Geronimus spectral transforms and measures on the complex plane

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To our friend Claude Brezinski on the occassion of his 65th birthday

Abstract

We analyze a special spectral transform of a measure μ supported on a compact subset C of the complex plane. A perturbation μ_1 of μ is said to be a Geronimus spectral transform if $\mathrm{d}\mu_1 = \frac{\mathrm{d}\mu}{|z-\alpha|^2}$ where $\alpha \notin C$. We focus our attention in the analysis of the Hessenberg matrix associated with the multiplication operator in terms of the orthogonal polynomial basis defined by the measure μ_1.

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1. Introduction

Rational spectral transforms of linear functionals have been considered in [2] in the framework of their connections with orthogonal polynomials and the representation of the multiplication operator with respect to such a basis, that is, a Jacobi matrix.

For linear functionals u associated with Jacobi matrices three canonical rational spectral transforms have been considered (see [2,13–15]).

(1) The canonical Christoffel transform $\tilde{u} = (x-\alpha)u$.
(2) The canonical Geronimus transform $\tilde{u} = (x-\alpha)^{-1}u + m\delta(x-\alpha)$.
(3) The canonical Uvarov transform $\tilde{u} = u + m\delta(x-\alpha)$.

Taking into account the LU and QR factorization of a Jacobi matrix, a simple explanation of the canonical rational spectral transforms using such factorizations is presented in [2]. Furthermore, the iteration of the Christoffel transform $\tilde{u} = (x-\alpha)^2 u$ yields a connection between the corresponding Jacobi matrices using the QR factorization of the Jacobi matrix associated with u (see [4]). Later on, some analogue of rational spectral transformations for Hermitian Toeplitz matrices is introduced in [5]. There, an Hermitian linear functional u is introduced in the linear space of the

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Laurent polynomials $\mathbb{L} = \text{span}\{z^n\}_{n\in\mathbb{Z}}$. Thus an inner product associated with u can be defined in the linear space $\mathbb{P}$ of polynomials with complex coefficients as follows:

$$\langle p, q\rangle_u = \langle u, p(z)\overline{q(\bar{z}^{-1})}\rangle, \quad p, q \in \mathbb{P}. \tag{1}$$

The Gram matrix T with respect to the canonical basis $\{z^n\}_{n\geqslant 0}$ is a Hermitian Toeplitz matrix. If the principal submatrices T_n of T are positive definite, then there exists a non-trivial probability measure μ supported on the unit circle (see [8,12]) such that:

$$\langle u, p\rangle = \int_0^{2\pi} p(e^{i\theta})\, d\mu(\theta), \quad p \in \mathbb{P}.$$

In such a case, there exists a sequence $\{\Phi_n\}_{n\geqslant 0}$ of monic polynomials such that:

$$\langle \Phi_n, \Phi_m\rangle_\mu = e_n(\mu)\delta_{n,m}, \quad n, m \in \mathbb{N}.$$

$\{\Phi_n\}_{n\geqslant 0}$ is said to be the sequence of monic polynomials orthogonal with respect to the Hermitian linear functional u. The polynomials $\{\varphi_n\}_{n\geqslant 0}$ with $\varphi_n = e_n(\mu)_n^{-1/2}\Phi_n$ are said to be orthonormal either with respect to u, or with respect to the inner product (1).

The multiplication operator $\underset{p(z)\to zp(z)}{\mathbb{P}\to\mathbb{P}}$ is represented in terms of the polynomial basis $\{\varphi_n\}_{n\geqslant 0}$ by a lower Hessenberg matrix $H(\mu)$. The algebraic analysis of the Hessenberg matrix associated with the multiplication operator has been presented in [10] in the framework of the state-space generators of general orthogonal polynomials.

In [5,11] we have analyzed the representation of such an operator when two kind of canonical perturbations of the linear functional u are considered

(1) *Canonical Christoffel transforms*, i.e., we have analyzed a new linear functional $\tilde{u}$ such that the corresponding inner product is

$$\langle p, q\rangle_{\tilde{u}} = \langle (z-\alpha)p, (z-\alpha)q\rangle_u, \quad p, q \in \mathbb{P}.$$

The iteration of the canonical Christoffel transform has been analyzed from an analytic point of view in [6,9].

(2) *Canonical Uvarov transforms*, i.e., we have introduced a new linear functional $\tilde{u}$ such that the corresponding inner product is

$$\langle p, q\rangle_{\tilde{u}} = \langle p, q\rangle_u + mp(\alpha)\overline{q(\bar{\alpha}^{-1})} + mp(\bar{\alpha}^{-1})\overline{q(\alpha)}$$

with $m \in \mathbb{R}_+$ and $\alpha \in \mathbb{C}$.

Again, the LU and QR factorizations play a central role.

More recently, in [3] some extensions of canonical Christoffel transforms have been considered for other kind of inner products. In particular, formal orthogonality on algebraic curves have been considered in the seminal paper [1]. Unfortunately, the algebraic study of the corresponding orthogonal polynomials (recurrence relations for instance) has not been done yet. But for a future work, the analysis of spectral canonical transforms for these inner products offers a new perspective.

The aim of our contribution is the analysis of a special example of Geronimus transform for measures supported on a compact subset of the complex plane. In some sense, this is an inverse transform of the Christoffel canonical one.

The structure of the manuscript is as follows. In Section 2 we introduce the basic concepts and the background for the study of the Geronimus transform μ_1 of a measure μ supported on an infinite compact subset of the complex plane. In particular, we will give the explicit expression of the polynomials orthogonal with respect to μ_1 in terms of the polynomials orthogonal with respect to μ. In Section 3, we obtain the corresponding Hessenberg matrices as well as the connections with the QR factorization. Finally, in Section 4, some examples are analyzed.

2. Background and preliminary results

Let μ be a non-trivial probability measure, i.e., a positive Borel measure, supported on a infinite compact subset C of the complex plane. Under such an assumption, there exists a sequence $\{\Phi_n(z;\mu)\}_{n\geqslant 0}$ of monic polynomials orthogonal with respect to μ such that:

$$\Phi_n(z;\mu) = \frac{1}{\Delta_{n-1}(\mu)} \begin{vmatrix} c_{0,0}(\mu) & c_{1,0}(\mu) & \cdots & c_{n,0}(\mu) \\ \vdots & \vdots & & \vdots \\ c_{0,n-1}(\mu) & c_{1,n-1} & \cdots & c_{n,n-1}(\mu) \\ 1 & z & \cdots & z^n \end{vmatrix},$$

where $c_{k,j} = \int_C z^k \bar{z}^j \, \mathrm{d}\mu$ and

$$\Delta_n(\mu) = \begin{vmatrix} c_{0,0}(\mu) & \cdots & c_{n,0}(\mu) \\ \vdots & & \vdots \\ c_{0,n}(\mu) & \cdots & c_{n,n}(\mu) \end{vmatrix}.$$

Notice that $\Delta_n(\mu) > 0$ for $n \geqslant 0$.

The sequence $\{\varphi_n(z;\mu)\}_{n\geqslant 0}$, given by $\varphi_n(z;\mu) = (e_n(\mu))^{-1/2}\Phi_n(z;\mu)$ is said to be the sequence of *orthonormal polynomials* associated with μ, where $e_n(\mu) = \dfrac{\Delta_n(\mu)}{\Delta_{n-1}(\mu)}$, $n \geqslant 0$, with the convention $\Delta_{-1}(\mu) = 1$.

The *nth reproducing kernel polynomial* associated with $\{\varphi_n(z;\mu)\}_{n\geqslant 0}$ is defined by

$$K_n(z,y;\mu) = \sum_{j=0}^{n} \overline{\varphi_j(y;\mu)}\varphi_j(z;\mu).$$

The functions

$$q_j(t) = \int_C \frac{\overline{\varphi_j(z;\mu)}}{t-z} \, \mathrm{d}\mu(z), \quad t \notin C, \quad j \geqslant 0 \tag{2}$$

are called *functions of second kind* associated with μ. We also denote

$$Q_j(t) = \int_C \frac{\overline{\Phi_j(z;\mu)}}{t-z} \mathrm{d}\mu(z) = (e_j(\mu))^{1/2} q_j(t).$$

We define

$$\mathrm{d}\mu_1 = \frac{\mathrm{d}\mu}{|z-\alpha|^2}, \quad \alpha \notin C. \tag{3}$$

This kind of perturbed measures has been studied in [7] as well as the corresponding sequence of monic orthogonal polynomials. It will be said a *canonical Geronimus* transform of the measure μ.

Let $\{\Phi_n(z;\mu_1)\}_{n\geqslant 0}$ be the sequence of monic orthogonal polynomials on C associated with μ_1 and $\{\varphi_n(z;\mu_1)\}_{n\geqslant 0}$ be the corresponding orthonormal polynomial sequence. Our first goal is to find an explicit expression for $\{\Phi_n(z;\mu_1)\}_{n\geqslant 0}$ in terms of the sequence $\{\Phi_n(z;\mu)\}_{n\geqslant 0}$.

Proposition 1. *Consider* $\varepsilon_n(\alpha) = \|\mu_1\| - \sum_{l=0}^{n} |q_l(\alpha)|^2$, $n \geqslant 0$. *Then*

$$\frac{\Delta_n(\mu_1)}{\Delta_{n-1}(\mu)} = \varepsilon_{n-1}(\alpha), \quad n \geqslant 1. \tag{4}$$

Notice that $\|\mu_1\| = \Delta_0(\mu_1)$.

Proof. If we consider the family $\{1, z-\alpha, \ldots, (z-\alpha)^n, \ldots\}$ and we define

$$d_{k,j}(\mu) = \int_C (z-\alpha)^k \overline{(z-\alpha)^j}\, \mathrm{d}\mu = \int_C (z-\alpha)^{k+1} \overline{(z-\alpha)^{j+1}}\, \mathrm{d}\mu_1 = d_{k+1,j+1}(\mu_1), \quad k, j \geqslant 0,$$

then the monic polynomial $\Phi_n(z;\mu)$ is given by

$$\Phi_n(z;\mu) = \frac{1}{\Delta_{n-1}(\mu)} \begin{vmatrix} d_{0,0}(\mu) & d_{1,0}(\mu) & \cdots & d_{n,0}(\mu) \\ \vdots & \vdots & & \vdots \\ d_{0,n-1}(\mu) & d_{1,n-1}(\mu) & \cdots & d_{n,n-1}(\mu) \\ 1 & z-\alpha & \cdots & (z-\alpha)^n \end{vmatrix}$$

as well as

$$\Delta_n(\mu_1) = \begin{vmatrix} d_{0,0}(\mu_1) & \cdots & d_{n,0}(\mu_1) \\ \vdots & & \vdots \\ d_{0,n}(\mu_1) & \cdots & d_{n,n}(\mu_1) \end{vmatrix} = \begin{vmatrix} d_{0,0}(\mu_1) & d_{1,0}(\mu_1) & \cdots & d_{n,0}(\mu_1) \\ d_{0,1}(\mu_1) & d_{0,0}(\mu) & \cdots & d_{n-1,0}(\mu) \\ \vdots & \vdots & & \vdots \\ d_{0,n}(\mu_1) & d_{0,n-1,}(\mu) & \cdots & d_{n-1,n-1}(\mu) \end{vmatrix}$$

If we use the determinantal Sylvester's identity for the above expression, then we get

$$\Delta_n(\mu_1)\Delta_{n-2}(\mu) = \Delta_{n-1}(\mu_1)\Delta_{n-1}(\mu) - e_{n-1}(\mu)\Delta_{n-2}^2(\mu)|q_{n-1}(\alpha)|^2, \quad n \geqslant 1.$$

From this, we deduce

$$\frac{\Delta_n(\mu_1)}{\Delta_{n-1}(\mu)} = \frac{\Delta_{n-1}(\mu_1)}{\Delta_{n-2}(\mu)} - |q_{n-1}(\alpha)|^2, \quad n \geqslant 1.$$

Therefore

$$\frac{\Delta_n(\mu_1)}{\Delta_{n-1}(\mu)} = \Delta_0(\mu_1) - \sum_{l=0}^{n-1} |q_l(\alpha)|^2, \quad n \geqslant 1.$$

Since $\Delta_0(\mu_1) = \|\mu_1\|$, our statement follows. $\square$

Corollary 2.

$$\frac{e_{n+1}(\mu_1)}{e_n(\mu)} = \frac{\varepsilon_n(\alpha)}{\varepsilon_{n-1}(\alpha)}, \quad n \geqslant 1.$$

Proposition 3. *The sequence of monic orthogonal polynomials* $\{\Phi_n(z;\mu_1)\}$ *can be obtained from*

$$\Phi_{n+1}(z;\mu_1) = (z-\alpha)\Phi_n(z;\mu) + \frac{\overline{Q_n(\alpha)}}{\varepsilon_{n-1}(\alpha)} S_n(z,\alpha;\mu), \quad n \geqslant 1,$$

$$\Phi_0(z;\mu_1) = 1, \quad \Phi_1(z;\mu_1) = z - \alpha + \frac{\overline{Q_0(\alpha)}}{\|\mu_1\|}, \tag{5}$$

where

$$S_n(z,\alpha;\mu) = \int_C \frac{z-t}{\alpha-t} K_{n-1}(z,t;\mu)\,\mathrm{d}\mu(t), \quad n \geqslant 1. \tag{6}$$

Here $K_n(z,\alpha;\mu)$ *is the nth reproducing kernel associated with* $\{\varphi_n(z;\mu)\}_{n\geqslant 0}$.

Proof. Since $\{(z-\alpha)\varphi_n(z;\mu)\}_{n\geqslant 0}$ is an orthonormal basis in $(z-\alpha)\mathbb{P}$ with respect to the inner product

$$\langle f, g\rangle_{\mu_1} = \int_C f(z)\overline{g(z)}\,\mathrm{d}\mu_1,$$

the Fourier expansion of the polynomial $\varphi_{n+1}(z;\mu_1)-\varphi_{n+1}(\alpha;\mu_1)$ in terms of the above basis yields:

$$\varphi_{n+1}(z;\mu_1)-\varphi_{n+1}(\alpha;\mu_1) = (z-\alpha)\sum_{j=0}^{n}\lambda_{n+1,j}\varphi_j(z;\mu) \quad \text{for every } n\geqslant 0.$$

Then

$$\begin{aligned}\lambda_{n+1,j} &= \int_C (\varphi_{n+1}(t;\mu_1)-\varphi_{n+1}(\alpha;\mu_1))\overline{(t-\alpha)\varphi_j(t;\mu)}\,\mathrm{d}\mu_1(t)\\ &= \int_C \varphi_{n+1}(t;\mu_1)\overline{(t-\alpha)\varphi_j(t;\mu)}\,\mathrm{d}\mu_1(t) - \varphi_{n+1}(\alpha;\mu_1)\int_C \overline{(t-\alpha)\varphi_j(t;\mu)}\,\mathrm{d}\mu_1(t)\\ &= \left(\frac{e_{n+1}(\mu_1)}{e_n(\mu)}\right)^{1/2}\delta_{n,j} + \varphi_{n+1}(\alpha;\mu_1)\int_C \frac{\overline{\varphi_j(t;\mu)}}{\alpha-t}\,\mathrm{d}\mu(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi_{n+1}(z;\mu_1) &= \varphi_{n+1}(\alpha;\mu_1)\left(1+(z-\alpha)\sum_{j=0}^{n-1}\int_C \frac{\overline{\varphi_j(t;\mu)}}{\alpha-t}\varphi_j(z;\mu)\,\mathrm{d}\mu(t)\right)\\ &\quad + \left[\left(\frac{e_{n+1}(\mu_1)}{e_n(\mu)}\right)^{1/2} + q_n(\alpha)\varphi_{n+1}(\alpha,\mu_1)\right](z-\alpha)\varphi_n(z;\mu)\\ &= \varphi_{n+1}(\alpha;\mu_1)\int_C \frac{z-t}{\alpha-t}K_{n-1}(z,t;\mu)\,\mathrm{d}\mu(t)\\ &\quad + \left[\left(\frac{e_{n+1}(\mu_1)}{e_n(\mu)}\right)^{1/2} + q_n(\alpha)\varphi_{n+1}(\alpha,\mu_1)\right](z-\alpha)\varphi_n(z;\mu).\end{aligned}$$

If we rewrite the above expression in terms of the monic orthogonal polynomials $\{\Phi_n(z;\mu)\}_{n\geqslant 0}$ and $\{\Phi_n(z;\mu_1)\}_{n\geqslant 0}$, we obtain

$$\Phi_{n+1}(z;\mu_1) = \Phi_{n+1}(\alpha;\mu_1)S_n(z,\alpha;\mu) + \frac{e_{n+1}(\mu_1)+Q_n(\alpha)\Phi_{n+1}(\alpha,\mu_1)}{e_n(\mu)}(z-\alpha)\Phi_n(z;\mu). \tag{7}$$

Notice that $S_n(z,\alpha;\mu)$ is a polynomial of degree n. For every $n\geqslant 0$, taking into account the leading coefficients in both the left- and right-hand sides of (7), we get

$$e_n(\mu)-e_{n+1}(\mu_1) = Q_n(\alpha)\Phi_{n+1}(\alpha;\mu_1), \tag{8}$$

and from Corollary 2,

$$\begin{aligned}\Phi_{n+1}(\alpha;\mu_1) &= \frac{e_{n+1}(\mu_1)}{e_n(\mu)}\frac{\overline{Q_n(\alpha)}}{\varepsilon_n(\alpha)}\\ &= \frac{\overline{Q_n(\alpha)}}{\varepsilon_{n-1}(\alpha)}.\end{aligned}$$

Finally, from Eq. (7), we obtain

$$\Phi_{n+1}(z;\mu_1) = (z-\alpha)\Phi_n(z;\mu) + \frac{\overline{Q_n(\alpha)}}{\varepsilon_{n-1}(\alpha)}S_n(z,\alpha;\mu). \qquad \square$$

Remark 4. If $C = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, using in (6) the Christoffel–Darboux formula (see [8,12]), then the sequence $\{\Phi_n(z; \mu_1)\}_{n \geqslant 0}$ satisfy

$$\Phi_{n+1}(z; \mu_1) = \left(z - \alpha \frac{\varepsilon_n(\alpha)}{\varepsilon_{n-1}(\alpha)} \right) \Phi_n(z; \mu) + \frac{\overline{q_n(\alpha)}\,\overline{q_n(\bar{\alpha}^{-1})}}{\varepsilon_{n-1}(\alpha)\alpha^{n+1}} \Phi_n^*(z; \mu), \quad n \geqslant 1. \tag{9}$$

This result can be seen with more detail in [7].

Now, from (3), we get

$$d\mu = |z - \alpha|^2 \, d\mu_1. \tag{10}$$

Let $H(\mu_1)$ be the lower Hessenberg matrix associated with the multiplication operator in terms of $\{\varphi_n(z, \mu_1)\}_{n \geqslant 0}$. In [5], we give a relation between $H(\mu)$ and $H(\mu_1)$

$$H(\mu_1) - \alpha I = LM \quad \text{and} \quad H(\mu) - \alpha I = ML, \tag{11}$$

where L is a lower triangular matrix such that $\varphi(z; \mu_1) = L\varphi(z; \mu)$ and M is a lower Hessenberg matrix so that $(z - \alpha)\varphi(z; \mu) = M\varphi(z; \mu_1)$. Here $\varphi(z; \tilde{\mu}) = [\varphi_0(z; \tilde{\mu}), \varphi_1(z; \tilde{\mu}), \ldots]^{\mathrm{T}}$ with $\tilde{\mu} = \mu, \mu_1$.

Our second goal is to find the explicit expression of the matrices M and L.

3. Hessenberg matrices and the Geronimus transform

Proposition 5. *The sequences* $\{\varphi_n(z, \mu)\}_{n \geqslant 0}$ *and* $\{\varphi_n(z, \mu_1)\}_{n \geqslant 0}$ *satisfy*

$$(z - \alpha)\varphi(z; \mu) = M\varphi(z; \mu_1),$$

where $\varphi(z; \mu) = [\varphi_0(z; \mu), \varphi_1(z; \mu), \ldots]^{\mathrm{t}}$, $\varphi(z; \mu_1) = \left[\varphi_0(z; \mu_1), \varphi_1(z; \mu_1), \ldots\right]^{\mathrm{t}}$, *and M is a lower Hessenberg matrix with entries*

$$m_{k,j} = \begin{cases} -\dfrac{\overline{q_k(\alpha)}}{\sqrt{\|\mu_1\|}} & \text{if } j = 0, k \geqslant 0, \\ \sqrt{\dfrac{\varepsilon_0(\alpha)}{\|\mu_1\|}} & \text{if } k = 0, j = 1, \\ -\dfrac{\overline{q_k(\alpha)} q_0(\alpha)}{\sqrt{\|\mu_1\|}\sqrt{\varepsilon_0(\alpha)}} & \text{if } j = 1, k \geqslant 1, \\ -\dfrac{\overline{q_k(\alpha)} q_{j-1}(\alpha)}{\sqrt{\varepsilon_{j-2}(\alpha)}\sqrt{\varepsilon_{j-1}(\alpha)}} & \text{if } 2 \leqslant j \leqslant k, \\ \sqrt{\dfrac{\varepsilon_k(\alpha)}{\varepsilon_{k-1}(\alpha)}} & \text{if } j = k + 1, \\ 0 & \text{if } j > k + 1. \end{cases} \tag{12}$$

Proof. Taking into account (5), for $n = 0$ we get

$$\Phi_1(z; \mu_1) - \frac{\overline{Q_0(\alpha)}}{\|\mu_1\|} \Phi_0(z; \mu_1) = (z - \alpha)\Phi_0(z; \mu)$$

and, as a consequence,

$$\left(\frac{e_1(\mu_1)}{e_0(\mu)} \right)^{1/2} \varphi_1(z; \mu_1) - \frac{\overline{q_0(\alpha)}}{\|\mu_1\|^{1/2}} \varphi_0(z; \mu_1) = (z - \alpha)\varphi_0(z; \mu).$$

Now, for $n = 1$,

$$\Phi_2(z;\mu_1) - \frac{\overline{Q_1(\alpha)}\|\mu_1\|}{\varepsilon_0(\alpha)\overline{Q_0(\alpha)}}\Phi_1(z;\mu_1) = (z-\alpha)\left[\Phi_1(z;\mu) - \left(\frac{\overline{Q_1(\alpha)}}{\overline{q_0(\alpha)}}\right)\varphi_0(z;\mu)\right]$$

i.e.,

$$\left(\frac{e_2(\mu_1)}{e_1(\mu)}\right)^{1/2}\varphi_2(z;\mu_1) - \frac{\|\mu_1\|}{\varepsilon_0(\alpha)}\frac{\overline{q_1(\alpha)}}{\overline{q_0(\alpha)}}\left(\frac{e_1(\mu_1)}{e_0(\mu)}\right)^{1/2}\varphi_1(z;\mu_1) = (z-\alpha)\left(\varphi_1(z;\mu) - \frac{\overline{q_1(\alpha)}}{\overline{q_0(\alpha)}}\varphi_0(z;\mu)\right).$$

Finally, for $n \geqslant 2$,

$$\begin{aligned}&\Phi_{n+1}(z;\mu_1) - \frac{\overline{Q_n(\alpha)}\varepsilon_{n-2}(\alpha)}{Q_{n-1}(\alpha)\varepsilon_{n-1}(\alpha)}\Phi_n(z;\mu_1)\\ &\quad = (z-\alpha)\left[\Phi_n(z;\mu) - \frac{\overline{Q_n(\alpha)}}{\varepsilon_{n-1}(\alpha)}\left(\frac{\sqrt{e_{n-1}(\mu)}\varepsilon_{n-2}(\alpha)}{\overline{Q_{n-1}(\alpha)}} - q_{n-1}(\alpha)\right)\varphi_{n-1}(z;\mu)\right],\end{aligned}$$

$$\Phi_{n+1}(z;\mu_1) - \left(\frac{e_n(\mu)}{e_{n-1}(\mu)}\right)^{1/2}\frac{\overline{q_n(\alpha)}}{\overline{q_{n-1}(\alpha)}}\frac{\varepsilon_{n-2}(\alpha)(\mu)}{\varepsilon_{n-1}(\alpha)(\mu_1)}\Phi_n(z;\mu_1) = (z-\alpha)\left[\Phi_n(z;\mu) - \frac{\sqrt{e_n(\mu)}\overline{q_n(\alpha)}}{\overline{q_{n-1}(\alpha)}}\varphi_{n-1}(z;\mu)\right].$$

Hence, taking into account Corollary 2

$$\left(\frac{\varepsilon_n(\alpha)}{\varepsilon_{n-1}(\alpha)}\right)^{1/2}\varphi_{n+1}(z;\mu_1) - \left(\frac{\varepsilon_{n-2}(\alpha)}{\varepsilon_{n-1}(\alpha)}\right)^{1/2}\frac{\overline{q_n(\alpha)}}{\overline{q_{n-1}(\alpha)}}\varphi_n(z;\mu_1) = (z-\alpha)\left[\varphi_n(z;\mu) - \frac{\overline{q_n(\alpha)}}{\overline{q_{n-1}(\alpha)}}\varphi_{n-1}(z;\mu)\right].$$

In a matrix form, we get

$$(z-\alpha)\tilde{M}\varphi(z;\mu) = \hat{M}\varphi(z;\mu_1), \tag{13}$$

where $\tilde{M}$ and $\hat{M}$ are lower and upper bidiagonal matrices, respectively, with entries

$$\tilde{m}_{k,j} = \begin{cases} 1, & j = k,\\ -\dfrac{\overline{q_k(\alpha)}}{\overline{q_{k-1}(\alpha)}}, & j = k-1,\\ 0 & \text{otherwise,}\end{cases} \qquad \hat{m}_{k,j} = \begin{cases} -\dfrac{\overline{q_0(\alpha)}}{\sqrt{\|\mu_1\|}}, & j = k = 0,\\ -\dfrac{\overline{q_1(\alpha)}}{\overline{q_0(\alpha)}}\sqrt{\dfrac{\|\mu_1\|}{\varepsilon_0(\alpha)}}, & j = k = 1,\\ -\dfrac{\overline{q_k(\alpha)}}{\overline{q_{k-1}(\alpha)}}\sqrt{\dfrac{\varepsilon_{k-2}(\alpha)}{\varepsilon_{k-1}(\alpha)}}, & j = k \geqslant 2,\\ \sqrt{\dfrac{\varepsilon_0(\alpha)}{\|\mu_1\|}}, & k = 0,\ j = 1,\\ \sqrt{\dfrac{\varepsilon_k(\alpha)}{\varepsilon_{k-1}(\alpha)}}, & j = k+1,\\ 0 & \text{otherwise.}\end{cases}$$

The matrix $\tilde{M}$ is non-singular. Thus Eq. (13) becomes

$$(z-\alpha)\varphi(z;\mu) = \tilde{M}^{-1}\hat{M}\varphi(z;\mu_1).$$

As a consequence,

$$M = \tilde{M}^{-1}\hat{M}. \tag{14}$$

$\tilde{M}^{-1}$ is a lower triangular matrix with entries $\tilde{m}_{k,j}^{(-1)}$ given by

$$\tilde{m}_{k,j}^{(-1)} = \begin{cases} \dfrac{\overline{q_k(\alpha)}}{\overline{q_j(\alpha)}}, & 0 \leqslant j \leqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the multiplication in (14) yields (12). $\square$

The matrix M is quasi-unitary (see [11]). Indeed

Proposition 6.

(i) $MM^* = I$.
(ii) $M^*M = I - \varepsilon_\infty(\alpha)\varphi(\alpha;\mu_1)\varphi(\alpha;\mu_1)^*$ *where*

$$\varepsilon_\infty(\alpha) = \lim_{n\to\infty}\varepsilon_n(\alpha) = \|\mu_1\| - \sum_{l=0}^{\infty}|q_l(\alpha)|^2 \geqslant 0,$$

where I denotes the unit matrix.

Proof.

(i)

$$\begin{aligned} I &= \langle\varphi(z;\mu), \varphi(z;\mu)^{\mathrm{t}}\rangle_\mu \\ &= \langle(z-\alpha)\varphi(z;\mu), (z-\alpha)\varphi(z;\mu)^{\mathrm{t}}\rangle_{\mu_1} \\ &= \langle M\varphi(z;\mu_1), \varphi(z;\mu_1)^{\mathrm{t}}M^{\mathrm{t}}\rangle_{\mu_1} \\ &= M\langle\varphi(z;\mu_1), \varphi(z;\mu_1)^{\mathrm{t}}\rangle_{\mu_1}M^* = MM^*. \end{aligned}$$

(ii)

$$(M^*)_{(0)}M^{(0)} = \frac{1}{e_0(\mu_1)}\sum_{l=0}^{\infty}|q_l(\alpha)|^2 = 1 - \frac{\varepsilon_\infty(\alpha)}{\|\mu_1\|},$$

$$\begin{aligned} (M^*)_{(1)}M^{(1)} &= \frac{1}{\varepsilon_0(\alpha)}\left(\varepsilon_0(\alpha)^2 + |q_0(\alpha)|^2\sum_{l=1}^{\infty}|q_l(\alpha)|^2\right) \\ &= \frac{1}{\|\mu_1\|\varepsilon_0(\alpha)}\left(\|\mu_1\|\varepsilon_0(\alpha) - \|\mu_1\||q_0(\alpha)|^2 + |q_0(\alpha)|^2\sum_{l=0}^{\infty}|q_l(\alpha)|^2\right) \\ &= 1 - \frac{|q_0(\alpha)|^2}{\|\mu_1\|\varepsilon_0(\alpha)}\varepsilon_\infty(\alpha). \end{aligned}$$

For $j \geqslant 2$,

$$(M^*)_{(j)} M^{(j)} = \frac{\varepsilon_{j-1}(\alpha)}{\varepsilon_{j-2}(\alpha)} \left(1 + \frac{|q_{j-1}(\alpha)|^2}{\varepsilon_{j-1}(\alpha)^2} \sum_{l=j}^{\infty} |q_l(\alpha)|^2 \right)$$

$$= \frac{1}{\varepsilon_{j-2}(\alpha)\varepsilon_{j-1}(\alpha)} \left(\varepsilon_{j-1}(\alpha)^2 + |q_{j-1}(\alpha)|^2 \sum_{l=j}^{\infty} |q_l(\alpha)|^2 \right)$$

$$= 1 - \frac{|q_{j-1}(\alpha)|^2}{\varepsilon_{j-2}(\alpha)\varepsilon_{j-1}(\alpha)} \varepsilon_\infty(\alpha),$$

$$(M^*)_{(0)} M^{(1)} = -\frac{q_0(\alpha)}{\|\mu_1\| \sqrt{\|\mu_1\| - |q_0(\alpha)|^2}} \varepsilon_\infty(\alpha).$$

For $j \geqslant 2$,

$$(M^*)_{(0)} M^{(j)} = -\frac{q_{j-1}(\alpha)}{\sqrt{\|\mu_1\|} \sqrt{\varepsilon_{j-2}(\alpha)\varepsilon_{j-1}(\alpha)}} \varepsilon_\infty(\alpha),$$

$$(M^*)_{(1)} M^{(j)} = -\frac{\overline{q_0(\alpha)} q_{j-1}(\alpha) \sqrt{\varepsilon_{j-1}(\alpha)}}{\sqrt{\|\mu_1\|} \sqrt{\varepsilon_0(\alpha)\varepsilon_{j-2}(\alpha)}} \left[1 - \frac{1}{\varepsilon_{j-1}(\alpha)} \sum_{l=j}^{\infty} |q_l(\alpha)|^2 \right]$$

$$= -\frac{\overline{q_0(\alpha)} q_{j-1}(\alpha)}{\sqrt{\|\mu_1\|} \sqrt{\varepsilon_0(\alpha)\varepsilon_{j-2}(\alpha)\varepsilon_{j-1}(\alpha)}} \varepsilon_\infty(\alpha).$$

Finally, for $2 \leqslant k < j$,

$$(M^*)_{(k)} M^{(j)} = -\frac{\overline{q_{k-1}(\alpha)} q_{j-1}(\alpha) \sqrt{\varepsilon_{j-1}(\alpha)}}{\sqrt{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)\varepsilon_{j-2}(\alpha)}} \left[1 - \frac{1}{\varepsilon_{j-1}(\alpha)} \sum_{l=j}^{\infty} |q_l(\alpha)|^2 \right]$$

$$= -\frac{\overline{q_{k-1}(\alpha)} q_{j-1}(\alpha)}{\sqrt{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)\varepsilon_{j-2}(\alpha)\varepsilon_{j-1}(\alpha)}} \varepsilon_\infty(\alpha).$$

Since $\Phi_{n+1}(\alpha; \mu_1) = \frac{\overline{Q_n(\alpha)}}{\varepsilon_{n-1}(\alpha)}$ for $n \geqslant 1$, we get

$$\varphi_{n+1}(\alpha; \mu_1) = \left(\frac{e_n(\mu)}{e_{n+1}(\mu_1)} \right)^{1/2} \frac{\overline{q_n(\alpha)}}{\varepsilon_{n-1}(\alpha)}$$

$$= \frac{\overline{q_n(\alpha)}}{\sqrt{\varepsilon_{n-1}(\alpha)\varepsilon_n(\alpha)}}, \quad n \geqslant 1. \qquad \square$$

Let L be the lower triangular matrix such that $\varphi(z;\mu_1) = L\varphi(z;\mu)$. Then, taking into account (13), we obtain

$$\hat{M}L = \tilde{M}(H(\mu) - \alpha I). \tag{15}$$

The entries $\tilde{h}_{k,j}$ of $\tilde{H} := \tilde{M}(H(\mu) - \alpha I)$ are

$$\tilde{h}_{k,j} = \begin{cases} -\sqrt{e_0(\mu)e_1(\mu)}\varphi_1(0)\overline{\varphi_0(0)}, & j = k = 0, \\ -\sqrt{\dfrac{e_k(\mu)}{e_{k-1}(\mu)}}\dfrac{\overline{q_k(\alpha)}}{\overline{q_{k-1}(\alpha)}} - \sqrt{e_k(\mu)e_{k+1}(\mu)}\varphi_{k+1}(0)\overline{\varphi_k(0)}, & j = k, \\ \sqrt{\dfrac{e_{k+1}(\mu)}{e_k(\mu)}}, & j = k+1, \\ \overline{\varphi_j(0)}\sqrt{e_k(\mu)}\left(\dfrac{\overline{q_k(\alpha)}}{\overline{q_{k-1}(\alpha)}}\varphi_k(0)\sqrt{e_{k-1}(\mu)} - \varphi_{k+1}(0)\sqrt{e_{k+1}(\mu)}\right), & j \leqslant k-1, \\ 0, & j > k+1, \end{cases}$$

and the entries $\hat{l}_{k,j}$ of $\hat{L} = \hat{M}L$ are

$$\hat{l}_{k,j} = \begin{cases} -\dfrac{\overline{q_0(\alpha)}}{\sqrt{\|\mu_1\|}}l_{0,0} + \sqrt{\dfrac{\varepsilon_1(\alpha)}{\varepsilon_0(\alpha)}}l_{1,0}, & j = k = 0, \\ -\dfrac{\overline{q_1(\alpha)}}{\overline{q_0(\alpha)}}\sqrt{\dfrac{\|\mu_1\|}{\varepsilon_0(\alpha)}}l_{1,1} + \sqrt{\dfrac{\varepsilon_1(\alpha)}{\varepsilon_0(\alpha)}}l_{2,1}, & j = k = 1, \\ \sqrt{\dfrac{\varepsilon_0(\alpha)}{\|\mu_1\|}}l_{1,1}, & k = 0, \quad j = 1, \\ -\dfrac{\overline{q_1(\alpha)}}{\overline{q_0(\alpha)}}\sqrt{\dfrac{\|\mu_1\|}{\varepsilon_0(\alpha)}}l_{1,0} + \sqrt{\dfrac{\varepsilon_1(\alpha)}{\varepsilon_0(\alpha)}}l_{2,0}, & k = 1, \quad j = 0, \\ \sqrt{\dfrac{\varepsilon_0(\alpha)}{\|\mu_1\|}}l_{1,1}, & k = 0, \quad j = 1, \\ \sqrt{\dfrac{\varepsilon_k(\alpha)}{\varepsilon_{k-1}(\alpha)}}l_{k+1,k+1}, & j = k+1, \\ -\dfrac{\overline{q_k(\alpha)}}{\overline{q_{k-1}(\alpha)}}\sqrt{\dfrac{\varepsilon_{k-2}(\alpha)}{\varepsilon_{k-1}(\alpha)}}l_{k,j} + \sqrt{\dfrac{\varepsilon_k(\alpha)}{\varepsilon_{k-1}(\alpha)}}l_{k+1,j}, & j \leqslant k, \\ 0, & j > k+1. \end{cases}$$

Since $l_{0,0}=\sqrt{\frac{\|\mu\|}{\|\mu_1\|}}$ and taking into account (15), the entries $l_{k,j}$ of L are given by

$$l_{1,0}=\sqrt{\frac{\|\mu_1\|}{\varepsilon_0(\alpha)}}\left(\tilde{h}_{0,0}+\frac{\overline{q_0(\alpha)}\sqrt{\|\mu\|}}{\|\mu_1\|}\right),$$

$$l_{2,0}=\sqrt{\frac{\varepsilon_0(\alpha)}{\varepsilon_1(\alpha)}}\left(\tilde{h}_{1,0}+\frac{\overline{q_1(\alpha)}}{\overline{q_0(\alpha)}}\sqrt{\frac{\|\mu_1\|}{\varepsilon_0(\alpha)}}l_{1,0}\right),$$

$$l_{k+1,0}=\sqrt{\frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_k(\alpha)}}\left(\tilde{h}_{k,0}+\frac{\overline{q_k(\alpha)}}{\overline{q_{k-1}(\alpha)}}\sqrt{\frac{\varepsilon_{k-2}(\alpha)}{\varepsilon_{k-1}(\alpha)}}l_{k,0}\right),\quad k\geqslant 2.$$

$$l_{1,1}=\sqrt{\frac{\|\mu_1\|}{\varepsilon_0(\alpha)}}\tilde{h}_{0,1},$$

$$l_{2,1}=\sqrt{\frac{\varepsilon_0(\alpha)}{\varepsilon_1(\alpha)}}\left(\tilde{h}_{1,1}+\frac{\overline{q_1(\alpha)}}{\overline{q_0(\alpha)}}\sqrt{\frac{\|\mu_1\|}{\varepsilon_0(\alpha)}}l_{1,1}\right),$$

$$l_{k+1,1}=\sqrt{\frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_k(\alpha)}}\left(\tilde{h}_{k,1}+\frac{\overline{q_k(\alpha)}}{\overline{q_{k-1}(\alpha)}}\sqrt{\frac{\varepsilon_{k-2}(\alpha)}{\varepsilon_{k-1}(\alpha)}}l_{k,1}\right),\quad k\geqslant 2.$$

$$l_{k+1,k+1}=\sqrt{\frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_k(\alpha)}}\tilde{h}_{k,k+1},\quad k\geqslant 1,$$

$$l_{k+1,j}=\sqrt{\frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_k(\alpha)}}\left(\tilde{h}_{k,j}+\frac{\overline{q_k(\alpha)}}{\overline{q_{k-1}(\alpha)}}\sqrt{\frac{\varepsilon_{k-2}(\alpha)}{\varepsilon_{k-1}(\alpha)}}l_{k,j}\right),\quad k\geqslant 1,\quad j\geqslant 2,\quad j\leqslant k+1. \tag{16}$$

More explicitly

$$l_{k,j}=\begin{cases}\sqrt{\dfrac{\|\mu\|}{\|\mu_1\|}} & \text{if } j=k=0,\\[2mm] \dfrac{\overline{q_{k-1}(\alpha)}}{\sqrt{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)}}\left(\sqrt{\|\mu\|}+\displaystyle\sum_{l=0}^{k-1}\frac{\varepsilon_{l-1}(\alpha)}{\overline{q_l(\alpha)}}\tilde{h}_{l,0}\right) & \text{if } j=0,\quad k>1,\\[2mm] \dfrac{\overline{q_{k-1}(\alpha)}}{\sqrt{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)}}\displaystyle\sum_{l=0}^{k-1}\frac{\varepsilon_{l-1}(\alpha)}{\overline{q_l(\alpha)}}\tilde{h}_{l,1} & \text{if } j=1,\quad k\geqslant 0,\\[2mm] \sqrt{\dfrac{\varepsilon_{k-2}(\alpha)}{\varepsilon_{k-1}(\alpha)}}\tilde{h}_{k-1,k} & \text{if } j=k\geqslant 1,\\[2mm] \dfrac{\overline{q_{k-1}(\alpha)}}{\sqrt{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)}}\displaystyle\sum_{l=j-1}^{k}\frac{\varepsilon_{l-1}(\alpha)}{\overline{q_l(\alpha)}}\tilde{h}_{l,j} & \text{if } j<k.\end{cases} \tag{17}$$

where $\varepsilon_{-1}(\alpha):=\|\mu_1\|$.

A result for the leading principal submatrices is the following

Proposition 7. *Let $(H(\mu)-\alpha I)_n$, M_n, and L_n be the leading principal submatrices of $H(\mu)-\alpha I$, M, and L, respectively. Then*

(i) $(H(\mu_1)-\alpha I)_n=L_nM_n$.
(ii) *The QR factorization of $(H(\mu_1)-\alpha I)_n^*$ is*

$$(H(\mu_1)-\alpha I)_n^*=\hat{Q}_n^*\hat{R}_n^*,$$

where $\hat{Q}_n = EM_n$, $\hat{R}_n = L_n E^{-1}$ and E is the diagonal matrix with entries $E_{k,k}$ given by

$$E_{k,k} = \begin{cases} 1, & 0 \leqslant k \leqslant n-1, \\ \dfrac{|q_n(\alpha)|}{\sqrt{\varepsilon_{n-1}(\alpha)}}, & k = n. \end{cases}$$

Proof. (i) If we use the first expression of (11) in block matrix, then we get

$$\begin{aligned} H(\mu_1) - \alpha I &= \left[\begin{array}{c|c} L_n & 0 \\ \hline L_{21} & L_{22} \end{array}\right] \left[\begin{array}{c|c} M_n & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \\ &= \left[\begin{array}{c|c} L_n M_n & L_n M_{12} \\ \hline L_{21} M_n + L_{22} M_{21} & L_{21} M_{12} + L_{22} M_{22} \end{array}\right] \end{aligned}$$

Thus, as a consequence $(H(\mu_1) - \alpha I)_n = L_n M_n$.

(ii) $\hat{R}_n$ is a lower triangular matrix, so that it is enough to prove that $\hat{Q}_n$ is a unitary matrix

$$(\hat{Q}_n^*)_{(0)} (\hat{Q}_n)^{(0)} = \frac{1}{\|\mu_1\|} \left(\sum_{l=0}^{n-2} |q_l(\alpha)|^2 + \varepsilon_{n-2}(\alpha) \right) = 1,$$

$$(\hat{Q}_n^*)_{(n-1)} (\hat{Q}_n)^{(n-1)} = \frac{\varepsilon_{n-2}(\alpha)}{\varepsilon_{n-3}(\alpha)} + \frac{|q_{n-2}(\alpha)|^2}{\varepsilon_{n-3}(\alpha)} = 1.$$

For $1 \leqslant k \leqslant n-2$, we get

$$\begin{aligned} (\hat{Q}_n^*)_{(k)} (\hat{Q}_n)^{(k)} &= \frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_{k-2}(\alpha)} + \frac{|q_{k-1}(\alpha)|^2}{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)} \left(\sum_{l=k}^{n-2} |q_l(\alpha)|^2 + \varepsilon_{n-2}(\alpha) \right) \\ &= \frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_{k-2}(\alpha)} + \frac{|q_{k-1}(\alpha)|^2}{\varepsilon_{k-2}(\alpha)} = 1, \end{aligned}$$

$$\begin{aligned} (\hat{Q}_n^*)_{(0)} (\hat{Q}_n)^{(k)} &= -\frac{q_{k-1}(\alpha)}{\|\mu_1\|} \left(\sqrt{\frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_{k-2}(\alpha)}} - \frac{1}{\sqrt{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)}} \left[\sum_{l=k}^{n-2} |q_l(\alpha)|^2 + \varepsilon_{n-2}(\alpha) \right] \right) \\ &= -\frac{q_{k-1}(\alpha)}{\|\mu_1\|} \left(\sqrt{\frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_{k-2}(\alpha)}} - \sqrt{\frac{\varepsilon_{k-1}(\alpha)}{\varepsilon_{k-2}(\alpha)}} \right) = 0. \end{aligned}$$

Now, for $1 \leqslant j < k \leqslant n-2$

$$\begin{aligned} (\hat{Q}_n^*)_{(k)} (\hat{Q}_n)^{(j)} &= -\frac{\overline{q_{j-2}(\alpha)} q_{j-1}(\alpha)}{\varepsilon_{j-2}(\alpha)} \left(\sqrt{\frac{\varepsilon_{j-1}(\alpha)}{\varepsilon_{j-3}(\alpha)}} - \frac{1}{\sqrt{\varepsilon_{j-3}(\alpha)\varepsilon_{j-1}(\alpha)}} \left[\sum_{l=j}^{n-2} |q_l(\alpha)|^2 + \varepsilon_{n-2}(\alpha) \right] \right) \\ &= -\frac{\overline{q_{j-2}(\alpha)} q_{j-1}(\alpha)}{\varepsilon_{j-2}(\alpha)} \left(\sqrt{\frac{\varepsilon_{j-1}(\alpha)}{\varepsilon_{j-3}(\alpha)}} - \frac{1}{\sqrt{\varepsilon_{j-3}(\alpha)\varepsilon_{j-1}(\alpha)}} \varepsilon_{j-1}(\alpha) \right) = 0. \end{aligned}$$

Finally,

$$(\hat{Q}_n^*)_{(n-1)} (\hat{Q}_n)^{(k)} = \frac{\overline{q_{k-1}(\alpha)} q_{n-2}(\alpha) \varepsilon_{n-2}(\alpha)}{\sqrt{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)\varepsilon_{n-3}(\alpha)\varepsilon_{n-2}(\alpha)}} - \sqrt{\frac{\varepsilon_{n-2}(\alpha)}{\varepsilon_{n-3}(\alpha)}} \frac{\overline{q_{k-1}(\alpha)} q_{n-2}(\alpha)}{\sqrt{\varepsilon_{k-2}(\alpha)\varepsilon_{k-1}(\alpha)}} = 0.$$

Thus, $\hat{Q}_n$ is a unitary matrix. □

4. Examples

4.1. Example 1

Consider on the unit circle the measure

$$\mathrm{d}\mu = |z-1|^2 \frac{\mathrm{d}\theta}{2\pi}, \quad z = \mathrm{e}^{\mathrm{i}\theta},$$

where $\mathrm{d}\theta$ is the Lebesgue measure. Then the monic orthogonal polynomial sequence on $\mathbb{T}=\{z \in \mathbb{C} : |z|=1\}$ associated with μ is (see [6,12])

$$\Phi_n(z;\mu) = \sum_{k=0}^{n} \frac{k+1}{n+1} z^k \quad \forall n \geqslant 0.$$

It is straightforward to verify that

$$e_n(\mu) = \int_{\mathbb{T}} |\Phi_n(z;\mu)|^2 \mathrm{d}\mu = \frac{n+2}{n+1},$$

hence $\|\mu\| = e_0(\mu) = 2$ and the sequence of polynomials orthonormal with respect to μ is

$$\varphi_n(z;\mu) = \frac{1}{\sqrt{(n+1)(n+2)}} \sum_{k=0}^{n} (k+1) z^k$$

$$= \frac{(n+1)z^{n+2} - (n+2)z^{n+1} + 1}{\sqrt{(n+1)(n+2)}(z-1)^2}, \quad n \geqslant 0.$$

Now, we define a Geronimus transformation of the measure μ as follows:

$$\mathrm{d}\mu_1 = \frac{\mathrm{d}\mu}{|z-\alpha|^2} = \frac{|z-1|^2}{|z-\alpha|^2} \frac{\mathrm{d}\theta}{2\pi}, \quad |\alpha| > 1, \quad z = \mathrm{e}^{\mathrm{i}\theta}.$$

Then, we obtain

$$\|\mu_1\| = \frac{2|\alpha|^2 - \alpha - \bar{\alpha}}{|\alpha|^2 - 1}. \tag{18}$$

From (2), we get

$$q_n(\alpha) = (1-\alpha^{-1})^2 \varphi_n(\alpha^{-1}), \tag{19}$$

and from (18) and (19)

$$\begin{aligned} \varepsilon_n(\alpha) &= \|\mu_1\| - \sum_{l=0}^{n} |q_l(\alpha)|^2 \\ &= \frac{2|\alpha|^2 - \alpha - \bar{\alpha}}{|\alpha|^2 - 1} - |1-\alpha^{-1}|^4 \sum_{l=0}^{n} |\varphi_l(\alpha^{-1})|^2 \\ &= \frac{2|\alpha|^2 - \alpha - \bar{\alpha}}{|\alpha|^2 - 1} - |1-\alpha^{-1}|^4 K_n(\alpha^{-1}, \alpha^{-1}) \\ &= \frac{1}{|\alpha|^2 - 1} \Bigg[2|\alpha|^2 - (\alpha + \bar{\alpha}) - \frac{1}{(n+1)(n+2)|\alpha|^{2(n+2)}} \Big(|(n+1)\alpha^{n+3} - (n+2)\alpha^{n+1} + \alpha|^2 \\ &\quad - |\alpha^{n+2} - (n+2)\alpha + n + 1|^2 \Big) \Bigg]. \end{aligned} \tag{20}$$

The entries $\tilde{h}_{k,j}$ of the matrix $\tilde{H} = \tilde{M}(H(\mu) - \alpha I)$ are

$$\tilde{h}_{k,j} = \begin{cases} -\dfrac{1}{2}, & j = k = 0, \\ -\dfrac{k}{k+1}\left(\dfrac{\bar{\alpha}^{k+2} - (k+2)\bar{\alpha} + k + 1}{\bar{\alpha}^{k+2} - (k+2)\bar{\alpha}^2 + k\bar{\alpha}} + \dfrac{1}{k(k+2)}\right), & j = k \geqslant 1, \\ \dfrac{\sqrt{(k+1)(k+3)}}{k+2}, & j = k+1, \\ \dfrac{\sqrt{k+1}(1-\bar{\alpha}^{-1})^2}{\sqrt{(j+1)(j+2)(k+2)}(\bar{\alpha}^{k+1} - (k+2)\bar{\alpha} + k)}, & j \leqslant k-1, \\ 0, & j > k. \end{cases} \tag{21}$$

The entries $m_{k,j}$ and $l_{k,j}$ of M and L, respectively, can be obtained, replacing (18)–(21) in (12) and (17). These expressions are cumbersome, and we will not show them.

4.2. Example 2

We consider on the unit circle the measure

$$\mathrm{d}\mu = \frac{\mathrm{d}\theta}{2\pi} + m\delta(z-1),$$

where $m > 0$. The monic orthogonal polynomial sequence on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ associated with μ is (see [5,12])

$$\Phi_n(z;\mu) = z^n - \frac{m}{1+nm}\sum_{k=0}^{n-1} z^k \quad \forall n \geqslant 1.$$

Then, we get

$$\begin{aligned} e_n(\mu) &= \int_0^{2\pi} \Phi_n(z;\mu)\bar{z}^n \frac{\mathrm{d}\theta}{2\pi} + m\Phi_n(1,\mu), \quad z = \mathrm{e}^{\mathrm{i}\theta} \\ &= \int_0^{2\pi}\left(z^n - \frac{m}{1+nm}\sum_{k=0}^{n-1} z^k\right)\bar{z}^n \frac{\mathrm{d}\theta}{2\pi} + \frac{m}{1+nm}, \quad z = \mathrm{e}^{\mathrm{i}\theta} \\ &= \frac{1+(n+1)m}{1+nm}, \end{aligned} \tag{22}$$

hence $\|\mu\| = e_0(\mu) = 1 + m$. The nth orthonormal polynomial with respect to μ is

$$\begin{aligned} \varphi_n(z;\mu) &= \sqrt{\frac{1+nm}{1+(n+1m)}}z^n - \frac{m}{\sqrt{(1+nm)(1+(n+1)m)}}\sum_{k=0}^{n-1} z^k \\ &= \frac{(1+nm)z^{n+1} - mz^n + m}{\sqrt{(1+nm)(1+(n+1)m)}(z-1)}, \quad n \geqslant 0. \end{aligned}$$

The entries $\tilde{h}_{k,j}$ of the matrix $\tilde{H} = \tilde{M}(H(\mu) - \alpha I)$ are

$$\tilde{h}_{k,j} = \begin{cases} -\dfrac{m^2}{1+m}, & j=k=0, \\ -\dfrac{1+(k-1)m}{1+km}\left(\dfrac{(1+(k+1)m)\bar{\alpha}-(1+km)}{(1+km)\bar{\alpha}^2-(1+(k-1)m)\bar{\alpha}}+\dfrac{m^2}{(1+(k-1)m)(1+(k+1)m)}\right), & j=k\geqslant 1, \\ \dfrac{\sqrt{(1+km)(1+(k+2)m)}}{1+(k+1)m}, & j=k+1, \\ \dfrac{m}{\sqrt{(1+jm)(1+(j+1)m)(1+km)(1+(k+1)m)}}\left(\dfrac{(1+(k+1)m)\bar{\alpha}-(1+km)}{(1+km)\bar{\alpha}^2-(1+(k-1)m)\bar{\alpha}}-m\right), & j\leqslant k-1, \\ 0, & j>k. \end{cases} \tag{23}$$

Consider a Geronimus transformation of the measure μ

$$\mathrm{d}\mu_2 = \frac{\mathrm{d}\mu}{|z-\alpha|^2} = \frac{1}{|z-\alpha|^2}\frac{\mathrm{d}\theta}{2\pi} + \frac{m}{|\alpha-1|^2}\delta(z-1), \quad |\alpha|>1, \ \ z=\mathrm{e}^{\mathrm{i}\theta}.$$

Then, we obtain

$$\|\mu_2\| = \frac{1}{|\alpha|^2-1} + \frac{m}{|\alpha-1|^2}. \tag{24}$$

The functions of the second kind associated with μ evaluated in α are

$$q_n(\alpha) = \frac{(1+(n+1)m)\alpha-(1+nm)}{\sqrt{(1+nm)(1+(n+1)m)}(\alpha-1)\alpha^{n+1}}. \tag{25}$$

From (24) and (25) we get

$$\varepsilon_n(\alpha) = \frac{1}{|\alpha|^2-1} + \frac{1}{|\alpha-1|^2|\alpha|^2}\left(m|\alpha|^2 - \sum_{k=0}^{n}\frac{|(1+(k+1)m)\alpha-(1+km)|^2}{(1+km)(1+(k+1)m)|\alpha|^{2k}}\right). \tag{26}$$

The expression of the entries of the matrices M and L are very complicated and they can be obtained using the expressions (22)–(26)) in (12) and (17).

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References

[1] C. Brezinski, Formal orthogonality on an algebraic curve, Ann. Numer. Math. 2 (1995) 21–33.
[2] M.I. Bueno, F. Marcellán, Darboux transformations and perturbation of linear functionals, Linear Algebra Appl. 384 (2004) 215–242.
[3] M.I. Bueno, F. Marcellán, Polynomial perturbations of bilinear functionals and Hessenberg matrices, Linear Algebra Appl. 414 (2006) 64–83.
[4] M. Buhmann, A. Iserles, On orthogonal polynomials transformed by the QR algorithm, J. Comput. Appl. Math. 43 (1992) 117–134.
[5] L. Daruis, J. Hernández, F. Marcellán, Spectral transformations for Hermitian Toeplitz matrices, J. Comput. Appl. Math. 202 (2007) 155–176.
[6] E. Godoy, F. Marcellán, An analogue of the Christoffel formula for polynomial modification of a measure on the unit circle, Boll. Un. Mat. Ital. A 5 (1991) 1–12.
[7] E. Godoy, F. Marcellán, Orthogonal polynomials and rational modification of measures, Canad. J. Math. 45 (1993) 930–943.
[8] U. Grenander, G. Szegő, Toeplitz Forms and their Applications, second ed., University of California Press, Berkeley, 1958, Chelsea, New York, 1984.
[9] M.E.H. Ismail, R.W. Ruedemann, Relation between polynomials orthogonal on the unit circle with respect to different weights, J. Approx. Theory 71 (1992) 39–60.

[10] T. Kailath, B. Porat, State-space generators for orthogonal polynomials, in: V. Mandrekar, H. Salehi, (Eds.), Prediction Theory and Harmonic Analysis, The Pesi Massani Volume, North-Holland, Amsterdam, 1983, pp. 131–163.
[11] F. Marcellán, J. Hernández, Christoffel transforms and Hermitian linear functionals, Mediterr. J. Math. 2 (2005) 451–458.
[12] B. Simon, Orthogonal polynomials on the unit circle, vols. 1 and 2, American Mathematical Society Colloquium Publication Series, vol. 54, American Mathematical Society, Providence, RI, 2005.
[13] V. Spiridonov, L. Vinet, A. Zhedanov, Spectral transformations, self-similar reductions and orthogonal polynomials, J. Phys. A: Math. Gen. 30 (1997) 7621–7637.
[14] G.J. Yoon, Darboux transforms and orthogonal polynomials, Bull. Korean Math. Soc. 39 (2002) 359–376.
[15] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997) 67–83.