Orthogonal polynomials and measures on the unit circle. The Geronimus transformations

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To professor William B. Gragg with occasion of his 70^{th} birthday

Abstract

In this paper we analyze a perturbation of a nontrivial positive measure supported on the unit circle. This perturbation is the inverse of the Christoffel transformation and is called the Geronimus transformation. We study the corresponding sequences of monic orthogonal polynomials as well as the connection between the associated Hessenberg matrices. Finally, we show an example of this kind of transformation.

Key words: Measures on the unit circle, orthogonal polynomials, spectral transformations, Hessenberg matrices.

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1 Introduction

The study of orthogonal polynomials with respect to a nontrivial positive Borel measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ was started by G. Szegő in several papers published from 1915 to 1925 (see [19]). Later on Y. Geronimus [4] extended this theory to a more general situation.

If υ is a linear functional in the linear space Λ of the Laurent polynomials ($\Lambda = span\{z^n\}_{n\in\mathbb{Z}}$) such that υ is Hermitian, i. e. $c_n = \langle \upsilon, z^n \rangle = \overline{\langle \upsilon, z^{-n} \rangle} = \bar{c}_{-n}$, $n \in \mathbb{Z}$, then a bilinear functional associated with υ can be introduced in the linear

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space $\mathbb{P}$ of polynomials with complex coefficient as follows

$$(p(z), q(z))_\upsilon = \left\langle \upsilon, p(z)\bar{q}(z^{-1}) \right\rangle \tag{1}$$

where $p, q \in \mathbb{P}$.

The Gram matrix associated with this bilinear functional in terms of the canonical basis $\{z^n\}_{n\geqslant 0}$ of $\mathbb{P}$ is

$$\mathbf{T} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}, \tag{2}$$

a Toeplitz matrix [8].

The linear functional is said to be quasi-definite if the principal leading submatrices of $\mathbf{T}$ are non-singular. If such matrices have positive determinant, then the linear functional is said to be positive definite. Every positive definite linear functional has an integral representation

$$\langle \upsilon, p(z) \rangle = \int_{\mathbb{T}} p(z) d\mu(z), \tag{3}$$

where μ is a nontrivial positive Borel measure supported on the unit circle (see [4], [8], [11], [17]).

If υ is a quasi-definite linear functional then a unique sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ such that

$$(P_n, P_m)_\upsilon = \mathbf{k}_n \delta_{n,m}, \tag{4}$$

can be introduced, where $\mathbf{k}_n \neq 0$ for every $n \geqslant 0$. It is said to be the monic orthogonal polynomial sequence associated with υ.

This polynomial sequence satisfies two equivalent recurrence relations due to G. Szegő (see [4], [8], [17], [19])

$$P_{n+1}(z) = zP_n(z) + P_{n+1}(0)P_n^*(z), \tag{5}$$

$$P_{n+1}(z) = \left(1 - |P_{n+1}(0)|^2\right) zP_n(z) + P_{n+1}(0)P_{n+1}^*(z), \tag{6}$$

the forward and backward recurrences, respectively, where $P_n^*(z) = z^n \bar{P}_n(z^{-1})$ is the so-called reversed polynomial. On the other hand, from (5) and (6) we deduce

$$zP_n(z) = \sum_{j=0}^{n+1} \lambda_{n,j} P_j(z), \tag{7}$$

with

$$\lambda_{n,j} = \begin{cases} 1 & \text{if } j = n+1, \\ \frac{\mathbf{k}_n}{\mathbf{k}_j} P_{n+1}(0)\overline{P_j(0)} & \text{if } j \leqslant n, \\ 0 & \text{otherwise}, \end{cases} \tag{8}$$

(see [12], [17]). Thus, the matrix representation of the linear operator $h : \mathbb{P} \mapsto \mathbb{P}$, the multiplication by z in terms of the basis $\{P_n\}_{n\geqslant 0}$ is

$$zP(z) = \mathbf{H}_P P(z),$$

where $P(z) = [P_0(z), P_1(z), \dots, P_n(z), \dots]^t$ and $\mathbf{H}_P$ is a lower Hessenberg matrix with entries $\lambda_{j,k}$ defined in (8).

Finally, in terms of the moments $\{c_n\}_{n\geqslant 0}$ an analytic function

$$C(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n} z^n \tag{9}$$

can be introduced. If υ is a positive definite linear functional, then C is analytic in the open unit disk and $\mathfrak{Re}\,(C(z)) > 0$ therein. In such a case C is said to be a Carathéodory function and it can be represented as a Riesz-Herglotz transform of the positive measure μ introduced in (3) (see [4], [11], [17])

$$C(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w).$$

Following some perturbations of the measure μ we have studied the behavior of the corresponding Carathéodory functions (see [14]) as well as the Hessenberg matrices associated with the corresponding sequence of orthogonal polynomials in three cases

(i) If $d\tilde{\mu} = |z-\alpha|^2 d\mu$, $|z| = 1$, then the so-called canonical Christoffel transformation appears. In [**?**] and [16] we have studied the connection between the associated Hessenberg matrices using the QR factorization. The iteration of the canonical Christoffel transformation has been analyzed in [5], [10], and [13].

(ii) If $d\tilde{\mu} = d\mu + \boldsymbol{m}\delta(z-z_0)$, $|z_0| = 1$, $\boldsymbol{m} \in \mathbb{R}_+$, then the so-called canonical Uvarov transformation appears. In [**?**] and [14] we have studied the connection between the corresponding sequences of monic orthogonal polynomials as well as the associated Hessenberg matrices using the LU and QR factorization. The iteration of the canonical Uvarov transformation has been studied in [4] and [13].

(iii) If $d\tilde{\mu} = \frac{1}{|z-\alpha|^2} d\mu$, $|z| = 1$, and $|\alpha| > 1$, then a special case of the Geronimus transform has been analyzed in [15]. In particular, the relation between the corresponding sequences of monic orthogonal polynomials and the associated Hessenberg matrices is stated. A more general framework is presented in [6].

These three examples of canonical spectral transforms are the analogues on the unit circle of the canonical spectral transforms on the real line considered by several authors (see [1], [18], [20] and [21]) in connection with bispectral problems and LU, UL, and QR factorization of Jacobi matrices, i. e. symmetric and tridiagonal matrices with real entries. An extension of the canonical Christoffel transformation for a general inner product is done in [2].

Notice that in these three cases, the corresponding Carathéodory functions are related by

$$\tilde{C}(z) = Q(z)C(z) + R(z),$$

with Q, R rational functions. This situation corresponds to the so-called *linear rational spectral transformations*. Other examples of pure rational spectral transformations have been summarized in [14].

Taking into account the linear functional $\tilde{\upsilon}$ associated with a Christoffel transformation satisfies

$$(p, q)_{\tilde{\upsilon}} = ((z-\alpha)p, (z-\alpha)q)_{\upsilon}, \quad |\alpha| \neq 1,$$

it seems to be natural to analyze the inverse transform $\tilde{\upsilon} \to \upsilon$. In such a case, in [14] we have shown that there are many solutions to this inverse problem. They are defined up to the addition of a trivial linear functional $\boldsymbol{m}\delta(z-\alpha) + \overline{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$ with $\boldsymbol{m} \in \mathbb{C} \setminus \{0\}$.

The aim of this paper is the study of the bilinear functional

$$(p, q) = \int_{\mathbb{T}} p(z)\overline{q(z)}\frac{d\mu}{|z-\alpha|^2} + \boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})} + \overline{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}, \tag{10}$$

where $|\alpha| > 1$ and $\boldsymbol{m} \in \mathbb{C} \setminus \{0\}$.

In particular, in Section 3 we give necessary and sufficient conditions for the existence of a sequence of monic polynomials orthogonal with respect to (10). Furthermore, the relation between the sequences of monic orthogonal polynomials with respect to μ and (10) is deduced. In Section 4, the connection between the corresponding Hessenberg matrices is obtained. Finally, in Section 5 an example of this Geronimus transformation when $d\mu = \dfrac{d\theta}{2\pi}$, i. e. the Lebesgue probability measure on the unit circle, is analyzed.

2 C-functions and linear spectral transforms

Let υ be a positive definite linear functional in Λ and consider the associated Carathéodory function (C-function) C given by

$$C(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n} z^n, \tag{11}$$

with $c_n = \langle \upsilon, z^n \rangle$. The linear functional υ induces a bilinear functional defined by

$$(p, q)_\upsilon = \left\langle \upsilon, p(z)\bar{q}(z^{-1}) \right\rangle.$$

We consider a linear rational spectral transformation $\tilde{C}$ of C

$$\tilde{C}(z) = A(z)C(z) + B(z), \tag{12}$$

where A, B are rational functions. The function $\tilde{C}$ is analytic at $z = 0$, i. e.

$$\tilde{C}(z) = \tilde{c}_0 + 2\sum_{n=0}^{\infty} \tilde{c}_{-n} z^n,$$

where $\tilde{c}_0 \in \mathbb{R}$, $\tilde{c}_{-k} \in \mathbb{C}$ for $k \in \mathbb{N}$, and $\limsup |\tilde{c}_k|^{\frac{1}{k}} < \infty$, and it can be associated with the linear functional $\tilde{\upsilon}$ in Λ defined by

$$\langle \tilde{\upsilon}, z^n \rangle = \tilde{c}_n, \;\; n \in \mathbb{Z},$$

with the convention $\tilde{c}_{-n} = \overline{\tilde{c}}_n$, $n \in \mathbb{N}$.

If $\{P_n\}_{n\geqslant 0}$ is a sequence of monic orthogonal polynomials with respect to υ, a natural problem is to analyze necessary and sufficient conditions for the existence of a monic orthogonal polynomial sequence with respect to $\tilde{\upsilon}$. This is an open problem, although some particular situations have studied in [14].

Next, we will show some examples of linear rational transformations.

2.1 Christoffel transformation

Let $\tilde{\upsilon}$ be a linear functional such that the associated bilinear functional satisfies

$$(p, q)_{\tilde{\upsilon}} = ((z-\alpha)p, (z-\alpha)q)_\upsilon, \;\; \alpha \in \mathbb{C}. \tag{13}$$

$\tilde{\upsilon} = C_\alpha(\upsilon)$ is the canonical Christoffel transformation of υ.

Thus, if $\tilde{C}(z) = \tilde{c}_0 + 2\sum_{n=1}^{\infty} \tilde{c}_{-n} z^n$, with $\tilde{c}_k = \left\langle \tilde{\upsilon}, z^k \right\rangle$ is a linear spectral transformation of C, then

$$\tilde{c}_k = \left\langle \tilde{\upsilon}, z^k \right\rangle = \left(z^k, 1\right)_{\tilde{\upsilon}} = \left(1 + |\alpha|^2\right) c_k - \alpha c_{k-1} - \bar{\alpha} c_{k+1}, \quad k \in \mathbb{Z},$$

and

$$\tilde{C}(z) = \left(1 + |\alpha|^2\right) C(z) - \bar{\alpha} \left(c_1 + 2\sum_{k=0}^{\infty} c_{-n} z^{n+1} \right) - \alpha \left(c_{-1} + 2\sum_{k=2}^{\infty} c_{-n} z^{n-1} \right).$$

As a consequence,

Proposition 1

$$\tilde{C}(z) = A(z)C(z) + B(z),$$

where

$$A(z) = \frac{(1 - \bar{\alpha} z)(z - \alpha)}{z}, \quad B(z) = \frac{-\bar{\alpha} c_0 z^2 + (\alpha c_{-1} - \bar{\alpha} c_1)\, z + \alpha c_0}{z}.$$

2.2 *Uvarov transformation*

Now, we consider $\tilde{\upsilon}$ such that the associated bilinear functional satisfies

$$(p, q)_{\tilde{\upsilon}} = (p, q)_{\upsilon} + \boldsymbol{m} p(\alpha)\overline{q(\bar{\alpha}^{-1})} + \overline{\boldsymbol{m}} p(\bar{\alpha}^{-1})\overline{q(\alpha)}, \tag{14}$$

with $|\alpha| > 1$ and $\boldsymbol{m} \in \mathbb{C} \setminus \{0\}$. $\tilde{\upsilon} = \mathcal{U}_{\alpha,\boldsymbol{m}}(\upsilon)$ is the *Uvarov canonical transform* of υ.

We will see that $\tilde{C}(z) = \tilde{c}_0 + 2\sum_{k=1}^{\infty} \tilde{c}_{-k} z^k$, with $\tilde{c}_k = \left\langle \tilde{\upsilon}, z^k \right\rangle$, is a linear spectral transformation of C.

$$\tilde{c}_k = \left(z^k, 1\right)_{\tilde{\upsilon}} = \left(z^k, 1\right)_{\upsilon} + \boldsymbol{m}\alpha^k + \overline{\boldsymbol{m}}\bar{\alpha}^{-k}.$$

Then,

Proposition 2

$$\tilde{C}(z) = A(z)C(z) + B(z),$$

where

$$A(z) = 1, \quad B(z) = \boldsymbol{m}\frac{\alpha + z}{\alpha - z} + \overline{\boldsymbol{m}}\frac{1 + \bar{\alpha} z}{1 - \bar{\alpha} z}.$$

2.3 Geronimus transformation

Consider the linear functional $\tilde{\upsilon}$ such that the associated bilinear functional satisfies

$$((z-\alpha)p,(z-\alpha)q)_{\tilde{\upsilon}} = (p,q)_{\upsilon}, \quad |\alpha|>1. \tag{15}$$

The linear functional $\tilde{\upsilon} = \mathcal{G}_{\alpha}(\upsilon)$ is said to be a *Geronimus canonical transform* of υ. Then

$$\begin{aligned} c_k &= \left(z^k,1\right)_{\upsilon} = \left(z^k(z-\alpha), z-\alpha\right)_{\tilde{\upsilon}} \\ &= \tilde{c}_k\left(1+|\alpha|^2\right) - \bar{\alpha}\tilde{c}_{k+1} - \alpha\tilde{c}_{k-1}, \quad k \geqslant 0. \end{aligned} \tag{16}$$

If $s_k = \frac{c_k}{\alpha^k}$ and $q_k = \frac{\tilde{c}_k}{\alpha^k} - \frac{\tilde{c}_{k-1}}{\alpha^{k-1}}$, then from the above expression we obtain

$$s_k = q_k - |\alpha|^2 q_{k+1}, \quad k \geqslant 0,$$

and, as a consequence,

$$q_k = \frac{q_0 - c_0 - \bar{\alpha}c_1 - \cdots - \bar{\alpha}^{k-1}c_{k-1}}{|\alpha|^{2k}}, \quad k \geqslant 1, \tag{17}$$

as well as,

$$\begin{aligned} q_0 &= \tilde{c}_0 - \alpha\bar{\tilde{c}}_1 \\ q_1 &= \frac{q_0 - c_0}{|\alpha|^2} = \frac{\tilde{c}_1}{\alpha} - \tilde{c}_0. \end{aligned}$$

Thus, q_0 is a free parameter. Therefore

$$\begin{aligned} \tilde{c}_0 - \bar{\alpha}\tilde{c}_1 &= \bar{q}_0 \\ \tilde{c}_0 - \frac{\tilde{c}_{-1}}{\alpha} &= \frac{c_0 - q_0}{|\alpha|^2} \end{aligned} \tag{18}$$

and

$$\left(1-|\alpha|^2\right)\tilde{c}_0 = 2\mathfrak{Re}\,(q_0) - c_0,$$

If we assume that $|\alpha|>1$, from (18) we get $\tilde{c}_1$. Furthermore

$$\frac{\tilde{c}_k}{\alpha^k} = \tilde{c}_0 + \sum_{j=1}^{k} q_j, \quad k \geqslant 2.$$

Therefore, we have a degree of freedom that is the choice of q_0. As a consequence, from (16) we get

Proposition 3

$$\tilde{C}(z) = A(z)C(z) + B(z), \tag{19}$$

where

$$A(z) = \frac{z}{(1-\bar{\alpha}z)(z-\alpha)}, \quad B(z) = \frac{\bar{\alpha}\tilde{c}_0 z^2 + 2i\Im\mathfrak{m}\,(q_0)\,z - \alpha\tilde{c}_0}{(1-\bar{\alpha}z)(z-\alpha)}.$$

From the above result, we deduce

$$\tilde{C}(z) = A(z)C(z) + \boldsymbol{m}\frac{\alpha+z}{\alpha-z} + \overline{\boldsymbol{m}}\frac{1+\bar{\alpha}z}{1-\bar{\alpha}z},$$

where $\boldsymbol{m} = \frac{1}{2}\frac{2q_0 - c_0}{1-|\alpha|^2}$.

In the positive definite case, an example of Geronimus transformation with $\boldsymbol{m} = 0$ has been analyzed in [6].

Proposition 4

(i) $\mathcal{G}_\alpha \circ \mathcal{C}_\alpha = \mathcal{U}_{\alpha,\widetilde{\boldsymbol{m}}}$.
(ii) $\mathcal{C}_\alpha \circ \mathcal{G}_\alpha = \mathcal{I}_d$.

Proof.

(i) Let G be analytic function associated with $\mathcal{C}_\alpha$. Then, from Proposition 1, we get

$$G(z) = \frac{\tilde{A}(z)}{z}C(z) - \left(\bar{\alpha}c_0 z - (\alpha\bar{c}_1 - \bar{\alpha}c_1) - \alpha c_0\frac{1}{z}\right)$$

where $\tilde{A}(z) = (1-\bar{\alpha}z)(z-\alpha)$.

If H is the analytic function associated with $(\mathcal{G}_\alpha \circ \mathcal{C}_\alpha)(\upsilon)$, then from Proposition 19,

$$\begin{aligned} H(z) &= \frac{z}{\tilde{A}(z)}G(z) + \boldsymbol{m}\frac{\alpha+z}{\alpha-z} + \overline{\boldsymbol{m}}\frac{1+\bar{\alpha}z}{1-\bar{\alpha}z} \\ &= C(z) + \widetilde{\boldsymbol{m}}\frac{\alpha+z}{\alpha-z} + \overline{\widetilde{\boldsymbol{m}}}\frac{1+\bar{\alpha}z}{1-\bar{\alpha}z} \end{aligned}$$

with

$$\widetilde{\boldsymbol{m}} = \boldsymbol{m} - \frac{1}{2}\left(c_0 + \frac{\alpha\bar{c}_1 - \bar{\alpha}c_1}{1-|\alpha|^2}\right).$$

(ii) Let $\tilde{C}$ be the analytic function associated with $(\tilde{\upsilon} = \mathcal{G}_\alpha)(\upsilon)$. Then, taking into account the Proposition 19

$$\tilde{C}(z) = \frac{z}{\tilde{A}(z)}C(z) + \boldsymbol{m}\frac{\alpha+z}{\alpha-z} + \overline{\boldsymbol{m}}\frac{1+\bar{\alpha}z}{1-\bar{\alpha}z}.$$

Now, if K is the analytic function associated with $(\mathcal{C}_\alpha \circ \mathcal{G}_\alpha)(\upsilon)$ then, from Proposition 1 we get

$$K(z) = \frac{\tilde{A}(z)}{z}\tilde{C}(z) - \left(\bar{\alpha}\tilde{c}_0 z + (\alpha\tilde{c}_1 - \bar{\alpha}\bar{v}_1) - \frac{\alpha\tilde{c}_0}{z}\right)$$
$$= C(z) + \tfrac{\tilde{A}(z)}{z}\left(\boldsymbol{m}\frac{\alpha + z}{\alpha - z} + \overline{\boldsymbol{m}}\frac{1 + \bar{\alpha}z}{1 - \bar{\alpha}z}\right) - \tfrac{\tilde{A}(z)}{z}\left(\boldsymbol{m}\frac{\alpha + z}{\alpha - z} + \overline{\boldsymbol{m}}\frac{1 + \bar{\alpha}z}{1 - \bar{\alpha}z}\right) = C(z).$$

■

3 Orthogonal polynomials and Geronimus transforms

Let μ be a nontrivial positive Borel measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. An inner product defined in the linear space $\mathbb{P}$ of polynomials with complex coefficients is associated with μ as follows

$$(p, q)_\mu = \int_{\mathbb{T}} p(z)\overline{q(z)}d\mu, \quad p, q \in \mathbb{P}.$$

The entries of the Gram matrix $\mathbf{T}$ associated with $(\cdot, \cdot)_\mu$ are given by

$$c_{k-l} = \left(z^k, z^l\right)_\mu, \quad k, l \in \mathbb{N}.$$

The matrix $\mathbf{T}$ is Hermitian with constant entries along the diagonals, i. e. $\mathbf{T}$ is a Toeplitz matrix. We will denote $\mathbf{T}_n$ the $(n+1) \times (n+1)$ leading principal submatrix of $\mathbf{T}$.

To the measure μ we can associate a sequence $\{\varphi_n\}_{n\geqslant 0}$ of *orthonormal polynomials* given by

$$\varphi_n(z) = \frac{1}{\sqrt{T_{n-1}T_n}} \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & c_0 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & c_{-1} \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix}, \tag{20}$$

where $T_n = \det \mathbf{T}_n$, $n \geqslant 0$, with the convention $T_{-1} = 1$. The leading coefficients are

$$\kappa_n(\mu) = \sqrt{\frac{T_{n-1}}{T_n}}. \tag{21}$$

We will denote the *monic orthogonal polynomials* with respect to μ by $P_n(z) = \frac{1}{\kappa_n(\mu)}\varphi_n(z)$. Then,

$$\mathbf{k}_n := (P_n, P_n)_\mu = \frac{1}{\kappa_n(\mu)^2} = \frac{T_n}{T_{n-1}}. \tag{22}$$

The *n-th reproducing kernel polynomial* associated with $\{\varphi_n\}_{n\geqslant 0}$ is defined by

$$K_n(z,y)=\sum_{j=0}^{n}\overline{\varphi_j(y;\mu)}\varphi_j(z;\mu)=\sum_{j=0}^{n}\frac{\overline{P_j(y)}P_j(z)}{\mathbf{k}_j}.$$

The functions

$$q_j(t)=\int_{\mathbb{T}}\frac{\overline{\varphi_j(z;\mu)}}{t-z}d\mu(z),\quad t\notin\mathbb{T},\quad j\geqslant 0, \tag{23}$$

are called *functions of second kind* associated with μ. We also denote

$$Q_j(t)=\int_{\mathbb{T}}\frac{\overline{P_j(z;\mu)}}{t-z}d\mu(z)=(\kappa_j(\mu))^{-1}q_j(t).$$

Now, we consider the Geronimus perturbation $\tilde{\mu}$ of μ as follows

$$d\tilde{\mu}=\frac{d\mu}{|z-\alpha|^2}+\boldsymbol{m}\delta_\alpha+\overline{\boldsymbol{m}}\delta_{\bar{\alpha}^{-1}},\quad \boldsymbol{m}\in\mathbb{C}, \tag{24}$$

where $|\alpha|>1$, as well as the associated bilinear functional

$$(p,q)_{\tilde{\mu}}:=\int_{\mathbb{T}}p(z)\overline{q(z)}\frac{d\mu}{|z-\alpha|^2}+\boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})}+\overline{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}.$$

$\tilde{\mu}$ is said to be the canonical Geronimus transform of μ.

Proposition 5 *The bilinear functional* $(p,q)_{\tilde{\mu}}=\int p(z)\overline{q(z)}\frac{d\mu}{|z-\alpha|^2}+\boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})}+\overline{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}$ *is quasi-definite if and only if*

$$\varepsilon_n(\alpha):=\|\tilde{\mu}\|-\sum_{j=0}^{n}\left|q_j(\alpha)+\boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{\varphi_j(\bar{\alpha}^{-1})}\right|^2\neq 0,\ \textit{for every } n\geqslant 0,$$

where $\|\tilde{\mu}\|=\int_{\mathbb{T}}\frac{d\mu}{|z-\alpha|^2}+\boldsymbol{m}+\overline{\boldsymbol{m}}$.

Proof. Let consider the basis $\{1,(z-\alpha),z(z-\alpha),\ldots,z^n(z-\alpha),\ldots\}$ of $\mathbb{P}$. Then the moments $\tilde{c}_{k,j}$ associated with $\tilde{\mu}$ in this basis are

$$\tilde{c}_{k,j}=\left(z^{k-1}(z-\alpha),z^{j-1}(z-\alpha)\right)_{\tilde{\mu}}=\left(z^{k-1},z^{j-1}\right)_{\mu}=c_{k-j},\ \ k,j=1,2,\ldots.$$

Thus, the Gram matrix $\widetilde{T}$ associated with $\tilde{\mu}$ has as the $(n+1)\times(n+1)$ leading principal submatrix

$$\widetilde{T}_n = \left|\begin{array}{c|ccc|c} \tilde{c}_{0,0} & \tilde{c}_{0,1} & \cdots & \tilde{c}_{0,n-1} & \tilde{c}_{0,n} \\ \hline \tilde{c}_{1,0} & & & & c_{-n+1} \\ \vdots & & \mathbf{T}_{n-2} & & \vdots \\ \tilde{c}_{n-1,0} & & & & c_{-1} \\ \hline \tilde{c}_{n,0} & c_{n-1} & \cdots & c_1 & c_0 \end{array}\right|, \tag{25}$$

with

$$\begin{aligned} \tilde{c}_{0,j} &= \left(1, z^{j-1}(z-\alpha)\right)_{\tilde{\mu}} = \left(\frac{1}{z-\alpha}, z^{j-1}\right)_{\mu} + \boldsymbol{m}\overline{(\bar{\alpha}^{-1}-\alpha)}\alpha^{-j+1}, \quad j = 1, 2, \ldots \\ \tilde{c}_{0,0} &= (1,1)_{\tilde{\mu}} = \left(\frac{1}{z-\alpha}, \frac{1}{z-\alpha}\right)_{\mu} + \boldsymbol{m} + \overline{\boldsymbol{m}}. \end{aligned}$$

Using the Sylvester identity (see [9], page 22) we get, for $n \geqslant 2$,

$$\widetilde{T}_n T_{n-2} = \widetilde{T}_{n-1} T_{n-1} - D_{n-1}\overline{D}_{n-1},$$

where

$$D_{n-1} = \left|\begin{array}{cccc} \tilde{c}_{0,1} & \tilde{c}_{0,2} & \cdots & \tilde{c}_{0,n} \\ c_0 & c_{-1} & \cdots & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_{-1} \end{array}\right|,$$

$$\begin{aligned} D_{n-1} &= (-1)^n T_{n-2} Q_{n-1}(\alpha) + (-1)^{n-1} T_{n-2} \overline{P_{n-1}(\bar{\alpha}^{-1})}\boldsymbol{m}(\alpha^{-1}-\bar{\alpha}) \\ &= (-1)^n T_{n-2}\left(Q_{n-1}(\alpha) + \boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{P_{n-1}(\bar{\alpha}^{-1})}\right). \end{aligned}$$

Then,

$$\widetilde{T}_n T_{n-2} = \widetilde{T}_{n-1} T_{n-1} - T_{n-2} T_{n-1} \left| q_{n-1}(\alpha) + \boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{\varphi_{n-1}(\bar{\alpha}^{-1})}\right|^2,$$

so we have

$$\frac{\widetilde{T}_n}{T_{n-1}} = \frac{\widetilde{T}_{n-1}}{T_{n-2}} - \left| q_{n-1}(\alpha) + \boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{\varphi_{n-1}(\bar{\alpha}^{-1})}\right|^2, \quad n \geqslant 2.$$

Furthermore, for $n = 1$ we have

$$\frac{\widetilde{T}_1}{T_0} = \tilde{c}_{0,0} - \frac{\left|\tilde{c}_{0,1}\right|^2}{c_0} = \|\tilde{\mu}\| - \left|q_0(\alpha) + \boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\overline{\varphi_0(\bar{\alpha}^{-1})}\right|^2.$$

Thus, we can recursively deduce

$$\frac{\widetilde{T}_n}{T_{n-1}} = \|\tilde{\mu}\| - \sum_{j=0}^{n-1} \left|q_j(\alpha) + \boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\overline{\varphi_j(\bar{\alpha}^{-1})}\right|^2 = \varepsilon_{n-1}(\alpha). \tag{26}$$

■

As a consequence, assuming the bilinear functional associated with $\tilde{\mu}$ is quasi-definite, we get

Corollary 6 *For $n \geqslant 0$*

$$\frac{\tilde{\mathbf{k}}_{n+1}}{\mathbf{k}_n} = \frac{\varepsilon_n(\alpha)}{\varepsilon_{n-1}(\alpha)},$$

with the convention $\varepsilon_{-1}(\alpha) = \|\tilde{\mu}\|$.

Proof. We have

$$\frac{\tilde{\mathbf{k}}_{n+1}}{\mathbf{k}_n} = \frac{\widetilde{T}_{n+1}}{\widetilde{T}_n}\frac{T_{n-1}}{T_n},$$

and the result follows. ■

Notice that $\{\varepsilon_n\}_{n\geqslant 0}$ is a decreasing sequence. According to cosine's theorem, we have

$$\left|q_j(\alpha) + \boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\overline{\varphi_j(\bar{\alpha}^{-1})}\right|^2 < 2\left(\left|q_j(\alpha)\right|^2 + \left|\boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\right|^2 \left|\varphi_j(\bar{\alpha}^{-1})\right|^2\right).$$

Then,

$$\varepsilon_n(\alpha) > \|\tilde{\mu}\| - 2\sum_{j=0}^{n} \left|q_j(\alpha)\right|^2 - 2\left|\boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\right|^2 K_n(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}),$$

but taking into account that the funcions $q_j(\alpha)$ are the Fourier coefficients of $\frac{1}{z-\alpha}$ with respect to the orthonormal sequence $\{\varphi_j\}_{n\geqslant 0}$, we have

$$\begin{aligned}\varepsilon_n(\alpha) &> \left\|\frac{1}{z-\alpha}\right\|_{\mu}^2 + \boldsymbol{m} + \overline{\boldsymbol{m}} - 2\sum_{j=0}^{n} \left|q_j(\alpha)\right|^2 - 2\left|\boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\right|^2 K_n(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}) \\ &> -\frac{1}{\|z-\alpha\|^2} + \boldsymbol{m} + \overline{\boldsymbol{m}} - 2\left|\boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\right|^2 K_n(\bar{\alpha}^{-1}, \bar{\alpha}^{-1}).\end{aligned}$$

Thus, if we define $\varepsilon_\infty(\alpha) = \lim_{n\to\infty} \varepsilon_n(\alpha)$, in order to have $\varepsilon_\infty(\alpha) > -\infty$, we need $\sum_{j=0}^{\infty} \left|\varphi_j(\bar{\alpha}^{-1})\right|^2 < \infty$, or equivalently, that the Szegő condition holds for μ (see [17], page 170).

Now, we assume that $(\cdot,\cdot)_{\tilde{\mu}}$ is quasi-definite or, equivalently, $\varepsilon_n(\alpha) \neq 0$, $n \geqslant 0$. Then there exists a monic orthogonal polynomial sequence $\left\{\tilde{P}_n\right\}_{n\geqslant 0}$ with respect to $(\cdot,\cdot)_{\tilde{\mu}}$.

We will give an explicit expression for the orthogonal polynomial sequence $\left\{\tilde{P}_n\right\}_{n\geqslant 0}$. Since $\{(z-\alpha)P_n(z)\}_{n\geqslant 0}$ is a basis in $(z-\alpha)\mathbb{P}$, which is orthogonal with respect to the bilinear functional $(\cdot,\cdot)_{\tilde{\mu}}$, then we can express $\tilde{P}_{n+1}(z) - \tilde{P}_{n+1}(\alpha) \in (z-\alpha)\mathbb{P}, n \geqslant 1$, in terms of the above basis, i.e.:

$$\tilde{P}_{n+1}(z) - \tilde{P}_{n+1}(\alpha) = (z-\alpha)P_n(z) + (z-\alpha)\sum_{j=0}^{n-1} \lambda_{n+1,j}P_j(z),$$

where

$$\begin{aligned}
\lambda_{n+1,j} &= \frac{\left(\tilde{P}_{n+1}(z) - \tilde{P}_{n+1}(\alpha), (z-\alpha)P_j(z)\right)_{\tilde{\mu}}}{\mathbf{k}_j} \\
&= -\frac{\tilde{P}_{n+1}(\alpha)}{\mathbf{k}_j}\left(1, (z-\alpha)P_j(z)\right)_{\tilde{\mu}} \\
&= -\frac{\tilde{P}_{n+1}(\alpha)}{\mathbf{k}_j}\left(\int_{\mathbb{T}} \frac{\overline{P_j(z)}}{z-\alpha}d\mu(z) + \boldsymbol{m}\overline{(\bar{\alpha}^{-1}-\alpha)P_j(\bar{\alpha}^{-1})}\right) \\
&= \frac{\tilde{P}_{n+1}(\alpha)}{\mathbf{k}_j}\left(Q_j(\alpha) + \boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{P_j(\bar{\alpha}^{-1})}\right), \quad j = 0, 1, \ldots, n-1.
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde{P}_{n+1}(z) &= (z-\alpha)P_n(z) + \tilde{P}_{n+1}(\alpha)\left(1 + (z-\alpha)\sum_{j=0}^{n-1}\frac{Q_j(\alpha)+\boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{P_j(\bar{\alpha}^{-1})}}{\mathbf{k}_j}P_j(z)\right) \\
&= (z-\alpha)P_n(z) + \tilde{P}_{n+1}(\alpha)\left(1 + (z-\alpha)\left[\int_{\mathbb{T}}\frac{K_{n-1}(z,t)}{\alpha-t}d\mu(t) + \right.\right. \\
&\quad \left.\left. + \boldsymbol{m}\left(\bar{\alpha}-\alpha^{-1}\right)K_{n-1}(z,\bar{\alpha}^{-1})\right]\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tilde{\mathbf{k}}_{n+1} &= \left(\tilde{P}_{n+1}(z), z^n(z-\alpha)\right)_{\tilde{\mu}} = (P_n(z), z^n)_\mu + \tilde{P}_{n+1}(\alpha)\left((1, z^n(z-\alpha))_{\tilde{\mu}} + \right. \\
&\quad \left. + \int_{\mathbb{T}}\frac{(K_{n-1}(z,t), z^n)_\mu}{\alpha-t}d\mu(t) + \boldsymbol{m}\left(\bar{\alpha}-\alpha^{-1}\right)\left(K_{n-1}(z,\bar{\alpha}^{-1}), z^n\right)_\mu\right) \\
&= \mathbf{k}_n - \tilde{P}_{n+1}(\alpha)\left(Q_n(\alpha) + \boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{P_n(\bar{\alpha}^{-1})}\right).
\end{aligned}$$

From Corollary 6, we get

$$\kappa_n(\mu)\left(q_n(\alpha)+\boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{\varphi_n(\bar{\alpha}^{-1})}\right)\tilde{P}_{n+1}(\alpha)=\frac{\left|q_n(\alpha)+\boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{\varphi_n(\bar{\alpha}^{-1})}\right|^2}{\varepsilon_{n-1}(\alpha)}.$$

Hence, for $n \geqslant 1$,

$$\tilde{P}_{n+1}(z) = (z-\alpha)P_n(z) + \frac{\overline{A}_n}{\varepsilon_{n-1}(\alpha)}\left(1 + (z-\alpha)\sum_{j=0}^{n-1}\frac{A_j}{\mathbf{k}_j}P_j(z)\right), \tag{27}$$

where $A_j = Q_j(\alpha) + \boldsymbol{m}(\bar{\alpha}-\alpha^{-1})\overline{P_j(\bar{\alpha}^{-1})},\ j = 0, 1, \ldots, n.$

Notice that $\tilde{P}_1(z) - \tilde{P}_1(\alpha) = z - \alpha$ and (27) also holds for $n = 0$, with the convention $\sum_{j=0}^{-1}\frac{A_j}{\mathbf{k}_j}P_j(z) = 0$.

4 Hessenberg matrices and Geronimus transforms

Proposition 7 *The sequences* $\{\tilde{P}_n\}_{n\geqslant 0}$ *and* $\{P_n\}_{n\geqslant 0}$ *satisfy*

$$(z-\alpha)P(z) = \mathbf{M}\tilde{P}(z), \tag{28}$$

where $P(z) = [P_0(z), P_1(z), \ldots]^t$, $\tilde{P}(z) = [\tilde{P}_0(z), \tilde{P}_1(z), \ldots]^t$, *and* $\mathbf{M}$ *is a lower Hessenberg matrix with entries*

$$m_{k,j} = \begin{cases} 1 & \text{if } j = k+1, \\ -\dfrac{\kappa_{j-1}(\mu)^2\overline{A}_k A_{j-1}}{\varepsilon_{j-1}(\alpha)} & \text{if } 1 \leqslant j \leqslant k, \\ -\dfrac{\overline{A}_k}{\|\tilde{\mu}\|} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We take

$$\tilde{P}_{n+1}(z) = (z-\alpha)P_n(z) + B_n\left[1 + (z-\alpha)\sum_{j=0}^{n-1}\frac{A_j}{\mathbf{k}_j}P_j(z)\right],$$

where $B_n = \dfrac{\bar{A}_n}{\varepsilon_{n-1}(\alpha)}$, for $n \geqslant 0$. Therefore, for $n = 0$, we have

$$\tilde{P}_1(z) = (z-\alpha)P_0(z) + B_0$$
$$\tilde{P}_1(z) - B_0\tilde{P}_0(z) = (z-\alpha)P_0(z).$$

For $n \geqslant 1$, we substract $B_n\tilde{P}_n(z)$ from $B_{n-1}\tilde{P}_{n+1}(z)$ and get

$$B_{n-1}\tilde{P}_{n+1}(z) - B_n\tilde{P}_n(z) = (z-\alpha)\left[B_{n-1}P_n(z) + \left(B_nB_{n-1}\frac{A_{n-1}}{\mathbf{k}_{n-1}} - B_n\right)P_{n-1}(z)\right].$$

Dividing by B_{n-1}, we finally obtain

$$\tilde{P}_{n+1}(z) - \frac{B_n}{B_{n-1}}\tilde{P}_n(z) = (z-\alpha)\left[P_n(z) - \frac{\overline{A}_n}{\overline{A}_{n-1}}P_{n-1}(z)\right].$$

In matrix form

$$(z-\alpha)\widehat{\mathbf{M}}P(z) = \widetilde{\mathbf{M}}\tilde{P}(z), \tag{29}$$

where $\widetilde{\mathbf{M}}$ and $\widehat{\mathbf{M}}$ are upper and lower bidiagonal matrices , respectively, with the following entries

$$\tilde{m}_{k,j} = \begin{cases} 1 & \text{if } j = k+1, \\ -B_0 & \text{if } j = k = 0, \\ -\frac{B_k}{B_{k-1}} & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases} \qquad \widehat{m}_{k,j} = \begin{cases} 1 & \text{if } j = k, \\ -\frac{\overline{A}_k}{\overline{A}_{k-1}} & \text{if } j = k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account that $\widehat{\mathbf{M}}$ is nonsingular, we can write

$$(z-\alpha)P(z) = \widehat{\mathbf{M}}^{-1}\widetilde{\mathbf{M}}\tilde{P}(z),$$

and, therefore,

$$\mathbf{M} = \widehat{\mathbf{M}}^{-1}\widetilde{\mathbf{M}}.$$

On the other hand, the entries of $\widehat{\mathbf{M}}^{-1}$are

$$\widehat{m}_{k,j} = \begin{cases} 1 & \text{if } j = k, \\ \frac{\overline{A}_k}{\overline{A}_j} & \text{if } j \leqslant k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Multiplying $\widehat{\mathbf{M}}^{-1}\widetilde{\mathbf{M}}$ the result follows. ■

Proposition 8 *The matrix* $\mathbf{M}$ *satisfies*

(*i*) $\mathbf{M}\widetilde{\mathbf{D}}\mathbf{M}^{*} = \mathbf{D}$.

(ii) $\mathbf{M}^*\mathbf{D}^{-1}\mathbf{M} = \widetilde{\mathbf{D}}^{-1} - \varepsilon_\infty(\alpha)\boldsymbol{b}\boldsymbol{b}^*$.

with $\mathbf{D} = \left(P, P^T\right)_\mu$, $\widetilde{\mathbf{D}} = \left(\tilde{P}, \tilde{P}^T\right)_{\tilde{\mu}}$, $\varepsilon_\infty(\alpha) = \lim_{n\to\infty} \varepsilon_n(\alpha)$, $\boldsymbol{b} = \left[\frac{1}{\|\tilde{\mu}\|}, \frac{\kappa_0(\mu)a_0}{\varepsilon_0(\alpha)}, \frac{\kappa_1(\mu)a_1}{\varepsilon_1(\alpha)}, \ldots\right]^t$, $a_j = \kappa_j(\mu)A_j$, *and the convention* $\mathbf{k}_{-1}(\mu) = A_{-1} = 1$.

Proof.

(i) We have

$$\mathbf{M}\widetilde{\mathbf{D}}\mathbf{M}^* = \mathbf{M}\left(\tilde{P}, \tilde{P}^T\right)_{\tilde{\mu}}\mathbf{M}^* = \left(\mathbf{M}\tilde{P}, (\mathbf{M}\tilde{P})^T\right)_{\tilde{\mu}}$$
$$= \left((z-\alpha)P, (z-\alpha)P^T\right)_{\tilde{\mu}} = \left(P, P^T\right)_\mu = \mathbf{D}.$$

(ii) Direct multiplication of $\mathbf{M}^*\mathbf{D}^{-1}\mathbf{M}$ yields

$$\left(\mathbf{M}^*\mathbf{D}^{-1}\mathbf{M}\right)_{0,0} = \frac{\|B_0\|^2}{\mathbf{k}_0} + \frac{1}{\|\tilde{\mu}\|^2}\sum_{l=1}^{\infty}\frac{|A_l|^2}{\mathbf{k}_l} = \frac{|a_0|^2}{\|\tilde{\mu}\|^2} + \frac{1}{\|\tilde{\mu}\|^2}\sum_{l=1}^{\infty}|a_l|^2 .$$
$$= \frac{1}{\tilde{\mathbf{k}}_0} - \varepsilon_\infty(\alpha)\frac{1}{(\varepsilon_{-1}(\alpha))^2}.$$

For the remaining diagonal entries,

$$\left(\mathbf{M}^*\mathbf{D}^{-1}\mathbf{M}\right)_{j,j} = \frac{1}{\mathbf{k}_{j-1}} + \frac{\kappa_{j-1}(\mu)^4\left|A_{j-1}\right|^2}{\varepsilon^2_{j-1}(\alpha)}\sum_{l=j}^{\infty}\frac{|A_l|^2}{\mathbf{k}_l}$$
$$= \frac{1}{\mathbf{k}_{j-1}}\left(1 + \frac{\left|a_{j-1}\right|^2}{\varepsilon_{j-1}(\alpha)}\right) - \varepsilon_\infty(\alpha)\frac{\kappa_{j-1}(\mu)^2\left|a_{j-1}\right|^2}{\varepsilon^2_{j-1}(\alpha)}$$
$$= \frac{1}{\tilde{\mathbf{k}}_j} - \varepsilon_\infty(\alpha)\frac{\kappa_{j-1}(\mu)^2\left|a_{j-1}\right|^2}{\varepsilon^2_{j-1}(\alpha)}.$$

Finally, for the non-diagonal entries, we have

$$\left(\mathbf{M}^*\mathbf{D}^{-1}\mathbf{M}\right)_{k,j} = -\frac{\kappa_{k-1}(\mu)^2\overline{A}_{k-1}A_{j-1}}{\mathbf{k}_{j-1}\varepsilon_{k-1}(\alpha)} + \frac{\kappa_{k-1}(\mu)^2\kappa_{j-1}(\mu)^2\overline{A}_{k-1}A_{j-1}}{\varepsilon_{k-1}(\alpha)\varepsilon_{j-1}(\alpha)}\sum_{l=j}^{\infty}\frac{|A_l|^2}{\mathbf{k}_l}$$
$$= \kappa_{k-1}(\mu)\kappa_{j-1}(\mu)\overline{a}_{k-1}a_{j-1}\left(-\frac{1}{\varepsilon_{k-1}(\alpha)} + \frac{\varepsilon_{j-1}(\alpha) - \varepsilon_\infty(\alpha)}{\varepsilon_{k-1}(\alpha)\varepsilon_{j-1}(\alpha)}\right)$$
$$= -\varepsilon_\infty(\alpha)\frac{\kappa_{k-1}(\mu)\kappa_{j-1}(\mu)\overline{a}_{k-1}a_{j-1}}{\varepsilon_{k-1}(\alpha)\varepsilon_{j-1}(\alpha)}.$$

■

Proposition 9 *Let* $\mathbf{M_n}$, $\mathbf{M_n^*}$, $\mathbf{D}_n$, *and* $\widetilde{\mathbf{D}}_n$ *be the* $(n+1)\times(n+1)$ *leading principal submatrices of* $\mathbf{M}$, $\mathbf{M}^*$, $\mathbf{D}$, *and* $\widetilde{\mathbf{D}}$, *respectively, and consider* $\boldsymbol{b}_n = \left[\frac{1}{\|\tilde{\mu}\|}, \frac{\kappa_0(\mu)a_0}{\varepsilon_0(\alpha)}, \ldots, \frac{\kappa_{n-1}(\mu)a_{n-1}}{\varepsilon_{n-1}(\alpha)}\right]^t$.

Then

(*i*) $\mathbf{M_n}\widetilde{\mathbf{D}}_\mathbf{n}\mathbf{M_n^*}= \mathbf{D}_n - \frac{\varepsilon_n(\alpha)}{\varepsilon_{n-1}(\alpha)}\mathbf{k}_n e_{n+1}e_{n+1}^t$, *where* $e_{n+1} = [0, \ldots, 0, 1] \in \mathbb{R}^{n+1}$.
(*ii*) $\mathbf{M_n^*}\mathbf{D_n^{-1}}\mathbf{M_n}=\widetilde{\mathbf{D}}_\mathbf{n}^{-1} - \varepsilon_n(\alpha)\boldsymbol{b}_n\boldsymbol{b}_n^*$.

Proof.

(*i*) For $k = 0, j \geqslant 1$, we have

$$\left(\mathbf{M}_n\widetilde{\mathbf{D}}_n\mathbf{M}_n^*\right)_{0,j} = \frac{B_0 A_j \tilde{\mathbf{k}}_0}{||\tilde{\mu}||} - \frac{\kappa_0(\mu)^2 A_j \overline{A}_0 \tilde{\mathbf{k}}_1}{\varepsilon_0(\alpha)}.$$

Taking into account Corollary 6, we get

$$\left(\mathbf{M}_n\widetilde{\mathbf{D}}_n\mathbf{M}_n^*\right)_{0,j} = A_j\overline{A}_0\left(\frac{1}{||\tilde{\mu}||} - \frac{\kappa_0(\mu)^2\varepsilon_0(\alpha)\mathbf{k}_0}{\varepsilon_0(\alpha)\varepsilon_{-1}(\alpha)}\right) = 0.$$

For the remaining non diagonal entries, $k \neq j$,

$$\begin{aligned}\left(\mathbf{M}_n\widetilde{\mathbf{D}}_n\mathbf{M}_n^*\right)_{k,j} &= \frac{\overline{A}_k A_j \tilde{\mathbf{k}}_0}{||\tilde{\mu}||^2} + \sum_{l=0}^{k-1}\frac{\kappa_l(\mu)^4 A_k\overline{A}_j\,|A_l|^2\,\tilde{\mathbf{k}}_{l+1}}{\varepsilon_l(\alpha)^2} - \frac{\kappa_k(\mu)^2\overline{A}_k A_j}{\varepsilon_k(\alpha)}\\ &= \overline{A}_k A_j\left(\frac{1}{||\tilde{\mu}||} + \sum_{l=0}^{k-1}\left(\frac{1}{\varepsilon_l(\alpha)} - \frac{1}{\varepsilon_{l-1}(\alpha)}\right) - \frac{1}{\varepsilon_{k-1}(\alpha)}\right) = 0.\end{aligned}$$

For the diagonal entries, $0 \leqslant j \leqslant n-1$,

$$\left(\mathbf{M}_n\widetilde{\mathbf{D}}_n\mathbf{M}_n^*\right)_{j,j} = \frac{\left|A_j\right|^2\tilde{\mathbf{k}}_0}{||\tilde{\mu}||^2} + \sum_{l=0}^{j-1}\frac{\kappa_l(\mu)^4\left|A_j\right|^2|A_l|^2\,\tilde{\mathbf{k}}_{l+1}}{\varepsilon_l(\alpha)^2} + \tilde{\mathbf{k}}_{j+1} \tag{30}$$

$$\begin{aligned}&= \left|A_j\right|^2\left(\frac{1}{||\tilde{\mu}||} + \sum_{l=0}^{k-1}\left(\frac{1}{\varepsilon_l(\alpha)} - \frac{1}{\varepsilon_{l-1}(\alpha)}\right)\right) + \frac{\mathbf{k}_j\varepsilon_j(\alpha)}{\varepsilon_{j-1}(\alpha)}\\ &= \left|A_j\right|^2\frac{1}{\varepsilon_{j-1}(\alpha)} + \frac{\mathbf{k}_j\varepsilon_j(\alpha)}{\varepsilon_{j-1}(\alpha)} = \mathbf{k}_j.\end{aligned} \tag{31}$$

Finally, for the last diagonal entry, $j = n$, notice that we get the same result as (30), up to the term $\tilde{\mathbf{k}}_{n+1}$, which is no longer than entry of $\mathbf{M}_n$. Therefore

$$\left(\mathbf{M}_n\widetilde{\mathbf{D}}_n\mathbf{M}_n^*\right)_{n,n} = \mathbf{k}_n - \tilde{\mathbf{k}}_{n+1} = \mathbf{k}_n - \frac{\varepsilon_n(\alpha)}{\varepsilon_{n-1}(\alpha)}\mathbf{k}_n.$$

(*ii*) In this case, we need the same calculations as in the proof of Proposition 8(*ii*), up to the fact that the sum is finite. Thus, we replace $\varepsilon_\infty(\alpha)$ by $\varepsilon_n(\alpha)$ in those calculations to get the result.

■

We will denote by $\mathbf{H}_P$ the lower Hessenberg matrix such that $zP(z) = \mathbf{H}_P P(z)$, i. e. $\mathbf{H}_P$ is the matrix associated with $\{P_n\}_{n\geqslant 0}$ for the multiplication operator $(hp)(z) = zp(z),\ p \in \mathbb{P}$. We now establish the relation between the matrices $\mathbf{H}_P$ and $\mathbf{H}_{\tilde{P}}$.

Proposition 10 *Let* $\mathbf{L}$ *be the lower triangular matrix with* 1 *as diagonal entries such that* $\tilde{P}(z) = \mathbf{L}P(z)$. *Then*

$$\mathbf{H}_P - \alpha\mathbf{I} = \mathbf{ML} \tag{32}$$

and

$$\mathbf{H}_{\tilde{P}} - \alpha\mathbf{I} = \mathbf{LM}. \tag{33}$$

Proof. From (28), we have

$$(\mathbf{H}_P - \alpha\mathbf{I})\, P(z) = (z-\alpha)P(z) = \mathbf{M}\tilde{P}(z) = \mathbf{ML}P(z).$$

In a similar way, we get

$$(\mathbf{H}_{\tilde{P}} - \alpha\mathbf{I})\, \tilde{P}(z) = (z-\alpha)\tilde{P}(z) = (z-\alpha)\mathbf{L}P(z) = \mathbf{LM}\tilde{P}(z).$$

■

From (29),

$$\widehat{\mathbf{M}}\,(\mathbf{H}_P - \alpha\mathbf{I}) = \widetilde{\mathbf{M}}\mathbf{L}. \tag{34}$$

As a consequence, the entries $l_{k,j},\ 0 \leqslant j \leqslant k,\ k = 1, 2, \ldots$ of $\mathbf{L}$ are given by

$$l_{k+1,j} = B_k \sum_{r=j-1}^{k} \frac{\hat{h}_{r,j}}{B_j}, \tag{35}$$

where $\hat{h}_{r,j}$ are the entries of the matrix $\hat{\mathbf{H}} = \widehat{\mathbf{M}}\,(\mathbf{H}_P - \alpha\mathbf{I})$. See also [7] where the *QR* algorithm for unitary Hessenberg matrices is studied.

5 Example

Next we will analyze the Geronimus transform $\tilde{\mu}$ of the Lebesgue measure, i. e., the bilinear functional

$$(p,q)_{\tilde{\mu}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{p(e^{i\theta})\overline{q(e^{i\theta})}}{|e^{i\theta}-\alpha|^2}d\theta + \boldsymbol{m}p(\alpha)\overline{q(\bar{\alpha}^{-1})} + \overline{\boldsymbol{m}}p(\bar{\alpha}^{-1})\overline{q(\alpha)}, |\alpha| > 1, \boldsymbol{m} \in \mathbb{C},$$

and we will determine the condition for $(\cdot,\cdot)_{\tilde{\mu}}$ to be quasi-definite (respectively, positive definite).

In the positive definite case, we need $\|\tilde{\mu}\| - \sum_{n=0}^{n-1} \left| q_j(\alpha) + \boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\overline{\varphi_j(\bar{\alpha}^{-1})} \right|^2 > 0$, i.e., $\|\tilde{\mu}\| > \sum_{n=0}^{n-1} \left| q_j(\alpha) + \boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\overline{\varphi_j(\bar{\alpha}^{-1})} \right|^2$.

But

$$\|\tilde{\mu}\| = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|z-\alpha|^2} + \boldsymbol{m} + \overline{\boldsymbol{m}} = \frac{1}{|\alpha|^2 - 1} + \boldsymbol{m} + \overline{\boldsymbol{m}}.$$

Thus, the bilinear functional is positive definite if and only if $\frac{1}{|\alpha|^2-1} + \boldsymbol{m} + \overline{\boldsymbol{m}} > \sum_{j=0}^{n-1} \left| a_j \right|^2$ for all $n \geqslant 0$ or, equivalently, $\varepsilon_\infty(\alpha) \geqslant 0$. This means

$$0 \geqslant |\boldsymbol{m}|^2 (|\alpha|^2 - 1),$$

and we get a positive definite case only if $\boldsymbol{m} = 0$. Now, we consider the quasi-definite case.

We need $\varepsilon_{n-1}(\alpha) \neq 0$ or, equivalently, $\|\tilde{\mu}\| \neq \sum_{n=0}^{n-1} \left| q_j(\alpha) + \boldsymbol{m}(\bar{\alpha} - \alpha^{-1})\overline{\varphi_j(\bar{\alpha}^{-1})} \right|^2$ for every $n \geqslant 1$.

Indeed, if

$$\begin{aligned} \frac{1}{|\alpha|^2 - 1} + \boldsymbol{m} + \overline{\boldsymbol{m}} &= \left| 1 + \boldsymbol{m}(|\alpha|^2 - 1) \right|^2 \frac{1}{|\alpha|^2} \sum_{n=0}^{n-1} \frac{1}{|\alpha|^{2n}} \\ &= \left(\frac{1}{|\alpha|^2 - 1} \boldsymbol{m} + \overline{\boldsymbol{m}} + |\boldsymbol{m}|^2 (|\alpha|^2 - 1)^2 \right) \frac{|\alpha|^{2n} - 1}{|\alpha|^{2n}}, \end{aligned}$$

then

$$\left| 1 + \boldsymbol{m}(|\alpha|^2 - 1) \right| = |\boldsymbol{m}| (|\alpha|^2 - 1) |\alpha|^n, \quad \text{for every } n \geqslant 1.$$

Thus, for a fixed α with $|\alpha| > 1$ we get a quasi-definite case if $\boldsymbol{m}$ satisfies

$$\frac{\ln \frac{\left|1 + \boldsymbol{m}(|\alpha|^2 - 1)\right|}{|\boldsymbol{m}|(|\alpha|^2 - 1)}}{\ln |\alpha|} \notin \mathbb{N}.$$

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