EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE BELLMAN EQUATION IN THE UNBOUNDED CASE

BY JUAN PABLO RINCÓN-ZAPATERO AND CARLOS RODRÍGUEZ-PALMERO[1]

We study the problem of the existence and uniqueness of solutions to the Bellman equation in the presence of unbounded returns. We introduce a new approach based both on consideration of a metric on the space of all continuous functions over the state space, and on the application of some metric fixed point theorems. With appropriate conditions we prove uniqueness of solutions with respect to the whole space of continuous functions. Furthermore, the paper provides new sufficient conditions for the existence of solutions that can be applied to fairly general models. It is also proven that the fixed point coincides with the value function and that it can be approached by successive iterations of the Bellman operator.

KEYWORDS: Dynamic programming, Bellman equation, fixed point theorem, k-local contraction.

1. INTRODUCTION

Many economic problems can be formulated as dynamic optimization models whose ultimate representation is a recursive dynamic program. Dynamic programming techniques and recursive methods, because of their wide applicability in such problems, have proven to be very important tools for solving different dynamic models arising from almost all branches of Economics: from Consumer Theory and Endogenous Growth, to Public Finance and Investment Theory, among others. All of these fields, which are central issues in Economics, allow us to understand efficient allocations in time of goods and resources.

In Stokey, Lucas, and Prescott (1989) the recursive approach is developed systematically, and is applied to dynamic economic problems where time plays an essential role. However, with few exceptions, the theory is based on the boundedness of the return function along feasible paths. In spite of this, there have been different studies allowing unbounded returns, such as Boyd (1990), Streufert (1990, 1998), and Becker and Boyd (1997) for dynamic programming with recursive utility; Alvarez and Stokey (1998) for the special class of homogeneous programs; Nakajima (1999) and the recent paper of Le Van and Morhaim (2002) for additive and separable utilities. It is important to disregard the hypothesis of boundedness since many interesting problems present unbounded return functions, such as the models of endogenous growth in Romer (1986), Lucas (1988), or Jones and Manuelli (1990).

[1]The research of J. P. Rincón-Zapatero was supported by Project VA108/01 of Junta de Castilla y León and Project BFM2002–00425 (FEDER) of Ministerio de Ciencia y Tecnología, Spain. C. Rodríguez-Palmero gratefully acknowledges financial support from Junta de Castilla y León, Project VA31/01. We are indebted to a co-editor and two anonymous referees for valuable comments and suggestions.

The approach adopted by Boyd (1990), and further developed by Durán (2000), is based on the introduction of a weighted norm in a certain space of continuous functions, thus obtaining the contraction property for the Bellman operator. Although this approach addresses the unbounded case, and can prove useful in specific problems, it seems difficult to apply in a general context and, more importantly, uniqueness of solutions is only obtained with respect to a limited class of continuous functions. Streufert (1990) introduces the notions of lower and upper convergence, leading to the concept of biconvergence. He defines the notion of admissibility and proves that under the assumption of biconvergence, the value function is the unique admissible solution to the Bellman equation. One limitation of his approach, however, is that he focuses exclusively on capital accumulation problems. Furthermore, when lower convergence fails, as in the unbounded below case, he only obtains upper semicontinuity for the value function.

The analysis carried out by Alvarez and Stokey (1998) applies to homogeneous programs. These authors present sufficient conditions for the existence of solutions to the Bellman equation, although the associated operator is not a contraction in the normed space they consider.

Nakajima (1999) also makes a contribution to the subject where the contraction or metric approach is avoided. It seems that the hypotheses proposed in his paper on convexity and monotonicity are more demanding than is required to prove the existence of a fixed point of the Bellman operator that coincides with the value function.

The method proposed in Le Van and Morhaim (2002) is based on the well known fact that the value function is a solution to the Bellman equation (see Stokey, Lucas, and Prescott (1989)). They impose conditions to obtain upper semicontinuity, and then further assumptions to achieve lower semicontinuity of the value function.

The consideration of the space of all continuous functions allows us to improve previous research in some ways. Firstly, our conditions for the existence of a fixed point are easier to test than those found in other papers. Secondly, in some instances we prove that the solution to the Bellman equation is unique in the whole class of continuous functions and finally, we give new existence theorems that can be applied to fairly general models.

Our approach is mainly based on metric fixed point theory. However, instead of considering normed space of functions, we focus on metric spaces, which are different depending on the characteristics of the problem. When the utility function is continuous on the technological set, we introduce two different complete metrics in the space of continuous functions by means of a numerable family of seminorms. It is then shown that the Bellman operator is, roughly speaking, a contraction. However, these metrics are not satisfactory in the unbounded below case as the contraction property is obtained only if the discounting factor is close to zero. In fact, the consideration of seminorms is clearly impossible in cases where the utility function takes the value $-\infty$ at some points. For our purpose, it is very convenient to distinguish between two different types of unbounded below programs: on one hand those programs where the utility function is not bounded below but is continuous on the technological set, and on the other hand, those where the value function can take the value $-\infty$ at some points. For the former case we have truncated the technology correspondence, approaching the fixed point by means of a sequence of fixed points of the truncated problems. For the latter, we return to the metric approach, but considering a numerable family of semidistances instead of seminorms, suitable for our purposes. If discounting and monotonicity are the main ingredients in the proof of the contraction properties when using seminorms, convexity and monotonicity are the suitable properties for the type of semidistances we consider. A relevant characteristic of our approach is that the utility function can be unbounded above and below simultaneously.

The common critique to the contraction approach in the unbounded below case is based on the two following points: (i) given that more than one solution to the Bellman equation could exist, the contraction techniques are meaningless and (ii) the curvature of the felicity function near problematic points could make the consideration of a norm on the space of functions impracticable. However, a more detailed analysis of the problem provides us with substantial information to argue against the aforementioned points. With respect to the first point, we can choose an appropriate set of functions to which the value function belongs and, as regards the second point, if the supremum norm is not adequate, we can still define another suitable metric. Thus, in our opinion the metric approach is very useful and efficient given that one can choose the adequate metric in the space of continuous functions, to make use of the right properties of the operators and then to apply the contraction technique to the operators.

Next we explain our main results. The main contributions of this paper are (i) to show existence and uniqueness of the solution to the Bellman equation in the class of continuous functions, whenever growth rates of the technology correspondence are bounded by one in the long run (Theorem 3), or if this last condition does not hold, when the discounting factor satisfies suitable bounds (Theorem 4), (ii) to prove existence of solutions when the state space is a closed, convex, and comprehensive subset of $\mathbb{R}^l_+$, the technology correspondence satisfies a property of monotonicity, and some technical assumptions are imposed on the instantaneous return function (Theorem 5), (iii) to show existence of solutions for problems where the return function can take the value $-\infty$ (Theorem 6), and (iv) to prove in all the above cases that the value function coincides with the fixed point and that it can be approximated by the sequence of successive iterations of the Bellman operator.

The paper is structured as follows: In Section 2 we present two fixed point theorems (Theorem 1 and Theorem 2) that are based on the Contraction Principle of Banach. In Section 3 we show that the aforementioned theorems can be applied to the Bellman operator in some circumstances. When the result is not applicable, or if the discounting factor is constrained, we can still recover existence of a fixed point with further hypotheses for the return function and the technology correspondence by means of two different approaches. The first is based on approximating the fixed point, whereas the second relies on the definition of a family of semidistances. Finally, Section 4 concludes with some additional remarks. All proofs can be found in the two appendixes.

2. TWO FIXED POINT THEOREMS

Through the paper X will be a topological space such that $X = \bigcup_j K_j$, where $\{K_j\}$ is a countable increasing sequence of nonempty and compact subsets of X such that for all compact subset K of X, there exists j with $K \subseteq K_j$. Let $C(X)$ denote the set of all continuous functions over X with images in $\mathbb{R}$. For each distance function (metric) $d_{\mathbb{R}}$ defined on $\mathbb{R}$ we can define a countable family of semidistances $\{d_j\}$ on $C(X)$ given by

$$d_j(f, g) = \max_{x \in K_j} d_{\mathbb{R}}(f(x), g(x)). \tag{1}$$

A set $A \subseteq C(X)$ is said to be bounded if there is a sequence $\{m_j\}$, $m_j < \infty$, such that $d_j(f, g) \leq m_j$ for all $f, g \in A$, and for all $j \in \mathbb{N}$. If $d_{\mathbb{R}}$ is a metric induced by a norm, then the above notion of boundedness coincides with the following. There is a sequence $\{m_j\}$, $m_j < \infty$, such that $d_j(f, 0) \leq m_j$ for all $f \in A$, for all $j \in \mathbb{N}$. The set A is closed on $C(X)$ if it is closed with respect to the topology generated by the family of semidistances $\{d_j\}$.

A metric d can be defined on $C(X)$ in terms of $\{d_j\}$ as follows:

$$d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f, g)}{1 + d_j(f, g)} \quad \text{for all } f, g \in C(X). \tag{2}$$

It is well known that the metric d induces the same topology on $C(X)$ as the family $\{d_j\}$. If $C(X)$ is complete with respect to the topology generated by $\{d_j\}$, then it is easy to verify that $(C(X), d)$ is a complete metric space. Moreover, the topology generated by the metric d is not normable. Furthermore, if $d_{\mathbb{R}}$ is the Euclidean distance, then the semidistances are in fact seminorms, and convergence in distance d means uniform convergence on compacta.[2]

Next, we introduce a definition characterizing the operators on $C(X)$ that we will consider throughout this paper.

DEFINITION 1: Given $k \in \{0, 1\}$, an operator $T : C(X) \to C(X)$ is a *k-Local Contraction (k-LC) relative to* A, $A \subseteq C(X)$, if and only if $d_j(Tf, Tg) \leq \beta_j d_{j+k}(f, g)$ for all $j \in \mathbb{N}$ and for all $f, g \in A$, where $0 \leq \beta_j < 1$ for all $j \in \mathbb{N}$. When $A = C(X)$ we simply call T a k-LC.

[2] All these facts can be found in Willard (1970).

By definition, it is clear that a 0-LC is also a 1-LC. Our main objective in this section is to prove that under suitable hypotheses a 0-LC operator T has a unique fixed point on $C(X)$, although T need not be a contraction on the metric generated by the family of semidistances. Proposition 1 below shows that a 0-LC is a nonexpansive mapping on $C(X)$ and is a contraction over bounded subsets of $C(X)$. The reader must be aware that it is not possible to apply the results of nonexpansive maps developed by Browder (1965) as $C(X)$ is not normable and hence is not a Banach space. However, whenever a 0-LC maps a closed and bounded subset of continuous functions into itself, we can assure the existence of a unique fixed point on $C(X)$, as we show in Theorem 1. On the other hand, the corresponding result for a 1-LC, which is stated in Theorem 2, is more limited and does not operate in the whole space of continuous functions. Instead, we define in such a case a metric that is well defined on a certain closed and bounded subset of continuous functions.

PROPOSITION 1: *Let $T : C(X) \to C(X)$ an operator.*
(a) *If T is a 0-LC, then for each $f, g \in C(X)$ there exists a constant $\alpha_{f,g} \in [0, 1)$, depending on f and g, such that*

$$d(Tf, Tg) \le \alpha_{f,g}\, d(f, g).$$

(b) *If T is a 0-LC relative to A, a bounded subset of $C(X)$, then there exists a constant $\alpha \in [0, 1)$, independent of f and g, such that*

$$d(Tf, Tg) \le \alpha\, d(f, g) \quad \text{for all } f, g \in A.$$

The bound $\alpha_{f,g}$ appearing in (a) of Proposition 1 is not uniform over $C(X)$; that is, it depends on the particular choice of f and g. Therefore, we cannot conclude that T is a contraction, but it only satisfies $d(Tf, Tg) < d(f, g)$ for all $f, g \in C(X)$. The following example illustrates this fact: Let us consider the set $X = \mathbb{R}$, and the operator $T : C(X) \to C(X)$ given by $Tf = \frac{1}{2}f$ for all $f \in C(X)$. Let $\{K_j\}$ be any countable increasing sequence of nonempty and compact subsets of $\mathbb{R}$ such that $\mathbb{R} = \bigcup_j K_j$. It is trivial that T is a 0-LC, and (a) of Proposition 1 holds. On the other hand, let us consider, for each $n \in \mathbb{N}$, the constant function $f \equiv n$; then it follows that $d(Tf, T0) = n/(2 + n)$ and $d(f, 0) = n/(1 + n)$, so for any $\sigma \in (0, 1)$, we have $d(Tf, T0) = ((1 + n)/(2 + n))d(f, 0) > \sigma d(f, 0)$ for n large enough. As a consequence, T is not a contraction on $C(\mathbb{R})$ although it presents a unique fixed point, the null function, in agreement with Theorem 1, which is the main result in this section and is stated next.

Notice that the closed and bounded subset $\{f \in C(X) : d_j(f, 0) \le d_j(T0, 0)_{1-\beta_j}$ for all $j \in \mathbb{N}\}$ is mapped into itself by any 0-LC. Hence, what the following result in fact shows is the existence of a unique fixed point on $C(X)$, for any 0-LC.

THEOREM 1: *Let* $T: C(X) \to C(X)$ *such that* T *is a* 0-*LC and* $T: A \to A$, *where* A *is a closed and bounded subset of* $C(X)$. *Then the following hold*:

(a) T *is a contraction on* A *and admits a fixed point* $\hat{f}$, $\hat{f} \in A$, *that is unique on* $C(X)$;

(b) *for any* $f \in C(X)$, $T^n f \xrightarrow{d} \hat{f}$ *as* $n \to \infty$.

REMARK 1: (i) One important aspect from a computational point of view is whether the fixed point can be approached by successive iterations of the operator T, over any point of $C(X)$. It is worth noting that the sequence of iterates of a 0-LC T over any function $f \in C(X)$ converges to the fixed point, although T need not be a contraction.

(ii) It is important to note that the existence of the fixed point is still guaranteed if T is only a 0-LC relative to a closed set $A \subseteq C(X)$ and not over the whole of $C(X)$. However, it is not then possible to assure uniqueness of the fixed point on $C(X)$. Also in this case, convergence of the successive approximations from an arbitrary element of $C(X)$ can fail.

Let us now consider a 1-LC operator T. Suppose that $T: A \to A$, with A a closed and bounded subset of $C(X)$. Let $\{m_j\}$ be a sequence of real numbers such that $d_j(f, g) \leq m_j$ for all $f, g \in A$, and for all $j \in \mathbb{N}$. For $c \in \mathbb{R}$ satisfying $c > 1$ and $\sum_j c^{-j} m_j < \infty$ we can define the distance

$$d_c(f, g) = \sum_{j=1}^{\infty} c^{-j} d_j(f, g),$$

which is well defined on A. It follows that (A, d_c) is a complete metric space and that convergence with respect to d_c means uniform convergence on compacta of X, whenever $d_{\mathbb{R}}$ is the Euclidean distance.

THEOREM 2: *Let* $T: A \to A$ *such that* T *is a* 1-*LC relative to* A, *where* A *is a closed and bounded subset of* $C(X)$ *such that* d_c *is well defined on* A *for some* $c > 1$, $\sup \beta_j = \beta < 1$ *and* $c\beta < 1$. *Then the following hold*:

(a) T *is a contraction on* A *and admits a unique fixed point* $\hat{f}$ *on* A;

(b) *for any* $f \in A$, $T^n f \xrightarrow{d_c} \hat{f}$ *as* $n \to \infty$.

REMARK 2: It is clear that we could have extended the definitions and results above to more general classes of functions other than continuous. For example, if we define the metric by means of a family of seminorms, we can consider functions (not necessarily continuous) that are bounded on bounded subsets of X. In this case we have to replace max by sup in the definition of the seminorms. It is easy to show that this new metric space is also complete, thus enabling us to rule out the hypothesis of continuity for the utility function defining the Bellman equation. Within the literature on this subject, interesting problems arise, such as those of multisector models, which do not meet the assumption of continuity (see Dutta and Mitra (1989)).

3. APPLICATIONS TO DYNAMIC PROGRAMMING

In this section we study dynamic programming problems setting as reduced form models. That is to say, the single-period reward depends only on the vector of state variables at the beginning and end of the period. The dynamic optimization problem consists in solving the following maximization problem:

$$\begin{aligned} & v^\star(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \quad \text{subject to} \\ & x_{t+1} \in \Gamma(x_t) \qquad\qquad (t = 0, 1, 2, \ldots), \\ & x_0 \in X \quad \text{fixed}, \end{aligned} \tag{3}$$

where X is a subset of $\mathbb{R}^l$, $U : \text{Graph}(\Gamma) \to \mathbb{R}$ is the return function, $\beta \in (0, 1)$ is the discounting factor, $\Gamma : X \to 2^X$ is the technological correspondence giving the set of admissible actions from any $x \in X$, $v^\star$ is the value function, and $v^\star(x_0)$ is the optimal value as a function of the initial condition x_0. Let us consider the space $Z = X \times X \times \cdots$, and define $\Pi : X \to Z$ by

$$\Pi(x_0) = \left\{\tilde{x} = (x_t) = (x_0, x_1, \ldots) \in Z \mid x_{t+1} \in \Gamma(x_t),\ t = 0, 1, \ldots\right\}, \qquad x_0 \in X.$$

For any $\tilde{x} \in \Pi(x_0)$, let $S(\tilde{x}) = \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1})$ be the total discounted returns. The following assumptions are typically made in this context:

(DP1) Γ *is nonempty, continuous and compact valued*.

(DP2) $U : \text{Graph}(\Gamma) \to \mathbb{R}$ *is continuous*.

The above hypotheses enable the application of Berge's Theorem of the Maximum, and consequently, the Bellman operator

$$\mathcal{B}f(x) = \max_{y \in \Gamma(x)} (U(x, y) + \beta f(y))$$

is well defined on the space of continuous functions on X. There is a close connection between a solution of the Bellman equation, $\mathcal{B}f = f$, and the value function. With suitable conditions, a fixed point of $\mathcal{B}$ is the value function of the problem (3), and, conversely, if the value function is upper semicontinuous and finite, then it is a solution of the Bellman equation as is shown in Stokey, Lucas, and Prescott (1989).

We will now proceed to introduce the notation that will be used in the analysis of the maximization problem (3). Given a return function U, and a technological correspondence Γ, we denote

$$\Gamma(K_j) = \bigcup_{x\in K_j} \Gamma(x).$$

It should be noted that, under assumptions (DP1), $\Gamma(K_j)$ is compact. We also define the function $\psi(x) = \max_{y\in\Gamma(x)} U(x, y)$, for all $x \in X$. Under the conditions (DP1) and (DP2), ψ is continuous, by Berge's Theorem of the Maximum.

3.1. *General Case*

In order to exploit the properties of the Bellman operator, monotonicity and discounting, we set as the real metric $d_{\mathbb{R}}$ the Euclidean distance, $d_{\mathbb{R}}(x, y) = |x - y|$. From this metric (norm) a family of semidistances (seminorms) $\{d_j\}$ on $C(X)$ is obtained, defined as in (1) by

$$d_j(f, g) = \max_{x\in K_j} |f(x) - g(x)| = \|f - g\|_{K_j} \qquad (d_j(f, 0) = \|f\|_{K_j}).$$

As we have already justified, $C(X)$ is a complete metric space with respect to the following metric d, defined as in (2) by

$$d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|f - g\|_{K_j}}{1 + \|f - g\|_{K_j}}.$$

A natural application of Theorem 1 for the operator appearing in the context of dynamic programming is the following result. The proof relies on the properties of monotonicity and discounting of the Bellman operator; conditions already used by Blackwell (1965) and Denardo (1967).

THEOREM 3: *Let $\mathcal{B}$ be a Bellman operator satisfying* (DP1) *and* (DP2) *so that there exists a countable increasing sequence $\{K_j\}$ of nonempty and compact subsets of X with $X = \bigcup_j K_j$ satisfying $\Gamma(K_j) \subseteq K_j$ for all $j \in \mathbb{N}$. Then the following hold*:

(a) *The Bellman equation has a unique solution $\hat{f}$ on $C(X)$. Furthermore, $\hat{f}$ satisfies*

$$\|\hat{f}\|_{K_j} \le \frac{\|\psi\|_{K_j}}{1 - \beta} \quad \textit{for all } j \in \mathbb{N}.$$

(b) *The value function $v^\star$ is continuous and coincides with the fixed point $\hat{f}$.*

(c) *For any $f \in C(X)$, $\mathcal{B}^n f \xrightarrow{d} v^\star$ as $n \to \infty$.*

The fulfillment of the assumption $\Gamma(K_j) \subseteq K_j$ for every j is valid in many cases provided we make a good choice of the family $\{K_j\}$. This is easy to see in a one-sector model. Let f denote the one-sector production function and assume that this technology is productive ($f(x) > x$ for some $x > 0$) and there is a maximum sustainable stock $b = f(b) > 0$. The correspondence Γ is defined by the relation $\Gamma(x) = [0, f(x)]$. It is clear that for any productive capital stock j where $f(j) > j$ and $K_j = [0, j]$, then it will be the case that $\Gamma(K_j) \nsubseteq K_j$. However, if $K_j = [0, bj]$ is chosen instead, then $\Gamma(K_j) \subseteq K_j$ is obtained. Another family of examples where $\Gamma(K_j) \subseteq K_j$ holds are those in which $\Gamma(x) = [0, x]$. In this case it is clear that such a property is true whenever $K_j = [0, j]$, for any continuous return function. Example 1 below shows how to apply Theorem 3 to more general contexts where the correspondence Γ is bounded from above by one in the long run.

The assumption $\Gamma(K_j) \subseteq K_j$ can sometimes be weakened in applications as the subsequent remarks and Example 2 demonstrate. In particular, the next remark is crucial in this respect.

REMARK 3: The proof of Theorem 3 shows

$$\|\mathcal{B}f - \mathcal{B}g\|_{K_j} \leq \beta \max_{y \in \Gamma(K_j)} |f(y) - g(y)| \quad \text{for all } f, g \in C(X).$$

This means that if we find a sequence $\{\alpha_j\}$ and a bounded and closed subset of continuous functions $A \subseteq C(X)$, such that

$$\max_{y \in \Gamma(K_j)} |f(y) - g(y)| \leq \alpha_j \, \|f - g\|_{K_j} \quad \text{for all } f, g \in A, \tag{4}$$

with $\beta \sup \alpha_j < 1$ and $\mathcal{B}$ maps A into A, then $\mathcal{B}$ is a 0-LC relative to A, and consequently it has a unique fixed point in A. Inequality (4) is the critical step in the proof of Theorem 3 and assuming $\Gamma(K_j) \subseteq K_j$ for all j is just a convenient means to that end, although it is not the only one. In the latter situations we show that there are other ways of showing that (4) holds, as in the familiar homogeneous case, which is studied in Example 2.

EXAMPLE 1 (Technological correspondence with growth rate bounded by one in the long run): The hypotheses of Theorem 3 are fulfilled in problems such that X is closed, and the correspondence Γ satisfies the following:

$$\text{There exists } R > 0 \text{ such that } \|x\| \geq R \Rightarrow \|y\| \leq \|x\| \text{ for all } y \in \Gamma(x).$$

In this case, to apply Theorem 3 it is sufficient enough to consider the countable increasing sequence $\{K_j\}$ of nonempty and compact subsets of X given by[3] $K_j = X \cap \overline{B(0, jr)}$, where $r = \max\{\|y\| : y \in \Gamma(\overline{B(0, R)} \cap X)\}$. In consequence, Theorem 3 is applicable to any dynamic optimization problem with

[3] $B(a, r)$ denotes the ball of $\mathbb{R}^l$ centered at a and radius $r > 0$. $\overline{B(a, r)}$ denotes its closure.

a technological correspondence with superlinear growth on a bounded subset of X, whenever the growth rate is bounded by one in the long run.

EXAMPLE 2 (Homogeneous case): Alvarez and Stokey (1998) study the case where U is homogeneous of degree $\theta \in \mathbb{R}$ and Graph(Γ) is a cone. We will distinguish the cases $\theta \geq 0$ and $\theta < 0$.

In the first case, we suppose that $\|y\| \leq \gamma\|x\|$ for all $y \in \Gamma(x)$, for all $x \in X$, for some $\gamma > 1$ satisfying $\beta\gamma^\theta < 1$ (when $\gamma \leq 1$, it is clear that $\Gamma(K_j) \subseteq K_j$ and Theorem 3 applies). A suitable countable family of compact sets is given by $K_j = X \cap \overline{B(0, j)}$. That way, it is straightforward to show that (4) is satisfied with $\alpha_j = \gamma^\theta$ for all $j \in \mathbb{N}$ and with the set A defined by

$$A = \Big\{ f \in C(X) : f \text{ is homogeneous of degree } \theta, \\ \|f\|_{K_j} \leq \frac{\|\psi\|_{K_j}}{(1-\gamma^\theta\beta)} \ \forall j \in \mathbb{N} \Big\}. \tag{5}$$

In the second case, we adopt the hypotheses $0 \notin X$, and $\|y\| \geq \xi\|x\|$ for all $y \in \Gamma(x)$, for all $x \in X$, for some $\xi > 0$ satisfying $\beta\xi^\theta < 1$. Notice that there is no need to adopt any assumption on the growth of the return function. Now the countable family of compact sets is given by $K_j = \{x \in X : 1/j \leq \|x\| \leq j\}$. Note that $\Gamma(K_j) \nsubseteq K_j$ because $0 \notin X$. An easy calculation once again shows that (4) is satisfied with the set A defined as in (5) and $\alpha_j = \xi^\theta$ for all $j \in \mathbb{N}$. To see this, let $x \in K_j$, $y \in \Gamma(x)$, and f is homogeneous of degree θ. Then

$$\begin{aligned}
|f(y)| &= (\|y\|j)^\theta \left| f\left(\frac{y}{\|y\|j}\right)\right| \\
&\leq (\xi\|x\|j)^\theta \left| f\left(\frac{y}{\|y\|j}\right)\right| && (\text{because } \|y\| \geq \xi\|x\| \text{ and } \theta < 0) \\
&\leq \xi^\theta \left| f\left(\frac{y}{\|y\|j}\right)\right| && \left(\text{due to } \|x\| \geq \frac{1}{j}\right) \\
&\leq \xi^\theta \|f\|_{K_j} && \left(\text{because } \frac{y}{\|y\|j} \in K_j\right).
\end{aligned}$$

Thus $\max_{y\in\Gamma(K_j)} |f(y)| \leq \xi^\theta \ \|f\|_{K_j}$, and hence $\max_{y\in\Gamma(K_j)} |f(y) - g(y)| \leq \xi^\theta\|f - g\|_{K_j}$ for all $f, g \in A$. In Subsection 3.3 we will give another result replacing the condition $\|y\| \geq \xi\|x\|$ for all $y \in \Gamma(x)$, for the weaker one: there exists some $y \in \Gamma(x)$ satisfying $\|y\| \geq \xi\|x\|$.

For any compact technological correspondence Γ, to build a sequence of increasing compact sets $\{K_j\}$ covering X and such that $\Gamma(K_j) \subseteq K_{j+1}$ is always possible. In other words, for any Bellman operator $\mathcal{B}$, the following property

$$\|\mathcal{B}f-\mathcal{B}g\|_{K_j} \le \beta \max_{y\in\Gamma(K_j)} f(y)-g(y)| \le \beta\, \|\mathcal{B}f-\mathcal{B}g\|_{K_{j+1}}$$

$$\text{for all } f, g \in A,$$

holds for an appropriate sequence of compact sets $\{K_j\}$, and consequently $\mathcal{B}$ is always[4] a 1-LC on $C(X)$. In order to apply Theorem 2 in such a case, we need to find a bounded and closed set of continuous functions A, which is mapped into itself by $\mathcal{B}$ and such that the metric d_c is well defined on it for some $c > 1$.

THEOREM 4: *Let $\mathcal{B}$ be a Bellman operator satisfying* (DP1) *and* (DP2) *so that there exists a countable increasing sequence $\{K_j\}$ of nonempty and compact subsets of X with $X = \bigcup_j K_j$ satisfying $\Gamma(K_j) \subseteq K_{j+1}$ for all $j \in \mathbb{N}$. Assume that the series $\sum_{j=1}^{\infty} c^{-j}\|\psi\|_{K_j}$ is convergent for some $c > 1$, satisfying $c\beta < 1$. Then the following hold*:

(a) *There exists a closed and bounded subset $A \subseteq C(X)$ such that the Bellman equation has a unique solution $\hat{f}$ on A. Furthermore, $\hat{f}$ satisfies*

$$\|\hat{f}\|_{K_j} \le \sum_{l=j}^{\infty} \beta^{l-j}\|\psi\|_{K_l} \quad \textit{for all } j \in \mathbb{N}.$$

(b) *The value function $v^\star$ is continuous and coincides with the fixed point $\hat{f}$.*

(c) *For any $f \in A$, $\mathcal{B}^n f \xrightarrow{d_c} v^\star$ as $n \to \infty$.*

REMARK 4: Boyd's Theorem (1990) is based on the existence of a continuous and positive function φ satisfying $\sup_{x\in X} \psi(x)/\varphi(x) = d < \infty$ and $\max_{y\in\Gamma(x)} \varphi(y) \le \gamma\varphi(x)$ for all $x \in X$, for some $\gamma > 0$ such that $\beta\gamma < 1$. Thus, in those cases where the construction of the sequence $\{K_j\}$ of nonempty and compact sets that cover the state space and that satisfy $\Gamma(K_j) = K_{j+1}$ being possible, the hypotheses of Boyd's Theorem are sufficient to apply Theorem 4. Actually, the bound

$$\|\psi\|_{K_{j+1}} \le d\,\|\varphi\|_{K_{j+1}} \le d\gamma\,\|\varphi\|_{K_j} \le \cdots \le d\gamma^j\,\|\varphi\|_{K_1}$$

holds for some constant $d > 0$, where $\beta\gamma < 1$. It is then obvious that there exists c with $1 < c < 1/\beta$ and such that the series in the statement of Theorem 4 converges.

[4]As an example, take the one-sector linear model with production function $f(x) = \gamma x$ for $\gamma > 1$. Let $K_j = [0, \gamma^{j-1}]$ for $j = 1, 2, \ldots$ and note that $\Gamma(K_j) = [0, \gamma^j] = K_{j+1}$. The rest of Example 4 holds for this model with appropriate restrictions on the return function.

EXAMPLE 3 (Bounded returns): When ψ is bounded—as when the instantaneous return function U is bounded—Theorem 4 shows that the solution to the Bellman equation is unique in the class of bounded functions. This is the well known classical result established by Blackwell (1965) and Denardo (1967).

EXAMPLE 4 (Returns that are bounded below but not above; Nakajima (1999)): Using Theorem 4, we can prove and extend the results of Nakajima regarding returns that are bounded below but not above. In addition to continuity of U and Γ, this author originally uses the following assumptions:

(a) U is increasing in x and decreasing in y.

(b) Γ is monotone increasing.

(c) $A \leq U(x,y) \leq B\ \|x\|^\theta + b$ for all $(x,y) \in \mathrm{Graph}(\Gamma)$, for some constant A, and some positive constants B, b, and θ, with $\theta < 1$.

(d) There exists a vector $x^u \in X$ such that for each $x \in X$, $x \neq 0$, with $\|x\| \leq \|x^u\|$, there exists $y \in \Gamma(x)$ satisfying $\|y\| \geq \alpha\|x\|$ for some $\alpha > 1$ with $\alpha^\theta\beta < 1$.

Under these conditions, Nakajima (1999) shows uniqueness of solutions to the Bellman equation—in fact, existence of the value function—on a certain space, and also convergence to it for the successive iterations of the Bellman operator from any initial function in such space. The approach of this author does not make use of the Contraction Mapping Theorem. As we shall see now, his assumptions allow us to apply Theorem 4, and consequently, the Bellman operator is a contraction in the metric space we have considered. Let us observe that neither the monotonicity of Γ and U, nor the restrictions $\theta < 1$ and $\alpha > 1$ are necessary requirements to obtain such a conclusion.

Our results permit us to extend the above sufficient conditions. Let us define the sequence $\{K_j\}$ of compact subsets by

$$K_j = \Gamma(K_0) \cup \overline{B(0, \|x_u\|\alpha^j)}, \quad \text{where} \quad K_0 = \overline{B(0, \|x_u\|)}.$$

It then follows that $\Gamma(K_j) \subseteq K_{j+1}$. Moreover, since $\alpha^\theta\beta < 1$, there exists $c > \alpha^\theta$ $(c > 1)$, such that $\alpha^\theta\beta < c\beta < 1$. For this value of c we have

$$\begin{aligned}\sum_{j=1}^{\infty} c^{-j}\|\psi\|_{K_j} &\leq \sum_{j=1}^{\infty} c^{-j}\max\bigl\{|A|, B\|x\|_{K_j}^\theta + b\bigr\} \\ &\leq \sum_{j=1}^{\infty} c^{-j}\max\bigl\{|A|, B\|x\|_{\Gamma(K_0)}^\theta + b, B\|x_u\|^\theta\alpha^{\theta j} + b\bigr\} < \infty.\end{aligned}$$

Hence Theorem 4 is applicable. It is also evident that the lower bound of the return function U can be removed and replaced by the following:

$|U(x, y)| \leq B\|x\|^{\theta} + b$, for all $(x, y) \in$ Graph(Γ). This constitutes a further generalization of Nakajima's result.

The approach we have followed up to now applies to rather general problems. However, in some instances it is not fully satisfactory, as in the so-called unbounded below programs, which are studied in the following subsections.

3.2. *Unbounded Below Case (Truncation Approach)*

Even if there is no sequence of compact sets $\{K_j\}$ satisfying the hypotheses of Theorem 3, in some cases we can prove the existence of a solution to the Bellman equation. Such a procedure consists of considering truncations Γ_i of the technological correspondence Γ, to apply Theorem 3 to each Γ_i and then to take limits as $i \to +\infty$ in the sequence of associated fixed points.

Let us suppose that X and Γ satisfy the following conditions:

(DP3) *X is a nonempty, closed, convex and comprehensive subset*[5] *of* $\mathbb{R}^l_+$.

(DP4) Graph(Γ) *satisfies*: $\hat{x} \geq x \Rightarrow (\hat{x}, y) \in$ Graph(Γ), *for all* $(x, y) \in$ Graph(Γ).

Note that (DP4) does not imply free disposal for Graph(Γ). In particular, $(0, 0)$ may not belong to Graph(Γ). (DP4) means that if an action is available today, then it is available forever. Let us define $X_i = \overline{B(0, i-1)} \cap X$, $i \in \mathbb{N}$. It is clear that $X_i \subseteq X_{i+1}$ for all $i \in \mathbb{N}$, and $X = \bigcup_i X_i$. For any $x \in X$, let us denote $P_{X_i}(x)$ the (unique) projection of x on the convex set X_i. For each $i \in \mathbb{N}$, we define the (truncated) correspondence Γ_i as follows:

$$\Gamma_i(x) = \begin{cases} \Gamma(x), & \text{if } x \in X_i, \\ \Gamma(P_{X_i}(x)), & \text{if } x \notin X_i. \end{cases}$$

From Berge's Theorem Γ_i is continuous and compact valued and $\Gamma_i(x) \subseteq \Gamma_{i+1}(x)$ for all $x \in X$, $i \in \mathbb{N}$. Furthermore, (DP3), (DP4) and some properties of the projection map on closed and convex subsets of $\mathbb{R}^l_+$ imply $\Gamma_i(x) \subseteq \Gamma(x)$ for all $x \in X$, $i \in \mathbb{N}$. This is proven in (i) of Appendix B.

To each Γ_i we can associate a truncated Bellman operator $\mathcal{B}_i$ on $C(X)$ defined by

$$\mathcal{B}_i f(x) = \max_{y \in \Gamma_i(x)} (U(x, y) + \beta f(y)).$$

From Example 1, $\mathcal{B}_i$ is a 0-LC on $C(X)$, with respect to a suitable sequence of compact sets $\{K^i_j\}$ depending on the index i (giving rise to different distances d_i on $C(X)$). Therefore, the conclusions of Theorem 3 are applicable to the operators $\mathcal{B}_i$. Let us denote f_i the unique fixed point of $\mathcal{B}_i$ on $C(X)$.

The following result establishes a necessary condition for the existence of fixed points of $\mathcal{B}$ in terms of the boundedness of the sequence $\{f_i\}$.

[5] X is comprehensive when the following is true: if $\hat{x} \in X$, $x \in \mathbb{R}^l_+$, and $x \leq \hat{x}$, then $x \in X$.

PROPOSITION 2: *Let X and $\mathcal{B}$ satisfy assumptions* (DP1) *to* (DP4). *If $\mathcal{B}$ has a fixed point on $C(X)$, then the sequence $\{f_i\}$ is bounded on $C(X)$.*

It can be easily proven that $\{f_i\}$ is an increasing family (see item (ii) of Appendix B). An immediate consequence of this and of the above proposition is the following result that might prove useful to conclude, at least heuristically, the nonexistence of solutions to the Bellman equation with the aid of computational methods.

COROLLARY 1: *Let X and $\mathcal{B}$ satisfy assumptions* (DP1) *to* (DP4). *If $\sup_{i\in\mathbb{N}} f_i(x) = \infty$ for some $x \in X$, then $\mathcal{B}$ has no fixed points on $C(X)$.*

It is important to note that the finiteness of $\hat{f} = \sup_{i\in\mathbb{N}} f_i = \lim_{i\to\infty} f_i$ is not sufficient for $\hat{f}$ to be a solution to the Bellman equation on $C(X)$ as this function is only lower semicontinuous as the supremum of continuous functions, and could be discontinuous. However, in (iii) of Appendix B, it is shown that it satisfies the equation

$$\hat{f}(x) = \sup_{y\in\Gamma(x)} \big(U(x, y) + \beta\hat{f}(y)\big), \tag{6}$$

whenever $\sup_{i\in\mathbb{N}} f_i$ is finite on X. Furthermore, the proof of Proposition 2 shows that $\hat{f}$ is the minimum function—continuous or not and in a pointwise sense—that satisfies (6). This property will be used in the proof of Theorem 5 below.

Our purpose is now to impose conditions such that the functions f_i converge uniformly over compact subsets of X to a fixed point of $\mathcal{B}$. In the following, for each $x_0 \in X$ and $i \in \mathbb{N}$, $\Pi_i(x_0)$ denotes the set of admissible paths (x_t) from x_0 such that $x_{t+1} \in \Gamma_i(x_t)$ for all t. $\Pi^0(x_0)$ stands for the subset of $\Pi(x_0)$ of all admissible paths $\tilde{x}$ from x_0 such that $S(\tilde{x})$ exists and $S(\tilde{x}) > -\infty$. The negative part of a real function f, denoted f^-, is defined by the expression $f^- = \min\{0, f\}$.

The norm of a function measures the size of the function in both positive and negative directions. Hence the growth of a function from below along the technological correspondence is as important as the growth from above. This motivates the constraint on the discount rate when we apply Theorem 4 to unbounded below programs. However, the Bellman operator is defined as a maximum, so intuitively the main difficulty in assuring the existence of a solution to the functional equation comes from the upper values of the return function. Theorem 5 avoids this kind of difficulty.

THEOREM 5: *Let X and $\mathcal{B}$ satisfy assumptions* (DP1) *to* (DP4), *as well as the further conditions*:

(i) *There exists an upper semicontinuous function g satisfying $f_i \le g$ for all $i \in \mathbb{N}$ and such that for all $\tilde{x} \in \Pi(x_0)$, all $x_0 \in X$, $\limsup_{t\to\infty} \beta^t g(x_t) \le 0$.*

(ii) *For each* $\tilde{x} = (x_t) \in \Pi^0(x_0)$, *all* $x_0 \in X$, *there exists* $a \in X$ *with* $a \in \Gamma(x_t) \cap \Gamma(a)$ *for all* t *large enough*, *and such that* $\lim_{t\to\infty} \beta^t U^-(x_t, a) = 0$.

Then the following hold:

(a) *The Bellman equation has a solution* $\hat{f}$ *on* $C(X)$. *Furthermore*, f_i *converge to* $\hat{f}$ *uniformly on compact subsets of* X.

(b) *The value function* $v^\star$ *is continuous and coincides with the fixed point* $\hat{f}$.

(c) $\hat{f} \leq f^\star$ *for any fixed point* $f^\star$ *of the Bellman operator*. *Actually*, $\hat{f}$ *is the unique solution satisfying* $\limsup_{t\to\infty} \beta^t \hat{f}(x_t) \leq 0$.

(d) *For any* $i \in \mathbb{N}$, $\mathcal{B}^n f_i$ *converges to* $v^\star$ *uniformly on compact subsets of* X *as* $n \to \infty$.

REMARK 5: (i) It is convenient to characterize those paths belonging to the set $\Pi^0(x_0)$. A necessary condition is that $\lim_{t\to\infty} \beta^t U^-(x_t, x_{t+1}) = 0$ and $\lim_{t\to\infty} \beta^t \psi^-(x_t) = 0$, which will be used in later examples. This observation is based on the inequalities $-\infty < \sum_{t=0}^\infty \beta^t U^-(x_t, x_{t+1}) \leq \sum_{t=0}^\infty \beta^t \psi^-(x_t)$.

(ii) A crucial assumption in Theorem 5 is the boundedness of the sequence $\{f_i\}$ by an upper semicontinuous function g satisfying (i). The existence of such a function can be asserted by means of a one side condition à la Boyd. Let us suppose that there exists an upper semicontinuous function $w : X \to \mathbb{R}_+$ satisfying

$$\psi \leq w, \quad \text{and}$$
$$\max_{y\in\Gamma(x)} w(y) \leq \gamma w(x) \quad \text{for all } x \in X,\ \gamma > 0, \quad \text{with } \beta\gamma < 1.$$

Then, if $f(x) \leq w(x)/(1-\beta\gamma)$, $\mathcal{B}f$ satisfies the same inequality

$$\begin{aligned}
\mathcal{B}f(x) &\leq \psi(x) + \beta \max_{y\in\Gamma(x)} f(y) \\
&\leq w(x) + \frac{\beta}{1-\beta\gamma} \max_{y\in\Gamma(x)} w(y) \\
&\leq \left(1 + \frac{\beta\gamma}{1-\beta\gamma}\right) w(x) \\
&= \frac{w(x)}{1-\beta\gamma}.
\end{aligned}$$

It is now obvious that each of the functions f_i are bounded in the same way, as the operator $\mathcal{B}_i$ is a contraction and the convergence of functions conserves the above property. We define the function g as $g(x) = w(x)/(1-\beta\gamma)$. Given $x_t \in \Gamma(x_{t-1})$, it follows $g(x_t) \leq \gamma g(x_{t-1})$, and hence by recurrence $\beta^t g(x_t) \leq (\beta\gamma)^t g(x_0)$ for $t \in \mathbb{N}$. Condition (i) in Theorem 5 is then obviously fulfilled. It is worth noting that the existence of a suitable function w bounding ψ does not require linear growth, either in the return function or in the technological correspondence.

(iii) Another useful observation is that in some problems the function f_1 is easily computed and so the successive iterates $\mathcal{B}^n f_1$ can be readily calculated. When $\Gamma(0) = \{0\}$, f_1 is given by $U(x, 0) + \beta f_1(0) = f_1(x)$ and $f_1(0) = U(0, 0)/(1 - \beta)$.

(iv) Of course, it would be of some interest to establish sufficient conditions in order for condition (ii) of the above theorem to hold. This is the case if either of the two following conditions holds:

- U is increasing in x. We have $U(x_t, 0) \geq U(0, 0)$. Hence for every $(x_t) \in \Pi(x_0)$, $0 \geq \lim_{t\to\infty} \beta^t U^-(x_t, 0) \geq \lim_{t\to\infty} \beta^t U^-(0, 0) = 0$.
- U is nonincreasing in y. In this case, $U(x_t, x_{t+1}) \leq U(x_t, 0)$. If $(x_t) \in \Pi^0(x_0)$, then $0 = \lim_{t\to\infty} \beta^t U^-(x_t, x_{t+1}) \leq \lim_{t\to\infty} \beta^t U^-(x_t, 0) \leq 0$.

Now we analyze some examples showing the scope of Theorem 5.

EXAMPLE 5: Let $U(x, y) = -mx + y$, with $m \geq 2$, $\Gamma(x) = [0, 2x]$, and $X = \mathbb{R}_+$. The conditions (DP1) to (DP4) are trivially fulfilled. The case $m = 2$ is rather pathological. The value function in this case is $v^\star \equiv 0$, but when $\beta > 1/2$, $\hat{f} = \sup f_i = -2x$. Truncation therefore provides an incorrect solution. Indeed, for all $x_0 > 0$, if we define $x_t = 2^t x_0$ for all t, the path (x_t) belongs to $\Pi^0(x_0)$ and $\lim_{t\to\infty} \beta^t U(x_t, a) = \lim_{t\to\infty} \beta^t(-m2^t x_0 + a) = -\infty$ for all $a \in \mathbb{R}_+$. Hence (ii) of Theorem 5 is not satisfied, as expected. However, the problem can be easily dealt with using Theorem 4, since in this case ψ is bounded—in fact, the zero function—so there are no constraints on β (see Example 3). Moreover, it is easy to check that Theorem 5 is applicable if $m > 2$ for every $\beta < 1$, although Theorem 4 needs $\beta < 1/2$. Let us observe that with Boyd's approach, a natural choice for φ is $\varphi(x) = 1 + x$, which imposes the limitation $\beta < 1/2$ over the discounting factor, a fact that can be easily computed.

Consider now the return function $U(x_1, x_2, y_1, y_2) = x_1^m - x_2^{2m} - y_1 - y_2$, $m \in \mathbb{N}$, and the technological correspondence $\Gamma(x_1, x_2) = [0, 2x_1] \times [0, x_2^2]$ defined on $X = \mathbb{R}_+^2$. In this example $\psi(x_1, x_2) = x_1^m - x_2^{2m}$ is unbounded below and above. Theorem 5 assures that the Bellman equation has a solution for all $\beta < 1/2^m$. Let us again compare this result with that obtained by means of Boyd's approach. A reasonable sensible selection for φ in this case seems to be $\varphi(x_1, x_2) = 1 + x_1^m + x_2^{2m}$. This choice implies $\beta = 0$. It can be seen that neither the results of Nakajima (1999) nor those of Le Van and Morhaim (2002) are applicable when $m \geq 2$, as the return function does not have linear growth.

EXAMPLE 6 (General quadratic return function): An important class of dynamic optimization problems are those in which the return function is quadratic. Linear-quadratic dynamic optimization programs appear, for instance, in many macroeconomics models (see Ljungqvist and Sargent (2000)), or as an approximation to the original decision model that allows us to compute an approximate solution. The attractive feature of a quadratic problem is that it is possible to give a closed form solution to the Bellman equation, whenever there are no constraints on the decision variables. The usual method is to postulate a quadratic functional form for the value function and then to determine the unknown coefficients. However, this method of guessing does not work in the presence of constraints, since the value function is no longer quadratic. Hence, it is of interest to decide whether there exists a solution to the Bellman equation.

Let us consider $U(x, y) = x'Ax + y'By + x'Cy + x'd + y'e$, $X = \mathbb{R}^n_+$, and Γ satisfying assumptions (DP1) to (DP4). The matrices A, B, C and the column vectors d, e are of order n. We make the following assumption:

$$\lim_{\|x\|\to\infty} \frac{\psi(x)}{\|x\|^2} = K < 0. \tag{7}$$

It turns out that ψ is bounded above because, in particular, it is counter-coercive (see Rockafellar and Wets (1998)) and so is the sequence $\{f_i\}$. Hence, item (i) in Theorem 5 holds. On the other hand, given $(x_t) \in \Pi^0(x_0)$, we know that $\lim_{t\to\infty} \beta^t \psi^-(x_t) = \lim_{t\to\infty} \beta^t \psi(x_t) = 0$, which, from (7), implies $\lim_{t\to\infty} \beta^t \|x_t\|^2 = 0$. It then follows that

$$\begin{aligned}\lim_{t\to\infty} \beta^t \|U(x_t, 0)\| &= \lim_{t\to\infty} \beta^t \|x_t' A x_t + x_t' d\| \\ &\leq \lim_{t\to\infty} \beta^t (\|x_t\|^2 \, \|A\| + \|x_t\| \, \|d\|) = 0,\end{aligned}$$

and consequently item (ii) in Theorem (5) holds as well. Hence, the hypotheses are fulfilled for all $\beta < 1$ and all the conclusions are applicable.

Let us observe that if we consider a weighted norm space where the Bellman operator is a contraction, the weighing function φ must be quadratic. In such a case, the condition appearing in Boyd's Theorem, $\beta \max_{y\in\Gamma(x)} \varphi(y) \leq \theta\varphi(x)$ for all $x \in X$, with $\theta < 1$, implies a constraint on β. For example, if we set the scalar case with $A = -(a + b)/2$, $B = -c/2$, $C = c$, $d = a$, $e = 0$, $a, b, c > 0$, and $\Gamma(x) = [0, 2x]$ (this is an example in Stokey, Lucas, and Prescott (1989, pp. 95, 96)), then Boyd's approach gives $\beta < 1/2$. At the same time, Theorem 4.14 in Stokey, Lucas, and Prescott (1989) is not of direct application in this case, as we do not know the functional form of the value function, and also since, with constraints, the successive iterates of the Bellman operator can be difficult to compute. On the other hand, to apply Theorem 5 to this problem, we need only to show that property (7) holds. Yet, $\psi(x) = ax - (b/2)\, x^2$, and therefore $\lim_{|x|\to\infty} \psi(x)/|x|^2 = -b/2 < 0$. Hence, all the conclusions of Theorem 5 are true, which in particular implies the existence of a fixed point coinciding with the value function, for all $\beta < 1$. Note that, if in this example we set $\Gamma(x) = [0, f(x)]$, with $f(x) \geq x$, then the same conclusions could be assured.

EXAMPLE 7 (Learning by doing): This is a very interesting example from an economic point of view and shares many features with the quadratic return problems that are unbounded below. Furthermore, it shows that another type of alternative truncation to those given can be possible in some problems. In this model a monopolist is producing a new product. The production function exhibits learning by doing, that is to say, the unit cost falls over time as the cumulative experience increases (we refer for details to Stokey, Lucas, and Prescott (1989) and references cited therein). The instantaneous profit function of the monopolist is given by

$$U(Q_t, Q_{t+1}) = (Q_{t+1} - Q_t)(\phi(Q_{t+1} - Q_t)) - \gamma(Q_{t+1} - Q_t, Q_t),$$

where Q_t denotes cumulative experience, $U : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is bounded above, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a stationary inverse demand function, and $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ relates unit cost to cumulative experience. The latter two functions are continuous and $\gamma(0, Q) = 0$ for all $Q \geq 0$. We replace the variables Q_{t+1} by y and Q_t by x. We first consider the Bellman operator associated to the problem as

$$\mathcal{B}f(x) = \sup_{y \geq x}\big((y - x)(\phi(y - x)) - \gamma(y - x, x) + \beta f(y)\big),$$

for f continuous and bounded above. We identify $X = \mathbb{R}_+$ and $\Gamma(x) = [x, \infty)$. Of course, Γ is not compact valued, so (DP1) is not fulfilled. On the other hand, (DP4) does not hold either. However, this is not a problem, as a few changes in the form of the truncations of the technology set allow us to obtain the desired conclusions. The truncations are now given by $\Gamma_i(x) = [x, i - 1]$, if $x \leq i - 1$ and $\Gamma_i(x) = \{x\}$ if $x \geq i - 1$, $i \in \mathbb{N}$. Notice that we do not need (DP4) to define these truncations. For every $i \in \mathbb{N}$ we can take the compact sets $K_j^i = [0, j(i - 1)]$, which in fact satisfy $\Gamma_i(K_j^i) = K_j^i$. The correspondences Γ_i verify (DP1) and the hypotheses in Theorem 3, so there exists a well defined sequence of continuous and bounded above functions $\{f_i\}$, such that the upper limit, $\hat{f}$, is lower continuous and satisfies the functional equation

$$\mathcal{B}\hat{f}(x) = \sup_{y \geq x}\big((y - x)(\phi(y - x)) - \gamma(y - x, x) + \beta \hat{f}(y)\big). \tag{8}$$

In order to prove continuity for the function $\hat{f}$, it is sufficient to justify that the hypotheses of Theorem 5 hold. First, it is clear that the property (i) is satisfied because U is bounded above, so the only remaining question is whether (ii) holds. However, in the proof of the aforementioned theorem, the constant action $a \in \Gamma(x_t)$ can be replaced in this case by the action $x_t \in \Gamma(x_t)$, whenever $\lim_{T\to\infty} \sum_{t=T+1}^{\infty} \beta^t U(x_t, x_t) \geq 0$, for every $(x_t) \in \Pi^0(x_0)$. This is obvious since $U(x_t, x_t) = 0$. Hence, the function $\hat{f}$ is continuous and satisfies the equation (8). Furthermore, since $f_1(x) = 0$ for all $x \in \mathbb{R}_+$, $\mathcal{B}^n 0$ converges to the value function, $v^\star$, uniformly on compact subsets of $\mathbb{R}_+$ as $n \to \infty$.

Finally, we prove that $\hat{f}$ satisfies the equation

$$\mathcal{B}\hat{f}(x) = \max_{y \geq x}\big((y-x)(\phi(y-x)) - \gamma(y-x, x) + \beta\hat{f}(y)\big).$$

Given that the maximum operation is not constrained to a compact subset, we need to impose some additional hypothesis over U, so that the maximum can be effectively attained and sup can be changed by max. To this end, we consider the following property

$$\text{(9)} \qquad \lim_{\|(x,y)\| \to +\infty} \frac{U(x, y)}{\|(x, y)\|} = -\infty,$$

whenever $(x, y) \in \text{Graph}(\Gamma)$. Then the upper sections $\{(x, y) \in \text{Graph}(\Gamma) : U(x, y) \geq r\}$ are compact and consequently U has a global maximum on the graph of the technological correspondence (see Rockafellar and Wets (1998)). The hypotheses put forward by Stokey, Lucas, and Prescott (1989) imply in particular this property for the benefit function, although in our framework there is no need to consider other concavity or smoothness assumptions. Taking property (9) into account, we know that given any $M > 0$, there exists $m > 0$ such that $\|(x, y)\| \geq m$ implies $U(x, y) < -M\|(x, y)\|$. Now consider the function $\mathcal{B}\hat{f}$ defined on X. Let $x \in X$, $x \geq m$; then

$$\begin{aligned}
\hat{f}(x) = \mathcal{B}\hat{f}(x) &= \sup_{y \geq x}\big(U(x, y) + \beta\hat{f}(y)\big) \\
&\leq \sup_{y \geq x}(-M\|(x, y)\| + \beta K) \quad (K \text{ is the upper bound for } \hat{f}) \\
&\leq -Mx + \beta K.
\end{aligned}$$

Thus, $\lim_{x \to +\infty} \mathcal{B}\hat{f}(x)/x = -\infty$, and therefore $\mathcal{B}\hat{f}$ has compact upper sections, so the maximum is effectively attained on $[x, \infty)$.

3.3. *Unbounded Below Case (Metric Approach)*

Our main concern in this subsection is to show that contraction techniques are still useful when the state space is $X = \mathbb{R}^l_+$, the return function is unbounded below and above and U can take the value $-\infty$ at some points of the technological set. These types of problems do not admit a useful truncation of the technological correspondence as in the previous subsection, so the approach given now is quite different. Actually, we return to the first approach of this study, that is to say, we try to prove that the Bellman operator is a contraction with respect to an adequate metric. Experience says that we cannot expect the operator to be a contraction if the metric is defined in terms of a norm or a family of seminorms, when the discounting factor is close to 1. This fact has already been noted by Boyd (1990), Alvarez and Stokey (1998), and Streufert (1998) among others.

The two main properties of the operator leading to the contraction property in the supremum norm are monotonicity and discount. However, there is another interesting property of the Bellman operator, that is convexity. The definition of the equation of dynamic programming as a maximization operation implies that the Bellman operator is—in a pointwise sense—convex. Convexity and monotonicity,[6] joined to the definition of a suitable distance in a certain space of functions, will be the main ingredients making it possible for the operator to be a contraction.

In the following, we will use this definition: given two strictly negative functions f, g, continuous on $X^\star = X \setminus \{0\}$, we say that $f/g = O(1)$ at $x = 0$ if and only if f/g is bounded in some neighborhood of $x = 0$.

Let $w_- \leq w_+ < w$ be three functions of $C(X^\star)$, such that

$$\frac{w_- - w}{w_+ - w} = O(1) \quad \text{at } 0. \tag{10}$$

We denote by A the subset of continuous functions on $X^\star$ defined as

$$A = \{f \in C(X^\star) : w_- \leq f \leq w_+\} = [w_-, w_+]. \tag{11}$$

We now consider a countable increasing sequence $\{K_j\}$ of nonempty and compact subsets of X satisfying $X = \bigcup_j K_j$, and the following semidistance on A:

$$d_j(f, g) = \sup_{x \in K_j} \left| \ln\left(\frac{f - w}{w_+ - w}(x)\right) - \ln\left(\frac{g - w}{w_+ - w}(x)\right) \right|, \qquad f, g \in A, \quad j \in \mathbb{N}. \tag{12}$$

The quantities d_j are well defined because of (10) and the definition (11) of A. Of course the function w appearing in (12) can be chosen in many ways. However, as we will see below, for our purposes, w must satisfy $\mathcal{B}w < w$. A metric d can be defined on A as in Section 2 by

$$d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f, g)}{1 + d_j(f, g)} \quad \text{for all } f, g \in A.$$

[6]The importance of convexity and monotonicity of functional operators has been recognized by several authors, as Krasnolselskii and Zabrieko (1984) or, more recently, Montrucchio (1998). The latter uses convexity in the framework of a functional equation involving strictly negative, homogeneous degree two and bounded real functions, his aim being to establish differentiability of the policy function in discrete time dynamic programs.

REMARK 6: If $\sup_j d_j(w_-, w_+) < \infty$, then it is possible to define a metric by taking the supremum on X instead of on each compact set K_j. This observation will be used afterwards. Notice also that the condition

$$\limsup_{\|x\|\to\infty, x\in X} \left(\frac{w_- - w}{w_+ - w}\right)(x) < \infty$$

implies $\sup_j d_j(w_-, w_+) < \infty$, since the quotient $(w_- - w)/(w_+ - w)$ is continuous except at $x = 0$ and bounded in a neighborhood of this point. In some applications, we will take $w \equiv 0$. Then, (10) is the same as $w_-/w_+ = O(1)$ at zero.

PROPOSITION 3: *If* (10) *holds, then* (A, d) *is a complete metric space.*

Now we introduce two new assumptions that will be used in this context and that replace (DP2) and (DP3), respectively.

(DP2′) $U : \mathrm{Graph}(\Gamma) \to \mathbb{R} \cup \{-\infty\}$ *is upper semicontinuous and continuous at every point where it is finite.*

(DP3′) *The following conditions hold true*:

(i) *For all* $x \in X^\star$, *there is a continuous selection* q *of* Γ *with* $U(x, q(x)) > -\infty$.

(ii) *There exist three continuous functions* w_-,w_+, *and* w *such that* $w_- \le w_+ < w$ *and*

(a) $\mathcal{B}w < w$, $\mathcal{B}w_- \ge w_-$, $\mathcal{B}w_+ \le w_+$, *and*

$$\frac{w_- - w}{w_+ - w} = O(1) \quad \text{at } 0.$$

(b) $\Pi^0(x_0) \neq \emptyset$ *for all* $x_0 \in X^\star$, *and for each* $\tilde{x} = (x_t) \in \Pi^0(x_0)$ *it follows* $\lim_{t\to\infty} \beta^t w_-(x_t) = \lim_{t\to\infty} \beta^t w_+(x_t) = 0$.

Assumption (DP3′) will permit us to find a closed and bounded set $A = [w_-, w_+]$ such that the Bellman operator maps this order interval into itself as a contraction. Moreover, the Bellman operator iterations will either be an increasing or decreasing sequence in $C(X^\star)$ that will always lie in the given order interval. This fact is important because, in general, the supremum or infimum of the sequence is not in the order interval, since ordinarily order intervals are not σ-Dedekind complete[7] in $C(X^\star)$. Thus, monotonicity of the Bellman operator alone is not sufficient to yield a fixed point. This technical problem is solved here by means of the convexity property of the operator,

[7]A set X is σ-Dedekind complete if the supremum or the infimum of any countable subset of X is an element of the set (see Aliprantis and Border (1998) for details).

allowing restriction of the local contraction approach to the order interval. Hence, the property of convexity allows us to overcome the failure of $C(X^\star)$ to be σ-Dedekind complete.

Although (DP3′) seems to be stringent and rather technical, we will show in the examples that follow the next theorem that many economic models share this property.

The following result is analogous to Theorem 3 but considering semidistances instead of seminorms. However, in the method of proof we use the convexity property of the Bellman operator instead of its discounting property.

THEOREM 6: *Let $\mathcal{B}$ be a Bellman operator satisfying* (DP1), (DP2′), *and* (DP3′) *so that there exists a countable increasing sequence $\{K_j\}$ of nonempty and compact subsets of X with $X = \bigcup_j K_j$ satisfying $\Gamma(K_j) \subseteq K_j$ for all $j \in \mathbb{N}$. Then the following hold*:

(a) *There exists a closed and bounded subset $A \subseteq C(X^\star)$ such that the Bellman equation has a unique solution $\hat{f}$ on A. Furthermore, $\hat{f}$ satisfies $w_- \leq \hat{f} \leq w_+$.*

(b) *The value function $v^\star$ is continuous in $X^\star$ and coincides with the fixed point $\hat{f}$.*

(c) *For any $f \in A$, $\mathcal{B}^n f \xrightarrow{d} v^\star$ as $n \to \infty$.*

REMARK 7: (i) Similar observations to those found in Remark 3 take place here. That is to say, the condition $\Gamma(K_j) \subseteq K_j$ can be dropped if the following inequality holds for all $j \in \mathbb{N}$:

$$\mu_j = \sup_{f \in A} d_j(f, \mathcal{B}w) \leq \mu. \tag{13}$$

This is because, in such a case, $d_j(\mathcal{B}f, \mathcal{B}g) \leq (1 - e^{-\mu})\, d_j(f, g)$ for all $j \in \mathbb{N}$ as can be seen in the proof of Theorem 6, and then $\mathcal{B}$ is a 0-LC although $\Gamma(K_j) \nsubseteq K_j$. In general, $\mathcal{B}w$ can be difficult to obtain. However, given that $\mathcal{B}w < w$, a sufficient condition for (13) to hold is $\sup_j d_j(w_-, w) < \infty$, which is a more workable condition. When $w \equiv 0$, (13) is just $\sup_j d_j(w_-, \mathcal{B}0) = \sup_j d_j(w_-, \psi) < \infty$.

(ii) When the function ψ is strictly negative, an obvious choice for w is the null function and for w_+ the function ψ itself. These considerations are based on the fact that, in such a case, $\mathcal{B}0 < 0$ and $\mathcal{B}\psi < \psi$.

(iii) As can be observed in the proof of Theorem 6, the discounting factor does not have any influence in the parameter of contraction with the metric considered in this section. However, it does play a very important role in the existence of a suitable closed and bounded subset of functions mapped into itself by the Bellman operator.

(iv) The convergence of the iterates of the Bellman operator beginning from any function belonging to the set A is understood in the metric considered. Yet it implies uniform convergence on compact subsets of X of

$\ln((\mathcal{B}^n f - w)/(w_+ - w))$ to the function $\ln((\hat{f} - w)/(w_+ - w))$. Hence, the sequence $(\mathcal{B}^n f - w)/(w_+ - w)$ converges to the function $(\hat{f} - w)/(w_+ - w)$, uniformly on compacta.

EXAMPLE 8 (Homogeneous negative case): Of particular interest is the homogeneous negative case, which we have already analyzed in Example 2. There we have proven that the operator is a contraction with respect to the family of seminorms whenever the discounting factor satisfies suitable bounds. As we shall show, Theorem 6 allows us to avoid this limitation. To begin with, let us assume that U is homogeneous of degree $\theta < 0$, $U(0,0) = -\infty$, and Graph(Γ) is a cone in $X = \mathbb{R}^l_+$. We also add the following assumptions:

(a) $U(x, y) \leq -a\|x\|^\theta$ for some $a > 0$, for all $(x, y) \in$ Graph(Γ).

(b) For all $x \in X$, there exists a continuous selection q of Γ with $\|q(x)\| \geq \alpha\|x\|$ for some $\alpha > 0$ such that $\beta\alpha^\theta < 1$, and $U(x, q(x)) \geq -b\|x\|^\theta$ for some $b > 0$.

Now we prove that (DP3′) is satisfied. The property (i) is obvious, by (b). Let us define the functions w_-, w_+, and w as follows:

$$w_-(x) = \frac{-b}{1-\beta\alpha^\theta}\,\|x\|^\theta,$$
$$w_+(x) = \psi(x),$$
$$w(x) = 0.$$

It is easily shown that w_-, w_+, and w satisfy the assumption (ii)-(a); in fact, we can drop the condition $\Gamma(K_j) \subseteq K_j$, as w_-/w_+ is bounded on X, since $-b\|x\|^\theta \leq \psi(x) \leq -a\|x\|^\theta$. Hence, it is obvious that (13) holds, and $\mathcal{B}$ is a 0-LC although $\Gamma(K_j) \nsubseteq K_j$ (see also the comments in Remark 6 and in (i) of Remark 7). We now check the property (ii)-(b) in (DP3′). First, given $x_0 \neq 0$, let (x_t) be the path satisfying $x_t = q(x_{t-1})$ for all $t \in \mathbb{N}$. Then (x_t) belongs to $\Pi^0(x_0)$, since $S((x_t)) \geq -\|x_0\|^\theta/(1-\alpha^\theta)$. Second,

$$\lim_{t\to\infty} \beta^t w_-(x_t) = \lim_{t\to\infty} \beta^t w_+(x_t) = 0$$

for all $(x_t) \in \Pi^0(x_0)$. This follows because $\beta^t\psi(x_t) = \beta^t w_+(x_t)$ tends to zero for all $(x_t) \in \Pi^0(x_0)$—see (i) of Remark 5—and hence $\beta^t\|x_t\|^\theta$ tends to zero as well, due to the upper bound on ψ. After this analysis, we can assert that $\mathcal{B}$ is a 0-LC in the metric considered for all β satisfying $\beta\alpha^\theta < 1$, that the value function is continuous and that it can be approached by means of iterations of the Bellman operator starting from any function lying between w_- and w_+. Actually, it can be approached from the zero function, since $\mathcal{B}0 = \psi$ belongs to this set of functions. It can be seen that neither homogeneity for the utility function, nor condition A5 in Alvarez and Stokey (1998) are necessary in order to obtain such a result—this fact is also mentioned in Le Van and Morhaim (2002).

EXAMPLE 9 (Returns that are bounded above but not below; Nakajima (1999)): Theorem 6 also covers the results of Nakajima regarding returns that are bounded above by zero but not below. In addition to continuity of U, where it is finite, and Γ, this author uses the following assumptions:

(a) U is nonpositive, concave, increasing in x, and decreasing in y.

(b) Γ is convex and monotone increasing.

(c) There exists a vector $x^l \in X^\star$ such that $U(x, y) \leq -C\,\|x\|^\theta - c$ for some $C > 0$, $c \geq 0$ and some $\theta < 0$, for all $(x, y) \in \text{Graph}(\Gamma)$ satisfying $\|x\| \leq \|x^l\|$.

(d) There exists a vector $x^u \in X$ such that for each $x \in X^\star$ with $\|x\| \leq \|x^u\|$ there exists a continuous selection q of Γ satisfying $0 \geq U(x, q(x)) \geq -D\|x\|^\theta - d$ for some $D > 0$, $d \geq 0$, and $\|q(x)\| \geq \alpha\,\|x\|$ for some $\alpha > 0$ with $\beta\alpha^\theta < 1$.

The conditions imposed imply that the curvature of the value function near zero is, roughly speaking, negatively homogeneous. Let us choose the functions w_-, w_+, and w as follows:

$$w_-(x) = \begin{cases} -\dfrac{d}{1-\beta} - \dfrac{D}{1-\beta\alpha^\theta}\,\|x\|^\theta, & \text{if } \|x\| \leq \|x^u\|, \\ -\dfrac{d}{1-\beta} - \dfrac{D}{1-\beta\alpha^\theta}\|x^u\|^\theta, & \text{if } \|x\| \geq \|x^u\|, \end{cases}$$

$$w_+(x) = \psi(x),$$

$$w(x) = 1.$$

It is straightforward to show both $\mathcal{B}w_+ \leq w_+$ and $\mathcal{B}w < w$. Making use of hypotheses (a) and (c), we also have $\mathcal{B}w_- \geq w_-$. As in the homogeneous negative case, it is easy to show that the remaining conditions of (DP3′) are satisfied. Assumption (i) is satisfied if we choose the path satisfying $x_t = q(x_{t-1})$ if $\|x_0\| < \|x^u\|$, and $x_t = x_0$ if $\|x_0\| \geq \|x^u\|$. Now, it is obvious that $w_-/w_+ = O(1)$ near 0, because (b) implies $\psi(x) \leq -C\|x\|^\theta - c$ for $\|x\| \leq \|x^l\|$. Actually, taking $w \equiv 1$, it follows that

$$\mu_j = d_j(w_-, w_+) = \sup_{K_j}\left|\ln\frac{w_- - w}{w_+ - w}\right| = \sup_{K_j}\left|\ln\frac{w_- - 1}{w_+ - 1}\right|$$

is bounded for every j and therefore the operator $\mathcal{B}$ is a 0-LC, although the technological correspondence presents growth rates greater than one. Hypothesis (ii)-(b) holds true as in Example 8. To finish, notice that the hypothesis of convexity is not needed in order to apply Theorem 6.

EXAMPLE 10 (Logarithmic utility function; technology with decreasing returns): Here the utility function is $U(x, y) = \ln(F(x) - y)$, $X = \mathbb{R}_+$ and $\Gamma(x) = [0, F(x)]$. We suppose that F is continuous on $[0, \infty)$, strictly increasing, $F(0) = 0$, and there exists $\bar{x} > 0$ with $F(\bar{x}) = \bar{x}$, $F(x) > x$ for all $x < \bar{x}$ and

$F(x) < x$ for all $x > \bar{x}$. We need to prove that (DP3′) holds. In order to find w_-, let us first define the continuous selection $q(x) = x/2$ if $x \le \bar{x}$, $q(x) = \bar{x}/2$ if $x \ge \bar{x}$. We have $q(x) \in \Gamma(x)$ for all $x > 0$, and $U(x, q(x)) > -\infty$; hence (i) of (DP3′) holds. Now we define the continuous function w_- as follows:

$$w_-(x) = \begin{cases} \dfrac{1}{(1-\beta)^2} \ln\dfrac{1}{2} + \dfrac{1}{1-\beta} \ln x, & \text{if } x \le \bar{x}, \\ \dfrac{1}{(1-\beta)^2} \ln\dfrac{1}{2} + \dfrac{1}{1-\beta} \ln \bar{x}, & \text{if } x \ge \bar{x}. \end{cases}$$

From this function, an easy computation shows that $\mathcal{B}w_- \ge w_-$. With respect to the continuous function w_+, it is defined next. Given that $\bar{x}^{1-\sigma}x^{\sigma}$ tends, as $\sigma \to 0$, to the discontinuous function $h(0) = 0 = F(0)$, $h(x) = \bar{x} = F(\bar{x})$, if $0 < x \le \bar{x}$, and because F is strictly increasing, it is obvious that there exists some $\sigma > 0$ small enough such that $\bar{x}^{1-\sigma}x^{\sigma} \ge F(x)$ for all $0 \le x \le \bar{x}$. Let us define the function w_+ by

$$w_+(x) = \begin{cases} \dfrac{\sigma}{1-\beta\sigma} \ln x + \dfrac{1-\sigma}{(1-\beta)(1-\beta\sigma)} \ln \bar{x}, & \text{if } x \le \bar{x}, \\ \dfrac{1}{1-\beta} \ln x, & \text{if } x \ge \bar{x}. \end{cases}$$

A tedious but straightforward computation shows that $\mathcal{B}w_+ \le w_+$. Now, let us consider the function w defined as follows: $w(x) = a + (1/(1-\beta)) \ln \bar{x}$ if $0 < x \le \bar{x}$ and $w(x) = a + w_+(x)$ for $x \ge \bar{x}$, where $a > 0$. Note that w is continuous, since $w_+(\bar{x}) = (1/(1-\beta)) \ln \bar{x}$ and that $w_+ < w$. Furthermore, $\mathcal{B}w < w$. The following step is to show that $(w_- - w)/(w_+ - w) = O(1)$ at $x = 0$. However, this is clear as w is constant near 0 and

$$\lim_{x\to 0^+} \frac{w_-(x)}{w_+(x)} = \frac{1-\beta\sigma}{\sigma\,(1-\beta)} > 1.$$

The last inequality implies $w_-(x) < w_+(x)$ for all x in a neighborhood of zero. The inequality $w_-(x) < w_+(x)$ for all $x \in X$ follows from the fact that $w_- < (1/(1-\beta)) \ln \bar{x}$, and the uniform convergence, as σ goes to zero, of the function w_+ to the function $(1/(1-\beta)) \ln \bar{x}$ on the compact subsets of $(0, \bar{x}]$. Thus we can conclude that (ii)-(a) is satisfied.

The property $\Pi^0(x_0) \neq \emptyset$ for all $x_0 > 0$ is obvious, since the path $(x_0, q(x_0), q(x_0), \ldots)$ belongs to $\Pi^0(x_0)$ for all $x_0 > 0$, hence (i) in (DP3′) holds. Now, let us consider $(x_t) \in \Pi^0(x_0)$. If $x_t \ge \bar{x}$ for some $t \in \mathbb{N}$, then $x_{t'} \le x_t$ for all $t' \ge t$, so any path in $\Pi^0(x_0)$ is bounded above, and then $\lim_{t\to\infty} \beta^t \ln x_t \le 0$. Moreover, since $F(x_t) \le \bar{x}^{1-\sigma} x_t^\sigma$ for all $x_t \le \bar{x}$, it follows

$$\begin{aligned}\lim_{t\to\infty} \beta^t \ln(\bar{x}^{1-\sigma} x_t^\sigma) = \lim_{t\to\infty} \beta^t \sigma \ln x_t &\ge \lim_{t\to\infty} \beta^t \ln F(x_t) \\ &\ge \lim_{t\to\infty} \beta^t \psi^-(x_t) = 0.\end{aligned}$$

Hence, $\lim_{t\to\infty} \beta^t \ln x_t = 0$, and therefore $\lim_{t\to\infty} \beta^t w_-(x_t) = \lim_{t\to\infty} \beta^t w_+(x_t) = 0$. After this analysis, we can assert that $\mathcal{B}$ is a 0-LC in the metric considered for all $\beta < 1$, that the value function is continuous, and that it can be approached starting from any function lying between w_- and w_+. Our hypotheses are weaker than those proposed in Le Van and Morhaim (2002), as we have not made any assumption on concavity or smoothness of the production function F.

EXAMPLE 11 (Homogeneous utility function; technology with decreasing returns): The return function is given by $U(x,y) = (F(x) - y)^\theta/\theta$, where $\theta < 0$. The state space is $X = \mathbb{R}_+$ and the technological correspondence is $\Gamma(x) = [0, F(x)]$ with F strictly increasing and continuously differentiable on $(0,\infty)$, $F'(0+) > 1$, $F(0) = 0$ and such that there exists $\bar{x} > 0$ with $F(\bar{x}) = \bar{x}$ and $F(x) < x$ if $x > \bar{x}$. The sets $K_j = [0, j\bar{x}]$ are compact and $\Gamma(K_j) \subseteq K_j$ holds. It can be seen that $\psi(x) = F(x)^\theta/\theta$ is strictly negative, hence $\mathcal{B}\psi \le \psi$ and we can take $w_+ = \psi$ and $w = 0$. The function w_- bounding for below the fixed point is constructed next. First, setting $x_1 \in (0, \bar{x})$, let us take the continuous selection $q(x) = x$ if $x \le x_1$, $q(x) = x_1$ if $x \ge x_1$. Obviously (i) of (DP3′) is fulfilled. Then, we define w_- by

$$w_-(x) = \begin{cases} \dfrac{1}{1-\beta} \dfrac{(F(x)-x)^\theta}{\theta}, & \text{if } x \le x_1, \\[2ex] \dfrac{1}{1-\beta} \dfrac{(F(x_1)-x_1)^\theta}{\theta}, & \text{if } x \ge x_1. \end{cases}$$

From this we obtain $w_- < w_+$ and $\mathcal{B}w_- \ge w_-$. Now we shall prove that $F(x)/(F(x)-x) = O(1)$ at 0. This will show that $w_+/w_- = O(1)$ at 0. By L'Hopital's rule, it follows that

$$1 \le \lim_{x\to 0^+} \frac{F(x)}{F(x)-x} = \lim_{x\to 0^+} \left(1 + \frac{x}{F(x)-x}\right) = 1 + \frac{1}{F'(0+)-1};$$

so we have

$$\frac{1}{1-\beta}\left(1 + \frac{1}{F'(0+)-1}\right)^\theta \le \lim_{x\to 0^+} \frac{w_+(x)}{w_-(x)} \le \frac{1}{1-\beta},$$

which proves the assertion. We have proved that (ii)-(a) holds. On the other hand, the verification that $\Pi^0(x_0) \ne \emptyset$ is obvious since the path $(x_0, q(x_0), \ldots,$

$q(x_0), \ldots)$ belongs to $\Pi^0(x_0)$. Finally, let $x_0 \neq 0$ and let $(x_t) \in \Pi^0(x_0)$. Notice that the path (x_t) is bounded. If (x_t) is bounded away from 0, then obviously

$$\lim_{t\to\infty} \beta^t w_-(x_t) = \lim_{t\to\infty} \beta^t w_+(x_t) = 0.$$

If (x_t) tends to 0, then, as we have shown above, $((F(x_t) - x_t)/F(x_t))^\theta = O(1)$. Taking into account that $\beta^t w_+(x_t) = \beta^t \psi(x_t) = \beta^t F(x_t)^\theta$ goes to zero as $t \to \infty$ for paths in $\Pi^0(x_0)$, then it must hold

$$\lim_{t\to\infty} \beta^t w_-(x_t) = \lim_{t\to\infty} \beta^t (F(x_t) - x_t)^\theta = 0.$$

Hence, the assumption (DP3′) is fulfilled, and all the conclusions in Theorem 6 hold. In particular, the value function can be approached starting from the zero function, as $\mathcal{B}0 = \psi = w_+$ belongs to $[w_-, w_+]$.

EXAMPLE 12 (Logarithmic utility function; the technology is a cone): We consider the set $X = \mathbb{R}^l_+$ and the return function $U(x, y) = \ln(\phi(x, y))$, where $\phi : \mathrm{Graph}(\Gamma) \to \mathbb{R}_+$ is continuous, $\phi(0, 0) = 0$, and $\mathrm{Graph}(\Gamma)$ is a cone. When ϕ is homogeneous of degree one, this model is known as the homogeneous of degree zero case and it was introduced by Alvarez and Stokey (1998). However, this last hypothesis is not necessary and can be eliminated. In fact, the usual assumptions used in the model can be weakened in other directions as shown below. Keeping in mind this idea, we assume the following conditions, which are weaker than the original.

(a) $\|y\| \leq \epsilon\|x\|$ for some $\epsilon > 0$, for all $(x, y) \in \mathrm{Graph}(\Gamma)$.

(b) $\phi(x, y) \leq B(\|x\|^n + \|y\|^n)$ for some $B, n > 0$, for all $(x, y) \in \mathrm{Graph}(\Gamma)$.

(c) For all $x \in X$, there exists a continuous selection q of Γ with $\|q(x)\| \geq \alpha\|x\|$ for some $\alpha > 0$, and such that $\phi(x, q(x)) \geq b\|x\|^n$ for some $b > 0$.

First, we consider the functions

$$\begin{aligned}
w_-(x) &= \frac{1}{1-\beta} \ln b + \frac{\beta n}{(1-\beta)^2} \ln \alpha + \frac{n}{1-\beta} \ln \|x\|, \\
w_+(x) &= \frac{1}{1-\beta}\ \ln(B(1+\epsilon^n)) + \frac{\beta n}{(1-\beta)^2} \ln \epsilon + \frac{n}{1-\beta} \ln \|x\|, \\
w(x) &= w_+(2x).
\end{aligned}$$

Conditions (i), (iii) and (ii)-(a) in (DP3′) are obviously fulfilled, so only (ii)-(b) needs proof. Let $x_0 \neq 0$. Then, the path (x_t), where $x_{t+1} = q(x_t)$ for all $t \in \mathbb{N}$, belongs to $\Pi^0(x_0)$, so $\Pi^0(x_0) \neq \emptyset$. Furthermore,

$$\lim_{t\to\infty} \beta^t n \ln \|x_t\| = \lim_{t\to\infty} \beta^t \ln(B(1+\epsilon^n)\|x_t\|^n) \geq \lim_{t\to\infty} \beta^t U^-(x_t, x_{t+1}) = 0,$$

and

$$\lim_{t\to\infty} \beta^t \ln \|x_t\| \leq \lim_{t\to\infty} \beta^t \ln(\epsilon^t \|x_0\|) = 0.$$

Thus $\lim_{t\to\infty}\beta^t \ln\|x_t\| = 0$, which implies $\lim_{t\to\infty}\beta^t w_-(x_t) = \lim_{t\to\infty}\beta^t \cdot w_+(x_t) = 0$. Now, since the quotient $(w_- - w)/(w_+ - w)$ is constant at every point of X, Theorem 6 is applicable although $\alpha > 1$.

EXAMPLE 13 (General logarithmic utility function): Let us consider the model as in the example above, where now Graph(Γ) is not necessarily a cone. We assume the following more general conditions.

(a) $\|y\| \le \epsilon\|x\|^m$ for some $\epsilon, m > 0$ such that $\beta m < 1$, for all $(x, y) \in$ Graph(Γ).

(b) $\phi(x, y) \le B(\|x\|^{n_1} + \|y\|^{n_2})$ for some $B, n_1, n_2 > 0$ such that $n_1 = mn_2$, for all $(x, y) \in$ Graph(Γ).

(c) For all $x \in X$, there exists a continuous selection q of Γ with $\|q(x)\| \ge \alpha\|x\|^m$ for some $\alpha > 0$, and such that $\phi(x, q(x)) \ge b\|x\|^{n_1}$ for some $b > 0$.

Let us consider the functions

$$\begin{aligned}
w_-(x) &= \frac{1}{1-\beta}\ln b + \frac{\beta n_1}{(1-\beta)(1-\beta m)}\ln\alpha + \frac{n_1}{1-\beta m}\ln\|x\|,\\
w_+(x) &= \frac{1}{1-\beta}\,\ln(B(1+\epsilon^{n_2})) + \frac{\beta n_1}{(1-\beta)(1-\beta m)}\ln\epsilon\\
&\quad + \frac{n_1}{1-\beta m}\,\ln\|x\|,\\
w(x) &= w_+(2x).
\end{aligned}$$

The proof of (DP3′) is completely similar to that given in Example 12, so we omit the details.

4. CONCLUDING REMARKS

In this paper we have provided new results regarding the existence and uniqueness of solutions to the Bellman equation in the case of unbounded returns. Our primary approach is a global one on the space of all continuous functions, with the aid of some metric fixed point theorems. The Boyd (1990) and Becker and Boyd approach (1990, 1997) contemplate only a subset of all continuous functions, obtaining uniqueness of solutions only with respect to this subset. On the other hand, they demand the existence of a continuous function satisfying certain properties. The construction of such a function is possible, as the previously mentioned authors show, for some parametric models, but does not appear obvious for a general dynamic programming problem. Other approaches, as in Streufert (1990, 1998), Nakajima (1999), and Le Van and Morhaim (2002) are based directly on the study of the value function. Our results enable us to cover fairly general models by means of the contraction techniques, with the important implications of convergence of successive iterations to the fixed point and uniqueness with regard to a certain class of functions.

We are able to prove existence and uniqueness whenever the associated operators are 0-LC, a concept to which Theorem 1 applies. If the operators are not suitable for application of the aforementioned theorem, we can still obtain a solution using Theorem 2 since the Bellman operator is always 1-LC. Theorem 5 allows us to analyze problems not covered in prior results. The flexibility and scope of our methodology is shown in the important case of unbounded below returns. In this case, instead of considering seminorms based on the supremum norm—which is clearly impossible—we choose an adequate family of semidistances. In this way, Theorem 6 is applicable to many models, as shown in the examples studied in the paper. In all the reported cases, the fixed point coincides with the value function.

It seems clear that our approach can be applied to stochastic dynamic programming with unbounded returns. Finally, we would like to mention that it can be applied to recursive utility and dynamic programming with recursive utility, as can be seen in Rincón-Zapatero and Rodríguez-Palmero (2003).

Dept. of Applied Economics (Mathematics), University of Valladolid, Avda. Valle de Esgueva 6, E-47011 Valladolid, Spain; zapatero@eco.uva.es

and

Dept. of Applied Economics (Mathematics), University of Valladolid, Avda. Valle de Esgueva 6, E-47011 Valladolid, Spain; cpalmero@eco.uva.es.

Manuscript received March, 2000; final revision received January, 2003.

APPENDIX A: PROOFS

PROPOSITION 1: (a) Let $f, g \in C(X)$. We have

$$\begin{aligned} d(Tf, Tg) &= \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(Tf, Tg)}{1 + d_j(Tf, Tg)} \\ &\le \sum_{j=1}^{\infty} 2^{-j} \frac{\beta_j d_j(f, g)}{1 + \beta_j d_j(f, g)} && \big(\text{since } T \text{ is a 0-LC and } \tfrac{x}{1+x} \text{ is increasing}\big) \\ &= \sum_{j=1}^{\infty} 2^{-j} a_j\, b_j && \Big(\text{set } a_j = \beta_j \tfrac{1+d_j(f,g)}{1+\beta_j d_j(f,g)} \text{ and } b_j = \tfrac{d_j(f,g)}{1+d_j(f,g)}\Big), \end{aligned}$$

where $a_j \in (0, 1)$ and $b_j \in [0, 1)$, for all $j \in \mathbb{N}$. To prove the statement, assume to the contrary that for all $\alpha \in [0, 1)$, $\sum_{j=1}^{\infty} 2^{-j} a_j b_j > \alpha \sum_{j=1}^{\infty} 2^{-j} b_j$. In this case we have $\sum_{j=1}^{\infty} 2^{-j} a_j b_j \ge \sum_{j=1}^{\infty} 2^{-j} b_j$, and consequently $\sum_{j=1}^{\infty} 2^{-j} b_j (a_j - 1) \ge 0$, which is a contradiction, since $a_j < 1$, for all $j \in \mathbb{N}$.

(b) Since A is a bounded subset of $C(X)$, there exists a sequence of uniform bounds $\{m_j\}$ such that $d_j(f, g) \le m_j$ for all $f, g \in A$, and for all $j \in \mathbb{N}$. Now let $f, g \in A$; we then have

$$\begin{aligned} d(Tf, Tg) &\le \sum_{j=1}^{\infty} 2^{-j} a_j\, b_j && (\text{from item (a)}) \\ &\le \sum_{j=1}^{\infty} 2^{-j} \beta_j \frac{1 + m_j}{1 + \beta_j m_j}\, b_j && \big(\text{since } f, g \in A \text{ and } \beta_j \tfrac{1+x}{1+\beta_j x} \text{ is increasing in } x\big) \\ &= \sum_{j=1}^{\infty} 2^{-j} a'_j\, b_j && \big(\text{set } a'_j = \beta_j \tfrac{1+m_j}{1+\beta_j m_j}\big), \end{aligned}$$

where $a'_j \in (0,1)$ and $b_j \in [0,1)$, for all $j \in \mathbb{N}$. Since a'_j does not depend on the particular choice of f and g in A, it is sufficient to prove there exists $\alpha \in [0,1)$ such that $\sum_{j=1}^{\infty} 2^{-j} a'_j b_j \le \alpha \sum_{j=1}^{\infty} 2^{-j} b_j$, for all sequences $\{b_j\}$ satisfying $b_j \in [0,1]$, for all $j \in \mathbb{N}$. To do so, suppose on the contrary that for each $\alpha \in [0,1)$ there exists a sequence $\{b_j^{\alpha}\}$ in $[0,1]$ such that

$$\sum_{j=1}^{\infty} 2^{-j} a'_j b_j^{\alpha} > \alpha \sum_{j=1}^{\infty} 2^{-j} b_j^{\alpha}. \tag{A.1}$$

Let $\{\alpha_n\}$ be a sequence such that $\alpha_n \in [0,1)$ for all $n \in \mathbb{N}$, and $\alpha_n \to 1$ as $n \to \infty$. Then, for each n, there exists a sequence $\tilde{b}_n = \{b_j^{\alpha_n}\} \in K = [0,1]^{\infty}$ satisfying (A.1). Since K is a compact set for the product topology, the sequence $\tilde{b}_1, \dots, \tilde{b}_n, \dots$ admits a convergent subsequence. Without loss of generality, we can thus assume $\tilde{b}_n \to \widetilde{B} = \{B_j\}$ in K, and therefore $b_j^{\alpha_n} \to B_j$ in $\mathbb{R}$, as $n \to \infty$, for each $j \in \mathbb{N}$. From this we obtain:

(i) $\sum_{j=1}^{\infty} 2^{-j} b_j^{\alpha_n} (a'_j - \alpha_n) > 0$, since $\tilde{b}_n$ verifies (A.1).
(ii) $b_j^{\alpha_n} (a'_j - \alpha_n) \to B_j (a'_j - 1)$, as $n \to \infty$.
(iii) $|2^{-j} b_j^{\alpha_n} (a'_j - \alpha_n)| \le 2^{-j+1}$ and $\sum_{j=1}^{\infty} 2^{-j+1} < \infty$.

Thus, from the Lebesgue Dominated Convergence Theorem, it follows $\sum_{j=1}^{\infty} 2^{-j} B_j (a'_j - 1) \ge 0$, which contradicts the fact that $a'_j < 1$ for all $j \in \mathbb{N}$. *Q.E.D.*

THEOREM 1: (a) By (b) of Proposition 1 the operator T is a contraction on A. Since A is closed, the Banach Theorem is applicable and therefore T has a unique fixed point $\hat{f}$ in A. The uniqueness of the fixed point on $C(X)$ follows from (a) of Proposition 1.

(b) Let $f \in C(X)$. Then, for all $j \in \mathbb{N}$

$$d_j(T^n f, \hat{f}) = d_j(T^n f, T^n \hat{f}) \le \beta_j^{n-1} d_j(f, \hat{f}), \quad n \in \mathbb{N}. \tag{A.2}$$

Moreover, given $\epsilon > 0$ there is a positive integer p such that $\sum_{j=p+1}^{\infty} 2^{-j} < \frac{\epsilon}{2}$. Furthermore, it is clear that

$$\sum_{j=1}^{p} 2^{-j} \frac{\beta_j^{n-1} d_j(f, \hat{f})}{1 + \beta_j^{n-1} d_j(f, \hat{f})} \to 0 \quad \text{as} \quad n \to \infty.$$

So, there exists a positive integer n_0 such that for all $n \ge n_0$

$$\sum_{j=1}^{p} 2^{-j} \frac{\beta_j^{n-1} d_j(f, \hat{f})}{1 + \beta_j^{n-1} d_j(f, \hat{f})} < \frac{\epsilon}{2}.$$

From this, and taking into account that (A.2) holds, it follows directly that for all $n \ge n_0$,

$$\begin{aligned} d(T^n f, \hat{f}) &= \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(T^n f, \hat{f})}{1 + d_j(T^n f, \hat{f})} \\ &\le \sum_{j=1}^{p} 2^{-j} \frac{\beta_j^{n-1} d_j(f, \hat{f})}{1 + \beta_j^{n-1} d_j(f, \hat{f})} + \sum_{j=p+1}^{\infty} 2^{-j} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $T^n f \xrightarrow{d} \hat{f}$ as $n \to \infty$. *Q.E.D.*

THEOREM 2: We only need to show that T is a contraction on A with respect to the metric d_c. To this end, let f, g be two functions in A. By definition of 1-LC, we have then the following contraction property:

$$d_c(Tf, Tg) = \sum_{j=1}^{\infty} c^{-j} d_j(Tf, Tg)$$

$$\leq \sum_{j=1}^{\infty} c^{-j}\, \beta\, d_{j+1}(f, g) = c\beta \sum_{j=2}^{\infty} c^{-j}\, d_j(f, g) \leq c\beta d_c(f, g). \qquad Q.E.D.$$

THEOREM 3: (a) There are two steps in the proof. The first is to show that $\mathcal{B}$ is a 0-LC. The second step is to show that $\mathcal{B}$ maps A into A, for some closed and bounded subset A of $C(X)$.

Step One. Let $f, g \in C(X)$. Since $f(y) \leq g(y) + \max_{y\in\Gamma(K_j)}|f(y) - g(y)|$ for all $y \in \Gamma(K_j)$, we have, for all $x \in K_j$,

$$\begin{aligned}\mathcal{B}f(x) &= \max_{y\in\Gamma(x)} (U(x, y) + \beta f(y)) \\ &\leq \max_{y\in\Gamma(x)} \Big(U(x, y) + \beta\Big(g(y) + \max_{y\in\Gamma(K_j)} |f(y) - g(y)|\Big)\Big) \\ &= \max_{y\in\Gamma(x)} (U(x, y) + \beta g(y)) + \beta \max_{y\in\Gamma(K_j)} |f(y) - g(y)| \\ &= \mathcal{B}g(x) + \beta \max_{y\in\Gamma(K_j)} |f(y) - g(y)|;\end{aligned}$$

therefore $\mathcal{B}f(x) - \mathcal{B}g(x) \leq \beta\max_{y\in\Gamma(K_j)}|f(y) - g(y)|$. Reversing the roles of f and g, and taking into account that $\Gamma(K_j) \subseteq K_j$, we obtain

$$\|\mathcal{B}f - \mathcal{B}g\|_{K_j} \leq \beta \max_{y\in\Gamma(K_j)} |f(y) - g(y)| \leq \beta\, \|f - g\|_{K_j}, \tag{A.3}$$

which means that $\mathcal{B}$ is a 0-LC.

Step Two. Let $A = \{f \in C(X) : \|f\|_{K_j} \leq \|\psi\|_{K_j}/(1 - \beta)$ for all $j \in \mathbb{N}\}$, which is a closed and bounded subset of $C(X)$. Let $f \in A$ and let $x \in K_j$. From (A.3), we obtain

$$\|\mathcal{B}f\|_{K_j} \leq \|\psi\|_{K_j} + \beta\|f\|_{K_j} \leq \|\psi\|_{K_j} + \beta\frac{\|\psi\|_{K_j}}{1-\beta} = \frac{\|\psi\|_{K_j}}{1-\beta} \quad \text{for all } j \in \mathbb{N},$$

which means that the Bellman operator maps A into A. Then by Theorem 1, $\mathcal{B}$ admits a fixed point $\hat{f} \in A$, which is unique on $C(X)$.

(b) Let $x_0 \in X$. We first show that $\Pi(x_0)$ is a compact subset of Z. Since Γ is upper hemicontinuous and closed-valued, then it is closed, which means that Π is closed-valued. Thus, $\Pi(x_0)$ is a closed subset of Z. Yet, $x_0 \in K_j$ for some $j \in \mathbb{N}$, and, since $\Gamma(K_j) \subseteq K_j$, it follows that $\Pi(x_0) \subseteq K_j^{\infty}$, which is compact in the product topology. Hence, $\Pi(x_0)$ is compact, as it is a nonempty and closed subset of a compact set. Further, under assumptions (DP1) and (DP2), the compact valued correspondence Π is also continuous in the product topology.

Now, we shall show that the total discount return function S is continuous in the product topology. Let $\{\tilde{x}^n\}$ be a sequence in $\Pi(x_0)$ such that $\tilde{x}^n = \{x_t^n\} \to \tilde{x} = \{x_t\}$ in the product topology, as $n \to \infty$. Then, for each $t \in \mathbb{N}$, we know that $x_t^n \to x_t$ pointwise as $n \to \infty$, and consequently, since U is continuous, it follows $U(x_t^n, x_{t+1}^n) \to U(x_t, x_{t+1})$ as n approaches infinity. In addition, using the continuity of U and the fact that (x_t^n, x_{t+1}^n) belongs to the compact set $K_j \times K_j$ for all $t, n \in \mathbb{N}$, we have

$$|U(x_t^n, x_{t+1}^n)| \leq K, \quad \text{for some } K \in \mathbb{R}.$$

From this we obtain $\sum_t \beta^t\ |U(x_t^n, x_{t+1}^n)| \leq \sum_t \beta^t\ K < +\infty$ and then, by the Lebesgue Dominated Convergence Theorem, it follows that $S(\tilde{x}_n) \to S(\tilde{x})$. Thus S is continuous in the product topology and, by Berge's Theorem of the Maximum, $v^\star$ is continuous. By Theorem 4.2 in Stokey, Lucas, and Prescott (1989), we know that $v^\star$ satisfies the Bellman Equation, so it must be equal to the unique fixed point $\hat{f}$.

(c) It follows from (b) of Theorem 1 and the above item. *Q.E.D.*

THEOREM 4: (a) As is explained in the previous paragraph to this theorem, $\mathcal{B}$ is always a 1-LC on $C(X)$. Let us consider the subset A of continuous functions f such that $\|f\|_{K_j} \leq m_j$, for all $j \in \mathbb{N}$, where the sequence $\{m_j\}$ is defined as

$$m_j = \sum_{l=j}^{\infty} \beta^{l-j} \|\psi\|_{K_l},$$

which is finite by hypothesis. Now it is easy to show that this sequence is nondecreasing and that the recursion $\|\psi\|_{K_j} + \beta m_{j+1} = m_j$ holds. Hence,

$$\|\mathcal{B}f\|_{K_j} \leq \|\psi\|_{K_j} + \beta \max_{y \in \Gamma(K_j)} |f(y)| \leq \|\psi\|_{K_j} + \beta\|f\|_{K_{j+1}} \leq m_j.$$

Therefore $\mathcal{B}$ maps A on A. Now we can apply Theorem 2 to conclude that $\mathcal{B}$ is a contraction in the metric d_c and therefore has a unique fixed point $\hat{f}$ on A.

(b) It is straightforward to show that $\lim_{t\to\infty} \beta^t \hat{f}(x_t) = 0$ for all $(x_t) \in \Pi(x_0)$, for every $x_0 \in X$. In fact,

$$\begin{aligned} \beta^t|\hat{f}(x_t)| &\leq \beta^t \sum_{l=t}^{\infty} \beta^{l-t} \|\psi\|_{K_l} && \text{(because } x_t \in K_t\text{)} \\ &= \sum_{l=t}^{\infty} \beta^l \|\psi\|_{K_l} \to 0, \ \text{as } t \to \infty && \Big(\text{because } \sum \beta^j \|\psi\|_{K_j} \text{ converges}\Big). \end{aligned}$$

Moreover, given $x_0 \in X$ and $(x_t) \in \Pi(x_0)$, $\sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \leq \sum_{t=1}^{\infty} \beta^j\ \|\psi\|_{K_j}$ is finite, so we can apply Theorem 4.3 in Stokey, Lucas, and Prescott (1989).

(c) It follows from Theorem 2 and the item above. *Q.E.D.*

PROPOSITION 2: Let $f \in C(X)$ be a fixed point of the Bellman operator $\mathcal{B}$, and $x \in X$. Then, for each $i \in \mathbb{N}$, it follows

$$\begin{aligned} \mathcal{B}_i f(x) &= \max_{y \in \Gamma_i(x)} \big(U(x, y) + \beta f(y)\big) \\ &\leq \max_{y \in \Gamma(x)} \big(U(x, y) + \beta f(y)\big) \quad && \text{(from } \Gamma_i(x) \subseteq \Gamma(x)\text{)} \\ &= \mathcal{B} f(x) \\ &= f(x) && \text{(since } f \text{ is a fixed point of } \mathcal{B}\text{)}. \end{aligned}$$

Thus $\mathcal{B}_i f \leq f$ and then, by the monotonicity of $\mathcal{B}_i$, we have $\mathcal{B}_i^n f \leq f$ for all $n \in \mathbb{N}$. From this, and taking into account that $\mathcal{B}_i$ satisfies (c) of Theorem 3, we have $f_i(x) = \lim_n \mathcal{B}_i^n f(x) \leq f(x)$. Hence, $f_i \leq f$ and the sequence $\{f_i\}$ is bounded on $C(X)$. *Q.E.D.*

THEOREM 5: (a) Condition (i) implies that $\sup_{i\in\mathbb{N}} f_i$ is well defined. As we have already noted, the function $\hat{f} = \sup_{i\in\mathbb{N}} f_i$ is lower semicontinuous as the supremum of continuous functions, and satisfies (6). Now we shall show it is upper semicontinuous, so we can consider max instead of sup in (6). We first claim that given $x_0 \in X$ and $\tilde{x}_0 \in \Pi^0(x_0)$, $\hat{f}(x_0) \geq S(\tilde{x}_0)$. Let us suppose $\tilde{x}_0 = (x_t^0) \in \Pi^0(x_0)$. For each $T \in \mathbb{N}$, there is an index $i_T \in \mathbb{N}$ such that the path $(x_0^0, x_1^0, \ldots, x_{T+1}^0, a, a, \ldots)$ belongs to $\Pi_{i_T}(x_0)$, so then

$$\begin{aligned}\hat{f}(x_0) &\geq f_{i_T}(x_0)\\ &\geq \sum_{t=0}^{T} \beta^t\, U(x_t^0, x_{t+1}^0) + \beta^{T+1}\, U(x_{T+1}^0, a) + \sum_{t=T+2}^{\infty} \beta^t\, U(a,a)\\ &\geq \sum_{t=0}^{T} \beta^t\, U(x_t^0, x_{t+1}^0) + \beta^{T+1}\, U^-(x_{T+1}^0, a) + \sum_{t=T+2}^{\infty} \beta^t\, U(a,a).\end{aligned}$$

Taking limits as T tends to $+\infty$, (ii) implies

$$\begin{aligned}\hat{f}(x_0) &\geq \sum_{t=0}^{\infty} \beta^t U(x_t^0, x_{t+1}^0) + \lim_{T\to\infty} \beta^{T+1} U^-(x_{T+1}^0, a)\\ &\geq \sum_{t=0}^{\infty} \beta^t\, U(x_t^0, x_{t+1}^0)\\ &= S(\tilde{x}_0).\end{aligned} \tag{A.4}$$

This proves our claim. Now, let $x_0 \in X$ and let $\{x_i\}$ be a sequence on X such that $\{x_i\} \to x_0$, as $i \to \infty$. To derive upper continuity for $\hat{f}$ we will prove that $\limsup_{i\to\infty} f_i(x_i) \leq \hat{f}(x_0)$. This result is based on the fact that if a sequence of functions f_i converges pointwise to a function $\hat{f}$ and $\limsup_{i\to\infty} f_i(x_i) \leq \hat{f}(x_0)$ for all $x_0 \in X$, for all $x_i \to x_0$, then $\hat{f}$ is upper semicontinuous (see Langen (1981)). In order to do so we proceed as follows. As f_i is the unique fixed point of the operator $\mathcal{B}_i$, we know that, for each i, there exists a path $\tilde{x}_i = (x_t^i) \in \Pi_i(x_i) \subseteq \Pi(x_i)$ such that

$$\begin{aligned}f_i(x_i) &= \sum_{t=0}^{T} \beta^t U(x_t^i, x_{t+1}^i) + \beta^{T+1}\, f_i(x_{T+1}^i)\\ &\leq \sum_{t=0}^{T} \beta^t U(x_t^i, x_{t+1}^i) + \beta^{T+1}\, g(x_{T+1}^i) \quad \text{for all } T \in \mathbb{N}.\end{aligned}$$

Since $\Pi(x_0)$ is compact in the product topology, the sequence $\{\tilde{x}_i\}$ admits a convergent subsequence, so we can assume without loss of generality that $\{\tilde{x}_i\}$ converges to the point $\tilde{x}_0 = (x_t^0) \in \Pi(x_0)$. Since U is continuous, g upper semicontinuous, and x_i^t converges to $x_t^0 \in X$ as $i \to \infty$, we obtain

$$\limsup_{i\to\infty} f_i(x_i) \leq \sum_{t=0}^{T} \beta^t U(x_t^0, x_{t+1}^0) + \beta^{T+1}\, g(x_{T+1}^0) \quad \text{for all } T \in \mathbb{N}.$$

Taking limits as T tends to ∞, this last conclusion yields

$$\begin{aligned}\limsup_{i\to\infty} f_i(x_i) &\leq \sum_{t=0}^{\infty} \beta^t U(x_t^0, x_{t+1}^0) + \limsup_{T\to\infty} \beta^{T+1} g(x_{T+1}^0)\\ &\leq \sum_{t=0}^{\infty} \beta^t U(x_t^0, x_{t+1}^0)\\ &= S(\tilde{x}_0),\end{aligned} \tag{A.5}$$

which in particular implies $\tilde{x}_0 \in \Pi^0(x_0)$. Next, combining (A.4) and (A.5),

$$\limsup_{i\to\infty} f_i(x_i) \le S(\tilde{x}_0) \le \hat{f}(x_0),$$

as required, so the function $\hat{f}$ is upper semicontinuous and, therefore continuous. Hence, it is a fixed point of the Bellman operator. Finally, the convergence of f_i to $\hat{f}$ uniformly on compact subsets of X follows from Dini's Theorem, as $\{f_i\}$ is an increasing sequence of continuous functions converging to a continuous function on a family of compact subsets covering X.

(b) Let $x_0 \in X$. Then $v^\star(x_0) = \max_{\tilde{x}\in\Pi(x_0)} S(\tilde{x}) \ge \max_{\tilde{x}\in\Pi_i(x_0)} S(\tilde{x}) = f_i(x_0)$, which implies $v^\star \ge f_i$ for all $i \in \mathbb{N}$, and consequently $v^\star \ge \hat{f}$. The other inequality has been proved in (A.4).

(c) Every solution $f^\star$ of the Bellman equation such that $\limsup_{t\to\infty} \beta^t f^\star(x_t) \le 0$ for every admissible path, satisfies $f^\star \le v^\star$. Now, from Proposition 2, the function $\hat{f} = v^\star$ is the minimum solution of the Bellman equation, so $f^\star = \hat{f}$ and the uniqueness statement follows.

(d) Given $i \in \mathbb{N}$, the sequence $\{\mathcal{B}^n f_i\}_n$ is increasing and is bounded by $\hat{f}$. Let us denote f as the limit function, where, of course, $f \le \hat{f}$. Now, it is easy to show that f satisfies (6), since:

$$\begin{aligned}
\sup_{y\in\Gamma(x)} (U(x,y) + \beta f(y)) &= \sup_{y\in\Gamma(x)} \sup_{n\in\mathbb{N}} \big(U(x,y) + \beta\mathcal{B}^n f_i(y)\big) \\
&= \sup_{n\in\mathbb{N}} \max_{y\in\Gamma(x)} \big(U(x,y) + \beta\mathcal{B}^n f_i(y)\big) \\
&= \sup_{n\in\mathbb{N}} \mathcal{B}\mathcal{B}^n f_i(x) \\
&= f(x).
\end{aligned}$$

Hence, $f \ge \hat{f}$ because $\hat{f}$ is the lowest function with this property. In consequence, $\hat{f} = f$ and the statement follows again from Dini's Theorem and item (a). *Q.E.D.*

PROPOSITION 3: That d_j is a semidistance and d a distance is obvious. Given a Cauchy sequence $\{f_n\}$ in A, it holds that $d_j(f_n, f_m) \to 0$ as $n, m \to \infty$, for every j. Hence, the sequence $\{\ln((f_n - w)/(w_+ - w))\}$ is Cauchy with respect to the supremum norm in the space of positive and bounded functions defined on K_j, continuous at $K_j \setminus \{0\}$. Since such space is complete, there exists a positive and bounded function g_j to which the sequence above converges uniformly on compact subsets of K_j. It is straightforward to show then that g_j is continuous on $K_j \setminus \{0\}$. An inductive argument on the index j gives rise to a continuous function g, globally defined on X, such that the sequence $\{\ln((f_n - w)/(w_+ - w))\}$ converges uniformly to g on compact subsets of X. Let us define $f = e^g(w_+ - w) + w$. It is clear that $f \in A$ and

$$d_j(f_n, f) = d_j(f_n, e^g(w_+ - w) + w) = \sup_{x\in K_j} \left| \ln\left(\frac{f_n - w}{w_+ - w}(x)\right) - g(x)\right|$$

tends to zero as $n \to \infty$. Therefore the space is complete. *Q.E.D.*

THEOREM 6: (a) Let us consider the bounded and closed subset of functions A defined by

$$A = \{f \in C(X^\star) : w_- \le f \le w_+,\ f(0) = -\infty\},$$

and let $f \in A$. We have $(f - w)/(w_+ - w) = O(1)$ at zero, since $w_- \le f \le w_+$ and $(w_- - w)/(w_+ - w) = O(1)$ at zero by hypothesis. Hence, the semidistances d_j are well defined on A. There are two steps in the proof. The first is to show that $\mathcal{B}$ is a 0-LC. The second step is to show that $\mathcal{B}(A) \subseteq A$.

Step One. Given $f, g \in C(X^\star)$ and $\lambda \in [0, 1]$, by using the definition of the Bellman operator it is easy to see that

$$\mathcal{B}(\lambda f + (1-\lambda)g)(x) \leq \lambda \mathcal{B} f(x) + (1-\lambda)\mathcal{B} g(x).$$

Moreover, for all $f, g \in A$, $x \in K_j$, we have

$$\ln\left(\frac{g-w}{w_+ - w}(x)\right) \leq \ln\left(\frac{f-w}{w_+ - w}(x)\right) + d_j(f, g).$$

Suppressing logarithms we obtain

$$\frac{g-w}{w_+ - w}(x) \leq e^{d_j(f,g)} \frac{f-w}{w_+ - w}(x).$$

Thus $(f - w) \leq e^{-d_j(f,g)}\ (g - w)$ for all nonzero $x \in K_j$ and then

$$f \leq e^{-d_j(f,g)} g + (1 - e^{-d_j(f,g)}) w. \tag{A.6}$$

Now, monotonicity and convexity of $\mathcal{B}$ imply

$$\begin{aligned} \mathcal{B} f(x) &\leq \mathcal{B}\big(e^{-d_j(f,g)} g + (1 - e^{-d_j(f,g)}) w\big)(x) \quad (\text{since } \Gamma(K_j) \subseteq K_j) \\ &\leq e^{-d_j(f,g)}\ \mathcal{B} g(x) + (1 - e^{-d_j(f,g)})\ \mathcal{B} w(x) \quad (\text{by convexity of } \mathcal{B}) \\ &\leq e^{-d_j(f,g)} \mathcal{B} g(x) + (1 - e^{-d_j(f,g)})\big(e^{-d_j(\mathcal{B}w, \mathcal{B}g)} \mathcal{B} g(x) + (1 - e^{-d_j(\mathcal{B}w, \mathcal{B}g)}) w(x)\big). \end{aligned} \tag{A.7}$$

The last inequality follows from (A.6) substituting f for $\mathcal{B}w$ and g for $\mathcal{B}g$. Rearranging terms, subtracting w, dividing by $w_+ - w$, and taking logarithms, we have the following inequality for all $x \in K_j$:

$$\ln\left(\frac{\mathcal{B}f - w}{w_+ - w}(x)\right) \geq \ln z_j + \ln\left(\frac{\mathcal{B}g - w}{w_+ - w}(x)\right),$$

where $z_j = (e^{-d_j(f,g)} + (1 - e^{-d_j(f,g)})\ e^{-d_j(\mathcal{B}w, \mathcal{B}g)})$. In (iv) of Appendix B it is shown that the inequality $\ln z_j \geq -(1 - e^{-d_j(\mathcal{B}w, \mathcal{B}g)})\ d_j(f, g)$ holds. Hence, for all $x \in K_j$

$$\begin{aligned} \ln\left(\frac{\mathcal{B}g - w}{w_+ - w}(x)\right) &\leq \ln\left(\frac{\mathcal{B}f - w}{w_+ - w}(x)\right) + (1 - e^{-d_j(\mathcal{B}w, \mathcal{B}g)}) d_j(f, g) \\ &\leq \ln\left(\frac{\mathcal{B}f - w}{w_+ - w}(x)\right) + (1 - e^{-\mu_j}) d_j(f, g), \end{aligned}$$

where $\mu_j = \sup_{f \in A}\ d_j(f, \mathcal{B}w)$. Finally, interchanging the roles of f and g we obtain

$$d_j(\mathcal{B}f, \mathcal{B}g) \leq (1 - e^{-\mu_j}) d_j(f, g),$$

and, consequently, the Bellman operator is a 0-LC as asserted.

Step Two. First, since assumptions (DP2′) and (i) of (DP3′) hold, Lemma 2 in Alvarez and Stokey (1998) assures that the Bellman operator maps continuous functions on $X^\star$ to continuous functions on $X^\star$. Of course $f(0) = -\infty$ implies $\mathcal{B} f(0) = -\infty$. Now, $\mathcal{B} f \in A$ whenever $f \in A$ by the properties of the bounding functions w_- and w_+ asserted in (DP3′).

(b) Let $(x_t) \in \Pi^0(x_0)$, $x_0 \in X^\star$, and let $\hat{f}$ be the fixed point whose existence is assured in the above item. Since $\hat{f} \in A$, we know that $w_-(x_t) \leq \hat{f}(x_t) \leq w_+(x_t)$ and so the statement follows from the hypotheses made on w_- and w_+. The proof that $\hat{f}$ coincides with the value function $v^\star$ is standard.

(c) It follows from (b) of Theorem 1 and the item above. *Q.E.D.*

APPENDIX B

This appendix is devoted to show the validity of some facts used in the paper.

(i) The truncated correspondences Γ_i satisfy $\Gamma_i(x) \subseteq \Gamma(x)$ for all $x \in X \subseteq \mathbb{R}^l_+$, for all $i \in \mathbb{N}$: Given $x \in X$, let $y \in \Gamma_i(x)$. If $x \in X_i$, then $\Gamma_i(x) = \Gamma(x)$, so suppose that $x \notin X_i$. In this case $y \in \Gamma(P_{X_i}(x))$ and $(P_{X_i}(x), y) \in \mathrm{Graph}(\Gamma)$. We claim that $x \geq P_{X_i}(x)$. Once this is proved, assumption (DP4) implies $(x, y) \in \mathrm{Graph}(\Gamma)$, that is to say, $y \in \Gamma(x)$. To prove the claim, notice that X_i is closed and convex, so the following inequality holds:

$$(x - z) \cdot (a - z) \leq 0 \tag{B.1}$$

for all $a \in X_i$ (see Luenberger (1969)), where "$\cdot$" denotes the scalar product of vectors and $z = P_{X_i}(x)$. Let us consider $J = \{j | x_j < z_j\}$ and $I = \{h | x_h \geq z_h\}$ and by way of contradiction suppose that J is nonempty. Let us define the vector a whose jth component is 0 if $j \in J$ and z_j otherwise; $a \in X$ since $a \leq x$ and X is comprehensive, hence $a \in X_i$ given that $\|a\| \leq \|z\| \leq i$. According to (B.1), $-(x_J - z_J) \cdot z_J \leq 0$, where x_J and z_J denote the components of the vectors x and z corresponding to J, respectively. This inequality contradicts the definition of J.

(ii) The sequence $\{f_i\}$ of approximations to the fixed point of the Bellman operator is increasing. We have

$$\begin{aligned} f_{i+1}(x) &= \mathcal{B}_{i+1} f_{i+1}(x) \\ &= \max_{y \in \Gamma_{i+1}(x)} (U(x, y) + \beta f_{i+1}(y)) \\ &\geq \max_{y \in \Gamma_i(x)} (U(x, y) + \beta f_{i+1}(y)) \quad (\text{since } \Gamma_i(x) \subseteq \Gamma_{i+1}(x)) \\ &= \mathcal{B}_i f_{i+1}(x). \end{aligned}$$

We know that $\mathcal{B}_i$ is a β-LC on $C(X)$ and that for all $f \in C(X)$, $\mathcal{B}_i^n f$ converges as $n \to \infty$ to f_i in some appropriated metric d. In particular, $\mathcal{B}_i^n f_{i+1} \xrightarrow{d} f_i$, as $n \to \infty$. By the monotonicity properties of $\mathcal{B}_i$, we obtain

$$f_{i+1} \geq \mathcal{B}_i f_{i+1} \geq \mathcal{B}_i^2 f_{i+1} \geq \cdots \geq \mathcal{B}_i^n f_{i+1} \geq \cdots.$$

In consequence, $f_i(x) \leq f_{i+1}(x)$. It then follows that $\hat{f} = \sup_{i \in \mathbb{N}} f_i$.

(iii) If $\hat{f} = \sup_{i \in \mathbb{N}} f_i$ is finite, then $\hat{f}(x) = \sup_{y \in \Gamma(x)} (U(x, y) + \beta \hat{f}(y))$. Similar arguments as in (ii) show that for $x \in X_i$ $f_i(x) \leq \sup_{y \in \Gamma(x)} (U(x, y) + \beta \hat{f}(y))$ for every $i \in \mathbb{N}$. From the above inequality we obtain $\hat{f}(x) \leq \sup_{y \in \Gamma(x)} (U(x, y) + \beta \hat{f}(y))$. Moreover, it is clear that

$$\begin{aligned} \mathcal{B} f_i(x) &= \mathcal{B}_i f_i(x) && (\text{from } \Gamma_i(x) = \Gamma(x) \text{ on } X_i) \\ &\leq \max_{y \in \Gamma_{i+1}(x)} (U(x, y) + \beta f_i(y)) && (\text{since } \Gamma_i(x) \subseteq \Gamma_{i+1}(x)) \\ &\leq \max_{y \in \Gamma_{i+1}(x)} (U(x, y) + \beta f_{i+1}(y)) && (\text{because } f_i \leq f_{i+1}) \\ &= f_{i+1}(x) \\ &\leq \hat{f}(x) && \Bigl(\text{because } \hat{f} = \sup_{i \in \mathbb{N}} f_i\Bigr). \end{aligned}$$

Taking the supremum in the above inequality we obtain $\sup_{y \in \Gamma(x)} (U(x, y) + \beta \hat{f}(y)) \leq \hat{f}$.

(iv) The inequality $\ln z_j \geq -(1 - e^{-d_j(\mathcal{B}\omega, \mathcal{B}g)})\, d_j(f, g)$ holds, where

$$z_j = \bigl(e^{-d_j(f,g)} + (1 - e^{-d_j(f,g)})\, e^{-d_j(\mathcal{B}\omega, \mathcal{B}g)}\bigr).$$

To prove this, let us consider the strictly convex function $z(x) = \ln(a + e^{-x}(1 - a))$, where $a = e^{-d_j(\mathcal{B}\omega, \mathcal{B}g)}$. Its second order Taylor expansion around zero gives $z(x) = -(1 - a)x + z''(\bar{x})x^2/2$, with $0 < \bar{x} < x$, hence $z(x) > -(1 - a)x$ which is the desired inequality when $x = d_j(f, g)$.

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