(1)}\phi(s), \\
&\psi_n(s) = [t]^{-1}\big(\Omega_{n+\sigma}(s+1)\Psi(s) + \Delta^{(1)}\Omega_{n+\sigma}(s) + d_n\phi(s+1)Q_{n+1}(s)\big), \quad n \geqslant 0, \\
&d_n = [n+1]\big(\langle v, Q_{n+1}^2\rangle k_n\big)^{-1}, \quad n \geqslant 0.
\end{aligned}$$

Notice that the polynomial $\tilde{\phi}$ is monic with $\deg\tilde{\phi} = t-1$.

Moreover, taking into account u is a quasi-definite linear functional, combining (62) and (63) we obtain $\tilde{\phi}(x)\Omega_{n+\sigma}(x) = \phi(x)\psi_n(x)$, and analyzing the highest degree of this relation, we get ψ_n is a monic polynomial with $\deg\psi_n = n+\sigma-1$. But, this contradicts the fact that $t = \deg\phi$ is the minimum non-negative integer such that v satisfies diagonal relations as (62). □

5.2. The q-Freud type polynomials

Let $\{P_n\}_{n\geqslant 0}$ be a SMOP with respect to a linear functional u such that $(u)_0 = \langle u, 1\rangle = 1$ and the following relation

$$\Delta^{(1)} P_n(s) = [n]P_{n-1}(s) + a_n P_{n-3}(s), \quad n \geqslant 2, \tag{64}$$

holds, where $P_{-1} \equiv 0$, $P_0 \equiv 1$, and $P_1(x) = x$, being $x \equiv x(s) = q^s$, i.e., $\omega = 0$.

We know that this family satisfies a TTRR, i.e., there exist two sequences of complex numbers $\{b_n\}_n$ and $\{c_n\}_n$, $c_n \neq 0$, such that

$$xP_n = P_{n+1} + b_n P_n + c_n P_{n-1}, \quad n \geqslant 1.$$

Furthermore, from a direct calculation we get $a_n = K(q)q^{-n}c_n c_{n-1} c_{n-2}$, $n \geqslant 2$. In fact, the parameters c_n satisfy the non-linear recurrence relation

$$q[n]c_{n-1} + K(q)q^{-n+1}c_n c_{n-1} c_{n-2} = [n-1]c_n + K(q)q^{-n-1}c_{n+1}c_n c_{n-1}, \quad n \geqslant 1,$$

with $c_0 = 0$, $c_1 = -P_2(0) \neq 0$, and $\lim_{q\uparrow 1} K(q) = 4$.

Moreover, from Proposition 3.2 we deduce that $\Phi \equiv 1$ and thus $\sigma = 2$. As a consequence, Ψ is a polynomial of degree 3. In other words, u is a q-semiclassical linear functional of class 2, i.e., u satisfies the following distributional equation:

$$\Delta^{(1)} u = \Psi u, \quad \deg \Psi = 3. \tag{65}$$

Lemma 5.2. $\Psi(x) = -K(q)q^{-3}P_3(x) - c_1^{-1}P_1(x)$.

So, (65) is the q-analog of the Pearson equation for the Freud case.

Proof. From our hypothesis Ψ is a polynomial of degree 3, so $\Psi(x) = e_0P_0 + e_1P_1 + e_2P_2 + e_3P_3$. Then, taking into account $d_n^2 = c_n d_{n-1}^2$, $n \geqslant 1$, and the value of a_n, $n \geqslant 3$, we get

$$\begin{aligned}
e_0 d_0^2 &= e_0\langle u, P_0^2\rangle = \langle \Psi u, P_0\rangle = -\langle u, \Delta^{(1)} P_0\rangle = 0,\\
e_1 d_1^2 &= e_1\langle u, P_1^2\rangle = \langle \Psi u, P_1\rangle = -\langle u, \Delta^{(1)} P_1\rangle = -1,\\
e_2 d_2^2 &= e_2\langle u, P_2^2\rangle = \langle \Psi u, P_2\rangle = -\langle u, \Delta^{(1)} P_2\rangle \overset{(64)}{=} -\langle u, [2]P_1\rangle = 0,\\
e_3 d_3^2 &= e_3\langle u, P_3^2\rangle = \langle \Psi u, P_3\rangle = -\langle u, \Delta^{(1)} P_3\rangle \overset{(64)}{=} -\langle u, [3]P_2 + a_3 P_0\rangle = -a_3. \qquad \square
\end{aligned}$$

From Theorem 3.2, we can write the second structure relation as follows

$$\begin{aligned}
&B_{n+2} + \xi_{n,n+1}B_{n+1} + \xi_{n,n}B_n + \xi_{n,n-1}B_{n-1} + \xi_{n,n-2}B_{n-2}\\
&\quad = B_{n+2}^{[1]} + \varsigma_{n,n+1}B_{n+1}^{[1]} + \varsigma_{n,n}B_n^{[1]}.
\end{aligned} \tag{66}$$

Using (64) we get

$$\begin{aligned}
\xi_{n,n+1} &= \varsigma_{n,n+1}, & \xi_{n,n} &= [n+3]^{-1}a_{n+3} + \varsigma_{n,n},\\
\xi_{n,n-1} &= [n+2]^{-1}\varsigma_{n,n+1}a_{n+2}, & \xi_{n,n-2} &= [n+1]^{-1}\varsigma_{n,n}a_{n+1}.
\end{aligned}$$

Moreover, combining both structure relations if $P_n(x) = \sum_{j=0}^{n} \lambda_{n,j} x^{n-j}$, then $\lambda_{n,2k+1} = 0$ for non-negative integers n, k such that $0 \leqslant k \leqslant (n-1)/2$, and

$$\lambda_{n,0} = 1, \qquad \lambda_{n,2k+2} = \frac{[n]c_{n-1}\lambda_{n-2,2k} + a_n\lambda_{n-3,2k}}{[n-2k-2]-[n]}, \quad 1 \leqslant k \leqslant n/2.$$

In fact, with these values, we obtain $c_n = \lambda_{n,2} - \lambda_{n+1,2}$, $b_n = \lambda_{n,1} - \lambda_{n+1,1} = 0$, and $\xi_{n,n+1} = \xi_{n,n-1} = \varsigma_{n,n+1} = 0$, $n \geqslant 0$. Hence, we can rewrite (66) as

$$\left(x^2 + \tilde{\upsilon}_{n,0}\right) B_n = B_{n+2}^{[1]} + \tilde{\rho}_n B_n^{[1]}, \tag{67}$$

where

$$\tilde{\upsilon}_{n,0} = \frac{a_{n+3}}{[n+3]} + \frac{q^{n+1}[n+1]}{K(q)c_{n+1}} - c_{n+1} - c_n \quad \text{and} \quad \tilde{\rho}_n = \frac{q^{n+1}[n+1]}{K(q)c_{n+1}}.$$

Lemma 5.3. *The moments of the linear functional u, $\{(u)_n\}_{n\geqslant 0}$, satisfy the following relation*

$$[n+1](u)_n = K(q)q^{-3}(u)_{n+4} + \left(\frac{1}{c_1} - \frac{[3]c_2 + a_3}{q(1+q)}\right)(u)_{n+2}, \quad n \geqslant 0, \tag{68}$$

where $(u)_0 = 1$.

Therefore, taking into account that $(u)_1 = (u)_3 = 0$, we can deduce u is a symmetric linear functional, i.e., $(u)_{2n+1} = \langle u, x^{2n+1} \rangle = 0$, $n \geqslant 0$.

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