Jacobi-Type orthogonal polynomials: holonomic equation and electrostatic interpretation

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Abstract

In this contribution we study some pertubation of the Jacobi weight function by adding a mass point at $x = 1$, one of the ends of the interval supporting such a measure. We find the explicit expression of the corresponding orthogonal polynomials as well as the holonomic equation that such polynomials satisfy. Next, we analyze the location of their zeros and, finally, we give an electrostatic interpretation of them.

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1 Introduction

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of complex numbers and μ a linear functional defined in the linear space $\mathbb{P}$ of polynomials with complex coefficients, where

$$\langle \mu, x^n \rangle = \mu_n, \ n \in \mathbb{N}.$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Moreover μ_n is called the *moment of order n* of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if:

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(i) The degree of P_n is n.

(ii) $\langle \mu, P_n(x)P_m(x)\rangle = 0$, $m \neq n$.

(iii) $\left\langle \mu, P_n^2(x)\right\rangle \neq 0$, $n \in \mathbb{N}$.

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be a sequence of *monic orthogonal polynomials.* It is clear that for every sequence of orthogonal polynomials there exists the corresponding family of monic orthogonal polynomials. In the sequel we will work with monic polynomials .

The next theorem, whose proof is given in [6], provides necessary and sufficient conditions for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1 *Let μ be the moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $\left[\mu_{i+j}\right]_{i,j\in\mathbb{N}}$ are nonsingular.*

A moment functional such that there exists the corresponding sequence of orthogonal polynomials is said to be *regular* or *quasi-definite.* See ([6]).

Now we show the three term recurrence formula that a sequence of monic orthogonal polynomials satisfies. It will be used in the next section. The proof is given in [6].

Theorem 2 *If μ is a regular moment functional and $\{P_n\}_{n\geq 0}$ is the corresponding sequence of monic orthogonal polynomials, then there exist sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$, with $\gamma_n \neq 0$ for every $n \in \mathbb{N}$, such that*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \tag{1}$$

with $P_0(x) = 1, P_1(x) = x - \beta_0$.

If $\phi(x)$ is a complex polynomial, we define the moment functional $\phi\mu$, the left multiplication by a polynomial ϕ, and $D\mu$, the usual distributional derivative of μ, as follows

$$\langle \phi\mu, p(x)\rangle = \langle \mu, \phi(x)\, p(x)\rangle, \ \ \langle D\mu, p(x)\rangle = -\langle \mu, p'(x)\rangle .$$

We will say that a sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ is *classical*, if there exist polynomials ϕ and ψ with $\deg\phi \leq 2$, $\deg\psi = 1$, such that μ satisfies the Pearson differential equation

$$D(\phi\mu) = \psi\mu.$$

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) are used in the literature taking into account their applications in Mathematical Physics, in particular in the study of problems involving hypergeometric differential equations (see [13], [14], and [10]).

The next theorem summarizes some properties of classical orthogonal polynomials. The proof is given in [11].

Theorem 3 *Let μ be a regular moment functional and $\{P_n\}_{n\geq 0}$ the corresponding sequence of monic orthogonal polynomials.*

1. *$\{P_n\}_{n\geq 0}$ is classical if and only if there exist sequences a_n, b_n, c_n, with $c_n \neq 0$ for every $n \in \mathbb{N}$, such that*

$$\phi(x)P_n'(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) \tag{2}$$

with $\deg \phi \leq 2$.

2. *$\{P_n\}_{n\geq 0}$ is classical if and only if, for every $n \in \mathbb{N}$, P_n is an eigenfunction of the differential operator*

$$\mathcal{L} = \phi D^2 + \psi D \tag{3}$$

where $\deg \phi \leq 2$, $\deg \psi = 1$, *and D denotes the standard derivative operator.*

3. *$\{P_n\}_{n\geq 0}$ is classical if and only if there exist sequences r_n and s_n such that*

$$P_n(x) = Q_n(x) + r_n Q_{n-1}(x) + s_n Q_{n-2}(x), \; n \geq 2, \tag{4}$$

with $Q_k(x) = \dfrac{P_{k+1}'(x)}{k+1}$, $k \geq 0$.

4. *(The Christoffel-Darboux identity)*

$$\sum_{k=0}^{n} \frac{P_k(x)P_k(u)}{\left\langle \mu, P_k^2 \right\rangle} = \frac{1}{\langle \mu, P_n^2 \rangle} \frac{P_{n+1}(x)P_n(u) - P_n(x)P_{n+1}(u)}{x-u}. \tag{5}$$

The polynomials $\{P_n\}_{n\geq 0}$ are *semiclassical orthogonal polynomials* if there exist polynomials ϕ and ψ with $\deg \psi \geq 1$, so that the corresponding moment functional μ satisfies $D(\phi\mu) = \psi\mu$.

Given a moment functional μ, if we add a Dirac mass we obtain a new moment functional $\widetilde{\mu}$. The next theorem, whose proof is given in [12], characterizes the sequences of monic orthogonal polynomials $\left\{\widetilde{P}_n\right\}_{n\geq 0}$ with respect to the moment functional $\widetilde{\mu}$.

Theorem 4 *Let $\{P_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials associated with the moment functional μ. We introduce the moment functional $\widetilde{\mu} = \mu + M\delta(x-a)$, where $\langle \delta(x-a), p(x)\rangle = p(a)$ is the Dirac moment functional supported at a. Then:*

1. *$\widetilde{\mu}$ is regular if and only if* $1 + MK_n(a,a) \neq 0$, $n \in \mathbb{N}$, *where*

$$K_n(x,a) = \sum_{j=0}^{n} \frac{P_j(a)}{\left\langle \mu, P_j^2 \right\rangle} P_j(x).$$

2. *For every $n \in \mathbb{N}$,*

$$\widetilde{P}_n(x) = P_n(x) - \frac{MP_n(a)}{1 + MK_{n-1}(a,a)} K_{n-1}(x,a). \tag{6}$$

As we mentioned before, a relevant class of classical orthogonal polynomials are the Jacobi polynomials. For these polynomials the corresponding moment functional satisfies a Pearson equation with $\phi(x) = 1 - x^2$ and $\psi_{\alpha,\beta}(x) = -(\alpha+\beta+2)\,x + (\beta-\alpha)$. The next proposition summarizes some properties of the Jacobi polynomials $P_n^{\alpha,\beta}(x)$. (See [1],[6],[10], [13], and [14]).

Proposition 1 *Let $\left\{P_n^{\alpha,\beta}\right\}_{n\geq 0}$ be the sequence of Jacobi monic orthogonal polynomials.*

1. *For every $n \in \mathbb{N}$,*

$$xP_n^{\alpha,\beta}(x) = P_{n+1}^{\alpha,\beta}(x) + \beta_n^{\alpha,\beta}P_n^{\alpha,\beta}(x) + \gamma_n^{\alpha,\beta}P_{n-1}^{\alpha,\beta}(x), \tag{7}$$

with

$$\begin{aligned} \beta_n^{\alpha,\beta} &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \\ \gamma_n^{\alpha,\beta} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \end{aligned}$$

$P_1^{\alpha,\beta}(x) = 1$, *and* $P_1^{\alpha,\beta}(x) = x + \dfrac{\alpha-\beta}{\alpha+\beta+2}$.

2. *For every $n \in \mathbb{N}$,*

$$\left(1-x^2\right)\left(P_n^{\alpha,\beta}(x)\right)' = a_n^{\alpha,\beta}P_{n+1}^{\alpha,\beta}(x) + b_n^{\alpha,\beta}P_n^{\alpha,\beta}(x) + c_n^{\alpha,\beta}P_{n-1}^{\alpha,\beta}(x), \tag{8}$$

where

$$\begin{aligned} a_n^{\alpha,\beta} &= -n, \\ b_n^{\alpha,\beta} &= \frac{2(\alpha-\beta)n(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \text{ and} \\ c_n^{\alpha,\beta} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}. \end{aligned}$$

3. *For every $n \in \mathbb{N}$, there exists a sequence of real numbers $\{\lambda_n\}_{n\geq 0}$ such that $P_n^{\alpha,\beta}(x)$ satisfies the second order linear differential equation*

$$\phi(x)y'' + \psi_{\alpha,\beta}(x)y' = \lambda_n^{\alpha,\beta}y \tag{9}$$

with $\lambda_n^{\alpha,\beta} = -n(n+1+\alpha+\beta)$.

4. *For every $n \in \mathbb{N}$,*

$$\left(P_n^{\alpha,\beta}(x)\right)' = nP_{n-1}^{\alpha+1,\beta+1}(x). \tag{10}$$

5. For every $n \in \mathbb{N}$ *there exists a constant* A_n *such that*

$$K_n(x,1) = A_n P_n^{\alpha+1,\beta}(x), \; A_n = \frac{P_n^{\alpha,\beta}(1)}{\langle \mu, 1\rangle \gamma_1\gamma_2...\gamma_n}. \tag{11}$$

6. For every $n \in \mathbb{N}$,

$$\begin{aligned} P_n^{\alpha,\beta}(1) &= 2^n \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+1)}, \\ P_n^{\alpha,\beta}(-1) &= (-2)^n \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+1)}. \end{aligned} \tag{12}$$

7. For every $n \in \mathbb{N}$,

$$\left\| P_n^{\alpha,\beta}(x) \right\|^2 = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)\, n!}{(2n+\alpha+\beta+1)\left(\Gamma(2n+\alpha+\beta+1)\right)^2}. \tag{13}$$

For instance, using (6) and (11) we can write

$$\widetilde{P}_n^{\alpha,\beta}(x) = P_n^{\alpha,\beta}(x) + d_n P_{n-1}^{\alpha+1,\beta}(x), \; d_n = -\frac{M P_n^{\alpha,\beta}(1)}{1 + M K_{n-1}(1,1)} A_{n-1}, \tag{14}$$

and, taking into account (11)

$$d_n = -\frac{M P_n^{\alpha,\beta}(1) P_{n-1}^{\alpha,\beta}(1)}{\left\langle \mu, \left(P_{n-1}^{\alpha,\beta}(x)\right)^2 \right\rangle + M P_{n-1}^{\alpha,\beta}(1) P_{n-1}^{\alpha+1,\beta}(1)}. \tag{15}$$

On the other hand, using (11) ,

$$\begin{aligned} P_n^{\alpha,\beta}(x) &= \frac{P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(1)}{P_n^{\alpha,\beta}(1)} \\ &= \frac{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{P_n^{\alpha,\beta}(1)} \left[K_n(x,1) - K_{n-1}(x,1) \right] \\ &= \frac{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{P_n^{\alpha,\beta}(1)} \left[A_n P_n^{\alpha+1,\beta}(x) - A_{n-1} P_{n-1}^{\alpha+1,\beta}(x) \right] \\ &= A_n \frac{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{P_n^{\alpha,\beta}(1)} P_n^{\alpha+1,\beta}(x) - A_{n-1} \frac{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{P_n^{\alpha,\beta}(1)} P_{n-1}^{\alpha+1,\beta}(x), \end{aligned}$$

and, taking into account $P_n^{\alpha,\beta}(x)$ is monic, according to (11) we get

$$P_n^{\alpha,\beta}(x) = P_n^{\alpha+1,\beta}(x) - A_{n-1} \frac{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{P_n^{\alpha,\beta}(1)} P_{n-1}^{\alpha+1,\beta}(x).$$

Using the last identity, from (14) we get

$$
\begin{aligned}
\widetilde{P}_n^{\alpha,\beta}(x) &= P_n^{\alpha+1,\beta}(x) + \left(d_n - A_{n-1} \frac{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{P_n^{\alpha,\beta}(1)} \right) P_{n-1}^{\alpha+1,\beta}(x) \\
&= P_n^{\alpha+1,\beta}(x) - \left(\frac{M P_n^{\alpha,\beta}(1)}{1 + M K_{n-1}(1,1)} + \frac{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{P_n^{\alpha,\beta}(1)} \right) A_{n-1} P_{n-1}^{\alpha+1,\beta}(x) \\
&= P_n^{\alpha+1,\beta}(x) - \left(\frac{M \left(P_n^{\alpha,\beta}(1) \right)^2 + (1 + M K_{n-1}(1,1)) \left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{(1 + M K_{n-1}(1,1))\, P_n^{\alpha,\beta}(1)} \right) A_{n-1} P_{n-1}^{\alpha+1,\beta}(x) \\
&= P_n^{\alpha+1,\beta}(x) - \frac{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2}{P_n^{\alpha,\beta}(1)} \left(\frac{\frac{M\left(P_n^{\alpha,\beta}(1)\right)^2}{\left\| P_n^{\alpha,\beta}(x) \right\|_{\alpha,\beta}^2} + (1 + M K_{n-1}(1,1))}{1 + M K_{n-1}(1,1)} \right) A_{n-1} P_{n-1}^{\alpha+1,\beta}(x) \\
&= P_n^{\alpha+1,\beta}(x) - \frac{1 + M K_n(1,1)}{1 + M K_{n-1}(1,1)} \frac{A_{n-1}}{A_n} P_{n-1}^{\alpha+1,\beta}(x) \\
&= P_n^{\alpha+1,\beta}(x) - \frac{\left\langle \mu, \left(P_n^{\alpha,\beta}(x) \right)^2 \right\rangle + M P_n^{\alpha,\beta}(1) P_n^{\alpha+1,\beta}(1)}{\left\langle \mu, \left(P_{n-1}^{\alpha,\beta}(x) \right)^2 \right\rangle + M P_{n-1}^{\alpha,\beta}(1) P_{n-1}^{\alpha+1,\beta}(1)} \frac{P_{n-1}^{\alpha,\beta}(1)}{P_n^{\alpha,\beta}(1)} P_{n-1}^{\alpha+1,\beta}(x).
\end{aligned}
$$

Let r_n and α_n be such that

$$
\begin{aligned}
\alpha_n &= \frac{\left\langle \mu, \left(P_n^{\alpha,\beta}(x) \right)^2 \right\rangle + M P_n^{\alpha,\beta}(1) P_n^{\alpha+1,\beta}(1)}{P_n^{\alpha,\beta}(1)} \\
r_n &= -\frac{\left\langle \mu, \left(P_n^{\alpha,\beta}(x) \right)^2 \right\rangle + M P_n^{\alpha,\beta}(1) P_n^{\alpha+1,\beta}(1)}{\left\langle \mu, \left(P_{n-1}^{\alpha,\beta}(x) \right)^2 \right\rangle + M P_{n-1}^{\alpha,\beta}(1) P_{n-1}^{\alpha+1,\beta}(1)} \frac{P_{n-1}^{\alpha,\beta}(1)}{P_n^{\alpha,\beta}(1)}
\end{aligned}
$$

then

Theorem 5 *For every $n = 1, 2...$*

$$
\widetilde{P}_n^{\alpha,\beta}(x) = P_n^{\alpha+1,\beta}(x) + r_n P_{n-1}^{\alpha+1,\beta}(x). \tag{16}
$$

where $r_n = -\dfrac{\alpha_n}{\alpha_{n-1}}$.

Using the definition of r_n, (12), (13), and making some calculations, we obtain the explicit expression of r_n

$$
r_n = -\frac{\Gamma(\alpha+1)\Gamma(\alpha+2) R(n,\alpha,\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)\, S(n,\alpha,\beta)},
$$

with

$$
\begin{aligned}
R(n,\alpha,\beta) &= 2^{\alpha+\beta+2} n!\Gamma(n+\beta+1) + 2M\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)\\
S(n,\alpha,\beta) &= 2^{\alpha+\beta+1}(n-1)!\Gamma(n+\beta)\Gamma(\alpha+1)\Gamma(\alpha+2) + M\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1),
\end{aligned}
$$

and, as a conclusion, the asymptotic expression

$$
r_n \sim -\frac{2\Gamma(\alpha+1)\Gamma(\alpha+2)(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \tag{17}
$$

2 The holonomic equation

We are going to find the second order linear differential equation satisfied by $\widetilde{P}_n^{\alpha,\beta}(x)$. Taking into account (9) if we change α for $\alpha+1$, then

$$
\begin{aligned}
\phi(x)\left(P_n^{\alpha+1,\beta}(x)\right)'' + \psi_{\alpha+1,\beta}(x)\left(P_n^{\alpha+1,\beta}(x)\right)' &= \lambda_n^{\alpha+1} P_n^{\alpha+1,\beta}(x),\\
\phi(x)\left(P_{n-1}^{\alpha+1,\beta}(x)\right)'' + \psi_{\alpha+1,\beta}(x)\left(P_{n-1}^{\alpha+1,\beta}(x)\right)' &= \lambda_{n-1}^{\alpha+1} P_{n-1}^{\alpha+1,\beta}(x).
\end{aligned}
$$

Adding to the first identity the second one multiplied by r_n, we obtain

$$
\phi(x)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)'' + \psi_{\alpha+1,\beta}(x)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' = \lambda_n^{\alpha+1} P_n^{\alpha+1,\beta}(x) + r_n\lambda_{n-1}^{\alpha+1} P_{n-1}^{\alpha+1,\beta}(x)
$$

as well as

$$
\begin{aligned}
&\phi(x)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)'' + \psi_{\alpha+1,\beta}(x)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)'\\
=\ & \lambda_n^{\alpha+1,\beta} P_n^{\alpha+1,\beta}(x) + r_n\lambda_{n-1}^{\alpha+1,\beta} P_{n-1}^{\alpha+1,\beta}(x) + r_n\lambda_n^{\alpha+1,\beta} P_{n-1}^{\alpha+1,\beta}(x) - r_n\lambda_n^{\alpha+1,\beta} P_{n-1}^{\alpha+1,\beta}(x)\\
=\ & \lambda_n^{\alpha+1,\beta}\left(P_n^{\alpha+1,\beta}(x) + r_n P_{n-1}^{\alpha+1,\beta}(x)\right) + r_n\left(\lambda_{n-1}^{\alpha+1,\beta} - \lambda_n^{\alpha+1,\beta}\right) P_{n-1}^{\alpha+1,\beta}(x)\\
=\ & \lambda_n^{\alpha+1}\widetilde{P}_n^{\alpha,\beta}(x) + r_n\left(\lambda_{n-1}^{\alpha+1,\beta} - \lambda_n^{\alpha+1,\beta}\right) P_{n-1}^{\alpha+1,\beta}(x).
\end{aligned}
$$

In order to write $P_{n-1}^{\alpha+1,\beta}(x)$ as a combination of $\widetilde{P}_n^{\alpha,\beta}(x)$ and $\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)'$, we use (10). Thus

$$
\begin{aligned}
\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' &= \left(P_n^{\alpha,\beta}(x)\right)' + d_n\left(P_{n-1}^{\alpha+1,\beta}(x)\right)'\\
\left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' &= (1-x)(1+x)\, nP_{n-1}^{\alpha+1,\beta+1}(x) + d_n\left(1-x^2\right)\left(P_{n-1}^{\alpha+1,\beta}(x)\right)'.
\end{aligned}
$$

On the other hand, using the Christoffel formula (see (5))

$$
(1+x)\,P_{n-1}^{\alpha+1,\beta+1}(x) = P_n^{\alpha+1,\beta}(x) + \sum_{n=0}^{n-1}\zeta_{n,j}P_j^{\alpha+1,\beta}(x)
$$

where

$$\zeta_{n,j} = \frac{\int_{-1}^{1}(1+x)\,P_{n-1}^{\alpha+1,\beta+1}(x)P_j^{\alpha+1,\beta}(x)\,(1-x)^{\alpha+1}\,(x+1)^{\beta}\,dx}{\left\|P_j^{\alpha+1,\beta}(x)\right\|_{\alpha+1,\beta}^2}$$
$$= \frac{\int_{-1}^{1}P_{n-1}^{\alpha+1,\beta+1}(x)P_j^{\alpha+1,\beta}(x)\,(1-x)^{\alpha+1}\,(x+1)^{\beta+1}\,dx}{\left\|P_j^{\alpha+1,\beta}(x)\right\|_{\alpha+1,\beta}^2} = 0,\ 0\le j\le n-2.$$

As a consequence

$$(1+x)\,P_{n-1}^{\alpha+1,\beta+1}(x) = P_n^{\alpha+1,\beta}(x) + t_n P_{n-1}^{\alpha+1,\beta}(x), \tag{18}$$

In order to compute t_n, we evaluate (18) in $x=-1$ and we obtain

$$t_n = -\frac{P_n^{\alpha+1,\beta}(-1)}{P_{n-1}^{\alpha+1,\beta}(-1)}.$$

Using (12) and making some calculations, we have

$$t_n = \frac{2\,(n+\beta)\,(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)\,(2n+\alpha+\beta)}.$$

From (18) and (8) , we deduce

$$\begin{aligned}
&\left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' \\
=\ &(1-x)\,n\left[P_n^{\alpha+1,\beta}(x) + t_n P_{n-1}^{\alpha+1,\beta}(x)\right] + \\
&+d_n\left[a_{n-1}^{\alpha+1,\beta}P_n^{\alpha+1,\beta} + b_{n-1}^{\alpha+1,\beta}P_{n-1}^{\alpha+1,\beta} + c_{n-1}^{\alpha+1,\beta}P_{n-2}^{\alpha+1,\beta}\right].
\end{aligned}$$

From (7)

$$P_{n-2}^{\alpha+1,\beta}(x) = \frac{1}{\gamma_{n-1}^{\alpha+1,\beta}}\left[\left(x-\beta_{n-1}^{\alpha+1,\beta}\right)P_{n-1}^{\alpha+1,\beta}(x) - P_n^{\alpha+1,\beta}(x)\right]$$

thus

$$\begin{aligned}
&\left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' \\
=\ &\left[(1-x)\,n + d_n a_{n-1}^{\alpha+1,\beta} - \frac{c_{n-1}^{\alpha+1,\beta}}{\gamma_{n-1}^{\alpha+1,\beta}}\right]P_n^{\alpha+1,\beta}(x) + \\
&+\left[(1-x)\,n t_n + b_{n-1}^{\alpha+1,\beta}d_n - \frac{c_{n-1}^{\alpha+1,\beta}}{\gamma_{n-1}^{\alpha+1,\beta}}\left(x-\beta_{n-1}^{\alpha+1,\beta}\right)\right]P_{n-1}^{\alpha+1,\beta}(x),
\end{aligned}$$

Using the values of $c_{n-1}^{\alpha+1,\beta}$ and $\gamma_{n-1}^{\alpha+1,\beta}$ given in Proposition 1, we have

$$\frac{c_{n-1}^{\alpha+1,\beta}}{\gamma_{n-1}^{\alpha+1,\beta}} = n+\alpha+\beta+1.$$

As a consequence

$$\begin{aligned} & \left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' \\ = \quad & -\left[xn - a_{n-1}^{\alpha+1,\beta} d_n + n+\alpha+\beta+1\right] P_n^{\alpha+1,\beta}(x) + \\ & + \left[-x\left(nt_n + n+\alpha+\beta+1\right) + nt_n + b_{n-1}^{\alpha+1,\beta} + (n+\alpha+\beta+1)\beta_{n-1}^{\alpha+1,\beta}\right] P_{n-1}^{\alpha+1,\beta}(x). \end{aligned}$$

If we define

$$v^{\alpha+1,\beta}(x) = -\left[xn - a_{n-1}^{\alpha+1,\beta} d_n + n+\alpha+\beta+1\right] \text{ and}$$

$$q^{\alpha+1,\beta}(x) = -x\left(nt_n + n+\alpha+\beta+1\right) + nt_n + b_{n-1}^{\alpha+1,\beta} + (n+\alpha+\beta+1)\beta_{n-1}^{\alpha+1,\beta},$$

we obtain

$$\left\{ \begin{array}{r} \widetilde{P}_n^{\alpha,\beta}(x) = P_n^{\alpha+1,\beta}(x) + r_n P_{n-1}^{\alpha+1,\beta}(x) \\ \left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' = v^{\alpha+1,\beta}(x) P_n^{\alpha+1,\beta}(x) + q^{\alpha+1,\beta}(x) P_{n-1}^{\alpha+1,\beta}(x) \end{array} \right. .$$

Thus,

$$P_{n-1}^{\alpha+1,\beta}(x) = \frac{\left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' - v^{\alpha+1,\beta}(x)\widetilde{P}_n^{\alpha,\beta}(x)}{q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)}. \tag{19}$$

As a conclusion

$$\begin{aligned} & \left[q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right]\left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)'' + \\ & + \psi_{\alpha+1,\beta}(x)\left[q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right]\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' \\ = \quad & \left[q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right] \lambda_n^{\alpha+1,\beta} \widetilde{P}_n^{\alpha,\beta}(x) + \\ & + r_n\left(\lambda_{n-1}^{\alpha+1,\beta} - \lambda_n^{\alpha+1,\beta}\right)\left[\left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' - v^{\alpha+1,\beta}(x)\widetilde{P}_n^{\alpha,\beta}(x)\right], \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \left[q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right]\left(1-x^2\right)\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)'' + \\ & + \left[\psi_{\alpha+1,\beta}(x)\left[q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right] + r_n\left(1-x^2\right)\left(\lambda_n^{\alpha+1,\beta} - \lambda_{n-1}^{\alpha+1,\beta}\right)\right]\left(\widetilde{P}_n^{\alpha,\beta}(x)\right)' \\ = \quad & \left[\left(q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right)\lambda_n^{\alpha+1,\beta} + r_n\left(\lambda_n^{\alpha+1,\beta} - \lambda_{n-1}^{\alpha+1,\beta}\right) v^{\alpha+1,\beta}(x)\right]\left(\widetilde{P}_n^{\alpha,\beta}(x)\right). \end{aligned}$$

So, we have proved

Theorem 6 *If* $\left\{P_n^{\alpha,\beta}\right\}_{n\geq 0}$ *are the Jacobi monic orthogonal polynomials and* $\left\{\widetilde{P}_n^{\alpha,\beta}\right\}_{n\geq 0}$ *are the monic orthogonal polynomials associated with the moment functional* $\widetilde{\mu} = \mu + M\delta(x-1)$, *then*

$$A(x;n;\alpha)\left(\widetilde{P}_n^{\alpha,\beta}\right)'' + B(x;n;\alpha)\left(\widetilde{P}_n^{\alpha,\beta}\right)' - C(x;n;\alpha)\widetilde{P}_n^{\alpha,\beta} = 0, \tag{20}$$

with
$A(x;n) = \left[q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right]\left(1-x^2\right)$,
$B(x;n;\alpha) = \psi_{\alpha+1,\beta}(x)\left[q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right] + r_n\left(1-x^2\right)\left(\lambda_n^{\alpha+1,\beta} - \lambda_{n-1}^{\alpha+1,\beta}\right)$, *and*
$C(x;n;\alpha) = \left[q^{\alpha+1,\beta}(x) - r_n v^{\alpha+1,\beta}(x)\right]\lambda_n^{\alpha+1} + r_n\left(\lambda_n^{\alpha+1,\beta} - \lambda_{n-1}^{\alpha+1,\beta}\right)v^{\alpha+1,\beta}(x)$.

3 The Zeros

We are going to have an explicit expression of $P_n^{\alpha,\beta}$ in terms of the generalized moments applying the Gram-Schmidt orthonormalization process to the family of polynomials $1, (1-x), (1-x)^2, ..., (1-x)^n$. Indeed, If

$$\begin{array}{rcl} \langle \mu, 1\rangle &=& \mu_0 = \zeta_0 \\ \langle \mu, 1-x\rangle &=& \zeta_1 \\ \left\langle \mu, (1-x)^2\right\rangle &=& \zeta_2 \\ . && . \\ . &=& . \\ . && . \\ \langle \mu, (1-x)^n\rangle &=& \zeta_n \end{array}$$

and

$$\Omega_n(\mu) = \begin{vmatrix} \zeta_0 & \zeta_1 & \cdot & \cdot & \cdot & \zeta_n \\ \zeta_1 & \zeta_2 & \cdot & \cdot & \cdot & \zeta_{n+1} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \zeta_{n-1} & \zeta_n & \cdot & \cdot & \cdot & \zeta_{2n-1} \\ \zeta_n & \zeta_{n+1} & \cdot & \cdot & \cdot & \zeta_{2n} \end{vmatrix}$$

then

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{\Omega_{n-1}(\mu)} \begin{vmatrix} \zeta_0 & \zeta_1 & \cdot & \cdot & \cdot & \zeta_n \\ \zeta_1 & \zeta_2 & \cdot & \cdot & \cdot & \zeta_{n+1} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \zeta_{n-1} & \zeta_n & \cdot & \cdot & \cdot & \zeta_{2n-1} \\ 1 & 1-x & . & . & . & (1-x)^n \end{vmatrix}. \tag{21}$$

If $\widetilde{\mu} = \mu + M\delta(x-1)$ then

$$\widetilde{P}_n^{\alpha,\beta}(x) = \frac{(-1)^n}{\Omega_{n-1}\left(\widetilde{\mu}\right)} \begin{vmatrix} \zeta_0 + M & \zeta_1 & \cdot & \cdot & \cdot & \zeta_n \\ \zeta_1 & \zeta_2 & \cdot & \cdot & \cdot & \zeta_{n+1} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \zeta_{n-1} & \zeta_n & \cdot & \cdot & \cdot & \zeta_{2n-1} \\ 1 & 1-x & \cdot & \cdot & \cdot & (1-x)^n \end{vmatrix}.$$

But

$$\begin{aligned} \Omega_{n-1}\left(\widetilde{\mu}\right) &= \begin{vmatrix} \zeta_0 + M & \zeta_1 & \cdot & \cdot & \cdot & \zeta_n \\ \zeta_1 & \zeta_2 & \cdot & \cdot & \cdot & \zeta_{n+1} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \zeta_{n-1} & \zeta_n & \cdot & \cdot & \cdot & \zeta_{2n-1} \\ \zeta_n & \zeta_{n+1} & \cdot & \cdot & \cdot & \zeta_{2n} \end{vmatrix} \\ &= \Omega_{n-1}(\mu) + M\Omega_{n-2}\left((1-x)^2\mu\right). \end{aligned}$$

As a consequence,

$$(-1)^n \widetilde{P}_n^{\alpha,\beta}(x) = \frac{(-1)^n \Omega_{n-1}(\mu) P_n^{\alpha,\beta}(x) + M(1-x)\Omega_{n-2}\left((1-x)^2\mu\right)(-1)^{n-1} P_{n-1}^{\alpha+2,\beta}(x)}{\Omega_{n-1}(\mu) + M\Omega_{n-2}\left((1-x)^2\mu\right)}.$$

$$\widetilde{P}_n^{\alpha,\beta}(x) = \frac{\Omega_{n-1}(\mu) P_n^{\alpha,\beta}(x) + M(x-1)\Omega_{n-2}\left((1-x)^2\mu\right) P_{n-1}^{\alpha+2,\beta}(x)}{\Omega_{n-1}(\mu) + M\Omega_{n-2}\left((1-x)^2\mu\right)}. \tag{22}$$

Taking into account (22)

$$\lim_{M\to\infty} \widetilde{P}_n^{\alpha,\beta}(x) = (x-1)\, P_{n-1}^{\alpha+2,\beta}(x) \tag{23}$$

Next we will need the following result whose proof can be founded in [3].

Proposition 2 *For $M > 0$, we get*

1. *The zeros of $\widetilde{P}_n^{\alpha,\beta}(x)$ are real, simple, and they are located in $(-1,1)$.*
2. *Since $r_n < 0$, we have*

$$\begin{aligned} x_{n,n}^{\alpha+1,\beta} &< x_{n,n}^{(M)} \text{ and} \\ x_{n,j}^{\alpha+1,\beta} &< x_{n,j}^{(M)} < x_{n-1,j}^{\alpha+1,\beta},\ j = 1, ..., n-1, \end{aligned}$$

where $\left\{x_{n,j}^{\alpha+1}\right\}_{j=1}^{n}$ are the zeros of the polynomial $P_n^{\alpha+1,\beta}(x)$ and $\left\{x_{n,j}^{(M)}\right\}_{j=1}^{n}$ the zeros of $\widetilde{P}_n^{\alpha,\beta}(x)$, respectively.

As a consequence,

$$\lim_{M\to\infty} x_{n,n}^{(M)} = 1, \lim_{M\to\infty} x_{n,k}^{(M)} = x_{n-1,k}^{\alpha+2}, \; k = 1, 2, ..., n-1.$$

Now, using (22), we conclude

$$\widetilde{P}_n^{\alpha,\beta}(1) = \frac{\Omega_{n-1}(\mu) P_n^{\alpha,\beta}(1)}{\Omega_{n-1}(\mu) + M\Omega_{n-2}\left((1-x)^2 \mu\right)}.$$

From this expression we get

$$\left(1 - x_{n,1}^{(M)}\right)\left(1 - x_{n,2}^{(M)}\right) ... \left(1 - x_{n,n}^{(M)}\right) = \frac{\Omega_{n-1}(\mu) P_n^{\alpha,\beta}(1)}{\Omega_{n-1}(\mu) + M\Omega_{n-2}\left((1-x)^2 \mu\right)}$$

and, as a consequence,

$$\begin{aligned} \lim_{M\to\infty} M\left(1 - x_{n,n}^{(M)}\right) &= \frac{\Omega_{n-1}(\mu) P_n^{\alpha,\beta}(1)}{P_{n-1}^{\alpha+2,\beta}(1)\Omega_{n-2}\left((1-x)^2 \mu\right)} \\ &= \frac{\Omega_{n-1}(\mu)}{\Omega_{n-2}\left((1-x)^2 \mu\right)} \frac{2(\alpha+1)(\alpha+2)}{(n+\alpha+1)(n+\alpha+\beta+1)}. \end{aligned}$$

On the other hand, taking into account (21),

$$\left\langle P_n^{\alpha,\beta}(x), (1-x)^n \right\rangle = (-1)^n \frac{\Omega_n(\mu)}{\Omega_{n-1}(\mu)},$$

i.e

$$\left\|P_n^{\alpha,\beta}(x)\right\|^2 = \frac{\Omega_n(\mu)}{\Omega_{n-1}(\mu)},$$

or, equivalently,

$$\Omega_n(\mu) = \left\|P_n^{\alpha,\beta}(x)\right\|^2 \Omega_{n-1}(\mu).$$

Using the last result, we have

$$\frac{\Omega_{n-1}(\mu)}{\Omega_{n-2}\left((1-x)^2 \mu\right)} = \frac{\left\|P_{n-1}^{\alpha,\beta}(x)\right\|^2 \left\|P_{n-2}^{\alpha,\beta}(x)\right\|^2 ... \left\|P_1^{\alpha,\beta}(x)\right\|^2 \left\|P_0^{\alpha,\beta}(x)\right\|^2}{\left\|P_{n-2}^{\alpha+2,\beta}(x)\right\|^2 \left\|P_{n-3}^{\alpha+2,\beta}(x)\right\|^2 ... \left\|P_0^{\alpha+2,\beta}(x)\right\|^2},$$

but, using (13)

$$\begin{aligned} \frac{\left\|P_{n-1}^{\alpha,\beta}(x)\right\|^2}{\left\|P_{n-2}^{\alpha+2,\beta}(x)\right\|^2} &= \frac{\dfrac{2^{2n+\alpha+\beta-1}\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(n+\alpha+\beta)(n-1)!}{(2n+\alpha+\beta-1)(\Gamma(2n+\alpha+\beta-1))^2}}{\dfrac{2^{2n+\alpha+\beta-1}\Gamma(n+\alpha+1)\Gamma(n+\beta-1)\Gamma(n+\alpha+\beta+1)(n-2)!}{(2n+\alpha+\beta-1)(\Gamma(2n+\alpha+\beta-1))^2}} \\ &= \frac{(n+\beta-1)(n-1)}{(n+\alpha)(n+\alpha+\beta)}. \end{aligned}$$

For instance

$$\frac{\Omega_{n-1}(\mu)}{\Omega_{n-2}\left((1-x)^2\mu\right)} =$$

$$\frac{(n+\beta-1)(n-1)(n+\beta-2)(n-2)\ldots(1+\beta)\left\|P_0^{\alpha,\beta}(x)\right\|^2}{(n+\alpha)(n+\alpha+\beta)(n+\alpha-1)(n+\alpha+\beta-1)\ldots(\alpha+2)(\alpha+\beta+2)} =$$

$$\frac{\frac{\Gamma(n+\beta)(n-1)!}{\Gamma(\beta+1)}}{\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)}}\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+1)}{(\alpha+\beta+1)(\Gamma(\alpha+\beta+1))^2} =$$

$$\frac{2^{\alpha+\beta+1}(n-1)!\Gamma(n+\beta)\Gamma(\alpha+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}.$$

Finally

$$\lim_{M\to\infty} M\left(1-x_{n,n}^{(M)}\right) = \frac{\Omega_{n-1}(\mu)}{\Omega_{n-2}\left((1-x)^2\mu\right)}\frac{2(\alpha+1)(\alpha+2)}{(n+\alpha+1)(n+\alpha+\beta+1)} =$$

$$\begin{aligned} & \frac{2^{\alpha+\beta+1}(n-1)!\Gamma(n+\beta)\Gamma(\alpha+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}\frac{2(\alpha+1)(\alpha+2)}{(n+\alpha+1)(n+\alpha+\beta+1)} \\ = & \frac{2^{\alpha+\beta+2}(n-1)!\Gamma(n+\beta)\Gamma(\alpha+1)\Gamma(\alpha+3)}{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}, \end{aligned}$$

finally, the following result concerning the greatest zero of $\widetilde{P}_n^{\alpha,\beta}$ holds

Theorem 7 *For $n \geq 1$, we get*

$$\lim_{M\to\infty} M\left(1-x_{n,n}^{(M)}\right) = \frac{2^{\alpha+\beta+2}(n-1)!\Gamma(n+\beta)\Gamma(\alpha+1)\Gamma(\alpha+3)}{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}. \tag{24}$$

4 Electrostatic model

Assume that $\left\{x_{n,k}^{(M)}\right\}_{k\geq 1}$ are the zeros of $\widetilde{P}_n^{\alpha,\beta}(x)$, and evaluate (20) in every zero. Thus

$$A(x_{n,k}^{(M)};n;\alpha)\left(\widetilde{P}_n^{\alpha,\beta}(x_{n,k}^{(M)})\right)'' + B(x_{n,k}^{(M)};n;\alpha)\left(\widetilde{P}_n^{\alpha,\beta}(x_{n,k}^{(M)})\right)' = 0,$$

and

$$\frac{\left(\widetilde{P}_n^{\alpha,\beta}(x_{n,k}^{(M)})\right)''}{\left(\widetilde{P}_n^{\alpha,\beta}(x_{n,k}^{(M)})\right)'} = -\frac{B(x_{n,k}^{(M)};n;\alpha)}{A(x_{n,k}^{(M)};n;\alpha)}.$$

But

$$
\begin{aligned}
-\frac{B(x_{n,k}^{(M)};n\,;\alpha)}{A(x_{n,k}^{(M)};n;\alpha)} &= -\frac{\psi_{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)\left[n^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)-r_n m^{\alpha+1,\beta}(x_{n,k}^{(M)})\right]+r_n\left(1-\left(x_{n,k}^{(M)}\right)^2\right)\left(\lambda_n^{\alpha+1,\beta}-\lambda_{n-1}^{\alpha+1,\beta}\right)}{\left[q^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)-r_n v^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)\right]\phi\left(x_{n,k}^{(M)}\right)}\\
&= -\frac{\psi_{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)}{\phi\left(x_{n,k}^{(M)}\right)}+\frac{r_n\left(\lambda_{n-1}^{\alpha+1,\beta}-\lambda_n^{\alpha+1,\beta}\right)}{q^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)-r_n v^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)}
\end{aligned}
$$

and making some calculations, we get

$$
\begin{aligned}
-\frac{\psi_{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)}{\phi\left(x_{n,k}^{(M)}\right)} &= \frac{(\alpha+\beta+3)\,x_{n,k}^{(M)}+(\alpha-\beta+1)}{1-\left(x_{n,k}^{(M)}\right)^2}\\
&= \frac{\alpha+2}{1-x_{n,k}^{(M)}}-\frac{\beta+1}{1+x_{n,k}^{(M)}}.
\end{aligned}
$$

From

$$
\frac{\left(\widetilde{P}_n^{\alpha,\beta}(x_{n,k}^{(M)})\right)''}{\left(\widetilde{P}_n^{\alpha,\beta}(x_{n,k}^{(M)})\right)'}=-2\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{x_{n,j}^{(M)}-x_{n,k}^{(M)}}
$$

we have

$$
\begin{aligned}
-2\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{x_{n,j}^{(M)}-x_{n,k}^{(M)}} = \frac{\alpha+2}{1-x_{n,k}^{(M)}}-\frac{\beta+1}{1+x_{n,k}^{(M)}}+\\
+\frac{r_n\left(\lambda_{n-1}^{\alpha+1}-\lambda_n^{\alpha+1}\right)}{q^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)-r_n v^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)}.
\end{aligned}
$$

Since $\lambda_n^{\alpha+1}=-n\,(n+\alpha+\beta+1)$ then

$$
\lambda_{n-1}^{\alpha+1,\beta}-\lambda_n^{\alpha+1,\beta}=2n+\alpha+\beta+1.
$$

Thus

$$
\begin{aligned}
\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{x_{n,j}^{(M)}-x_{n,k}^{(M)}}+\frac{\alpha+2}{2\left(1-x_{n,k}^{(M)}\right)}-\frac{\beta+1}{2\left(1+x_{n,k}^{(M)}\right)}+\\
+\frac{r_n\,(2n+\alpha+\beta+1)}{q^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)-r_n v^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)}=0.
\end{aligned}
\tag{25}
$$

Taking into account

$$
\begin{aligned}
& q^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right) - r_n v^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right) \\
= \; & x_{n,k}^{(M)}\left(nr_n - nt_n - n - \alpha - \beta - 1\right) + \\
& + nt_n + b_{n-1}^{\alpha+1,\beta} + (n+\alpha+\beta+1)\beta_{n-1}^{\alpha+1,\beta} \\
& \left(-a_{n-1}^{\alpha+1,\beta} d_n + n + \alpha + \beta + 1\right) r_n,
\end{aligned}
$$

if we define l_n and s_n as follows

$$
\begin{aligned}
l_n &= \frac{r_n\left(2n+\alpha+\beta+1\right)}{\left(nr_n - nt_n - n - \alpha - \beta - 1\right)}, \\
s_n &= -\frac{nt_n + b_{n-1}^{\alpha+1,\beta} + (n+\alpha+\beta+1)\beta_{n-1}^{\alpha+1,\beta} + \left(-a_{n-1}^{\alpha+1,\beta} d_n + n + \alpha + \beta + 1\right) r_n}{\left(nr_n - nt_n - n - \alpha - \beta - 1\right)}.
\end{aligned}
$$

Then

$$
\frac{r_n\left(2n+\alpha+\beta+1\right)}{q^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right) - r_n v^{\alpha+1,\beta}\left(x_{n,k}^{(M)}\right)} = \frac{l_n}{x_{n,k}^{(M)} - s_n}.
$$

In other words, (25) becomes

$$
\sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{x_{n,j}^{(M)} - x_{n,k}^{(M)}} + \frac{\alpha+2}{2\left(1 - x_{n,k}^{(M)}\right)} - \frac{\beta+1}{2\left(1 + x_{n,k}^{(M)}\right)} + \frac{l_n}{x_{n,k}^{(M)} - s_n} = 0. \tag{26}
$$

Thus, the following electrostatic interpretation for the location of zeros of $\widetilde{P}_n^{\alpha,\beta}$ can be stated. If we consider n charges located in the real line under a logarithmic interaction with an external field

$$
\varphi(x) = -\frac{(\alpha+2)}{2}\ln|x-1| - \frac{\beta+1}{2}\ln|x+1| + l_n \ln|x - s_n|,
$$

this equation means that the gradient of the total energy

$$
E(X) = -\sum_{1\leq k<j\leq n} \ln\left|x_k - x_j\right| + \sum_{j=1}^{n}\varphi\left(x_j\right)
$$

with $X = (x_1, x_2, ..., x_n)$ vanishes at $\left(x_{n,1}^{(M)}, x_{n,2}^{(M)}, ..., x_{n,n}^{(M)}\right)$. In other words, it is a critical point.

Notice that l_n and s_n have the asymptotic behavior

$$
\begin{aligned}
l_n &\sim \frac{2\Gamma(\alpha+1)\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(\alpha+2)+3} > 0 \\
s_n &\sim \frac{1 - \Gamma(\alpha+1)\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(\alpha+2)+3} = -\frac{\Gamma(\alpha+1)\Gamma(\alpha+2) - 1}{\Gamma(\alpha+1)\Gamma(\alpha+2)+3}.
\end{aligned}
$$

Notice that l_n and s_n do not depend on M.

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