Perturbations of Laguerre-Hahn functional. Modification by the derivative of a Dirac Delta

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Abstract

In this paper we consider a perturbation of a Laguerre-Hahn functional by adding a derivative of a Dirac Delta. This transformation leaves invariant the family of Laguerre-Hahn linear functionals. We shall also analyze the class of the perturbed linear functional. Finally, we illustrate these perturbations by adding the derivative of a Dirac Delta to the first kind associated functional of the classical Laguerre functional. The expression of the new orthogonal polynomials is obtained.

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1 Introduction

The study of the modification of a linear functional by the addition of the derivatives of a Dirac Delta is related with the theory of the rational approximation of Markov functions (see [12]). In the last years some applications in boundary problems for linear differential equations of fourth order (see [9]), as well as some extensions of the Gaussian quadrature rules have been considered.

In [6] the first approach to such a kind of perturbations is presented. There necessary and sufficient conditions for the regularity of the modified functional are obtained.

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In the same work, when the initial functional is semi-classical, an exhaustive study of the corresponding sequence of orthogonal polynomials is done.

Later on in [2] and [3] a generalization of the classical Laguerre polynomials has been treated when the addition of the first derivative of a Dirac Delta at the point $z = 0$ to the classical Laguerre functional is considered. In particular, the hypergeometric character of these new polynomials is deduced.

The aim of our contribution is to analyze how the Laguerre-Hahn character for the perturbed linear functional is preserved under such a kind of perturbations as well as to determine the class of the perturbed functional. In some previous work (see [1]) other examples of perturbations of Laguerre-Hahn linear functionals have been analized. The class of the perturbed linear functionals is also studied.

The structure of the paper is as follows. Section 2 contains some introductory results and notations concerning linear functionals. Section 3 is devoted to the characterization of the Laguerre-Hahn linear functionals and the definition of the class. In Section 4 a modification of a Laguerre-Hahn functional is carried out by adding the first derivative of a Dirac Delta and we study the class of the new Laguerre-Hahn functional. The above perturbation has been carried out for the first kind associated functional of the classical ones (Hermite, Laguerre, Jacobi and Bessel), showing the equations that the new linear functional satisfies, as well as the Ricatti differential equation which fulfils the corresponding Stieltjes function fulfils.

Finally, nn Section 5 we obtain the explicit expression of the polynomials orthogonal with respect to such a perturbation when the first kind associated Laguerre functional is considered.

2 Preliminaries

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of complex numbers and μ a linear functional defined in the linear space $\mathbb{P}$ of the polynomials with complex coefficients, such that

$$\langle \mu, x^n \rangle = \mu_n, \; n = 0, 1, 2...$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Moreover μ_n is said to be the $n - th$ *moment* of the functional μ.

If $\phi(x)$ is a complex polynomial, we define the moment functional $\phi\mu$, the left multiplication by a polynomial ϕ, and $D\mu$, the usual distributional derivative of μ, as follows

$$\langle \phi\mu, p(x) \rangle = \langle \mu, \phi(x)\, p(x) \rangle\,, \;\; \langle D\mu, p(x) \rangle = -\langle \mu, p'(x) \rangle\,, \;\; p \in \mathbb{P}.$$

On the other hand, δ_c will denote the Dirac delta linear functional $\langle \delta_c, p \rangle = p(c)$. In particular, we will use the notation $\delta_0 = \delta$.

A sequence of orthogonal polynomials $\{P_n\}_{n\geq 0}$ is said to be *classical* if there exist polynomials ϕ and ψ, with $\deg \phi \leq 2$ and $\deg \psi = 1$, such that μ satisfies the Pearson differential equation

$$D(\phi\mu) = \psi\mu.$$

Let μ be a linear functional on the linear space $\mathbb{P}$ of polynomials with complex coefficients and let $S(\mu)(z)$ be its Stieltjes function defined by

$$S(\mu)(z) = -\sum_{n\geq 0} \frac{\mu_n}{z^{n+1}} \tag{1}$$

where $\mu_n = \langle \mu, x^n \rangle$, $n \geq 0$, are the moments of μ. In the sequel, we will assume, that $\mu_0 = 1$.

Let $\mathbb{P}'$ be the algebraic dual space of $\mathbb{P}$ and $\Delta = span\{D^n\delta\}_{n\in\mathbb{N}}$. We consider the isomorphism $F : \Delta \to \mathbb{P}$ given as follows:

For

$$\upsilon = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \mu_n D^n \delta,$$

we get

$$F(\upsilon)(z) = \sum_{n=0}^{N} \mu_n z^n.$$

Then,

$$S(\mu)(z) = -z^{-1} F(\mu)(z^{-1}).$$

If

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

then we define for every $p \in \mathbb{P}$

$$\begin{aligned} (\mu p)(z) &= \sum_{m=0}^{n} \left(\sum_{j=m}^{n} a_j \mu_{j-m} \right) z^m, \\ (\theta_0 p)(z) &= \frac{p(z) - p(0)}{z}. \end{aligned}$$

Thus

$$S(p\mu)(z) = p(z)S(\mu)(z) + (\mu\theta_0 p)(z).$$

The functional $x^{-1}\mu$ and the product of two linear functionals are defined respectivly as follows

$$\left\langle x^{-1}\mu, p \right\rangle = \langle \mu, \theta_0 p \rangle\,; \;\; \langle \mu\upsilon, p \rangle = \langle \mu, \upsilon p \rangle\,, \;\; p \in \mathbb{P}.$$

If $\mu_0 = 1$, we define the linear functional μ^{-1} by

$$\mu\mu^{-1} = \delta_0.$$

Then it is straightforward to prove that

1. $x\left(x^{-1}\mu\right) = \mu$
2. $x^{-1}(x\mu) = \mu - \delta_0$
3. $x^{-2}(x^2\mu) = x^{-1}\left(x^{-1}\mu\right) = \mu - \delta_0 + \mu_1 D\delta_0.$

We give the following auxiliary results (see [14] and [15] for a more comprehensive approach)

Lemma 1 *For $p \in \mathbb{P}$ and $\mu, \upsilon \in \mathbb{P}'$, we have*

1. $S'(\mu)(z) = S(D\mu)(z).$
2. $S(\mu\upsilon)(z) = -zS(\mu)(z)S(\upsilon)(z).$
3. $S(x^{-1}\mu)(z) = \frac{1}{z}S(\mu)(z).$

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if

(i) The degree of P_n is n.

(ii) $\langle \mu, P_n(x)P_m(x)\rangle = 0,\ m \neq n.$

(iii) $\left\langle \mu, P_n^2(x)\right\rangle \neq 0,\ n = 0, 1, 2, ...$

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be the sequence of *monic orthogonal polynomials* with respect to the linear functional μ. It is clear that for every sequence of orthogonal polynomials there exists the corresponding family of monic orthogonal polynomials.

The next theorem, whose proof appears in [10] , gives necessary and sufficient conditions for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 2 *Let μ be a linear functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to μ if and only if the leading principal submatrices of the Hankel matrix $\left[\mu_{i+j}\right]_{i,j\in\mathbb{N}}$ are non singular.*

A linear functional such that there exists the correspondient sequence of orthogonal polynomials is said to be *regular* or *quasi-definite.*

Now we show the three term recurrence relation that a sequence of monic orthogonal polynomials satisfies. The proof is given in [10].

Theorem 3 *If μ is a quasi-definite linear functional and is $\{P_n\}_{n\geq 0}$ the corresponding sequence of monic orthogonal polynomials, then there exist sequences of complex numbers $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$, with $\gamma_n \neq 0$ for every $n \in \mathbb{N}$, such that:*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \tag{2}$$

with $P_0(x) = 1, P_1(x) = x - \beta_0$. Conversely, given two sequences of complex numbers $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ with $\gamma_n \neq 0$ for every $n \geq 1$, there exists a unique linear functional μ such that the polynomials defined in (2) constitute the sequence of monic polynomials orthogonal with respect to μ.

Definition 1 *([10], [16].) Let* $\{P_n\}_{n\geq 0}$ *be a sequence of monic orthogonal polynomials with respect to a quasi-definite functional* μ.. *The sequence of monic polynomials* $\left\{P_n^{(1)}\right\}_{n\geq 0}$ *defined by*

$$P_n^{(1)}(x) = \left\langle \mu_\xi, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi} \right\rangle, \ n \geq 0, \tag{3}$$

is said to be the associated sequence of first order for the sequence $\{P_n\}_{n\geq 0}$.

Notice that the sequence $\left\{P_n^{(1)}\right\}_{n\geq 0}$ satisfies the three term recurrence relation

$$xP_n^{(1)}(x) = P_{n+1}^{(1)}(x) + \beta_{n+1} P_n^{(1)}(x) + \gamma_{n+1} P_{n-1}^{(1)}(x),$$

with $P_0^{(1)}(x) = 1$, $P_1^{(1)}(x) = x - \beta_1$.

According to Theorem 3, we shall note by $\mu^{(1)}$ the normalized functional, $\left(\mu^{(1)}\right)_0 = \mu_0^{(1)} = 1$, such that the sequence $\left\{P_n^{(1)}\right\}_{n\geq 0}$ is the corresponding sequence of monic orthogonal polynomials.

Theorem 4 *Let* μ *be a linear functional. Then*

$$\gamma_1 \mu^{(1)} = -x^2 \mu^{-1}.$$

In a general way, the associated sequence of $r - th$ order, $r \in \mathbb{N}$, $\left\{P_n^{(r)}\right\}_{n\geq 0}$ is defined by the recurrence relation

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r})\, P_n^{(r)}(x) - \gamma_{n+r} P_{n-1}^{(r)}(x), \ n \geq 1, \tag{4}$$

$$P_1^{(r)}(x) = x - \beta_r, P_0^{(r)}(x) = 1.$$

This corresponds to a forward shifted perturbation in the cofficients of the three term recurrence relation.

3 The Laguerre-Hahn linear functionals

Definition 2 *([1], [13], [14].) A linear functional* μ *on the linear space* $\mathbb{P}$ *belongs to the Laguerre-Hahn family if its Stieltjes function satisfies a Riccati equation*

$$\Phi(z) S'(\mu)(z) = B(z) S^2(\mu)(z) + C(z) S(\mu)(z) + D(z) \tag{5}$$

where $\Phi(z), B(z), C(z)$, *and* $D(z)$ *are polynomials with complex coefficients. Here* $\Phi(z) \neq 0$, $B(z) \neq 0$, *and* $D(z) = [(D\mu)\theta_0\Phi](z) + (\mu\theta_0 C)(z) - \left(\mu^2\theta_0 B\right)(z)$.

Remark 5 *When* $B(z) = 0$, *we get a first order linear differential equation* $\Phi(z)S'(\mu)(z) = C(z)S(\mu)(z) + D(z)$ *and the linear functional is said to be an affine Laguerre-Hahn functional. It is also called a semiclassical linear functional (see [14]).*

Definition 3 *Let* $\{P_n\}_{n\geq 0}$ *be a sequence of monic orthogonal polynomials with respect to a quasi-definite linear functional.* $\{P_n\}_{n\geq 0}$ *belongs to the Laguerre-Hahn family if* μ *is a Laguerre-Hahn linear functional.*

The proof of the next theorem can be found in [15].

Theorem 6 *Let μ be a quasi-definite and normalized functional and let $\{P_n\}_{n\geq 0}$ be the corresponding sequence of monic orthogonal polynomials. The following statements are equivalent*

1. *μ is a Laguerre-Hahn functional.*
2. *μ satisfies the functional equation $D[\Phi\mu]+\Psi\mu+B\left(x^{-1}\mu^2\right)=0$, where $\Phi(x)$, $B(x)$, and $C(x)$ are the polynomials defined in (5), and*
$$\Psi(x) = -\left[\Phi'(x) + C(x)\right]. \tag{6}$$
3. *μ satisfies the functional equation*
$$D[x\Phi\mu] + (x\Psi - \Phi)\mu + B\mu^2 = 0 \tag{7}$$
with the additional condition $\langle \mu, \Psi\rangle + \left\langle \mu^2, \theta_0 B\right\rangle = 0$ where $\Phi(x)$, $\Psi(x)$, and $B(x)$ are the same polynomials as in (6).
4. *Every polynomial $P_n(x)$, $n \geq 0$, satisfies the so-called structure relation*
$$\Phi(x)P'_{n+1}(x) - B(x)P_n^{(1)}(x) = \sum_{\mu=n-s}^{n+d} \theta_{n,\mu}P_\mu(x),\ n \geq s+1$$
where $\Phi(x)$ and $B(x)$ are the same polynomials as in (5) and $\left\{P_n^{(1)}\right\}_{n\geq 0}$ is the sequence of associated orthogonal polynomials of first order for $\{P_n\}_{n\geq 0}$, where $t = \deg\Phi$, $P = \deg\Psi \geq 1$, $r = \deg B$, $s = \max\{p-1, d-2\}$, and $d = \max\{t, r\}$.

3.1 Determination of the order of the class

In the characterization (7), we must notice that the representation is not unique. In fact, it is enough to multiply by any polynomial in both sides of the equation. On the other hand, uniqueness is deduced by imposing a minimality condition on the degrees of the polynomials involved in (7). We will discuss this question below.

If μ satisfies the equation $D[\Phi\mu] + \Psi\mu + B\left(x^{-1}\mu^2\right) = 0$, then multiplying it by a polynomial $q(x)$, we get
$$D[\Phi^*\mu] + \Psi^*\mu + B^*\left(x^{-1}\mu^2\right) = 0, \tag{8}$$
where, $\Phi^* = q\Phi$, $\Psi^* = q\Psi - q'\Phi$, and $B^* = qB$.

Thus we can associate with the linear functional μ the set of nonnegative integer numbers

$$\begin{aligned} H(\mu) \quad = \quad & \{\max\{p-1, d-2\},\ \text{being } d = \max\{t, r\},\ \text{where } t = \deg\Phi^*, p = \deg\Psi^*, \\ & \text{and } r = \deg B^*, \text{among all the choices of } \Phi^*, \Psi^*,\ \text{and } B^* \\ & \ \text{such that (8) holds}\}. \end{aligned}$$

Definition 4 *The class of the Laguerre-Hahn functional μ is the minimum of $H(\mu)$.*

Theorem 7 *The class of the Laguerre-Hahn functional μ is s if and only if*

$$\prod_{a\in Z_\Phi}\left(\left|\langle\mu,\Psi_a\rangle+\left\langle\mu^2,\theta_0 B_a\right\rangle\right|+|r_a|+|s_a|\right)\neq 0.$$

where Z_Φ is the set of zeros of $\Phi(x)$. Here the polynomials Φ_a, Ψ_a, and B_a as well as the numbers r_a and s_a are defined by

$$\begin{aligned}\Phi(x) &= (x-a)\Phi_a(x),\\ \Psi(x)+\Phi_a(x) &= (x-a)\Psi_a(x)+r_a,\\ B(x) &= (x-a)B_a(x)+s_a.\end{aligned}$$

As a consequence of the previous result we can give a characterization of the class of a Laguerre-Hahn linear functional in terms of the polynomials $B(x)$, $C(x)$, and $D(x)$ defined in (5) using the Stieltjes function.

Corollary 1 *Let μ be a quasi-definite Laguerre-Hahn linear functional satisfying (5). s is the class of μ if and only if*

$$\prod_{a\in Z_\Phi}(|C(a)|+|B(a)|+|D(a)|)\neq 0$$

i.e., the polynomials $\Phi, B, C,$ and D are coprime.

4 Modification by a derivative of a Dirac Delta

Proposition 1 *Let μ be a Laguerre-Hahn linear functional and let M and c be arbitrary complex numbers. Then*

$$\widetilde{\mu}=\mu+M\delta_c'$$

is a Laguerre-Hahn linear functional.

Proof. Let $S=S(\mu)(z)$ be the Stieltjes function corresponding to the functional μ such that

$$\Phi(z)S'=B(z)S^2+C(z)S+D(z) \tag{9}$$

and let $\widetilde{S}=S(\widetilde{\mu})(z)$ be the Stieltjes function associated with $\widetilde{\mu}$. Then

$$\langle\widetilde{\mu},x^n\rangle=\mu_n-Mnc^{n-1},\ n\geq 0.$$

Thus

$$S(z)=\widetilde{S}(z)-\frac{M}{(z-c)^2}$$

and substituting the above expression in (9), we get

$$(z-c)^4\Phi(z)\widetilde{S}' = (z-c)^4 B\widetilde{S}^2 + \left((z-c)^4 C - 2M(z-c)^2 B\right)\widetilde{S} \tag{10}$$
$$+\left(M^2 B - 2M(z-c)\Phi - M(z-c)^2 C + (z-c)^4 D\right).$$

As a consequence, $\widetilde{\mu}$ is a Laguerre-Hahn functional satisfying the distributional equation

$$D\left[(x-c)^4\Phi\widetilde{\mu}\right] + \left[(x-c)^4\Psi + 2\mu(x-c)^2 B - 4(x-c)^3\Phi\right]\widetilde{\mu} + (x-c)^4 B\left(x^{-1}\widetilde{\mu}^2\right) = 0.$$

This means that the family of Laguerre-Hahn linear functionals remains invariant under such a kind of perturbations. ■

4.1 Determination of the Class

In the sequel, we will assume that μ is a Laguerre-Hahn linear functional of class s.

Proposition 2 *Let μ be a Laguerre-Hahn linear functional of class s and $\widetilde{\mu} = \mu + M\delta_c'$. Then $\widetilde{\mu}$ is a Laguerre Hahn linear functional of class $\widetilde{s}$ such that $s-4 \leq \widetilde{s} \leq s+4$.*

Proof. Let Φ, Ψ, and B be as in Theorem 6. Let

$$D\left[\Phi^*\widetilde{\mu}\right] + \Psi^*\widetilde{\mu} + B^*\left(x^{-1}\widetilde{\mu}^2\right) = 0, \tag{11}$$

be the equation which fulfils $\widetilde{\mu}$, where

$$\Phi^*(x) = (x-c)^4\Phi(x),\ \Psi^*(x) = (x-c)^2\left((x-c)^2\Psi(x) - 4(x-c)\Phi(x) + 2\mu B(x)\right), \tag{12}$$

$$B^*(x) = (x-c)^4 B(x). \tag{13}$$

Then

$$\begin{aligned} \deg\Phi^* &= t^* \leq s+6,\\ \deg\Psi^* &= p^* \leq s+5,\\ \deg B^* &= r^* \leq s+6. \end{aligned}$$

As a consequence, $d^* = \max\{t^*, r^*\} \leq s+6$ and $\widetilde{s} = \max\{p^*-1, d^*-2\} \leq s+4$.

On the other hand, since $\mu = \widetilde{\mu} - M\delta_c'$ then $s \leq \widetilde{s} + 4$. ■

Proposition 3 *Let $\widetilde{\mu}$ be a Laguerre-Hahn linear functional such that the equation (11) holds. Then for every zero of $\Phi^*(x)$ different from c, the equation (11) cannot be simplified by division of the polynomial coefficients.*

Proof. From the assumption about the linear functional μ, $S(\mu)(z)$ fulfils (5), where the polynomials Φ, B, C, and D are coprime.

Let Φ^* and B^* be as in Proposition 2 and

$$\begin{aligned} C^*(z) &= (z-c)^4 C(z) - 2M(z-c)^2 B(z)\\ D^*(z) &= M^2 B(z) - 2M(z-c)\Phi(z) - M(z-c)^2 C(z) + (z-c)^4 D(z). \end{aligned}$$

Assume a is a zero of Φ^* different from c. Three different situations can be analyzed

1. If $B(a) \neq 0$, then $B^*(a) \neq 0$.

2. If $B(a) = 0$ and $C(a) \neq 0$, then we get $C^*(a) \neq 0$.

3. If $B(a) = C(a) = 0$ then taking into account $D(a) \neq 0$, we get $D^*(a) \neq 0$, and, as a consequence,
$$|B^*(a)| + |C^*(a)| + |D^*(a)| \neq 0.$$

As a conclusion, the equation (5) cannot be simplified. ■

In order to analyze the class of μ we will study the behaviour of the polynomials B, C, and D at $z = c$.

Proposition 4 *Let Φ, B, C, and D the polynomials defined in (5). For $\widetilde{\mu} = \mu + M\delta'_c$, let $\widetilde{s}$ and s be the class of $\widetilde{\mu}$ and μ, respectively. Then we get*

1. $\widetilde{s} = s + 4$ *if* $B(c) \neq 0$.

2. $\widetilde{s} = s + 3$ *if*
$$B(c) = 0, \tag{14}$$
and $MB'(c) - 2\Phi(c) \neq 0$

3. $\widetilde{s} = s + 2$ *if the condition (14) is satisfied together with*
$$MB'(c) - 2\Phi(c) = 0, \tag{15}$$
and $\frac{1}{2}MB''(c) - 2\Phi'(c) - C(c) \neq 0$.

4. $\widetilde{s} = s + 1$ *if some of the following two cases hold*

   4.1. *(14), (15),*
   $$\frac{1}{2}MB''(c) - 2\Phi'(c) - C(c) = 0, \tag{16}$$
   and $B'(c) \neq 0$.

   4.2. *(14), (15), (16),*
   $$B'(c) = 0, \tag{17}$$
   and $\frac{M}{3!}B'''(c) - \Phi''(c) - C'(c) \neq 0$.

5. $\widetilde{s} = s$ *if some of the following three cases hold*

   5.1. *(14), (15), (16), (17),*
   $$\frac{M}{3!}B'''(c) - \Phi''(c) - C'(c) = 0, \tag{18}$$
   and $\Phi(c) \neq 0$.

   5.2. *(14), (15), (16), (17), (18),*
   $$\Phi(c) = 0, \tag{19}$$
   and $C(c) - MB''(c) \neq 0$.

5.3. (14), (15), (16), (17), (18), (19),

$$C(c) - MB''(c) = 0, \tag{20}$$

and $\frac{M^2}{4!}B^{(4)}(c) - \frac{2M}{3!}\Phi'''(c) - \frac{M}{2}C''(c) + D(c) \neq 0.$

6. $\widetilde{s} = s - 1$ *if some of the following three cases hold*

6.1. (14), (15), (16), (17), (18), (19), (20),

$$\frac{M^2}{4!}B^{(4)}(c) - \frac{2M}{3!}\Phi'''(c) - \frac{M}{2}C''(c) + D(c) = 0, \tag{21}$$

and $\Phi'(c) \neq 0.$

6.2. (14), (15), (16), (17), (18), (19), (20), (21),

$$\Phi'(c) = 0, \tag{22}$$

and $C'(c) - \frac{2M}{3!}B'''(c) \neq 0.$

6.3. (14), (15), (16), (17), (18), (19), (20), (21), (22),

$$C'(c) - \frac{2M}{3!}B'''(c) = 0 \tag{23}$$

and $\frac{M^2}{5!}B^{(5)}(c) - \frac{2M}{4!}\Phi^{(4)}(c) - \frac{M}{3!}C'''(c) + D'(c) \neq 0.$

7. $\widetilde{s} = s - 2$ *if some of the following four cases hold*

7.1. (14), (15), (16), (17), (18), (19), (20), (21), (22), (23),

$$\frac{M^2}{5!}B^{(5)}(c) - \frac{2M}{4!}\Phi^{(4)}(c) - \frac{M}{3!}C'''(c) + D'(c) = 0, \tag{24}$$

and $\Phi''(c) \neq 0.$

7.2. (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24),

$$\frac{M^2}{5!}B^{(5)}(c) - \frac{2M}{4!}\Phi^{(4)}(c) - \frac{M}{3!}C'''(c) + D'(c) = 0, \Phi''(c) = 0, \tag{25}$$

and $B''(c) \neq 0.$

7.3. (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25),

$$B''(c) = 0, \tag{26}$$

and $\frac{1}{2}C''(c) - \frac{2M}{4!}B^{(4)}(c) \neq 0.$

7.4. (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26),

$$\frac{1}{2}C''(c) - \frac{2M}{4!}B^{(4)}(c) = 0, \tag{27}$$

and $\frac{M^2}{6!}B^{(6)}(c) - \frac{2M}{5!}\Phi^{(5)}(c) - \frac{M}{4!}C^{(4)}(c) + \frac{1}{2}D''(c) \neq 0.$

8. $\widetilde{s} = s - 3$ *if some of the following four cases hold*

8.1. *(14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27),*

$$\frac{M^2}{6!}B^{(6)}(c) - \frac{2M}{5!}\Phi^{(5)}(c) - \frac{M}{4!}C^{(4)}(c) + \frac{1}{2}D''(c) = 0 \tag{28}$$

and $\Phi'''(c) \neq 0$.

8.2. *(14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28),*

$$\Phi'''(c) = 0, \tag{29}$$

and $B'''(c) \neq 0$.

8.3. *(14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29),*

$$B'''(c) = 0, \tag{30}$$

and $\frac{1}{3!}C'''(c) - \frac{2M}{5!}B^{(5)}(c) \neq 0$.

8.4. *(14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30),*

$$\frac{1}{3!}C'''(c) - \frac{2M}{5!}B^{(5)}(c) = 0, \tag{31}$$

and $\frac{M^2}{7!}B^{(7)}(c) - \frac{2M}{6!}\Phi^{(6)}(c) - \frac{M}{5!}C^{(5)}(c) + \frac{1}{3!}D'''(c) \neq 0$.

9. $\widetilde{s} = s - 3$ *if (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30), (31), and*

$$\frac{M^2}{7!}B^{(7)}(c) - \frac{2M}{6!}\Phi^{(6)}(c) - \frac{M}{5!}C^{(5)}(c) + \frac{1}{3!}D'''(c) \neq 0. \tag{32}$$

10. $\widetilde{s} = s - 4$ *if (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30), (31), and*

$$\frac{M^2}{7!}B^{(7)}(c) - \frac{2M}{6!}\Phi^{(6)}(c) - \frac{M}{5!}C^{(5)}(c) + \frac{1}{3!}D'''(c) = 0. \tag{33}$$

Proof. In a general way, $\widetilde{S}(z) = S(\widetilde{\mu})(z)$ fulfils the equation (10), with $\widetilde{s} \leq s + 4$. Notice that if $B(c) \neq 0$ then $\widetilde{s} = s + 4$.

In the sequel, we will use the following notation. If $T_k(z)$ is a polynomial of degree k, c a real number, and k a non negative integer number, then $T_{c,k}(z)$ will be the polynomial such that

$$T_{c,k}(z) = (z - c)T_{c,k+1}(z) + t_c^{(k)},$$

with the initial condition $T_{c,0}(z) = T(z)$.

If $B(c) = 0$ holds, then in (10) we can divide by $z - c$, and

$$\begin{aligned}(z-c)^3\Phi\widetilde{S}' &= (z-c)^3 B\widetilde{S}^2 + \Big((z-c)^3C - 2M(z-c)B\Big)\widetilde{S} \\ &\quad + \Big(M^2B_{c,1} - 2M\Phi - M(z-c)C + (z-c)^3D\Big),\end{aligned}$$

thus $\widetilde{s} \leq s+3$. In particular, if $MB_{c,1}(c) - 2\Phi(c) \neq 0$, $\widetilde{s} = s+3$.

If $MB_{c,1}(c) - 2\Phi(c) = 0$, then $MB'(c) - 2\Phi(c) = 0$. Dividing by $(z-c)$ in the above expression, we get

$$(z-c)^2\Phi\widetilde{S}' = (z-c)^2 B\widetilde{S}^2 + \Big((z-c)^2C - 2MB\Big)\widetilde{S} \\ + \Big(M^2B_{c,2} - 2M\Phi_{c,1} - MC + (z-c)^2D\Big), \tag{34}$$

and then $\widetilde{s} \leq s+2$. In particular if $MB_{c,2}(c) - 2\Phi_{c,1}(c) - C(c) \neq 0$ then $\widetilde{s} = s+2$.

If $MB_{c,2}(c) - 2\Phi_{c,1}(c) - C(c) = 0$, then $\frac{1}{2}MB''(c) - 2\Phi'(c) - C(c) = 0$. Dividing by $(z-c)$ in (34),

$$(z-c)\Phi\widetilde{S}' = (z-c)B\widetilde{S}^2 + ((z-c)C - 2MB_{c,1})\widetilde{S} \\ + \Big(M^2B_{c,3} - 2M\Phi_{c,2} - MC_{c,1} + (z-c)D\Big), \tag{35}$$

thus $\widetilde{s} \leq s+1$. In particular if $B'(c) = 0$ then $\widetilde{s} = s+1$.

If $B_{c,1}(c) = 0$ and $MB_{c,3}(c) - 2\Phi_{c,2}(c) - C_{c,1}(c) \neq 0$, then $\widetilde{s} = s+1$. If $B^{'}(c) = 0$ and $\frac{M}{3!}B'''(c) - \Phi''(c) - C'(c) = 0$, once again, we can divide by $(z-c)$, in (35),

$$\Phi\widetilde{S}' = B\widetilde{S}^2 + (C - 2MB_{c,2})\widetilde{S} + \Big(M^2B_{c,4} - 2M\Phi_{c,3} - MC_{c,2} + D\Big),$$

then $\widetilde{s} \leq s$. Assume $\Phi(c) \neq 0$. Then the above equation cannot be simplified. If either $\Phi(c) = 0$ and $B(c) \neq 0$, or if $\Phi(c) = 0, B(c) = 0$ and $C(c) - 2MB_{c,2}(c) \neq 0$ or if $\Phi(c) = B(c) = 0 = C(c) - 2MB_{c,2}(c) = 0$ but $M^2B_{c,4}(c) - 2M\Phi_{c,3}(c) - MC_{c,2}(c) + D \neq 0$ we cannot simplify.

Now, if the following four conditions hold

1. $\Phi(c) = 0$,
2. $B(c) = 0$,
3. $C(c) - 2MB_{c,2}(c) = 0$, or equivalently, $C(c) - MB''(c) = 0$,
4. $M^2B_{c,4}(c) - 2M\Phi_{c,3}(c) - MC_{c,2}(c) + D(c) = 0$, or equivalently, $\frac{M^2}{4!}B^{(4)}(c) - \frac{2M}{3!}\Phi'''(c) - \frac{M}{2}C''(c) + D(c) = 0$,

then

$$\Phi_{c,1}\widetilde{S}' = B_{c,1}\widetilde{S}^2 + (C_{c,1} - 2MB_{c,3})\widetilde{S} + \Big(M^2B_{c,5} - 2M\Phi_{c,4} - MC_{c,3} + D_{c,1}\Big).$$

This means that $\widetilde{s} \leq s-1$. Thus if at least one of the above conditions does not hold and $\Phi'(c) = 0$, then we cannot simplify and $\widetilde{s} = s$.

Again, if the following four conditions are satisfied

1. $\Phi_{c,1}(c) = 0$, i.e. $\Phi'(c) = 0$,
2. $B_{c,1}(c) = 0$, i.e. $B'(c) = 0$,
3. $C_{c,1}(c) - 2MB_{c,3}(c) = 0$, i.e. $C'(c) - \frac{2M}{3!}B'''(c) = 0$,

4. $M^2B_{c,5}(c) - 2M\Phi_{c,4}(c) - MC_{c,3}(c) + D_{c,1}(c) = 0$, i.e. $\frac{M^2}{5!}B^{(5)}(c) - \frac{2M}{4!}\Phi^{(4)}(c) - \frac{M}{3!}C'''(c) + D'(c) = 0$,

then

$$\Phi_{c,2}\widetilde{S}' = B_{c,2}\widetilde{S}^2 + (C_{c,2} - 2MB_{c,4})\widetilde{S} + \left(M^2B_{c,6} - 2M\Phi_{c,5} - MC_{c,4} + D_{c,2}\right),$$

which means that $\widetilde{s} \leq s-2$. In particular, if one the above conditions does not hold then the class is $s-1$.

If the following conditions are satisfied

1. $\Phi_{c,2}(c) = 0$, i.e. $\Phi''(c) = 0$.
2. $B_{c,2}(c) = 0$, i.e. $B''(c) = 0$,
3. $C_{c,2}(c) - 2MB_{c,4}(c) = 0$ i.e. $\frac{1}{2}C''(c) - \frac{2M}{4!}B^{(4)}(c) = 0$,
4. $M^2B_{c,6}(c) - 2M\Phi_{c,5}(c) - MC_{c,4}(c) + D_{c,2}(c) = 0$ i.e. $\frac{M^2}{6!}B^{(6)}(c) - \frac{2M}{5!}\Phi^{(5)}(c) - \frac{M}{4!}C^{(4)}(c) + \frac{1}{2}D''(c) = 0$,

then

$$\Phi_{c,3}\widetilde{S}' = B_{c,3}\widetilde{S}^2 + (C_{c,3} - 2MB_{c,5})\widetilde{S} + \left(M^2B_{c,7} - 2M\Phi_{c,6} - MC_{c,5} + D_{c,3}\right),$$

which means that $\widetilde{s} \leq s-3$. In particular, if one of the above conditions does not hold the class is $s-2$.

Finally, if the following conditions are satisfied

1. $\Phi_{c,3}(c) = 0$ i.e. $\Phi'''(c) = 0$
2. $B_{c,3}(c) = 0$ i.e. $B'''(c) = 0$,
3. $C_{c,3}(c) - 2MB_{c,5}(c) = 0$ i.e. $\frac{1}{3!}C'''(c) - \frac{2M}{5!}B^{(5)}(c) = 0$,
4. $M^2B_{c,7}(c) - 2M\Phi_{c,6}(c) - MC_{c,5}(c) + D_{c,3}(c)$ i.e. $\frac{M^2}{7!}B^{(7)}(c) - \frac{2M}{6!}\Phi^{(6)}(c) - \frac{M}{5!}C^{(5)}(c) + \frac{1}{3!}D'''(c) = 0$,

then

$$\Phi_{c,4}\widetilde{S}' = B_{c,4}\widetilde{S}^2 + (C_{c,4} - 2MB_{c,6})\widetilde{S} + \left(M^2B_{c,8} - 2M\Phi_{c,7} - MC_{c,6} + D_{c,4}\right),$$

which means that $\widetilde{s} = s-4$. If one of the above conditions does not hold $\widetilde{s} = s-3$.

■

4.2 Examples

In the next examples we will describe the equation that $\widetilde{\mu} = \mu^{(1)} + M\delta_c'$ satisfies when $\mu^{(1)}$ is the associated functional of the first kind for the classical orthogonal polynomials (Hermite, Laguerre and Jacobi). We also find the Riccati equation that the corresponding Stieltjes function $\widetilde{S}(z) = S(\widetilde{\mu})(z)$, satisfies.

Because $\mu^{(1)}$ is a Laguerre-Hahn functional of class $s = 0$ (see [6] and [8]), the corresponding Stieltjes function satisfies the Riccati differential equation (5). The polynomials Φ, Ψ, B, C, and D are listed in Table 1.

	$H_n^{(1)}$	$L_n^{\alpha,(1)}$	$P_n^{\alpha,\beta,(1)}$
$\Phi(z)$	1	z	$z^2 - 1$
$\Psi(z)$	$2z$	$z - \alpha - 3$	$-(\alpha+\beta+4)z - \dfrac{\alpha^2-\beta^2}{\alpha+\beta+2}$
$B(z)$	-1	$-\alpha - 1$	$\dfrac{4(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}$
$C(z)$	$-2z$	$-z + \alpha + 2$	$(\alpha+\beta+2)z - \dfrac{\alpha^2-\beta^2}{\alpha+\beta+2}$
$D(z)$	-2	-1	$\alpha+\beta+3$

Table 1

According to Definition (2),

4.2.1 Hermite polynomials

In this case $B(c) = -1 \neq 0$ and $\widetilde{\mu}$ fulfils the equation

$$D\left[(x-c)^4\widetilde{\mu}\right] + 2(x-c)^2\left((x-c)^2x - N - 2(x-c)\right)\widetilde{\mu} - (x-c)^4\left(x^{-1}\widetilde{\mu}^2\right) = 0.$$

The class $\widetilde{s}$ of the linear functional $\widetilde{\mu}$ is $\widetilde{s} = 4$. Furthermore $\widetilde{S}(z)$ satisfies the Riccati equation

$$(z-c)^4\widetilde{S}' = -(z-c)^4\widetilde{S}^2 + 2(z-c)^2\left(-z(z-c)^2 + N\right)\widetilde{S} + \left(-N^2 - 2N(z-c)^2 + 2z(z-c)^2 - 2(z-c)^4\right).$$

4.2.2 Laguerre polynomials

$B(c) = -\alpha - 1 \neq 0$ and $\widetilde{\mu}$ fulfils the equation

$$\begin{aligned} D\left[x(x-c)^4\widetilde{\mu}\right] &+ (x-c)^2\left((x-\alpha-3)(x-c)^2 - 2N(a+1) - 4x(x-c)\right)\widetilde{\mu} \\ &- (\alpha+1)(x-c)^4\left(x^{-1}\widetilde{\mu}^2\right) = 0. \end{aligned}$$

The class of the linear functional $\widetilde{\mu}$ is $\widetilde{s} = 4$. Furthermore $\widetilde{S}(z)$ satisfies the Riccati equation.

$$\begin{aligned} z(z-c)^4\widetilde{S}' &= -(\alpha+1)(z-c)^4\widetilde{S}^2 \qquad (36) \\ &+(z-c)^2\left(2N(\alpha+1) + (-z+\alpha+2)(z-c)^2\right)\widetilde{S} \\ &+\left(-N^2(\alpha+1) - 2Nz(z-c) - N(-z+\alpha+2)(z-c)^2 - (z-c)^4\right). \end{aligned}$$

4.2.3 Jacobi polynomials

$B(c) \neq 0$, because when $\alpha+\beta+1=0$ we get a semiclassical case. Therefore $\widetilde{s}=4$. On the other hand, $\widetilde{S}$ satisfies,

$$(z-c)^4(z^2-1)\widetilde{S}' =$$

$$(z-c)^4\frac{4(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\widetilde{S}^2 + (z-c)^2\left(-2N\frac{4(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2} + \right.$$
$$\left. +(z-c)^2\left((\alpha+\beta+2)z - \frac{\alpha^2-\beta^2}{\alpha+\beta+2}\right)\right)\widetilde{S} +$$
$$+\left(-2N(z^2-1)(z-c) + N^2\frac{4(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\right.$$
$$\left. -N(z-c)^2\left((\alpha+\beta+2)z - \frac{\alpha^2-\beta^2}{\alpha+\beta+2}\right) + (z-c)^4(\alpha+\beta+3)\right),$$

and $\widetilde{\mu}$ satisfies the distributional equation

$$D\quad\left[(x-c)^4(x^2-1)\widetilde{\mu}\right] + (x-c)^2\left(\left(-(\alpha+\beta+4)x + \frac{\alpha^2-\beta^2}{\alpha+\beta+2}\right)(x-c)^2\right.$$
$$\left. + 2N\frac{(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2} - 4(x-c)(x^2-1)\right)\widetilde{\mu} +$$
$$4(x-c)^4\frac{(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\left(x^{-1}\widetilde{\mu}^2\right) = 0.$$

5 Some results on perturbed first kind associated Laguerre polynomials by the derivative of a Dirac Delta.

In this section we deal with the first order associated Laguerre polynomials $(L_n^{\alpha})^{(1)}(x) = L_n(\alpha,1,x)$. These polynomials satisfy the following fourth order linear differential equation (see [11]).

$$x^2L_{n+1}^{(iv)}(\alpha,1,x) + 5xL_{n+1}'''(\alpha,1,x) + \left(-x^2+2(n+\alpha+3)x-\alpha^2+4\right)L_{n+1}''(\alpha,1,x) \tag{37}$$

$$+3(-x+\alpha+n+3)L_{n+1}'(\alpha,1,x) + (n+3)(n+1)L_{n+1}(\alpha,1,x) = 0,\ n \geq 0.$$

They are orthogonal with respect to the linear functional μ, (see [5]).

$$\langle\mu, p(x)\rangle = \int_0^{\infty}\frac{p(x)x^{\alpha}e^{-x}dx}{\left|\Psi(\alpha, 1-\alpha, xe^{-\pi i})\right|^2},\ \alpha > 2, \tag{38}$$

where

$$
\begin{aligned}
\Psi(a,b,x) &= 1+\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n}x^n,\\
(a)_n &= a(a+1)\cdots(a+n-1),\ (a)_0=1.
\end{aligned}
$$

The orthogonality relation for monic polynomials becomes

$$
\int_0^{\infty}\frac{L_n(\alpha,1,x)L_m(\alpha,1,x)x^{\alpha}e^{-x}dx}{\left|\Psi(\alpha,1-\alpha,xe^{-\pi i})\right|^2}=\Gamma(\alpha+n+2)(n+1)!\delta_{mn}. \tag{39}
$$

Also the following structure relation holds

$$
xL_n'(\alpha,1,x)=-(n+2)L_n(\alpha,1,x)-L_{n+1}(\alpha,1,x)+L_{n+1}(\alpha,0,x) \tag{40}
$$

where $L_n(\alpha,0,x)$ will denote the classical Laguerre polynomials.

The first kind associated polynomials for the classical Laguerre polynomials can be represented in terms of hypergeometric series

$$
L_n(\alpha,1,x)=(-1)^n(n+1)!(\alpha+2)_n\sum_{k=0}^{n}\frac{(-n)_k x^k}{(2)_k(\alpha+2)_k}\,{}_3F_2\left(k-n,1,\alpha+1;\alpha+k+2,k+2;1\right),
$$

where

$$
{}_nF_m(a_1,a_2,...,a_n;b_1,b_2,...,b_m;x)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k,...,(a_n)_k}{(b_1)_k(b_2)_k,...,(b_m)_k}\frac{x^k}{k!}.
$$

(see [4] and [5]).

The polynomials $\{L_n(\alpha,1,x)\}_{n\geq 0}$ satisfy the Christoffel-Darboux formula,

$$
\sum_{m=0}^{n-1}\frac{L_m(\alpha,1,x)L_m(\alpha,1,y)}{(m+1)!\Gamma(\alpha+m+2)}=\frac{1}{x-y}\frac{L_n(\alpha,1,x)L_{n-1}(\alpha,1,y)-L_{n-1}(\alpha,1,x)L_n(\alpha,1,y)}{n!\Gamma(\alpha+n+1)}.
$$

Now we consider $\widetilde{\mu}=\mu+M\delta_c'$. Thus

$$
\langle\widetilde{\mu},p(x)\rangle=\langle\mu,p(x)\rangle-Mp'(c). \tag{41}
$$

We denote $L_n(\alpha,1,M,x)$ the monic polynomials which are orthogonal with respect to $\widetilde{\mu}$. Thus

Proposition 5 *For every $n\in\mathbb{N}$,*

$$
L_n(\alpha,1,M,x)=R(x;\alpha,n,M,c)L_n(\alpha,1,x)-S(x;\alpha,n,M,c)L_{n-1}(\alpha,1,x) \tag{42}
$$

where

$$
R(x;\alpha,n,M,c)=
$$

$$1-\frac{MB(\alpha,n,M,c)L_{n-1}(\alpha,1,c)}{D(\alpha,n,M,c)\,(x-c)\,\Gamma(\alpha+n+1)n!}-\frac{MA(\alpha,n,M,c)L_{n-1}(\alpha,1,c)}{D(\alpha,n,M,c)(x-c)^2\Gamma(\alpha+n+1)n!}$$
$$-\frac{MA(\alpha,n,M,c)L'_{n-1}(\alpha,1,c)}{D(\alpha,n,M,c)(x-c)\Gamma(\alpha+n+1)n!},$$

$$S\,(x;\alpha,n,M,c)=$$

$$\frac{MB(\alpha,n,M,c)L_{n}(\alpha,1,c)}{D(\alpha,n,M,c)\,(x-c)\,\Gamma(\alpha+n+1)n!}+\frac{MA(\alpha,n,M,c)L_{n}(\alpha,1,c)}{D(\alpha,n,M,c)(x-c)^2\Gamma(\alpha+n+1)n!}$$
$$+\frac{MA(\alpha,n,M,c)L'_{n}(\alpha,1,c)}{D(\alpha,n,M,c)(x-c)\Gamma(\alpha+n+1)n!},$$

$$\begin{aligned}
A(\alpha,n,M,c) &= L_n(\alpha,1,c)\left(1-M\frac{L''_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L''_{n-1}(\alpha,1,c)L_{n-1}(\alpha,1,c)}{2\Gamma(\alpha+n+1)n!}\right)\\
&\quad +M\frac{L'_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L'_{n-1}(\alpha,1,c)L_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!}L'_n(\alpha,1,c),\\
B(\alpha,n,M,c) &= \left(1-M\frac{L''_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L''_{n-1}(\alpha,1,c)L_{n-1}(\alpha,1,c)}{2\Gamma(\alpha+n+1)n!}\right)L'_n(\alpha,1,c)\\
&\quad +\frac{M}{6\Gamma(\alpha+n+1)n!}\Big(L'''_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L'''_{n-1}(\alpha,1,c)L_n(\alpha,1,c)\\
&\quad +3\left(L''_n(\alpha,1,c)L'_{n-1}(\alpha,1,c)-L''_{n-1}(\alpha,1,c)L'_n(\alpha,1,c)\right)\Big)L_n(\alpha,1,c),\\
D(\alpha,n,M,c) &= \left(1-M\frac{L''_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L''_{n-1}(\alpha,1,c)L_{n-1}(\alpha,1,c)}{2\Gamma(\alpha+n+1)n!}\right)^2\\
&\quad -\frac{L'_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L'_{n-1}(\alpha,1,c)L_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!}\times\\
&\quad \frac{M^2}{6\Gamma(\alpha+n+1)n!}\Big(L'''_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L'''_{n-1}(\alpha,1,c)L_n(\alpha,1,c)\\
&\quad +3\left(L''_n(\alpha,1,c)L'_{n-1}(\alpha,1,c)-L''_{n-1}(\alpha,1,c)L'_n(\alpha,1,c)\right)\Big)L_n(\alpha,1,c).
\end{aligned}$$

Proof. We consider the Fourier expansion

$$L_n(\alpha,1,M,x)=L_n(\alpha,1,x)+\sum_{k=0}^{n-1}a_{n,k}L_k(\alpha,1,x),$$

where

$$a_{n,k}=\frac{\langle\mu,L_n(\alpha,1,M,x)L_k(\alpha,1,x)\rangle}{||L_k(\alpha,1,x)||_\mu^2}.$$

In order to find the coefficients $a_{n,k}$, we can use the orthogonality of polynomials $L_n(\alpha,1,M,x)$ with respect to $\widetilde{\mu}$. i.e.,

$$\langle\widetilde{\mu},L_n(\alpha,1,M,x)L_k(\alpha,1,x)\rangle=0,\ 0\le k\le n-1.$$

From (41)

$$\begin{aligned}\langle\widetilde{\mu}, L_n(\alpha,1,M,x)L_k(\alpha,1,x)\rangle &= \langle\mu, L_n(\alpha,1,M,x)L_k(\alpha,1,x)\rangle \\ &\quad -ML_n'(\alpha,1,M,c)L_k(\alpha,1,c)-ML_n(\alpha,1,M,c)L_k'(\alpha,1,c)\end{aligned}$$

and, as a consequence,

$$a_{n,k}=\frac{ML_n'(\alpha,1,M,c)L_k(\alpha,1,c)+ML_n(\alpha,1,M,c)L_k'(\alpha,1,c)}{\|L_k(\alpha,1,x)\|_\mu^2}.$$

Thus

$$\begin{aligned}L_n(\alpha,1,M,x) &= L_n(\alpha,1,x)+ML_n'(\alpha,1,M,c)\sum_{k=0}^{n-1}\frac{L_k(\alpha,1,c)L_k(\alpha,1,x)}{\Gamma(\alpha+k+2)(k+1)!} \\ &\quad +ML_n(\alpha,1,M,c)\sum_{k=0}^{n-1}\frac{L_k'(\alpha,1,c)L_k(\alpha,1,x)}{\Gamma(\alpha+k+2)(k+1)!}.\end{aligned}$$

Using the notation

$$\begin{aligned}K_n(x,y) &= \sum_{k=0}^{n}\frac{L_k(\alpha,1,x)L_k(\alpha,1,y)}{\|L_k(\alpha,1,x)\|_{\mu^{(1)}}^2}, \\ \frac{\partial^{j+k}\left(K_n(x,y)\right)}{\partial^j x\partial^k y} &= K_n^{(j,k)}(x,y)\end{aligned}$$

we get

$$L_n(\alpha,1,M,x)=L_n(\alpha,1,x)+ML_n'(\alpha,1,M,c)K_{n-1}(x,c)+ML_n(\alpha,1,M,c)K_{n-1}^{(0,1)}(x,c),$$

and we have the following system of linear equations

$$\left\{\begin{array}{l}\left(1-MK_{n-1}^{(1,0)}(c,c)\right)L_n(\alpha,1,M,c)-MK_{n-1}(c,c)L_n'(\alpha,1,M,c)=L_n(\alpha,1,c),\\ -MK_{n-1}^{(1,1)}(c,c)L_n(\alpha,1,M,c)+\left(1-MK_{n-1}^{(1,0)}(c,c)\right)L_n'(\alpha,1,M,c)=L_n'(\alpha,1,c).\end{array}\right. \tag{43}$$

In order to find $L_n(\alpha,1,M,c)$ and $L_n'(\alpha,1,M,c)$ we will need the values, $K_n(c,c)$, $K_{n-1}^{(0,1)}(c,c)$, and $K_{n-1}^{(1,1)}(c,c)$.

Using the Christoffel-Darboux formula we get

$$
\begin{aligned}
K_{n-1}(x,c) &= \frac{1}{x-c}\frac{L_n(\alpha,1,x)L_{n-1}(\alpha,1,c)-L_{n-1}(\alpha,1,x)L_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!},\\
K^{(0,1)}_{n-1}(x,c) &= \frac{1}{(x-c)^2}\frac{L_n(\alpha,1,x)L_{n-1}(\alpha,1,c)-L_{n-1}(\alpha,1,x)L_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!}\\
&\quad+\frac{1}{(x-c)}\frac{L_n(\alpha,1,x)L'_{n-1}(\alpha,1,c)-L_{n-1}(\alpha,1,x)L'_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!},\\
K^{(1,1)}_{n-1}(x,c) &= \frac{-2}{(x-c)^3}\frac{L_n(\alpha,1,x)L_{n-1}(\alpha,1,c)-L_{n-1}(\alpha,1,x)L_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!}\\
&\quad+\frac{1}{(x-c)^2}\frac{L'_n(\alpha,1,x)L_{n-1}(\alpha,1,c)-L'_{n-1}(\alpha,1,x)L_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!}\\
&\quad-\frac{1}{(x-c)^2}\frac{L_n(\alpha,1,x)L'_{n-1}(\alpha,1,c)-L_{n-1}(\alpha,1,x)L'_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!}\\
&\quad\frac{1}{(x-c)}\frac{L'_n(\alpha,1,x)L'_{n-1}(\alpha,1,c)-L'_{n-1}(\alpha,1,x)L'_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!}.
\end{aligned}
$$

As a consequence,

$$
K_{n-1}(c,c)=\lim_{x\to c}K_{n-1}(x,c)=\frac{L'_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L'_{n-1}(\alpha,1,c)L_n(\alpha,1,c)}{\Gamma(\alpha+n+1)n!},
$$

$$
\begin{aligned}
K^{(0,1)}_{n-1}(c,c) &= K^{(1,0)}_{n-1}(c,c)=\lim_{x\to c}K^{(0,1)}_{n-1}(x,c)\\
&= \frac{L''_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L''_{n-1}(\alpha,1,c)L_{n-1}(\alpha,1,c)}{2\Gamma(\alpha+n+1)n!},
\end{aligned}
$$

and

$$
\begin{aligned}
K^{(1,1)}_{n-1}(c,c) &= \frac{1}{6\Gamma(\alpha+n+1)n!}\Big(L'''_n(\alpha,1,c)L_{n-1}(\alpha,1,c)-L'''_{n-1}(\alpha,1,c)L_n(\alpha,1,c)\\
&\quad+3\left(L''_n(\alpha,1,c)L'_{n-1}(\alpha,1,c)-L''_{n-1}(\alpha,1,c)L'_n(\alpha,1,c)\right)\Big).
\end{aligned}
$$

From (43) we get

$$
\begin{aligned}
L_n(\alpha,1,M,c) &= \frac{A(\alpha,n,M,c)}{D(\alpha,n,M,c)},\\
L'_n(\alpha,1,M,c) &= \frac{B(\alpha,n,M,c)}{D(\alpha,n,M,c)}.
\end{aligned}
$$

Therefore

$$
L_n(\alpha,1,M,x)=R(x;\alpha,n,M,c)L_n(\alpha,1,x)-S(x;\alpha,n,M,c)L_{n-1}(\alpha,1,x),
$$

■

We will like to show how we could find some particular expressions of (42). When $c = 0$, we need to calculate, $L_n(\alpha, 1, 0)$, $L'_n(\alpha, 1, 0)$, $L''_n(\alpha, 1, 0)$, and $L'''_n(\alpha, 1, 0)$. Using the structure relation (40) evaluated in $c = 0$, in the cases of $n = 0$ and $n = 1$, we get two expressions from which we obtain

$$L_2(\alpha, 1, 0) = a_2 - 3a_1 + 2.3,$$

where $a_1 = L_1(\alpha, 1, 0)$, $a_2 = L_2(\alpha, 1, 0)$. We suppose that

$$L_n(\alpha, 1, 0) = \sum_{k=0}^{n} \frac{(-1)^k (n+1)! a_{n-k}}{(n+1-k)!}, \tag{44}$$

with $a_n = L_n(\alpha, 1, 0)$, $n \in \mathbb{N}$. From (40), we get

$$(n+2)L_n(\alpha, 1, 0) + L_{n+1}(\alpha, 1, 0) = a_{n+1},$$

then, from (44) we obtain

$$\begin{aligned} L_{n+1}(\alpha, 1, 0) &= a_{n+1} - (n+2) \sum_{k=0}^{n} \frac{(-1)^k (n+1)! a_{n-k}}{(n+1-k)!} \\ &= a_{n+1} - \sum_{k=0}^{n} \frac{(-1)^k (n+2)! a_{n-k}}{(n+1-k)!} \\ &= a_{n+1} - \sum_{k=1}^{n} \frac{(-1)^{k-1} (n+2)! a_{n+1-k}}{(n+2-k)!} \\ &= \sum_{k=1}^{n+1} \frac{(-1)^k (n+2)! a_{n+1-k}}{(n+2-k)!}, \end{aligned}$$

therefore the expression (44) holds.

If we take derivatives three times (42), and evaluate each one of the derivatives in $c = 0$ then, we get the following expressions

$$\begin{aligned} (n+3)L'_n(\alpha, 1, 0) + L'_{n+1}(\alpha, 1, 0) &= L'_{n+1}(\alpha, 1, 0), \\ (n+4)L''_n(\alpha, 1, 0) + L''_{n+1}(\alpha, 1, 0) &= L''_{n+1}(\alpha, 1, 0), \\ (n+5)L'''_n(\alpha, 1, 0) + L'''_{n+1}(\alpha, 1, 0) &= L'''_{n+1}(\alpha, 1, 0). \end{aligned}$$

Applying the same method that we use in order to prove (44) in each one of the previous equations we obtain

Proposition 6 *For every $n \in \mathbb{N}$,*

$$L_n(\alpha,1,0) = \sum_{k=0}^{n} \frac{(-1)^n (n+1)!\Gamma(n+\alpha+1-k)}{(n+1-k)!\Gamma(\alpha+1)}, \tag{45}$$

$$L_n'(\alpha,1,0) = \sum_{k=0}^{n-1} \frac{(-1)^{n-1} (n+2)!(n-k)\Gamma(n+\alpha+1-k)}{(n+2-k)!\Gamma(\alpha+2)}, \tag{46}$$

$$L_n''(\alpha,1,0) = \sum_{k=0}^{n-2} \frac{(-1)^n (n+3)!(n-k-1)_2\Gamma(n+\alpha+1-k)}{(n+3-k)!\Gamma(\alpha+3)}, \tag{47}$$

$$L_n'''(\alpha,1,0) = \sum_{k=0}^{n-3} \frac{(-1)^{n-1} (n+4)!(n-k-2)_3\Gamma(n+\alpha+1-k)}{(n+4-k)!\Gamma(\alpha+4)}. \tag{48}$$

References

[1] J. Alaya, *Quelques résultats nouveaux dans la théorie des polynômes de Laguerre-Hahn.*, Thèse de Doctorat, Université de Tunis II. (1996)

[2] R. Alvarez-Nodarse, J. Arvesú and F. Marcellán, *A generalization of the classical Laguerre polynomials*, Rend.Circ.Mat. Palermo Serie 2, **44** (1995), 315–329.

[3] R. Alvarez-Nodarse and F. Marcellán, *A generalization of the classical Laguerre polynomials: Asymptotic properties and Zeros,* Appl. Anal, **62** (1996), 349–366.

[4] R. Askey, *Orthogonal Polynomials and Special Functions*, Reg. Conf. in Appl. Math, **21**. SIAM, Philadelphia, (1975).

[5] R. Askey and J. Wimp, *Associated Laguerre and Hermite polynomials*, Proc. of Roy. Soc. Edin, **96** (1984), 15–37.

[6] S. Belmehdi and F. Marcellán, *Orthogonal polynomials associated with some modifications of a linear functional*, Appl. Anal. **46** (1992), 1–24.

[7] H. Bouakkaz, *Les polynômes orthogonaux de Laguerre-Hahn de classe zéro*, Thèse Troisième Cycle. Université Pierre et Marie Curie, Paris (1990).

[8] H. Bouakkaz and P. Maroni, *Description des polynômes orthogonaux de Laguerre-Hahn de classe zéro,* In Orthogonal Polynomials and their Applications, C. Brezinski et al editors, IMACS Ann. Comput. Appl. Math. **9**. J.C Baltzer AG, Basel (1991), 189–194.

[9] E. Buendia, J. Dehesa and F. Galvez *The distribution of zeros of the polynomial eigenfunctions of ordinary differential operators of arbitrary order*, Lect. Notes in Math. **1329**. Springer Verlag. Berlin. (1988), 222–235.

[10] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, (1978).

[11] J. Dini, *Sur les formes linéaires et les polynômes orthogonaux de Laguerre-Hahn*, Thèse Troisième Cycle. Université Pierre et Marie Curie, Paris (1988).

[12] G. López,*Convergence of Padé approximants of Stieltjes type meromorphic functions and comparative asymptotic for orthogonal polynomials*, USSR Math. Sborn. **64** (1989), 207–227.

[13] A. Magnus, *Riccati acceleration of the Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials,* In Padé Approximation and its Applications, H. Werner and H.T Burger editors, Lect. Notes in Math. **1071**. Springer Verlag, Berlin (1984), 213–230.

[14] P. Maroni, *Une théorie algébrique des polynômes orthogonaux: Applications aux polynômes orthogonaux semi-classiques*, In Orthogonal Polynomials and their Applications. C. Brezinski et al. eds. IMACS Ann. Comp. Appl. Math. **9** J.C Baltzer AG, Basel (1991), 95–130.

[15] E. Prianes and F. Marcellán, *Orthogonal polynomials and Stieltjes functions:the Laguerre-Hahn case*, Rendi.di Mat. **16** (1996), 117–141.

[16] J. Sherman, *On the numerators of the convergents of the Stieltjes continued fractions*, Trans. Amer. Math. Soc, **35** (1933), 64–87.