When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials?

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Dedicated to Professor Jesús S. Dehesa on the occasion of his 60th birthday

Abstract

Given $\{P_n\}$ a sequence of monic orthogonal polynomials, we analyze their linear combinations with constant coefficients and fixed length, i.e.,
$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \cdots + a_k P_{n-k}$, $a_k \neq 0$, $n > k$.
Necessary and sufficient conditions are given for the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ as well as an interesting interpretation in terms of the Jacobi matrices associated with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$.

Moreover, in the case $k = 2$, we characterize the families $\{P_n\}_{n\geq 0}$ such that the corresponding polynomials $\{Q_n\}_{n\geq 0}$ are also orthogonal.

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1 Introduction and basic definitions

Given a linear functional u on the linear space $\mathbb{P}$ of polynomials with real coefficients, a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ with deg $P_n = n$ is said to be

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orthogonal with respect to u if $\langle u, P_n P_m \rangle = 0$ for every $n \neq m$, and $\langle u, P_n^2 \rangle \neq 0$ for every $n = 0, 1, \dots$.

A linear functional u is said to be quasi–definite (respectively positive definite) if the leading principal submatrices H_n of the Hankel matrix $H = (u_{i+j})_{i,j \geq 0}$ associated with u, where $u_k = \langle u, x^k \rangle$, $k \geq 0$ are nonsingular (respectively positive definite) for every n, (see [4]).

A very well known result (Favard's theorem, see [4] for instance) gives a characterization of a quasi–definite (respectively positive definite) linear functional in terms of the three–term recurrence relation that the sequence $\{P_n\}_{n \geq 0}$ satisfies, i.e.

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \\ P_0(x) &= 1, \quad P_1(x) = x - \beta_0, \end{aligned} \tag{1.1}$$

whith $\gamma_n \neq 0$ (respectively $\gamma_n > 0$).

In particular, if u is a positive definite linear functional then there exists a positive Borel measure μ supported on an infinite subset of $\mathbb{R}$ such that $\langle u, q \rangle = \int_{\mathbb{R}} q \, d\mu$ for every $q \in \mathbb{P}$. In such a situation, the zeros of P_n are real, simple, and they are located in the convex hull of the support of the measure μ. Furthermore, the zeros of P_{n-1} interlace with those of P_n. Actually, this is a relevant fact in numerical quadrature, i.e. in the discrete representation

$$\int_{\mathbb{R}} q \, d\mu \sim \sum_{k=1}^{n} \lambda_k q(c_k), \quad q \in \mathbb{P}. \tag{1.2}$$

If we choose as $(c_k)_{k=1}^n$ the zeros of P_n then (1.2) is exact for every polynomial of degree at most $2n-1$ and, as a consequence of the interlacing property aforementioned, the Christoffel-Cotes numbers $(\lambda_k)_{k=1}^n$ are positive numbers.

In general, given the pair (q, μ) with $q(x) = \prod_{k=1}^{n} (x - c_k)$ and letting $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_k = \int_{\mathbb{R}} \frac{q(x)}{q'(c_k)(x - c_k)} \, d\mu(x)$, $1 \leq k \leq n$, there exists an integer number $d(q, \mu)$ with $n - 1 \leq d(q, \mu) \leq 2n - 1$, so that (1.2) is exact for the polynomials of degree $\leq d(q, \mu)$ but not for all polynomials of degree $d(q, \mu) + 1$. The number $d(q, \mu)$ is said to be the degree of precision of (q, μ).

Shohat, in [12], proved that (q, μ) has degree of precision $2n - 1 - k$ if and only if $q = P_n + a_1 P_{n-1} + \cdots + a_k P_{n-k}$ where $a_k \neq 0$ and $\{P_n\}$ is the sequence of monic polynomials orthogonal with respect to the measure μ.

Moreover, when $\operatorname{supp} \mu = (-1, 1)$, Peherstorfer addresses in [11] sufficient conditions on the real numbers $\{a_j\}_{j=1}^k$ under which the polynomial $q = P_n + a_1 P_{n-1} + \cdots + a_k P_{n-k}$ has n simple zeros in $(-1, 1)$ and whose Christoffel-Cotes numbers are positive.

In [12] a discussion about the zeros of the polynomial $q = P_n + a_1 P_{n-1}$ is given in terms of $\operatorname{sign} \mathrm{a}_1$: they are real and simple and at most one of them lies outside $\operatorname{supp} \mu$. Moreover, the zeros of the polynomial $q = P_n + a_1 P_{n-1} + a_2 P_{n-2}$ are studied. If $a_2 < 0$, all the zeros are real and simple and at most two of them do not belong to the supp μ. In addition, in [3] it is proved that if $a_2 < 0$ then the zeros of P_{n-1} interlace with the zeros of q. The position of the least and greatest zero of q in terms of the least and greatest zero of P_n is also analyzed.

In [1] the positivity of Christoffel-Cotes numbers and the distribution of zeros of linear combinations $R = P_m + \cdots + a_s P_s$ where $a_s \neq 0\,, 1 \le s \le m \le n$ and $m \le d(q,\mu)$ is analyzed. Here $q(x) = \prod_{k=1}^{n}(x - c_k)$ with $c_1 < \cdots < c_n$. If all the Christoffel-Cotes numbers are positive, then either R is a non–zero scalar multiple of q or at least N of the intervals (c_k, c_{k+1}) contain a zero of R where $N = \min\{s, d(q,\mu) + 1 - m\} \ge 1$.

Grinshpun, in [6], studied the orthogonality of special linear combinations of polynomials orthogonal with respect to a weight function supported on an interval of the real line. Such families of orthogonal polynomials come up in some extremal problems of Zolotarev–Markov type as well as in problems of least deviating from zero. He proved that the Bernstein–Szegő polynomials can be represented as a linear combination of the Chebyshev polynomials of the same kind. Nevertheless, the special feature of this representation is that the coefficients do not depend on n. The relevant question is if this property characterizes Bernstein–Szegő polynomials. Theorem 3.1 in [6] gives a positive answer in the sense that Bernstein–Szegő polynomials and just them can be represented as a linear combination of Chebyshev polynomials with constant coefficients independent of n and fixed length. In other words, $\{Q_n\}_{n\ge0}$ with $Q_n = P_n + a_1 P_{n-1} + \cdots + a_k P_{n-k}\,, n > k$, where $\{P_n\}_{n\ge0}$ is the Chebyshev sequence of j–th kind ($j = 1,2,3,4$) and $a_k \neq 0$, is a sequence of orthogonal polynomials with respect to a weight $\widetilde{\omega}$ if and only if $\widetilde{\omega}(x) = \dfrac{\mu_j(x)}{h_k(x)}$, where h_k is a polynomial of degree k positive on $(-1,1)$ and μ_j is the Chebyshev weight of j–th kind, ($j = 1,2,3,4$).

The aim of this work is to analyze linear combinations with constant coefficients $Q_n = P_n + a_1 P_{n-1} + \cdots + a_k P_{n-k}\,, n > k$, of a sequence of orthogonal polynomials $\{P_n\}_{n\ge0}$. In Section 2 we find necessary and sufficient conditions so that the sequence $\{Q_n\}_{n\ge0}$ is orthogonal with respect to a linear functional v. Moreover, we discuss the matrix representation for the multiplication operator in terms of the bases $\{P_n\}_{n\ge0}$ and $\{Q_n\}_{n\ge0}$ respectively. Such a matrix is a monic tridiagonal (Jacobi) matrix. We prove that the leading principal submatrix associated with $\{Q_n\}_{n\ge0}$ is similar to a rank–one perturbation of the leading principal submatrix associated with $\{P_n\}_{n\ge0}$. Also, we give a simple algorithm to compute the polynomial h_k of degree k appearing in the relation between the two functionals, $u = h_k v$.

In Section 3, the case $k = 2$ is addressed, describing all the families $\{P_n\}_{n\ge0}$ orthogonal with respect to a linear functional such that the corresponding $\{Q_n\}_{n\ge0}$ is also orthogonal, obtaining explicit expressions for the recurrence parameters $\{\beta_n\}_{n\ge0}$ and $\{\gamma_n\}_{n\ge1}$ of the sequence $\{P_n\}_{n\ge0}$. Finally, in Section 4 we present some remarks and examples of such sequences $\{P_n\}_{n\ge0}$.

2 Orthogonality and Jacobi matrices

In the sequel $\{P_n\}_{n\geq 0}$ denotes a sequence of monic polynomials orthogonal (SMOP) with respect to a quasi–definite linear functional u.

Let $\{Q_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg Q_n = n$ such that, for $n \geq k+1$,

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \cdots + a_k P_{n-k}(x) \tag{2.1}$$

where the coefficients $\{a_j\}_{j=1}^k$ are independent of n and $a_k \neq 0$.

Our aim will be to deduce necessary and sufficient conditions in order to the sequence $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to a quasi–definite linear functional v and to give the relation between the linear functionals u and v, via Jacobi matrices.

Proposition 2.1 *Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials with recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ ($\gamma_n \neq 0$) and let $\{Q_n\}_{n\geq 0}$ be a sequence of monic polynomials such that, for $n \geq k+1$,*

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \cdots + a_k P_{n-k}(x)$$

where $\{a_j\}_{j=1}^k$ are constant coefficients and $a_k \neq 0$. Then $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to a quasi–definite linear functional if and only if the following conditions hold

(i) For each j, $1 \leq j \leq k$, the polynomials Q_j satisfy a three term recurrence relation $xQ_j(x) = Q_{j+1}(x) + \widetilde{\beta}_j Q_j(x) + \widetilde{\gamma}_j Q_{j-1}(x)$, with $\widetilde{\gamma}_j \neq 0$.

(ii) For $n \geq k+2$

$$\begin{aligned}
&\gamma_n + a_1(\beta_{n-1} - \beta_n) = \gamma_{n-k}\,, \\
&a_{j-1}(\gamma_{n-k} - \gamma_{n-j+1}) = a_j(\beta_{n-j} - \beta_n)\,, \quad 2 \leq j \leq k\,.
\end{aligned}$$

(iii)

$$\begin{aligned}
&\gamma_{k+1} + a_1(\beta_k - \beta_{k+1}) \neq 0 \\
&a_j\gamma_{k-j+1} + a_{j+1}(\beta_{k-j} - \beta_{k+1}) = a_j^{(k)}[\gamma_{k+1} + a_1(\beta_k - \beta_{k+1})]\,,\ 1 \leq j \leq k-1\,, \\
&a_k\gamma_1 = a_k^{(k)}[\gamma_{k+1} + a_1(\beta_k - \beta_{k+1})]\,,
\end{aligned}$$

where $a_j^{(k)}$, $j = 1, \ldots, k$, denotes the coefficient of P_{k-j} in the Fourier expansion of Q_k in terms of the orthogonal system $\{P_j\}_{j=0}^k$.

Moreover, denoting by $\widetilde{\beta_n}$ and $\widetilde{\gamma_n}$ the coefficients of the three-term recurrence relation for the polynomials Q_n we have for $n \geq k+1$

$$\widetilde{\beta_n} = \beta_n, \quad \widetilde{\gamma_n} = \gamma_n + a_1(\beta_{n-1} - \beta_n)\,, \tag{2.2}$$

Proof. According to Favard's theorem, the sequence $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to a quasi–definite linear functional if and only if, for every n, it satisfies a three–term recurrence relation

$$xQ_n(x) = Q_{n+1}(x) + \widetilde{\beta_n} Q_n(x) + \widetilde{\gamma_n} Q_{n-1}(x)$$

where $\widetilde{\gamma_n} \neq 0$, $n \geq 1$. So, condition (i) follows.

Let $n \geq k+2$. From $xQ_n(x) = xP_n(x) + \sum_{j=1}^{k} a_j x P_{n-j}(x)$ and a little work involving (2.1) and the recurrence relation for the polynomials P_n it follows that

$$\begin{aligned} xQ_n(x) &= Q_{n+1}(x) + \beta_n Q_n(x) + [\gamma_n + a_1(\beta_{n-1} - \beta_n)] Q_{n-1}(x) \\ &+ \sum_{j=2}^{k} \{a_j(\beta_{n-j} - \beta_n) - a_{j-1}[\gamma_n - \gamma_{n-j+1} + a_1(\beta_{n-1} - \beta_n)]\} P_{n-j}(x) \\ &- a_k[\gamma_n - \gamma_{n-k} + a_1(\beta_{n-1} - \beta_n)] P_{n-(k+1)}(x). \end{aligned}$$

Then, whenever $n \geq k+2$, Q_n satisfies a three-term recurrence relation if and only if the coefficient of Q_{n-1} in the above formula is different from 0 and the coefficients of the polynomials $\{P_j\}_{j=n-(k+1)}^{n-2}$ vanish, i.e.,

$$\gamma_n + a_1(\beta_{n-1} - \beta_n) \neq 0 \tag{2.3a}$$

$$a_{j-1}[\gamma_n - \gamma_{n-j+1} + a_1(\beta_{n-1} - \beta_n)] = a_j(\beta_{n-j} - \beta_n), \quad j = 2, \ldots, k \tag{2.3b}$$

$$\gamma_n + a_1(\beta_{n-1} - \beta_n) = \gamma_{n-k}. \tag{2.3c}$$

Notice that, since $\gamma_n \neq 0, n \geq 1$, (2.3a) is a consequence of (2.3c). Moreover, using (2.3c), the formula (2.3b) can be rewritten in the form

$$a_{j-1}(\gamma_{n-k} - \gamma_{n-j+1}) = a_j(\beta_{n-j} - \beta_n), \quad j = 2, \ldots, k.$$

So, (ii) holds.

Next, we study the case $n = k+1$. Let $Q_k(x) = P_k(x) + \sum_{j=1}^{k} a_j^{(k)} P_{k-j}(x)$ be the Fourier expansion of Q_k in terms of the orthogonal system $\{P_n\}$. Handling in the same way as above we have

$$\begin{aligned} xQ_{k+1}(x) &= Q_{k+2}(x) + \beta_{k+1} Q_{k+1}(x) + [\gamma_{k+1} + a_1(\beta_k - \beta_{k+1})] Q_k(x) \\ &+ \sum_{j=1}^{k-1} \left[a_{j+1}(\beta_{k-j} - \beta_{k+1}) - a_j^{(k)}[\gamma_{k+1} + a_1(\beta_k - \beta_{k+1})] + a_j \gamma_{k-j+1} \right] P_{k-j}(x), \\ &+ [a_k \gamma_1 - a_k^{(k)}(\gamma_{k+1} + a_1(\beta_k - \beta_{k+1}))] P_0(x)\,, \end{aligned}$$

and arguing as in the proof of (ii), (iii) holds.

Finally, (2.2) is an immediate consequence of the precedent results. □

Remark. Let us to point out that, because of (iii), the coefficients $\{a_j^{(k)}\}_{j=1}^{k}$ are determined by the recurrence parameters $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ as well as the constants $\{a_j\}_{j=1}^k$. So, the relation (2.1) and the orthogonality of $\{Q_n\}_{n\geq k+1}$ fix the polynomial Q_k. As a consequence, in the particular case $k=1$, the sequence $\{Q_n\}_{n\geq 0}$ is completely determined by (2.1) and the orthogonality property.

Now, we consider two families of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ with respect to the quasi-definite linear functionals u and v, respectively, satisfying the condition (2.1). It is well known (see, e.g., [10]) that the relation between the two linear functionals is $u = h_k v$ where h_k is a polynomial of degree k.

Writing $\mathbf{P} = (P_0, P_1, \ldots, P_n, \ldots)^T$ and $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_n, \ldots)^T$ for the column vectors associated with these orthogonal families, and $\mathbf{J}_P$ and $\mathbf{J}_Q$ for the corresponding Jacobi matrices, we get

$$x\,\mathbf{P} \,=\, \mathbf{J}_P\,\mathbf{P}\,, \quad x\,\mathbf{Q} \,=\, \mathbf{J}_Q\,\mathbf{Q}\,. \tag{2.4}$$

If $\mathbf{M}$ denotes the matrix associated with the change of bases $\mathbf{Q} = \mathbf{MP}$, then $\mathbf{M}$ is a lower triangular matrix with diagonal entries equal to 1 and zero subdiagonals from the $(k+1)$–th one.

From (2.4) it follows $\mathbf{M}\,\mathbf{J}_P\,\mathbf{P} \,= x\,\mathbf{M}\,\mathbf{P} \,=\, \mathbf{J}_Q\,\mathbf{M}\,\mathbf{P}$ and, therefore

$$\mathbf{M}\,\mathbf{J}_P \,=\, \mathbf{J}_Q\,\mathbf{M}\,. \tag{2.5}$$

From this simple relation it follows straightforward the entries of the matrix $\mathbf{J}_Q$.

Moreover, from the equations (2.4) we get

$$x(\mathbf{P})_n \,=\, (\mathbf{J}_P)_n(\mathbf{P})_n + P_{n+1}e_{n+1} \tag{2.6}$$

$$x(\mathbf{Q})_n \,=\, (\mathbf{J}_Q)_n(\mathbf{Q})_n + Q_{n+1}e_{n+1} \tag{2.7}$$

where $e_{n+1} = (0, \ldots, 0, 1)^T \in \mathbb{R}^{n+1}$. Here, the symbol $(\mathbf{A})_n$ stands for the truncation of any infinite matrix $\mathbf{A}$ at level $n+1$. Using the relation (2.1), the representation of the change of bases $(\mathbf{Q})_n = (\mathbf{M})_n\,(\mathbf{P})_n$ and (2.7), we deduce

$$x(\mathbf{M})_n(\mathbf{P})_n = (\mathbf{J}_Q)_n(\mathbf{M})_n(\mathbf{P})_n + P_{n+1}e_{n+1} + \mathbf{L}_n(\mathbf{P})_n$$

where

$$\mathbf{L}_n = \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 \\ . & \ldots & . & . & \ldots & . \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ . & \ldots & . & . & \ldots & . \\ 0 & \ldots & 0 & a_k & \ldots & a_1 \end{pmatrix} \in \mathbb{R}^{(n+1,n+1)}\,.$$

Thus,

$$x(\mathbf{P})_n = (\mathbf{M})_n^{-1}\left[(\mathbf{J}_Q)_n(\mathbf{M})_n + \mathbf{L}_n\right](\mathbf{P})_n + P_{n+1}e_{n+1}\,.$$

Comparing this formula with (2.6) we get

$$(\mathbf{J}_P)_n = (\mathbf{M})_n^{-1}\left[(\mathbf{J}_Q)_n(\mathbf{M})_n + \mathbf{L}_n\right]\,,$$

that is

$$(\mathbf{J}_Q)_n = (\mathbf{M})_n\left[(\mathbf{J}_P)_n - \mathbf{L}_n\right](\mathbf{M})_n^{-1}\,.$$

This last expression means that $(\mathbf{J}_Q)_n$ is similar to a rank–one perturbation of the matrix $(\mathbf{J}_P)_n$ and this perturbation is given by the matrix $\mathbf{L}_n$. In particular, the zeros of the polynomial Q_n are the zeros of the characteristic polynomial of the matrix $(\mathbf{J}_P)_n - \mathbf{L}_n$.

Next, we are going to describe an explicit algebraic relation between the Jacobi matrices $\mathbf{J}_P$ and $\mathbf{J}_Q$, keeping in mind basically the relationship between the linear functionals u and v, that is $u = h_k v$.

To do this, we first observe that $\mathbf{Q}\mathbf{Q}^T = \mathbf{M}\mathbf{P}\mathbf{P}^T\mathbf{M}^T$. Writing $\mathbf{D}_P = \langle u, \mathbf{P}\mathbf{P}^T\rangle$ and $\mathbf{D}_Q = \langle v, \mathbf{Q}\mathbf{Q}^T\rangle$ we have

$$\langle v, h_k\mathbf{Q}\mathbf{Q}^T\rangle = \langle h_k v, \mathbf{Q}\mathbf{Q}^T\rangle = \langle u, \mathbf{Q}\mathbf{Q}^T\rangle = \mathbf{M}\langle u, \mathbf{P}\mathbf{P}^T\rangle\mathbf{M}^T = \mathbf{M}\mathbf{D}_P\mathbf{M}^T.$$

Since $\langle v, h_k\mathbf{Q}\mathbf{Q}^T\rangle = \langle v, h_k(\mathbf{J}_Q)\mathbf{Q}\mathbf{Q}^T\rangle = h_k(\mathbf{J}_Q)\mathbf{D}_Q$, then

$$h_k(\mathbf{J}_Q) = \mathbf{M}\mathbf{D}_P\mathbf{M}^T\mathbf{D}_Q^{-1}\,. \tag{2.8}$$

On the other hand, from (2.5) it follows

$$h_k(\mathbf{J}_Q) = \mathbf{M}h_k(\mathbf{J}_P)\mathbf{M}^{-1}\,. \tag{2.9}$$

From (2.8) and (2.9) we deduce

$$h_k(\mathbf{J}_P) = \mathbf{D}_P\mathbf{M}^T\mathbf{D}_Q^{-1}\mathbf{M}\,. \tag{2.10}$$

So, we have a simple algorithm to compute the polynomial h_k.

(1) From the data $\mathbf{M}$ and $\mathbf{J}_P$, we have (2.5) and we can deduce $\mathbf{J}_Q$.

(2) From $\mathbf{J}_P$ and $\mathbf{J}_Q$ we deduce $\mathbf{D}_P$ and $\mathbf{D}_Q$, respectively.

(3) Using (2.10) and taking into account that h_k is a polynomial of degree k, $h_k(x) = c_0 + c_1 x + \cdots + c_k x^k$, we get

$$h_k(\mathbf{J}_P) = c_0 I + c_1\mathbf{J}_P + \cdots + c_k\mathbf{J}_P^k = \mathbf{D}_P\mathbf{M}^T\mathbf{D}_Q^{-1}\mathbf{M}\,,$$

which is a system of linear equations with $k+1$ unknowns. Notice that the matrices of the first and second terms are $2k+1$ diagonal.

If the monic polynomials $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ would be replaced by the corresponding orthonormal polynomials $\{\widetilde{P}_n\}_{n\geq 0}$ and $\{\widetilde{Q}_n\}_{n\geq 0}$, similar computations would have led to

$$h_k(\mathbf{J}_{\widetilde{P}}) = \widetilde{\mathbf{M}}^T\widetilde{\mathbf{M}}\,, \qquad h_k(\mathbf{J}_{\widetilde{Q}}) = \widetilde{\mathbf{M}}\widetilde{\mathbf{M}}^T\,,$$

where $\widetilde{\mathbf{M}}$ denotes the matrix of the change of bases, that is $\widetilde{\mathbf{Q}} = \widetilde{\mathbf{M}}\widetilde{\mathbf{P}}$. This gives us an interesting interpretation of the matrix operation involving the linear combination of the orthogonal polynomials $Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \cdots + a_k P_{n-k}(x), \quad n \geq k+1$.

3 The Case $k = 2$

Among the classical orthogonal polynomial families, the Chebyshev polynomials are the unique families such that the sequence of polynomials $\{Q_n\}_{n\geq 0}$ defined by (2.1) is orthogonal (see for example [2]). But, what happens if the sequence $\{P_n\}_{n\geq 0}$ is not a classical one?

In this Section, our main goal will be to describe, for the case $k = 2$, all the families of monic polynomials $\{P_n\}_{n\geq 0}$ orthogonal with respect to a quasi–definite linear functional such that the new families $\{Q_n\}_{n\geq 0}$ are also orthogonal.

Theorem 3.1 *Let $\{P_n\}_{n\geq 0}$ be a SMOP with respect to a quasi–definite linear functional. Assume that a_1 and a_2 are real numbers with $a_2 \neq 0$ and Q_n the monic polynomials defined by*

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + a_2 P_{n-2}(x)\,, \quad n \geq 3\,. \tag{3.1}$$

Then the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ depends on the choice of a_1 and a_2. More precisely, $\{Q_n\}_{n\geq 0}$ is a SMOP if and only if $\gamma_3 + a_1(\beta_2 - \beta_3) \neq 0$, and

(i) if $a_1 = 0$, for $n \geq 4$, $\beta_n = \beta_{n-2}$ and $\gamma_n = \gamma_{n-2}$.

(ii) if $a_1 \neq 0$ and $a_1^2 = 4a_2$, then for $n \geq 2$,

$$\beta_n = A + Bn + Cn^2\,, \qquad \gamma_n = D + En + Fn^2, \tag{3.2}$$

with $a_1 C = 2F\,,\quad a_1 B = 2E - 2F\,,\quad (A, B, C, D, E, F \in \mathbb{R})$.

(iii) if $a_1 \neq 0$ and $a_1^2 > 4a_2$, then for $n \geq 2$,

$$\beta_n = A + B\lambda^n + C\lambda^{-n}\,, \qquad \gamma_n = D + E\lambda^n + F\lambda^{-n}\,,$$

with $a_1 C = (1+\lambda)F\,,\quad a_1\lambda B = (1+\lambda)E\,,\quad (A, B, C, D, E, F \in \mathbb{R})\,,$ where λ is the unique solution in $(-1, 1)$ of the equation $a_1^2\lambda = a_2(1+\lambda)^2$.

(iv) if $a_1 \neq 0$ and $a_1^2 < 4a_2$, and let $\lambda = e^{i\theta}$ be the unique solution of the equation $a_1^2\lambda = a_2(1+\lambda)^2$ with $\theta \in (0, \pi)$, then for $n \geq 2$

$$\beta_n = A + Be^{in\theta} + \overline{B}e^{-in\theta}\,, \qquad \gamma_n = D + Ee^{in\theta} + \overline{E}e^{-in\theta}\,,$$

with $a_1\lambda\, B = (1+\lambda)\, E,\quad (A, D \in \mathbb{R}\,, B, E \in \mathbb{C})\,.$

Proof. Applying Proposition 2.1 to the particular case $k = 2$, we have that $\{Q_n\}_{n\geq 0}$ is a SMOP if and only if $\gamma_3 + a_1(\beta_2 - \beta_3) \neq 0$ and, for $n \geq 4$,

$$a_1(\gamma_{n-2} - \gamma_{n-1}) = a_2(\beta_{n-2} - \beta_n), \tag{3.3}$$

$$\gamma_n - \gamma_{n-2} = a_1(\beta_n - \beta_{n-1}). \tag{3.4}$$

Observe that $i)$ follows directly.

In the sequel, we will assume $a_1 \neq 0$. From (3.3) and (3.4), we deduce that β_n and γ_n are solutions of the difference equation with constant coefficients

$$y_n + \left(1 - \frac{a_1^2}{a_2}\right) y_{n-1} - \left(1 - \frac{a_1^2}{a_2}\right) y_{n-2} - y_{n-3} = 0\,, \quad n \geq 5\,. \tag{3.5}$$

According to the solutions of the associated characteristic equation

$$(\lambda - 1)\left[\lambda^2 + \left(2 - \frac{a_1^2}{a_2}\right)\lambda + 1\right] = 0\,, \tag{3.6}$$

we can analyze three cases (see, for instance, [5]).

(ii) If $a_1^2 = 4a_2$, then $\lambda = 1$ is a root with multiplicity 3 and therefore

$$\beta_n = A + Bn + Cn^2\,, \quad \gamma_n = D + En + Fn^2\,, \quad n \geq 5\,.$$

Note that the obtained expressions for β_n and γ_n hold also for $n \geq 2$, just applying (3.5) for n equal to 7, 6, and 5.

Inserting these expressions of β_n and γ_n in (3.3) and (3.4) we have

$$n[2a_1F - a_1^2C] = \frac{1}{2}a_1^2B - a_1E + a_1F, \quad n \geq 4\,,$$

$$n[4F - 2a_1C] = a_1B - a_1C - 2E + 4F\,, \quad n \geq 4\,,$$

which is equivalent to

$$a_1C - 2F = 0\,, \qquad a_1B - 2E + 2F = 0\,.$$

Moreover, since $\beta_n, \gamma_n \in \mathbb{R}$, $n \geq 1$, it is easy to check that $A, B, C, D, E, F \in \mathbb{R}$.

Conversely, the values of β_n and γ_n given by (3.2), and the above relations lead, trough (3.3) and (3.4), to the orthogonality of the sequence $\{Q_n\}$.

(iii) and (iv) If $a_1^2 \neq 4a_2$, then

$$\beta_n = A + B\lambda^n + C\lambda^{-n}\,, \quad \gamma_n = D + E\lambda^n + F\lambda^{-n}\,, \quad n \geq 5\,,$$

where λ is the unique solution of the equation (3.6) such that $\lambda \in (-1, 1)$ if $a_1^2 > 4a_2$ and $\lambda = e^{i\theta}$ with $\theta \in (0, \pi)$, if $a_1^2 < 4a_2$.

Upon applying the same reasoning as in the case (ii) we get that the previous formulas are also valid for $n \geq 2$.

Inserting these values of β_n and γ_n in (3.3) and (3.4) we have

$$\lambda^{2n-2}[a_1E - a_2B(\lambda + 1)] = a_1F\lambda - a_2C(\lambda + 1)\,, \quad n \geq 4\,,$$

$$\lambda^{2n-2}[a_1B\lambda - (\lambda + 1)E] = a_1C - (\lambda + 1)F\,, \quad n \geq 4\,.$$

Then, since λ is a solution of the equation $a_1^2\lambda = a_2(1 + \lambda)^2$, we have that the above both formulas are equivalent to the following system

$$a_1C = (\lambda + 1)F\,, \qquad a_1\lambda B = (\lambda + 1)E\,.$$

Again, the conditions $\beta_n, \gamma_n \in \mathbb{R}$, $n \geq 1$, yield A, B, C, D, E, F are real numbers in the case (iii) and, in the case (iv), $A, D \in \mathbb{R}$ and B, C, E, F are complex numbers with $C = \overline{B}, \quad F = \overline{E}$. $\square$

4 Further remarks and comments

After the work of Section 3 it is natural to ask us the following question: it is possible to give explicitly the SMOP $\{P_n\}_{n\geq 0}$, as well as their orthogonality measure, such that the sequence $\{Q_n\}_{n\geq 0}$ defined by (3.1) is also a SMOP? This problem might be quite hard. In this Section we make some remarks concerning to it and we show some examples.

First, we point out a difference between the cases $k = 1$ and $k = 2$. Let Q_n be the monic polynomials defined by

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x), \quad n \geq 2$$

with $a_1 \neq 0$. From Proposition 2.1 written for $k = 1$, it follows that $\{Q_n\}_{n\geq 0}$ is a SMOP (see [8] in a more general setting), if and only if

$$\begin{aligned} &\gamma_2 + a_1(\beta_1 - \beta_2) \neq 0 \\ &\gamma_n - \gamma_2 = a_1(\beta_n - \beta_2)\,, \quad n \geq 3\,. \end{aligned} \tag{4.1}$$

Thus, in the case $k = 1$, for any sequence of $\{\gamma_n\}_{n\geq 1}$ with $\gamma_n \neq 0$, if we take $\beta_0, \beta_1 \in \mathbb{R}$, and β_n $(n \geq 2)$ satisfying (4.1), we obtain all the SMOP $\{P_n\}_{n\geq 0}$ such that $\{Q_n\}_{n\geq 0}$ is also a SMOP. However, in the case $k = 2$, Theorem 3.1 implies that the recurrence coefficients γ_n and β_n have to be solutions of the equation (3.5). Therefore, although in both cases we get that β_n and γ_n have a similar asymptotic behaviour, roughly speaking, for $k = 2$ there are much less families $\{P_n\}_{n\geq 0}$.

Examples. According to Theorem 3.1, all the SMOP $\{P_n\}_{n\geq 0}$ such that the sequence $\{Q_n\}_{n\geq 0}$ where $Q_n = P_n + a_2 P_{n-2}, n \geq 3$ with $a_2 \neq 0$ is again a SMOP, satisfy for $n \geq 4$, $\beta_n = \beta_{n-2} \quad and \quad \gamma_n = \gamma_{n-2},$.

The families of monic orthogonal polynomials which fulfill these conditions were explicitly given in terms of Chebyshev polynomials in [9, Example 2, p. 109]. Observe that this situation corresponds to the case $a_1 = 0$. However, in the case $a_1 \neq 0$, the explicit description of all sequences $\{P_n\}_{n\geq 0}$ remains still open. Besides the four Chebyshev families, we have identify some explicit solutions, for instance, the continuous big q-Hermite polynomials (see [7]).

Whenever $k = 1$, an interesting case arises when $\beta_n = \beta_0$, for all n and $\gamma_n = \gamma_1, n \geq 2$. In particular, it follows that the only symmetric orthogonal polynomials $\{P_n\}$ such that the sequence $P_n + a_1 P_{n-1}$ is also an SMOP are the Chebyshev polynomials (up to a variable change).

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