Linear Spectral Transformations and Laurent Polynomials

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Abstract. In this manuscript we analyze some linear spectral transformations of a Hermitian linear functional using the multiplication by some class of Laurent polynomials. We focus our attention in the behavior of the Verblunsky parameters of the perturbed linear functional. Some illustrative examples are pointed out.

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1. Introduction and preliminary results

Let $\mathcal{L}$ be a linear functional in the linear space $\Lambda = \text{span}\,\{z^n\}_{n\in\mathbb{Z}}$ of the Laurent polynomials such that $\mathcal{L}$ is Hermitian, i.e. $c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \bar{c}_{-n}$, $n \in \mathbb{Z}$. Then, we can introduce a bilinear functional associated with $\mathcal{L}$ in the linear space $\mathbb{P}$ of polynomials with complex coefficients as follows (see [7],[12])

$$\langle p(z), q(z) \rangle_{\mathcal{L}} = \left\langle \mathcal{L}, p(z)\bar{q}(z^{-1}) \right\rangle \tag{1.1}$$

where $p, q \in \mathbb{P}$.

In terms of the canonical basis $\{z^n\}_{n\geqslant 0}$ of $\mathbb{P}$, the Gram matrix associated with this bilinear functional is

$$\mathbf{T} = \begin{bmatrix} c_0 & c_{-1} & \cdots & c_{-n} & \cdots \\ c_1 & c_0 & \cdots & c_{-(n-1)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_n & c_{n-1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix} \tag{1.2}$$

i.e., a Toeplitz matrix [10].

The linear functional is said to be quasi-definite if the principal leading submatrices of $\mathbf{T}$ are non-singular. In this case, a unique sequence of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ such that

$$\langle \Phi_n, \Phi_m \rangle_{\mathcal{L}} = \mathbf{k}_n \delta_{n,m}, \tag{1.3}$$

can be introduced, where $\mathbf{k}_n \neq 0$ for every $n \geqslant 0$. It is said to be the monic orthogonal polynomial sequence associated with $\mathcal{L}$.

These polynomials satisfy the following recurrence relations (see [7], [10], [17], [19])

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \qquad n = 0, 1, 2, \dots \tag{1.4}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) + \overline{\Phi_{n+1}(0)} z\Phi_n(z), \qquad n = 0, 1, 2, \dots \tag{1.5}$$

Here $\Phi_n^*(z) = z^n \overline{\Phi}_n(1/z)$ is the reversed polynomial associated with $\Phi_n(z)$ (see [17]), and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$, with $|\Phi_n(0)| \neq 1, n \geqslant 1$, are called reflection (or Verblunsky) parameters.

$K_n(z, y)$, the n-th reproducing kernel associated with $\{\Phi_n\}_{n\geqslant 0}$, is defined by

$$K_n(x, y) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}\Phi_j(x)}{\mathbf{k}_j} = \frac{\overline{\Phi_{n+1}^*(y)}\Phi_{n+1}^*(x) - \overline{\Phi_{n+1}(y)}\Phi_{n+1}(x)}{\mathbf{k}_{n+1}(1 - \bar{y}x)}.$$

Notice that

$$\Phi_n^*(z) = \mathbf{k}_n K_n(z, 0).$$

Moreover,

$$K_n^*(x, y) := \frac{1}{\mathbf{k}_{n+1}} \frac{\Phi_{n+1}(x)\Phi_{n+1}^*(y) - \Phi_{n+1}^*(x)\Phi_{n+1}(y)}{x - y},$$

i.e. this is the Bézoutian of Φ_{n+1}, Φ_{n+1}^* up to a constant factor.

On the other hand, from the recurrence relations we deduce

$$z\Phi_n(z) = \sum_{j=0}^{n+1} \lambda_{n,j}\Phi_j(z), \tag{1.6}$$

with

$$\lambda_{n,j} = \begin{cases} 1 & \text{if } j = n+1, \\ \frac{\mathbf{k}_n}{\mathbf{k}_j}\Phi_{n+1}(0)\overline{\Phi_j(0)} & \text{if } j \leqslant n, \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

(see [13],[17]). Thus, the matrix representation of the linear operator $h : \mathbb{P} \mapsto \mathbb{P}$, the multiplication by z, in terms of the basis $\{\Phi_n\}_{n\geqslant 0}$ is

$$z\Phi(z) = \mathbf{H}_\Phi \Phi(z),$$

where $\Phi(z) = [\Phi_0(z), \Phi_1(z), \dots, \Phi_n(z), \dots]^t$ and $\mathbf{H}_\Phi$ is a lower Hessenberg matrix with entries $\lambda_{n,k}$ defined in (1.7).

If the leading principal submatrices of $\mathbf{T}$ have a positive determinant, then the linear functional is said to be positive definite. Every positive definite linear functional has an integral representation

$$\langle \mathcal{L}, p(z) \rangle = \int_{\mathbb{T}} p(z) d\sigma(z), \tag{1.8}$$

where σ is a nontrivial probability Borel measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (see [7],[10],[12],[17]).

Then, there exists a sequence $\{\varphi_n\}_{n\geqslant 0}$ of orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + \dots, \quad \kappa_n > 0,$$

such that

$$\int_{-\pi}^{\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})} d\sigma(\theta) = \delta_{m,n}, \quad m, n \geqslant 0. \tag{1.9}$$

Notice that

$$\Phi_n(z) = \frac{\varphi_n(z)}{\kappa_n},$$

as well as $|\Phi_n(0)| < 1$ for every $n \geqslant 1$.

It is well known that if σ is a nontrivial probability measure supported on the unit circle, then there exists a unique sequence of Verblunsky parameters $\{\Phi_n(0)\}_{n\geqslant 1}$ associated with σ. The converse is also true, i.e., given a sequence of complex numbers $\{a_n\}_{n\geqslant 1}$, with $a_n \in \mathbb{D}$, there exists a nontrivial probability measure on the unit circle such that if $\{\Phi_n\}_{n\geqslant 0}$ is the corresponding sequence of monic orthogonal polynomials then $a_n = \Phi_n(0)$.

The family of Verblunsky parameters provides a quantitative information about the measure and the corresponding sequence of orthogonal polynomials.

The measure σ can be decomposed into a part that is absolutely continuous with respect to the Lebesgue measure $\frac{d\theta}{2\pi}$ and a singular measure. Thus, if $\omega = \sigma'$

$$d\sigma(\theta) = \omega(\theta)\frac{d\theta}{2\pi} + d\sigma_s(\theta).$$

Definition 1.1 ([17],[19]). Suppose the Szegő condition,

$$\int_{\mathbb{T}} \log(\omega(\theta))\frac{d\theta}{2\pi} > -\infty, \tag{1.10}$$

holds. Then, the Szegő function, D(z), is defined in $\mathbb{D}$ by

$$D(z) = \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(\omega(\theta))d\theta\right). \tag{1.11}$$

The Szegő condition (1.10) is equivalent to $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$. On the other hand, the measure σ is said to be of bounded variation if

$$\sum_{n=0}^{\infty} |\Phi_{n+1}(0) - \Phi_n(0)| < \infty$$

holds.

Finally, in terms of the moments $\{c_n\}_{n\geqslant 0}$ an analytic function

$$F(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n} z^n \tag{1.12}$$

can be introduced. If $\mathcal{L}$ is a positive definite linear functional, then $F(z)$ is analytic in the open unit disk and $\mathfrak{Re}\,(F(z)) > 0$ therein. In such a case $F(z)$ is said to be a Carathéodory function and it can be represented as a Riesz-Herglotz transform of the nontrivial probability measure σ introduced in (1.8) (see [7],[12],[17])

$$F(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\sigma(w).$$

Following are some perturbations of the measure σ, for which we have studied the behavior of the corresponding Carathéodory functions (see [15]) as well as the Hessenberg matrices associated with the corresponding sequence of orthogonal polynomials in three cases

- *(i)* If $d\tilde{\sigma} = |z-\alpha|^2 d\sigma$, $|z| = 1$, then the so-called canonical Christoffel transformation appears. In [4] and [16] we have studied the connection between the associated Hessenberg matrices using the QR factorization. The iteration of the canonical Christoffel transformation has been analyzed in [8],[11], and [14]. See also [2] for a more general framework.
- *(ii)* If $d\tilde{\sigma} = d\sigma + \boldsymbol{m}\delta(z-\alpha)$, $|\alpha| = 1$, $\boldsymbol{m} \in \mathbb{R}_+$, then the so-called canonical Uvarov transformation appears. In [4] and [15] we have studied the connection between the corresponding sequences of monic orthogonal polynomials as well as the associated Hessenberg matrices using the LU and QR factorization. The iteration of the canonical Uvarov transformation has been studied in [7] and [14]. Asymptotic properties for the corresponding sequences of orthogonal polynomials have been studied in [20].
- *(iii)* If $d\tilde{\sigma} = \frac{1}{|z-\alpha|^2} d\sigma + \boldsymbol{m}\delta(z-\alpha) + \bar{\boldsymbol{m}}\delta(z-\bar{\alpha}^{-1})$, $|z| = 1$, $\boldsymbol{m} \in \mathbb{C}$, and $|\alpha| \neq 1$, then a special case of the Geronimus transform has been analyzed in [5]. In particular, the relation between the corresponding sequences of monic orthogonal polynomials and the associated Hessenberg matrices is stated. A more general framework is presented in [9].

Notice that the above transformations constitute the analogue on the unit circle of the canonical linear spectral transforms on the real line (see [1] and [21]).

The aim of our contribution is to analyze a new perturbation of a Hermitian linear functional $\mathcal{L}$, such that the Christoffel transformation is a particular case.

In Section 2, we introduce two perturbations of $\mathcal{L}$ by using the left multiplication by the real and imaginary part of a complex polynomial, respectively. Necessary and sufficient conditions on order to preserve the quasi-definiteness of a linear functional under such perturbations are known in the literature (see [3] and [18]). We also determine the relation between the associated Hessenberg matrices. In Section 3 we prove that these perturbations belong to the family of linear spectral transformations. In Section 4, we deduce the explicit expression for the

Verblunsky parameters associated with these perturbations and, as a consequence, the invariance of the Szegő class of bounded variation measures follows. Finally, in Section 5 we show some illustrative examples.

2. Transformations $\mathcal{L}_R = \frac{1}{2}\mathfrak{Re}[P_n(z)]\mathcal{L}$ and $\mathcal{L}_I = \frac{1}{2i}\mathfrak{Im}[P_n(z)]\mathcal{L}$

Let consider the following transformations of a hermitian linear functional $\mathcal{L}$.

Definition 2.1. Given a hermitian linear functional $\mathcal{L}$ and a monic polynomial $P_n(z) = \sum_{i=0}^{n} \alpha_i z^i$, $\alpha_n = 1$, we will denote by $\mathcal{L}_R$ and $\mathcal{L}_I$ the linear functionals such that

(i) $\langle \mathcal{L}_R, q\rangle = \left\langle \mathcal{L}, \frac{1}{2}(P_n(z) + \bar{P}_n(z^{-1}))q \right\rangle$,
(ii) $\langle \mathcal{L}_I, q\rangle = \left\langle \mathcal{L}, \frac{1}{2i}(P_n(z) - \bar{P}_n(z^{-1}))q \right\rangle$.

Notice that $\mathcal{L}_R$ and $\mathcal{L}_I$ are also hermitian. If $\mathcal{L}$ is quasi-definite, necessary and sufficient conditions for $\mathcal{L}_R$ and $\mathcal{L}_I$ to be also quasi-definite have been studied in [3] and [18], when $P_1(z) = z - \alpha$. Moreover, explicit expressions for the sequences of monic polynomials orthogonal with respect to $\mathcal{L}_R$ and $\mathcal{L}_I$ in terms of $\{\Phi_n\}_{n\geqslant 0}$ are also shown. Indeed,

Proposition 2.2 ([18]).

(i) *If $|\mathfrak{Re}(\alpha)| \neq 1$, and b_1, b_2 are the zeros of the polynomial $z^2 - (\alpha + \bar{\alpha})z + 1$, then $\mathcal{L}_R$ is quasi-definite if and only if $K_n^*(b_1, b_2) \neq 0$, $n \geqslant 0$. In addition, if $\{Y_n\}_{n\geqslant 0}$ denotes the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_R$, then*

$$Y_{n-1}(z) = \frac{\Phi_n(z)K_{n-1}^*(b_1, b_2) - K_{n-1}^*(z, b_2)\Phi_n(b_1)}{K_{n-1}^*(b_1, b_2)(z - b_1)}, \quad n \geqslant 1, \tag{2.1}$$

and

$$Y_{n-1}(0) = \frac{\Phi_n(b_1)\Phi_{n-1}(b_2) - \Phi_n(b_2)\Phi_{n-1}(b_1)}{K_{n-1}^*(b_1, b_2)(b_1 - b_2)\mathbf{k}_{n-1}}, \quad n \geqslant 1. \tag{2.2}$$

(ii) *If $|\mathfrak{Re}(\alpha)| = 1$, and b is the double zero of the polynomial $z^2 - (\alpha + \bar{\alpha})z + 1$, then $\mathcal{L}_R$ is quasi-definite if and only if $K_n^*(b, b) \neq 0$, $n \geqslant 0$. In addition,*

$$Y_{n-1}(z) = \frac{\Phi_n(z)K_{n-1}^*(b, b) - K_{n-1}^*(z, b)\Phi_n(b)}{K_{n-1}^*(b, b)(z - b)}, \quad n \geqslant 1, \tag{2.3}$$

and

$$Y_{n-1}(0) = -b\frac{\Phi_n(0)K_{n-1}^*(b, b)\mathbf{k}_{n-1} - \Phi_{n-1}(b)\Phi_n(b)}{K_{n-1}^*(b, b)\mathbf{k}_{n-1}}, \quad n \geqslant 1. \tag{2.4}$$

Proposition 2.3 ([18]).

(i) If $|\mathfrak{Im}(\alpha)| \neq 1$, and $\tilde{b}_1, \tilde{b}_2$ are the zeros of the equation $z^2 + (\bar{\alpha} - \alpha)z - 1$, then $\mathcal{L}_I$ is quasi-definite if and only if $K_n^*(\tilde{b}_1, \tilde{b}_2) \neq 0$, $n \geqslant 0$. In addition, if $\{y_n\}_{n\geqslant 0}$ denotes the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_I$, then

$$y_{n-1}(z) = \frac{\Phi_n(z)K_{n-1}^*(\tilde{b}_1, \tilde{b}_2) - K_{n-1}^*(z, \tilde{b}_2)\Phi_n(\tilde{b}_1)}{K_{n-1}^*(\tilde{b}_1, \tilde{b}_2)(z - \tilde{b}_1)}, \quad n \geqslant 1, \tag{2.5}$$

and

$$y_{n-1}(0) = \frac{\Phi_n(\tilde{b}_1)\Phi_{n-1}(\tilde{b}_2) - \Phi_n(\tilde{b}_2)\Phi_{n-1}(\tilde{b}_1)}{K_{n-1}^*(\tilde{b}_1, \tilde{b}_2)(\tilde{b}_1 - \tilde{b}_2)\mathbf{k}_{n-1}}, \quad n \geqslant 1. \tag{2.6}$$

(ii) If $|\mathfrak{Im}(\alpha)| = 1$, and $\tilde{b}$ is the double zero of the equation $z^2 + (\bar{\alpha} - \alpha)z - 1$, $\mathcal{L}_I$ is quasi-definite if and only if $K_n^*(\tilde{b}, \tilde{b}) \neq 0$, $n \geqslant 0$. In addition,

$$y_{n-1}(z) = \frac{\Phi_n(z)K_{n-1}^*(\tilde{b}, \tilde{b}) - K_{n-1}^*(z, \tilde{b})\Phi_n(\tilde{b})}{K_{n-1}^*(\tilde{b}, \tilde{b})(z - \tilde{b})}, \quad n \geqslant 1, \tag{2.7}$$

and

$$y_{n-1}(0) = \tilde{b}\frac{\Phi_n(0)K_{n-1}^*(\tilde{b}, \tilde{b})\mathbf{k}_{n-1} - \Phi_{n-1}(\tilde{b})\Phi_n(\tilde{b})}{K_{n-1}^*(\tilde{b}, \tilde{b})\mathbf{k}_{n-1}}, \quad n \geqslant 1. \tag{2.8}$$

Notice that, if $\alpha = a + ci$, then $b_1 = a + \sqrt{a^2 - 1}$ and $b_2 = b_1^{-1} = a - \sqrt{a^2 - 1}$, as well as $\tilde{b}_1 = \sqrt{1 - c^2} + ci$ and $\tilde{b}_2 = -\tilde{b}_1^{-1}$.

There is another equivalent condition for the quasi-definiteness of $\mathcal{L}_R$ and $\mathcal{L}_I$ and, as a consequence, an expression for the corresponding families of Verblunsky coefficients follows

Proposition 2.4 ([3]). *The linear functionals $\mathcal{L}_R$ and $\mathcal{L}_I$ are quasi-definite if and only if $\Pi_n(b_1) \neq 0$, $\Pi_n(\tilde{b}_1) \neq 0$, $n \geqslant 0$, respectively, where*

$$\Pi_n(x) = \begin{vmatrix} x\Phi_n(x) & \Phi_n^*(x) \\ x^{-1}\Phi_n(x^{-1}) & \Phi_n^*(x^{-1}) \end{vmatrix}.$$

Moreover, the families of Verblunsky parameters $\{Y_n(0)\}_{n\geqslant 1}$, $\{y_n(0)\}_{n\geqslant 1}$, associated with $\mathcal{L}_R$ and $\mathcal{L}_I$, respectively, are given by

$$Y_n(0) = (b_1 - b_1^{-1})\frac{\Phi_n(b_1)\Phi_n(b_1^{-1})}{\Pi_n(b_1)}, \qquad n \geqslant 1 \tag{2.9}$$

$$y_n(0) = (\tilde{b}_1 + \tilde{b}_1^{-1})\frac{\Phi_n(\tilde{b}_1)\Phi_n(-\tilde{b}_1^{-1})}{\Pi_n(\tilde{b}_1)}, \qquad n \geqslant 1. \tag{2.10}$$

Remark 2.5. We recover the Christoffel transformation when $|\mathfrak{Re}(\alpha)| \geqslant 1$. This transformation was studied in [4],[16].

Now, we study the relation between the Hessenberg matrix associated with $\mathcal{L}_R$, which will be denoted by $\mathbf{H}_Y$, and the Hessenberg matrix associated with $\mathcal{L}$.

Assume $|\alpha| \neq 1$. From (2.1),

$$\begin{aligned}(z-b_1)Y_n(z) &= \Phi_{n+1}(z) - \frac{\Phi_{n+1}(b_1)}{K_n^*(b_1,b_2)} K_n^*(z,b_2) \\ &= \Phi_{n+1}(z) - \frac{\Phi_{n+1}(b_1)}{K_n^*(b_1,b_2)} \left[\frac{1}{\mathbf{k}_n} \frac{z\Phi_n(z)\Phi_n^*(b_2) - b_2\Phi_n^*(z)\Phi_n(b_2)}{z-b_2} \right] \\ &= \Phi_{n+1}(z) - \frac{\Phi_{n+1}(b_1)}{\mathbf{k}_n K_n^*(b_1,b_2)} \left[\frac{\Phi_{n+1}(z)\Phi_n^*(b_2) - \Phi_{n+1}(b_2)\Phi_n^*(z)}{z-b_2} \right].\end{aligned}$$

Thus,

$$\begin{aligned}&(z-b_1)(z-b_2)Y_n(z) \\ &= (z-b_2)\Phi_{n+1}(z) - \frac{\Phi_{n+1}(b_1)\Phi_n^*(b_2)}{\mathbf{k}_n K_n^*(b_1,b_2)} \Phi_{n+1}(z) \\ &\quad + \frac{\Phi_{n+1}(b_1)\Phi_{n+1}(b_2)}{K_n^*(b_1,b_2)} \sum_{j=0}^{n} \frac{\overline{\Phi_j(0)}\Phi_j(z)}{\mathbf{k}_j} \\ &= z\Phi_{n+1}(z) - b_2\Phi_{n+1}(z) - \frac{\Phi_{n+1}(b_1)\Phi_n^*(b_2)}{\mathbf{k}_n K_n^*(b_1,b_2)} \Phi_{n+1}(z) \\ &\quad + \frac{\Phi_{n+1}(b_1)\Phi_{n+1}(b_2)}{K_n^*(b_1,b_2)} \sum_{j=0}^{n} \frac{\overline{\Phi_j(0)}\Phi_j(z)}{\mathbf{k}_j} \\ &= \Phi_{n+2}(z) - \Phi_{n+2}(0)\Phi_{n+1}^*(z) - b_2\Phi_{n+1}(z) \\ &\quad - \frac{\Phi_{n+1}(b_1)}{\mathbf{k}_n K_n^*(b_1,b_2)} \left(\Phi_n^*(b_2)\Phi_{n+1}(z) - \Phi_{n+1}(b_2) \sum_{j=0}^{n} \frac{\overline{\Phi_j(0)}\Phi_j(z)}{\mathbf{k}_j} \right) \\ &= \Phi_{n+2}(z) - \left(b_2 + \Phi_{n+2}(0)\overline{\Phi_{n+1}(0)} + \frac{\Phi_{n+1}(b_1)\Phi_n^*(b_2)}{\mathbf{k}_n K_n^*(b_1,b_2)} \right) \Phi_{n+1}(z) \\ &\quad + \left(\frac{\Phi_{n+1}(b_1)\Phi_{n+1}(b_2)}{K_n^*(b_1,b_2)} - \Phi_{n+2}(0)\mathbf{k}_{n+1} \right) \sum_{j=0}^{n} \frac{\overline{\Phi_j(0)}\Phi_j(z)}{\mathbf{k}_j}.\end{aligned}$$

In matrix form the above expression reads

$$(z-b_1)(z-b_2)Y(z) = \mathbf{M}_R \Phi(z), \tag{2.11}$$

where $\mathbf{M}_R$ is a matrix with entries

$$\tilde{m}_{n,j} = \begin{cases} 1 & \text{if } j = n+2, \\ b_2 + \Phi_{n+2}(0)\overline{\Phi_{n+1}(0)} + \frac{\Phi_{n+1}(b_1)\Phi_n^*(b_2)}{\mathbf{k}_n K_n^*(b_1,b_2)} & \text{if } j = n+1, \\ \left(\frac{\Phi_{n+1}(b_1)\Phi_{n+1}(b_2)}{K_n^*(b_1,b_2)} - \Phi_{n+2}(0)\mathbf{k}_{n+1} \right) \overline{\Phi_j(0)} & \text{if } j \leqslant n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

and $Y(z) = [Y_0(z), Y_1(z), \dots]^t$, $\Phi(z) = [\Phi_0(z), \Phi_1(z), \dots]^t$. Notice that (2.11) can also be written as

$$z[\mathfrak{Re}\{P_1(z)\}Y(z)] = \mathbf{M}_R\Phi(z). \tag{2.13}$$

On the other hand, we have $zY(z) = \mathbf{H}_Y Y(z)$, and then from (2.11) we get

$$(\mathbf{H}_Y - b_1\mathbf{I})(\mathbf{H}_Y - b_2\mathbf{I})Y(z) = \mathbf{M}_R\Phi(z) = \mathbf{M}_R\mathbf{L}_{Y\Phi}Y(z),$$

where $\mathbf{L}_{Y\Phi}$ is a lower triangular matrix such that $\Phi(z) = \mathbf{L}_{Y\Phi}Y_(z)$, i.e. a matrix of change of basis. Therefore,

$$(\mathbf{H}_Y - b_1\mathbf{I})(\mathbf{H}_Y - b_2\mathbf{I}) = \mathbf{M}_R\mathbf{L}_{Y\Phi}.$$

It is not so difficult to show that the entries of $\mathbf{L}_{Y\Phi}$ are

$$l_{n,j} = \begin{cases} 1 & \text{if } j = n, \\ -\frac{\mathbf{k}_n}{\tilde{\mathbf{k}}_{n-1}}(\Phi_{n+1}(0)\overline{Y_j(0)} - 1) & \text{if } j = n-1, \\ -\frac{\mathbf{k}_n}{\tilde{\mathbf{k}}_j}\Phi_{n+1}(0)\overline{Y_j(0)} & \text{if } j \leqslant n-2, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

where $Y_k(0)$ can be calculated using (2.9) and $\tilde{\mathbf{k}}_{n-1} = -\dfrac{\mathbf{k}_n K_n^*(b_1, b_2)}{2K_{n-1}^*(b_1, b_2)}$.

3. Carathéodory functions

Let σ be a nontrivial probability measure supported on the unit circle and consider the transformation $d\tilde{\sigma} = \mathfrak{Re}(P_n)d\sigma$, where P_n is some polynomial in z of degree n. If $F(z)$ is the Carathéodory function associated with σ, we want to find $F_R(z)$, the Carathéodory function associated with $\tilde{\sigma}$.

Proposition 3.1. *Let σ be a nontrivial probability Borel measure supported on the unit circle. Consider a perturbation to σ defined by $d\tilde{\sigma} = (\mathfrak{Re}P_n)d\sigma$, where P_n is a polynomial on z of degree n, i.e. $P_n(z) = z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_n$. Let $F(z)$ be the Carathéodory function associated with σ. Then, $F_R(z)$, the Carathéodory function associated with $\tilde{\sigma}$, is a linear spectral transformation of $F(z)$ given by*

$$F_R(z) = \frac{[P_n(z) + \overline{P}_n(1/z)]F(z) + Q_n(z) - \overline{Q}_n(1/z)}{2}$$

where $Q_n(z) = \displaystyle\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}[P_n(e^{i\theta}) - P_n(z)]d\sigma.$

Proof. We have

$$\begin{aligned}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}P_n(e^{i\theta})d\sigma &= \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}[P_n(e^{i\theta}) - P_n(z)]d\sigma + P_n(z)F(z) \\ &= P_n(z)F(z) + Q_n(z),\end{aligned}$$

where

$$Q_n(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}[P_n(e^{i\theta}) - P_n(z)]d\sigma.$$

On the other hand,

$$\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\overline{P_n(e^{i\theta})}d\sigma = \int_0^{2\pi} \frac{\frac{1}{z}e^{-i\theta}}{\frac{1}{z}-e^{-i\theta}}[\overline{P}_n(e^{-i\theta})-\overline{P}_n(1/z)]d\sigma + P_n(1/z)F(z)$$
$$= \overline{P}_n(1/z)F(z) - \overline{Q}_n(1/z).$$

Therefore

$$F_R(z) = \frac{[P_n(z)+\overline{P}_n(1/z)]F(z)+Q_n(z)-\overline{Q}_n(1/z)}{2}. \tag{3.1}$$

□

Remark 3.2. Notice that this result was proved in [3], where (3.1) was obtained using the relation between the moments associated with $\mathcal{L}$ and $\mathcal{L}_R$.

4. Verblunsky parameters

In the rest of the manuscript, we assume that σ is a probability measure of the Szegő class, i.e. (1.10) holds, as well as it is a measure of bounded variation.

Proposition 4.1. *The family of Verblunsky parameters* $\{Y_n(0)\}_{n\geqslant 1}$ *can be given in terms of the family* $\{\Phi_n(0)\}_{n\geqslant 1}$ *by*

$$Y_n(0) = A_n(b_1)\Phi_{n+1}(0) + B_n(b_1), \tag{4.1}$$

with

$$A_n(b_1) = \frac{\Phi_n(b_1^{-1})\Phi_n^*(b_1) - \Phi_n(b_1)\Phi_n^*(b_1^{-1})}{\Phi_{n+1}(b_1)\Phi_n^*(b_1^{-1}) - \Phi_{n+1}(b_1^{-1})\Phi_n^*(b_1)}, \tag{4.2}$$

$$B_n(b_1) = \frac{\Phi_{n+1}(b_1)\Phi_n(b_1^{-1}) - \Phi_{n+1}(b_1^{-1})\Phi_n(b_1)}{\Phi_{n+1}(b_1)\Phi_n^*(b_1^{-1}) - \Phi_{n+1}(b_1^{-1})\Phi_n^*(b_1)}. \tag{4.3}$$

Proof. From the recurrence relation and (2.9), we have

$$Y_n(0)$$
$$= \frac{[\Phi_{n+1}(b_1)-\Phi_{n+1}(0)\Phi_n^*(b_1)]\Phi_n(b_1^{-1}) - [\Phi_{n+1}(b_1^{-1})-\Phi_{n+1}(0)\Phi_n^*(b_1^{-1})]\Phi_n(b_1)}{[\Phi_{n+1}(b_1)-\Phi_{n+1}(0)\Phi_n^*(b_1)]\Phi_n^*(b_1^{-1}) - [\Phi_{n+1}(b_1^{-1})-\Phi_{n+1}(0)\Phi_n^*(b_1^{-1})]\Phi_n^*(b_1)},$$

and the result follows by a rearrangement of their terms. □

Now, we study the behavior of $A_n(b_1)$ and $B_n(b_1)$ when $n\to\infty$. For $|b_1|<1$, the division by $\Phi_{n+1}(b_1^{-1})$ in the numerator and denominator of $A_n(b_1)$ yields

$$A_n(b_1) = \frac{[\Phi_n(b_1^{-1})\Phi_n^*(b_1) - \Phi_n(b_1)\Phi_n^*(b_1^{-1})]\frac{1}{\Phi_{n+1}(b_1^{-1})}}{[\Phi_{n+1}(b_1)\Phi_n^*(b_1^{-1}) - \Phi_{n+1}(b_1^{-1})\Phi_n^*(b_1)]\frac{1}{\Phi_{n+1}(b_1^{-1})}},$$

Thus, when $n\to\infty$, we get

$$\lim_{n\to\infty} A_n(b_1) = -\lim_{n\to\infty}\frac{b_1\Phi_n^*(b_1)}{\Phi_n^*(b_1)} = -b_1,$$

since $\lim_{n\to\infty} \frac{\Phi_{n+1}(z)}{\Phi_n(z)} = z$, for $z \in \mathbb{C} \setminus \mathbb{D}$, and if $|b_1| < 1$, then

$$\lim_{n\to\infty} \frac{\Phi_n(b_1)\Phi_n^*(b_1^{-1})}{\Phi_{n+1}(b_1^{-1})} = \lim_{n\to\infty} \frac{\Phi_n(b_1)b_1^{-n}\overline{\Phi_n(\bar{b}_1)}}{\Phi_{n+1}(b_1^{-1})} = 0.$$

In a similar way, for $|b_1| > 1$, dividing by $\Phi_{n+1}(b_1)$ in the numerator and denominator of $A_n(b_1)$, we obtain $\lim_{n\to\infty} A_n(b_1) = -b_1^{-1}$.

On the other hand, if $|b_1| < 1$, we get

$$B_n(b_1) = \frac{[\Phi_{n+1}(b_1)\Phi_n(b_1^{-1}) - \Phi_{n+1}(b_1^{-1})\Phi_n(b_1)]/\Phi_n(b_1^{-1})}{[\Phi_{n+1}(b_1)\Phi_n^*(b_1^{-1}) - \Phi_{n+1}(b_1^{-1})\Phi_n^*(b_1)]/\Phi_n(b_1^{-1})}.$$

Notice than $\Phi_n(b_1^{-1})$ never vanishes if $|b_1| < 1$ and thus the denominator only vanishes on $b_1 = \pm 1$. When $n \to \infty$ the numerator of $B_n(b_1)$ becomes

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\Phi_{n+1}(b_1)\Phi_n(b_1^{-1}) - \Phi_{n+1}(b_1^{-1})\Phi_n(b_1)}{\Phi_n(b_1^{-1})} \\
&= \lim_{n\to\infty} \Phi_n(b_1) \left[\frac{\Phi_{n+1}(b_1)}{\Phi_n(b_1)} - \frac{\Phi_{n+1}(b_1^{-1})}{\Phi_n(b_1^{-1})} \right] \\
&= (b_1 - b_1^{-1}) \lim_{n\to\infty} \Phi_n(b_1) = 0.
\end{aligned}$$

For the denominator, we have

$$\lim_{n\to\infty} \frac{\Phi_{n+1}(b_1)\Phi_n^*(b_1^{-1}) - \Phi_{n+1}(b_1^{-1})\Phi_n^*(b_1)}{\Phi_n(b_1^{-1})} = \lim_{n\to\infty} [-b_1^{-1}\Phi_n^*(b_1)] = -b_1^{-1}.$$

In a similar way, when $|b_1| > 1$, dividing the numerator and denominator of $B_n(b_1)$ by $\Phi_n(b_1)$ and calculating the limit, we obtain the same result. Therefore, $\lim_{n\to\infty} B_n(b_1) = 0$ for all $b_1 \in \mathbb{R} \setminus 0$, except for $b_1 = \pm 1$. As a conclusion, we have the following result.

Proposition 4.2. *Suppose that* $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$ *and* $\sum_{n=0}^{\infty} |\Phi_{n+1}(0) - \Phi_n(0)| < \infty$. *Then, for* $|\alpha| \in \mathbb{C} \setminus \{0, 1, -1\}$,

(*i*) $\sum_{n=0}^{\infty} |Y_n(0)|^2 < \infty$.

(*ii*) $\sum_{n=0}^{\infty} |Y_{n+1}(0) - Y_n(0)| < \infty$.

Remark 4.3. If $b_1 = \pm 1$, then we get a result proved in [6].

5. Examples

5.1. A Bernstein-Szegő case

We study a spectral linear transformation of a Bernstein-Szegő measure given by $d\tilde{\sigma} = (z - \alpha + z^{-1} - \bar{\alpha})\frac{1-|\beta|^2}{|z-\beta|^2}\frac{d\theta}{2\pi}$, with $\alpha \in \mathbb{C} \setminus \{0, 1, -1\}$ and $|\beta| < 1$. It is well known that

$$\Phi_n(z) = z^n - \beta z^{n-1} \quad \text{and} \quad \Phi_n^*(z) = 1 - \bar{\beta} z, \qquad n \geqslant 1.$$

In this case, the condition for the existence of the sequence of monic orthogonal polynomials $\{Y_n\}_{n\geqslant 0}$ is

$$\begin{aligned} 0 &\neq b_1 \Phi_n(b_1)\Phi_n^*(b_1^{-1}) - b_1^{-1}\Phi_n(b_1^{-1})\Phi_n^*(b_1) \\ &= b_1^n(b_1 - \beta)(1 - \bar{\beta} b_1^{-1}) - b_1^{-n}(b_1^{-1} - \beta)(1 - \bar{\beta} b_1), \end{aligned}$$

and thus

$$b_1^{2n} \neq \frac{(b_1^{-1} - \beta)(1 - \bar{\beta} b_1)}{(b_1 - \beta)(1 - \bar{\beta} b_1^{-1})} = \frac{\Phi_1(b_1^{-1})\Phi_1^*(b_1)}{\Phi_1(b_1)\Phi_1^*(b_1^{-1})}.$$

So we have a quasi-definite case if and only if

$$\frac{\ln \frac{\Phi_1(b_1^{-1})\Phi_1^*(b_1)}{\Phi_1(b_1)\Phi_1^*(b_1^{-1})}}{2 \ln b_1} \notin \mathbb{N}.$$

If $\beta = 0$, i.e. a transformation of the Lebesgue measure, then the above condition becomes

$$b_1^{2n} \neq \frac{1}{b_1^2},$$

i.e.,

$$b_1 \neq e^{\frac{k\pi i}{n+1}}, \qquad 1 \leqslant k \leqslant n.$$

Next, we obtain the expression for the family of Verblunsky parameters associated with the perturbed linear functional. From (4.2),

$$\begin{aligned} A_n(b_1) &= \frac{b_1^{-n+1}(b_1^{-1} - \beta)(1 - \bar{\beta} b_1) - b_1^{n-1}(b_1 - \beta)(1 - \bar{\beta} b_1^{-1})}{b_1^n(b_1 - \beta)(1 - \bar{\beta} b_1^{-1}) - b_1^{-n}(b_1^{-1} - \beta)(1 - \bar{\beta} b_1)} \\ &= \frac{b_1^{-n+1}\Phi_1(b_1^{-1})\Phi_1^*(b_1) - b_1^{n-1}\Phi_1(b_1)\Phi_1^*(b_1^{-1})}{b_1^n\Phi_1(b_1)\Phi_1^*(b_1^{-1}) - b_1^{-n}\Phi_1(b_1^{-1})\Phi_1^*(b_1)} \\ &= \frac{b_1\Phi_1(b_1^{-1})\Phi_1^*(b_1) - b_1^{2n-1}\Phi_1(b_1)\Phi_1^*(b_1^{-1})}{b_1^{2n}\Phi_1(b_1)\Phi_1^*(b_1^{-1}) - \Phi_1(b_1^{-1})\Phi_1^*(b_1)}. \end{aligned}$$

Notice that

$$\begin{aligned} \lim_{n\to\infty} A_n(b_1) &= -b_1, \qquad |b_1| < 1, \\ \lim_{n\to\infty} A_n(b_1) &= -b_1^{-1}, \qquad |b_1| > 1. \end{aligned}$$

On the other hand, from (4.3),

$$\begin{aligned} B_n(b_1) &= \frac{b_1^n(b_1-\beta)b_1^{-n+1}(b_1^{-1}-\beta) - b_1^{-n}(b_1^{-1}-\beta)b_1^{n-1}(b_1-\beta)}{b_1^n(b_1-\beta)(1-\bar{\beta}b_1^{-1}) - b_1^{-n}(b_1^{-1}-\beta)(1-\bar{\beta}b_1)} \\ &= \frac{b_1\Phi_1(b_1)\Phi_1(b_1^{-1}) - b_1^{-1}\Phi_1(b_1)\Phi_1(b_1^{-1})}{b_1^n\Phi_1(b_1)\Phi_1^*(b_1^{-1}) - b_1^{-n}\Phi_1(b_1^{-1})\Phi_1^*(b_1)} \\ &= \frac{b_1^n(b_1-b_1^{-1})\Phi_1(b_1)\Phi_1(b_1^{-1})}{b_1^{2n}\Phi_1(b_1)\Phi_1^*(b_1^{-1}) - \Phi_1(b_1^{-1})\Phi_1^*(b_1)}. \end{aligned}$$

Therefore, for large n, if $|b_1| < 1$, then

$$Y_n(0) = A_n(b_1)\Phi_{n+1}(0) + B_n(b_1) \sim N_1(b_1)b_1^n,$$

with $N_1(b_1) = -\frac{(b_1-b_1^{-1})\Phi_1(b_1)\Phi_1(b_1^{-1})}{\Phi_1(b_1^{-1})\Phi_1^*(b_1)}$.

If $|b_1| > 1$, then

$$Y_n(0) \sim N_2(b_1)b_1^{-n},$$

with $N_2(b_1) = \frac{(b_1-b_1^{-1})\Phi_1(b_1)\Phi_1(b_1^{-1})}{\Phi_1(b_1)\Phi_1^*(b_1^{1})}$.

Finally, for $\beta = 0$,

$$\begin{aligned} A_n(b_1) &= \frac{1-b_1^{2n}}{b_1^{2n+1}-b_1^{-1}}, \\ B_n(b_1) &= \frac{b_1^n(b_1-b_1^{-1})}{b_1^{2n+1}-b_1^{-1}}. \end{aligned}$$

So, in this case we get the following asymptotic behavior for the Verblunsky parameters

$$\begin{aligned} Y_n(0) &\sim b_1^n, \qquad |b_1| < 1, \\ Y_n(0) &\sim b_1^{-n}, \qquad |b_1| > 1. \end{aligned}$$

The behavior of such parameters for some specific values of b_1 is shown in Fig. 1.

5.2. The case $d\tilde{\sigma} = (z - \alpha + z^{-1} - \bar{\alpha})|z-1|^2\frac{d\theta}{2\pi}$

Now we study a transformation of the measure $d\sigma = |z-1|^2\frac{d\theta}{2\pi}$, $z = e^{i\theta}$ (see [16]). Is is well known that if $\{\Phi_n\}_{n\geqslant 0}$ denotes the sequence of monic polynomials orthogonal with respect to σ, then

$$\Phi_n(z) = \frac{1}{z-1}\left(z^{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} z^j\right), \qquad n \geqslant 1, \tag{5.1}$$

or, equivalently,

$$\Phi_n^*(z) = \frac{1}{1-z}\left(1 - \frac{1}{n+1}\sum_{j=0}^{n} z^{j+1}\right), \qquad n \geqslant 1. \tag{5.2}$$

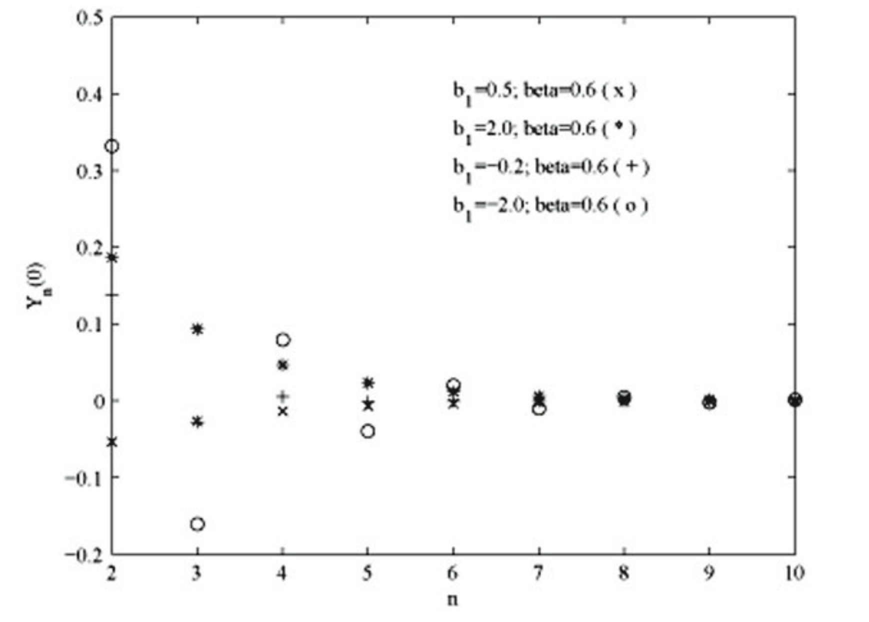

FIGURE 1

Notice that

$$\Phi_n(0) = \frac{1}{n+1}, \qquad n \geqslant 1. \tag{5.3}$$

Then, the perturbed linear functional is quasi-definite if and only if

$$b_1 \frac{1}{b_1 - 1}\left(b_1^{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} b_1^j\right)\frac{1}{1-b_1^{-1}}\left(1 - \frac{1}{n+1}\sum_{j=0}^{n} b_1^{-j-1}\right)$$

$$- b_1^{-1}\frac{1}{b_1^{-1}-1}\left(b_1^{-n-1} - \frac{1}{n+1}\sum_{j=0}^{n} b_1^{-j}\right)\frac{1}{1-b_1}\left(1 - \frac{1}{n+1}\sum_{j=0}^{n} b_1^{j+1}\right) \neq 0,$$

$$b_1^{n+2} - b_1^{-n-2} - \frac{1}{n+1}\sum_{j=0}^{n} b_1^{j+1} - \frac{b_1^{n+2}}{n+1}\sum_{j=0}^{n} b_1^{-j-1} + \frac{1}{n+1}\sum_{j=0}^{n} b_1^{-j-1}$$

$$+ \frac{b_1^{-n-2}}{n+1}\sum_{j=0}^{n} b_1^{j+1} \neq 0,$$

$$b_1^{n+2} - b_1^{-n-2} - \frac{2}{n+1}\sum_{j=0}^{n} b_1^{j+1} + \frac{2}{n+1}\sum_{j=0}^{n} b_1^{-j-1} \neq 0.$$

In other words

$$b_1^{2n+4}-1-\frac{2b_1^{n+2}-2}{n+1}\sum_{j=0}^{n}b_1^{j+1}\neq 0,$$
$$b_1^{n+2}+1-\frac{2b_1}{n+1}\frac{b_1^{n+1}-1}{b_1-1}\neq 0,$$
$$(n+1)(b_1-1)(b_1^{n+2}+1)-2b_1(b_1^{n+1}-1)\neq 0,\qquad \text{for every } n\in\mathbb{N}.$$

On the other hand, from (2.9), the Verblunsky parameters are

$$Y_n(0)$$
$$=\frac{(b_1-b_1^{-1})\dfrac{1}{b_1-1}\left(b_1^{n+1}-\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n}b_1^{j}\right)\dfrac{1}{b_1^{-1}-1}\left(b_1^{-n-1}-\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n}b_1^{-j}\right)}{\dfrac{1}{(b_1-1)(1-b_1^{-1})}\left(b_1^{n+2}-b_1^{-n-2}-\dfrac{2}{n+1}\displaystyle\sum_{j=0}^{n}b_1^{j+1}+\dfrac{2}{n+1}\displaystyle\sum_{j=0}^{n}b_1^{-j-1}\right)}$$
$$=\frac{(b_1^{-1}-b_1)\left(1-\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n}b_1^{j+1}-\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n}b_1^{-j-1}+\dfrac{1}{(n+1)^2}\displaystyle\sum_{j=0}^{n}b_1^{j}\sum_{j=0}^{n}b_1^{-j}\right)}{b_1^{n+2}-b_1^{-n-2}-\dfrac{2}{n+1}\displaystyle\sum_{j=0}^{n}b_1^{j+1}+\dfrac{2}{n+1}\displaystyle\sum_{j=0}^{n}b_1^{-j-1}}$$
$$=\frac{(b_1^{-1}-b_1)\left[b_1^{n+2}-\dfrac{b_1(b_1^{n+2}+1)}{n+1}\dfrac{b_1^{n+1}-1}{b_1-1}+\dfrac{b_1^2}{(n+1)^2}\left(\dfrac{b_1^{n+1}-1}{b_1-1}\right)^2\right]}{b_1^{2n+4}-1-\dfrac{2b_1(b_1^{n+2}-1)}{n+1}\dfrac{b_1^{n+1}-1}{b_1-1}},$$

and, therefore, for n large enough, if $|b_1|<1$

$$Y_n(0)\sim\frac{(b_1^{-1}-b_1)\left[\dfrac{b_1}{(n+1)(b_1-1)}+\dfrac{b_1^2}{(n+1)^2(b_1-1)^2}\right]}{-1-\dfrac{2b_1}{(n+1)(b_1-1)}}$$
$$\sim(b_1-b_1^{-1})\frac{b_1}{(n+1)(b_1-1)}\sim\frac{1}{n+1}.$$

On the other hand, if $|b_1|>1$, then

$$Y_n(0)=\frac{(b_1^{-1}-b_1)\left[-\dfrac{1}{(n+1)(b_1-1)}+\dfrac{1}{(n+1)^2(b_1-1)^2}\right]}{1-\frac{2}{(n+1)(b_1-1)}}$$
$$\sim(b_1-b_1^{-1})\frac{1}{(n+1)(b_1-1)}\sim\frac{1}{n+1}.$$

Finally, we show the behavior of $Y_n(0)$ for some specific values of b_1 in Fig. 2.

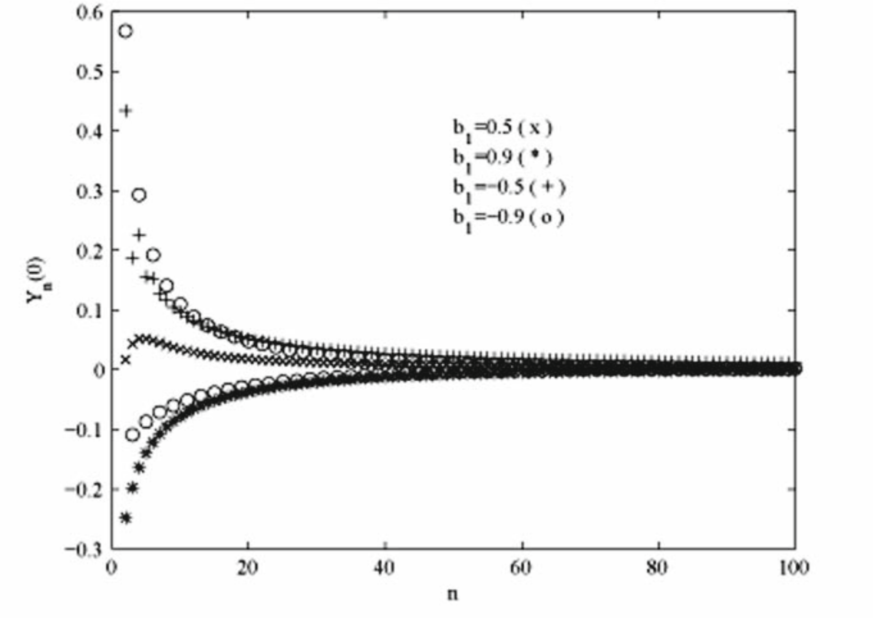

FIGURE 2

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