(SUB-) OPTIMAL ENTRY FEES*

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A B S T R A C T

We extend Myerson's (1981) model by allowing for uncertainty about the number of bidders. In such extension the Revenue Equivalence Theorem still holds and the optimal allocation rule remains the same. Hence, the optimal auction can be implemented with an appropriate reserve price. Nonetheless, we show that entry fees are sub-optimal. The reasons are heterogeneity in bidders' beliefs about the number of bidders, and auctioneer's uncertainty about the optimum entry fee, if any. Our result implies a reversal of the revenue ranking by Milgrom and Weber (1982) which is consistent with many real life situations: auction houses, internet auctions,...

KEYWORDS: Optimal Auction; Random Number of Bidders; Reserve Price; Entry Fee.

1 Introduction

One stylised fact is the widespread use of reserve prices to enhance auctioneer's expected revenue, whereas, entry fees are very scarcely used. This fact is specially true for what we can call small or popular auctions, for instance, traditional auction houses like Christie's or Sotheby's, the newborn internet auctions like eBay or Yahoo, or some second hand markets organised through auctions like those in London for second hand cars or second hand houses. Another illustration of this observation is that an specialised book such as Cassady (1967) dedicates several pages to the effect of a reserve price but it provides no comment with respect to entry fees.

This empirical fact contrasts with the theoretical literature that tend to suggest that the auctioneer can improve his expected revenue by reducing the reserve price and increasing or introducing an entry fee.[1] This is for instance the case of the celebrated model of Milgrom and Weber (1982), but also of Engelbrecht-Wiggans (1993), or Levin and Smith (1994). Note that these results hold in spite of the revenue equivalence established by Myerson (1981). The reason is that once we enrich the model of Myerson (Milgrom and Weber, for example, allow for affiliation in valuations) the equivalence between the use of entry fees and reserve prices breaks down.

We argue in this paper that it is not necessarily the case that reality and theory are at odds. We show that the revenue ranking provided by Milgrom and Weber (and the other papers in the latter paragraph) can be reversed. We provide an explanation based on a quite common feature of the examples we mention above: the uncertainty about the number of bidders that both the bidders and the auctioneer have. The former at the stage of deciding their participation in the auction and the latter when he decides the auction characteristics. Other authors like Cassady (1967), and McAfee and McMillan (1987) have argued about the importance of this fact.

We model such uncertainty assuming that the number of bidders is drawn according to a quite general exogenous stochastic process. Under this assumption, we show that the revenue maximising auction can be implemented with a reserve price, for instance in a second price auction. We also show that entry fees can implement the revenue maximising auction only in non generic cases.

The intuition of our results is better understood from the Revenue Equivalence Theorem: Myerson (1981) shows that under independent private information about valuations, risk neutrality, and fixed number of bidders, the auctioneer's expected revenue is determined by the allocation rule and the expected utility of the bidders with less willingness to pay. He also shows that the optimal allocation rule is quite simple: to sell to the bidder with highest valuation provided that her valuation is above a certain cut-off. Thus, Myerson concludes that several mechanism are optimal, among them auctions with reserve prices and entry fees appropriately chosen.

We show that a direct consequence of the results by McAfee and McMillan (1987) is that the Revenue Equivalence Theorem and the structure of the optimal

[1] An exception is Waehrer, Harstad, and Rothkopf (1998).

allocation rule remains the same when the number of bidders is not fixed but rather exogenously stochastic. Such result directly implies that an appropriate auction set-up, as a second price auction or an English auction, with a concrete reserve price is expected revenue maximising. However, we cannot argue as Myerson that auctions with entry fees are optimal.

The reason why an auction with an appropriate reserve price can implement the optimal auction but not with an entry fee is subtle. We can argue as Myerson that there is always an entry fee that induce a given bidder to take the same entry decisions as with an optimal reserve price. Hence, it seems that we could appeal to the Revenue Equivalence Theorem to claim that there must exist an optimal entry fee.

Nonetheless, the entry fee that induces the same bidder's entry decisions as a given reserve price will depend on the bidder's beliefs about the actual number of bidders. The more bidders she expects to meet in the auction, the less incline she will be to pay the entry fee. Thus, if there is some heterogeneity in these beliefs, we cannot find an entry fee that induce all the bidders simultaneously to take the same entry decisions as an optimal reserve price. Moreover, even if the bidders hold the same beliefs, the auctioneer will not be able to compute the optimal entry fee unless he knows the bidders' beliefs. In a world with uncertainty about the number of bidders this can be a quite restrictive assumption.

In reality, for instance, quite often there are experienced bidders that have clear ideas about the number of active bidders, and less experienced bidders with more blurred beliefs. Similarly, bidders can have access to some pieces of information (magazines, car queues close to the auction house, a bunch of colleagues that are considering to enter the auction,...) that were not available to the auctioneer when he announced the entry fee.

We also argue that the situations in which entry fees are optimal are quite unstable. Suppose that there is uncertainty about the number of actual bidders, but bidders have homogenous beliefs known by the auctioneer. Under such conditions an entry fee appropriately chosen in a second price auction is optimal. Suppose now a slightly different game in which bidders have the possibility of acquiring secretly some additional information about the number of actual bidders. Then, we show that each bidder has strict incentives to acquire this additional information when all the other bidders play the equilibrium strategies of the model with no information acquisition.

Hence, if the cost of information acquisition is sufficiently low, some bidders will acquire information. Then, entry fees will be no longer optimal. There will be heterogeneity among the bidders' beliefs because either the additional information differs across bidders or because not all the bidders have acquired new information. Moreover, if the auctioneer does not know the source of this additional information he will be uncertain about the bidders' beliefs.

In practice, for instance, this will happen if bidders are initially equally uninformed. Then, we could argue that entry fees make sense as bidders' beliefs about the number of active bidders can be somewhat homogeneous. However, the mere fact that entry fees are used prompts the bidders to seek out information. Hence, if some of the bidders get additional information, it will create the aforementioned heterogeneity in bidders' beliefs.

Our results above are achieved when the auctioneer has no private information about the number of active bidders. We show that similar conditions although less restrictive are required when the auctioneer has some private information. For instance, the auctioneer can have some information that appropriately revealed to the bidders vanishes the heterogeneity in their beliefs. However, as we illustrate in the paper, quite often the auctioneer cannot make use of his private information to use optimal entry fees because he cannot credibly reveal it. Note that a bidder is more willing to pay high entry fees if she believes that there are fewer other bidders. Hence, the auctioneer has incentives to mislead the bidders making them belief that there are less bidders than there are in reality.

The most closely related paper is that by McAfee and McMillan (1987). They also characterise the optimal auction in a model in which bidders have uncertainty about the number of bidders. McAfee and McMillan, however, do not compare in terms of expected revenue auctions with reserve prices and/or entry fees. Moreover, our model generalises McAfee and McMillan results to the case in which the auctioneer holds uncertainty about the number of bidders and the bidders hierarchies of beliefs about the number of bidders.

We have also referred to some other papers in which entry fees are compared with reserve prices. The model by Milgrom and Weber (1982) has been already explained. Engelbrecht-Wiggans (1993), and Levin and Smith (1994) study a model in which the number of bidders varies endogenously. The former paper points out that an entry fee can implement full surplus extraction, and the latter shows that in fact, entry fees strictly dominate reserve prices. Note, however, what they call an entry fee should more appropriately be called an inspection fee: bidders only learn their valuation after paying it. Hence, an entry fee can extract the bidders' surplus because bidders have no private information at the time when they have to pay the inspection fee. A reserve price cannot do so well because it is paid only once the bidders have their private information and then the auctioneer cannot extract the bidders' informational rents. Finally, Waehrer, Harstad, and Rothkopf (1998) provide a result in line with our ranking result. They show that a risk averse auctioneer can achieve higher expected utility using an appropriate reserve price than with any entry fee.

The paper is organised as follows. In Section 2 we present the set of assumptions that describe the basic features of our model. We move in the third section to our main results: characterisation of the optimal auction, optimality of reserve prices and sub-optimality of entry fees. In the fourth section, we consider the incentives of bidders to acquire information. Section 5 extends the analysis of section 3 to the case in which the auctioneer has private information. Section 6 concludes.

2 The Model

An expected revenue[2] maximising auctioneer puts up for sale a single unit of a non divisible good. A set of N bidders that we denote by $B \subset \mathbb{N}$ and call the *active bidders* is interested in this good. Their preferences are characterised by a von Neumann-Morgenster utility function equal to $v_i - p$ when bidder i gets the good and pays p, and equal to $-p$ when she does not get the good but pays p. We shall refer to v_i as bidder's i *valuation*. Bidders' valuations are assumed to be private information and to follow each[3] an independent distribution F with support $[0, 1]$ and density f.

Our independence assumption goes around two problems. As Cremer and McLean (1988) have shown, optimal auctions when bidders' valuations have strict correlation are counterintuitive and quite sensitive to the primitives of the model. Moreover, Milgrom and Weber (1982) show that if the joint distribution of bidders' valuations satisfies the affiliation inequality, reserve prices are dominated by entry fees. In this case our independence assumption allows to provide a clear-cut set-up in which our results do not mix with Milgrom and Weber's result.

We restrict to what Myerson (1981) has called the *regular* case. This is that the function $v - \frac{1-F(v)}{f(v)}$ is strictly increasing in $v \in [0, 1]$. Such assumption is quite standard in auction theory and it is satisfied by many distribution functions (e.g. the uniform). We shall denote by v^* the unique solution to $v^* - \frac{1-F(v^*)}{f(v^*)} = 0$ and assume for simplicity that v^* exists and belongs to the interval $(0, 1)$.

We allow for bidders and auctioneer uncertainty about the number of bidders, although consistent with a common prior. This common prior is modelled[4] assuming that the set of active bidders B is drawn from a set of *potential bidders* according to an exogenous random process.[5] In our model, the fact of being active conveys the bidder private information about B. On top of this private information, we allow bidders to have some additional private information about B. To do so, we assume that the exogenous process that selects active bidders also generates a signal per active bidder. We denote by S_i the private signal that corresponds to bidder i, and by $\mathcal{S}_i$ its support. In some instances, we shall also allow the auctioneer to have private information about B, we shall refer to this additional information by S_0, with support $\mathcal{S}_0$. For simplicity we assume that the signals' support is countable.[6]

[2] Our results would also hold if the auctioneer puts some reserve value in retaining the good but in two trivial cases: when the reserve value is so low that the auctioneer wants to sell always; and when it is so high that the auctioneer never wants to sell.

[3] We consider a private value set-up for simplicity. Our results directly generalise to a common value set-up under two assumptions: the common value is a linear function of the private signals of the bidders, and the bidders' private information about the value of the object is statistically independent.

[4] We make explicit the procedure through which the number of bidders is selected in order to be able to construct in a Bayesian fashion bidder's beliefs about the number of active bidders. McAfee and McMillan (1987) use a similar construction to model exogenous uncertainty about the number of active bidders.

[5] More formally, we assume that the support of B is a countable subset of $2^{\mathbb{N}}$, and $\mathcal{B}$ is the union of all the sets in this support.

[6] That the support of S_i is countable is restrictive. Note, however, that Mertens and Zamir

Our model to generate beliefs about the set of active bidders is that a direct generalisation Mc Affe and Mc Millan's (1987) model. We generalise it by allowing private signals about the set of active bidders. The reason of such extension is that we are interested in the issue of information acquisition about the set of active bidders. Note that this extension makes our model differ in that the identity of the bidder does not convey the beliefs of the bidder about the set of active bidders as in the model of Mc Affe and Mc Millan.

We refer to the probability measure that describes the common prior that generates B and the signals S_i's and S_0 with Pr[.]. We also denote by Pr[.|.] to the probability of the event on the left side of the vertical bar conditional on the event on the right side of the vertical bar. Finally, denote by E[.] and E[.|.] the respective expected values associated to the common prior. We shall assume that that for any set A with positive probability (with respect to the probability measure Pr) $E[N|A] < \infty$.

Recall that we have implicitly assumed that bidders' information about their valuations is orthogonal to the process that captures the information about the set of active bidders. There is little hope that our results hold under more general information structures but in rather specific examples. To see why note that this assumption avoids two problems: First, if bidders' signals provide some information about the other bidders' valuations, we can expect that active bidders' beliefs about the other active bidders' valuations will differ unless we make strong symmetry assumptions. This would imply working with "asymmetric" auctions whose analysis is complex and does not provide clear-cut predictions in general (some partial results have been provided by Maskin and Riley (2000)). Second, the assumption that bidders' valuations are statistically independent is less plausible since they could (although not necessarily) be statistically related through the set of active bidders.

3 Optimal Auctions

In this section we show that the revenue maximising auction can be implemented in general through a second price auction with an appropriate reserve price, but not with an entry fee. To provide such results we start characterising the set of optimal auctions:

Proposition 1. *An auction is optimal if and only if the following conditions hold*[7] *a.s.: (i) The good is allocated to the active bidder with highest valuation if this is higher than v^*; otherwise, the auctioneer retains the good. (ii) Each active bidder that has value* 0, *and for each possible realisation of their private signal S_i, gets zero expected utility.*

Proof. Our proof makes use of the results of McAfee and McMillan (1987). Their analysis of the optimal auction differs from us in only one aspect: they assume that

(1985) have proved that a general belief space can be arbitrarily approximated by a finite belief space, hence a fortiori by a countable belief space.

[7] We write "a.s." for almost surely with respect to the probability measure denoted by Pr.

bidders do not hold private signals. This assumption in principle can be reinterpret assuming that bidders with different private signals are different potential bidders. Note that under our assumptions the new set of potential bidders is also countable as required by McAffe and McMillan's assumptions. However, by doing this trick we miss one important detail of our model. This assumption implies that the bidder's identity tells the auctioneer the bidder's beliefs about the set of active bidders (and hierarchies of beliefs). We show that this assumption turns out to be unrestrictive. Theorem 4 by McAfee and McMillan implies that the optimal mechanism must satisfy the conditions provided in Proposition 1 when the auctioneer can infer bidder's beliefs from bidder's identity. But, this result implies the proposition since the former optimal mechanism can be implemented even if the auctioneer does not know the bidders' beliefs. For instance, a second price auction with reserve price v^* implements the optimum (see below). ■

The proof of this proposition is based on McAfee and McMillan (1987). They apply an adaptation of Myerson's (1981) proof to the case in which the set of active bidders is stochastic. They show that bidders risk neutrality plus independency of bidders' valuations imply the revenue equivalence. From this, it is easy to deduce that condition (i) characterises the optimal allocation rule, and condition (ii) the zero profit condition of the minimum type. Note that condition (i) imposes the same restrictions on the optimal allocation as in Myerson's model, whereas condition (ii) is just a direct generalisation of an equivalent condition of optimality in Myerson's model.

Note that our proof generalises McAfee and McMillan's result by allowing bidders to have some additional private information about the set of active bidders. We show that this new addition does not affect to the set of optimal auctions. This is so because a second price auction with a reserve price equal to v^* implements the optimal auction. It is weakly dominant for the bidders to: (a) enter the auction if and only if their valuations are above v^* and (b) bid the true valuation conditional on entering. Hence,

Corollary 1. *A second price auction (or an English auction) with reserve price v^* and no entry fee implements the optimal auction.*

Since the revenue equivalence holds, we could expect that entry fees appropriately chosen are also optimal with generality. We show below that this conjecture is not true. We provide such proof considering entry fees in any auction mechanism, and even if combined with a reserve price. To provide such general result we define an *entry fee* as a front payment to the auctioneer that each of the bidders must do in order to participate in the auction.[8] We also define a *reserve price* as the price that the winner of the auction pays (on top of the entry fee) when no other bidder submits a serious bid. Note that these definitions cover entry fees and reserve prices in standard auction mechanisms (first price auctions,[9] second price auctions,...)

[8] Obviously, this payment must be non refundable, so we disregard mechanisms in which front payments are (partially) refund by whatever means to the losers of the auction. For instance, an all pay auction that allows for negative bids.

[9] If the number of bidders that will submit serious bids is observed before the bids are submitted.

For the sake of simplicity we focus in this section in the case in which the auctioneer does not have private information about the set of active bidders or their respective signals S_i. We show in Section 5 how to modify Proposition 2 when the auctioneer has some private information.

Proposition 2. *If the auctioneer has no private information, an entry fee is optimal only if:*

(a) "Active bidders have homogeneous beliefs" in the sense that for all $i, j \in \mathcal{B}$,

$$E[F(v^*)^{N-1}|S_i, i \in B] = E[F(v^*)^{N-1}|S_j, j \in B],\ a.s.$$

(b) "The auctioneer knows the common beliefs of the potential bidders" in the sense that there exists a $k \in [0,1]$ such that for all $i \in \mathcal{B}$,

$$E[F(v^*)^{N-1}|S_i, i \in B] = k,\ a.s.$$

Proof. We start providing an *indifference condition* that must be satisfied by each bidder if condition (i) in Proposition 1 hold in equilibrium. We then show that this indifference condition can be satisfied simultaneously by all the active bidders only if (a) holds. Then, we show that (b) is also necessary since the auctioneer can compute the optimal entry fee only if he knows $\mathrm{E}[F(v^*)^{N-1}|S_i, i \in B]$.

If condition (i) holds active bidders with valuation v^* must be indifferent between entering the auction or not. Suppose, for instance, that an active bidder with valuation v^* strictly prefers to stay out of the auction. Then, the continuity[10] of the active bidders expected utility with respect to valuations implies that she would not participate in the auction if she had a type close to v^*. This means that with positive probability this active bidder will not get the object when she has the highest valuation and this valuation is above v^*, i.e. condition (i) does not hold with positive probability.

When condition (i) verifies, a generic active bidder i with valuation v^* wins the auction if and only if no other active bidder has a valuation above v^*. Bidder i puts probability $\mathrm{E}[F(v^*)^{N-1}|S_i, i \in B]$ on this event. In this case, she pays the reserve price, say r. This means that our indifference condition implies that the entry fee e must be such that:

$$e = (v^* - r)\mathrm{E}[F(v^*)^{N-1}|S_i, i \in B],\ \text{a.s.} \tag{1}$$

This condition can be satisfied simultaneously by all active bidders only if (a) holds. Note also that (b) must hold as otherwise the auctioneer would not be able to compute the optimal entry fee either by itself or combined with a reserve price. ■

[10] We do not provide the proof that bidder's expected utility is continuous with respect to the bidder's valuation. This proof follows from Equation (47) in McAfee and McMillan (1987).

The reason why there is no optimal entry fee with generality is somewhat subtle. It is true that we can always find an entry fee that can induce a given bidder to take the same entry decisions as with an optimal reserve price. Hence, we could think of appealing to the revenue equivalence to claim that there must exist an optimal entry fee. However, the entry fee that corresponds to the optimal reserve price will depend on the bidder's beliefs about the number of active bidders. Thus, if these beliefs differ in a particular sense across bidders, we cannot find an entry fee that produces the same entry level as an optimal reserve price. Another difficulty is that even if all the active bidders had the same beliefs, the auctioneer could be uncertain about which are the bidders' beliefs and hence, the optimal entry fee. Since we assume that the auctioneer does not have private information, this will happen when the bidders' beliefs are common but random and this randomness shifts in a particular sense the common beliefs.[11]

Proposition 2 specifies the particular sense in which if beliefs differ there is no optimal entry fee. This sense is that active bidders put different probability on winning the auction when they have a valuation v^*, all the active bidders follow entry strategies characterised by the cut-off v^*, and the outcome of the auction is such that bidders with higher valuations outbid bidders with lower valuations. Similarly, Proposition 2 specifies in a parallel way the sense in which shifts of an active bidder's beliefs can make entry fees sub-optimal.

We only find two scenarios in which the necessary conditions for optimal entry fees are naturally satisfied: when the number of active bidders is common knowledge, and in a "fully symmetric" model in which all bidders have the same beliefs about the number of active bidders and the auctioneer knows these beliefs. Once we admit that there is some kind of heterogeneity or auctioneer's uncertainty about the beliefs about the number of bidders, the general result will be that entry fees are sub-optimal.

For instance, if some active bidders beliefs about the number of active bidders dominate the beliefs of some other active bidders in the sense of strict first order stochastic dominance, condition (a) of Proposition 2 will be violated. The reason is that $F(v^*)^{N-1}$ is a strictly decreasing function. Hence, the active bidders with dominated beliefs in the above sense will put strictly higher probability on the expected value of $F(v^*)^{N-1}$ than the active bidders with dominant beliefs. The same thing happens when all the active bidders' beliefs are common but they randomly shift in the sense of first order stochastic dominance.

3.1 An Example with Heterogeneity among Bidders

We provide a simple example in which active bidders do not receive any private signal S_i. The source of heterogeneity on bidders' beliefs will come from the fact

[11] We could think that the auctioneer can avoid this problem by fixing a "contingent" entry fee that varies with the number of bidders that submit bids. But, this is actually what a reserve price is: an entry fee that it is paid only if one bidder enters the auction. In fact, it can be shown that when entry fees are sub-optimal, the only contingent entry fee that implements the optimum is a reserve price.

that being active provides different bidders different information. More precisely, we shall assume that the probabilities that bidders are active are correlated, and this correlation is asymmetric. For simplicity, we assume a very simple model of correlation. There are two groups of bidders G_1 and G_2, each with $\bar{N}_1$ and $\bar{N}_2$ potential bidders respectively. All the bidders in a given group are active with a probability $\rho \in (0,1)$ and inactive with probability $1-\rho$. These probabilities are identical and independent for both groups. The asymmetries will come from the fact that we assume that $\bar{N}_1 > \bar{N}_2$.

Figure 1 shows that the distribution of N conditional on bidder $i \in G_1$ active $(N|G_1)$ dominates the distribution of N conditional on bidder $j \in G_2$ active $(N|G_2)$ in the sense of strict first-order stochastic dominance. Hence, the expected value of $F(v^*)^{N-1}$ conditional on bidder $i \in G_1$ active will be strictly lower than the same expected value conditional on bidder $j \in G_2$ active. Since this situation will happen with positive probability, the conditions of Proposition 2 are violated and thus, entry fees are sub-optimal.

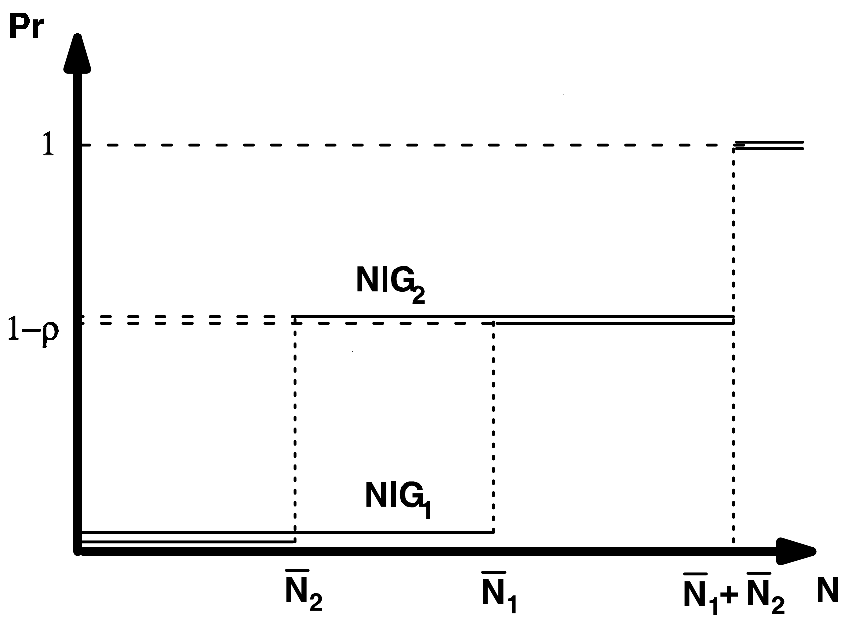

Figure 1: Distribution functions of N conditional on a bidder of group G_1 active $(N|G_1)$ and conditional on a bidder of group G_2 active $(N|G_2)$.

3.2 An Example with Auctioneer's Uncertainty

We illustrate auctioneer's uncertainty assuming that the each bidder knows the number of active bidders but the auctioneer does not know it. More formally, we assume that B has a non degenerate prior and that each active bidder's signal informs of the set of active bidders, i.e. S_i equals B for all $i \in B$. Under this assumption the first condition of Proposition 2 is trivially satisfied. However, the auctioneer does not know the active bidders' beliefs in the sense required by Proposition 2. Hence, entry fees will be sub-optimal because of the auctioneer's uncertainty.

4 Acquisition of Information and Optimal Entry Fees

In this section, we provide an example that illustrates that the situations in which entry fees are optimal are quite unstable. We show that if the auctioneer uses an entry fee, bidders have strict incentives to acquire secretly *additional information* about the number of active bidders. Once, some bidders acquire this additional information with positive probability, in general it is no longer optimal to fix an entry fee.

For this purpose we assume that the auctioneer first makes an announcement of the auction structure, and then bidders upon observing[12] this announcement can decide to acquire or not some private signal. We start assuming that all the bidders are ex ante symmetric in the sense that there exists[13] a distribution function G such that the distribution of N conditional on bidder i active equals G for all potential bidders $i \in \mathcal{B}$. We also assume that active bidders do not observe any private signal. Under these assumptions, the necessary conditions for optimality of entry fees of Proposition 2 are satisfied. In fact, it is easy to show that under such conditions a second price auction with an appropriate entry fee is optimal (for instance, an entry fee that satisfies equation (1)).

Suppose that the auctioneer fixes an optimal entry fee in a second price auction. Suppose also that the common distribution G is non degenerate. This means that active bidders hold some uncertainty about the number of active bidders. We then study the incentives of a generic bidder i to reduce this uncertainty by acquiring secretly a signal S_i when all the other bidders follow the entry decisions that correspond to the model with no information acquisition, i.e. enter if and only if $v_j \geq v^*$ $(j \neq i)$. For simplicity we assume that S_i is a dicothomic signal such that it shifts the conditional distribution of the number of active bidders in the sense of strict first order stochastic dominance. More precisely, we assume that for s_l and s_h the two realisations of S_i, the distribution of N conditional on $i \in B$ and $S_i = s_h$ dominates the distribution of N conditional on $i \in B$ and $S_i = s_h$ in the sense of strict first order stochastic dominance. The labelling of s_l and s_h is without loss of generality.

Proposition 3. *Bidder i has strict incentives to acquire secretly the additional information S_i.*

Proof. We start recalling that bidder i's expected utility when she has a valuation v^*, enters the auction, and all the other bidders enter the auction if and only if $v_j \geq v^*$ $(j \neq i)$, equals $(v^* - r)\mathrm{E}[F(v^*)^{N-1}|\mathrm{Inf}] - e$ where r is the reserve price, e the entry fee,

[12] The analysis can also be extended to the case in which the bidders take the decision before the auctioneer makes the announcement. In this case, we could argue that if the bidders forecast that the auctioneer is going to announce a positive entry fee, they have strict incentives to acquire some additional information.

[13] Example 1 by McAfee and McMillan (1987) shows how to construct the active bidders selection rule to assure that all the active bidders have the same beliefs about the number of active bidders.

and "Inf" is bidder i's private information. Recall also that in an optimal auction a bidder with valuation v^* must be indifferent between entering the auction or not, this is $(v^* - r)\mathrm{E}[F(v^*)^{N-1}|i \in B] - e = 0$. Finally, a consequence of strict first order stochastic dominance is that $\mathrm{E}[F(v^*)^{N-1}|i \in B, \hat{S}_i = s_h] < \mathrm{E}[F(v^*)^{N-1}|i \in B] < \mathrm{E}[F(v^*)^{N-1}|i \in B, \hat{S}_i = s_l]$. Therefore, $(v^* - r)\mathrm{E}[F(v^*)^{N-1}|i \in B, \hat{S}_i = s_h] - e < 0 < (v^* - r)\mathrm{E}[F(v^*)^{N-1}|i \in B, \hat{S}_i = s_l] - e$. This is, if bidder i acquires signal S_i and enter the auction with a valuation v^* (or close to v^* by continuity), she gets strictly negative expected utility if $S_i = s_h$ and strictly positive expected utility if $S_i = s_l$. Thus, bidder i can strictly improve by acquiring secretly the signal $\hat{S}_i$ and revising his entry strategy when her valuation equals v^* (or it is close to v^*): not entering the auction if $S_i = s_h$ and entering if $S_i = s_l$. ■

If at least one bidder acquires secretly her corresponding signal S_i the conditions of Proposition 2 do not hold. Thus, we can figure out that that even if the conditions of Proposition 2 are satisfied, if the cost of information acquisition is sufficiently low, entry fees cannot be optimal.

5 The Auctioneer Has Private Information

In this section we assume that the auctioneer privately observes a signal S_0 informative of the number of active bidders and of the bidders' hierarchies of beliefs associated, this is of B and the S_i's signals. The signal S_0 was already introduced in Section 2.

Private information allows the auctioneer to make entry fees optimal even if the conditions of Proposition 2 are not satisfied. There are two ways by which a privately informed auctioneer can make entry fees optimal. First, by (partially) revealing his private information the auctioneer can eliminate the heterogeneity of active bidders' beliefs in the sense of Proposition 2. Second, the auctioneer will have more information to compute the optimal entry fee.

To restate the necessary conditions of Proposition 2, we need to formalise information revelation. We introduce an independent random variable Z with uniform distribution function on $[0, 1]$. Then, we model the information that the auctioneer reveals with what we call *reporting functions*. These are functions Π that assign to each realisation of S_0 and Z a value π that we call a report.[14] We assume that this function and its realisations are common knowledge. Note that such reporting functions allow for pooling of information and for noisy revelation of information, or even no information revelation.

We also make the perfect Bayesian equilibrium assumption that bidders learn the information on which the auctioneer conditions his auction rules. For instance, if the auctioneer chooses a given entry fee e if and only if he observes a given realisation of his private information $S_0 = s_0$, the bidders will learn that $S_0 = s_0$ upon observing e.[15] This is equivalent to say that the auctioneer can condition his

[14] These functions has also been used by Milgrom and Weber (1982).

[15] More precisely, we think of a two stage game. In a first stage, the auctioneer makes an announcement of his auction mechanism contingent on his private information (and possibly he also

auction mechanism only on the information that he conveys, i.e. on the realisations of his report function.

Proposition 4. *Given a reporting function* Π*, an auctioneer can implement the optimal auction conditional on a realisation* π *in the support of* Π*, only if:*

(a) "Active bidders have conditional homogeneous beliefs" in the sense that for all $i, j \in \mathcal{B}$*,*

$$E[F(v^*)^{N-1}|S_i, i \in B, \Pi(S_0, Z) = \pi] = E[F(v^*)^{N-1}|S_j, j \in B, \Pi(S_0, Z) = \pi], \text{ a.s.}$$

(b) "The auctioneer knows the common conditional beliefs of the active bidders" in the sense that there exists a constant k *such that for all* $i \in \mathcal{B}$*,*

$$E[F(v^*)^{N-1}|S_i, i \in B, \Pi(S_0, Z) = \pi] = k, \text{ a.s.}$$

Proof. The above conditions are the necessary conditions in Proposition 2 conditional on the information revealed by the report function. ■

Corollary 2. *The auctioneer can find optimal to fix an entry fee for some realisations of his private signal only if there exists a reporting function that verifies the conditions of Proposition 4.*

Although the fact that the auctioneer has some private information makes more feasible that the he can implement the optimum with an entry fee, the conditions are still quite restrictive. Note that only if the auctioneer's information can vanish the active bidders' heterogeneity in the sense of the proposition, entry fees can be optimal. A natural example where the conditions of Proposition 4 hold is when the auctioneer knows the number of active bidders. Example 5.1 illustrates a simple case in which the conditions of Proposition 4 are not satisfied.

Moreover, the above conditions are sufficient for entry fees appropriately chosen to be optimal only if the auctioneer has full commitment power. This commitment refers not only to the mechanism rules, as in the standard set-up, but also to the veracity of the information that the auctioneer conveys, i.e. to the report function. Example 5.2 shows that the auctioneer can have incentives to mislead the bidders by revealing false information.

5.1 An Example with Conditional Heterogeneity among Bidders

We consider a similar case to the example in sub-section 3.1. We assume that active bidders do not receive any private signal S_i. The source of heterogeneity are the asymmetries among bidders. We assume that there are two groups of bidders G_1

reveals part of his private information). In a second stage, the bidders observe the auction mechanism announced by the auctioneer and take the bidding decisions. The perfect Bayesian equilibrium restrictions imply that the bidders update their beliefs in a consistent way with the auctioneer's map from his private information to the set of auction mechanisms along the equilibrium path.

and G_2, each with $\bar{N}_1$ and $\bar{N}_2$ potential bidders respectively. All the bidders in the first group are always active, whereas all the bidders in the second group are simultaneously active with a probability $\rho \in (0, 1)$, and simultaneously inactive with probability $1 - \rho$.

Figure 2 shows that the distribution of N conditional on bidder $i \in G_2$ active $(N|G_2)$ dominates the distribution of N conditional on bidder $j \in G_1$ active $(N|G_1)$ in the sense of strict first-order stochastic dominance. The difference is that whereas bidders of group G_2 know the number of active bidders to be $\bar{N}_1 + \bar{N}_2$ when they are active, bidders of group G_1 do not know whether the bidders of G_2 are active, and then $N = \bar{N}_1 + \bar{N}_2$, or the bidders of G_2 are not active and then $N = \bar{N}_1$. This implies that condition (ii) of Proposition 2 is not satisfied, and hence, entry fees cannot be optimal.

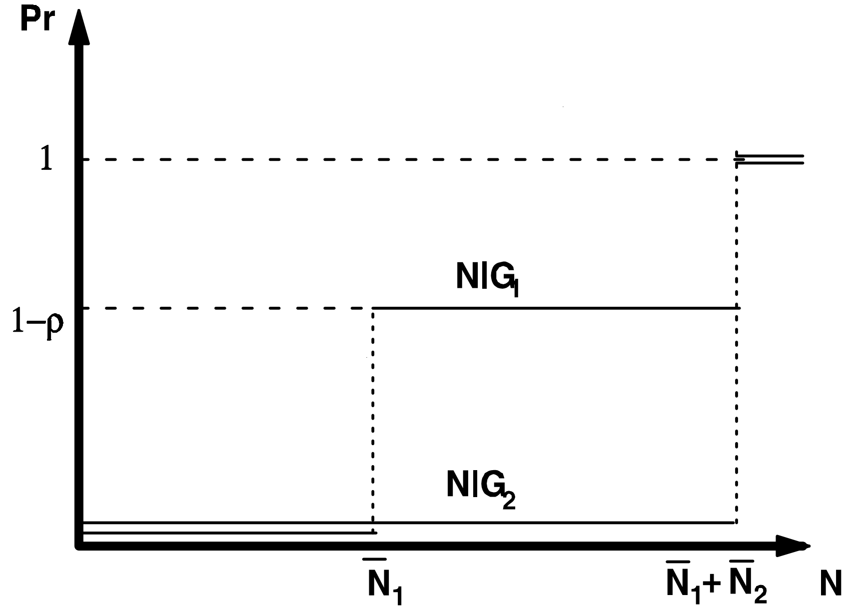

Figure 2: Distribution functions of N conditional on a bidder of group G_1 active $(N|G_1)$ and conditional on a bidder of group G_2 active $(N|G_2)$.

Bidders in the second group always know (when they are active) the number of active bidders. Hence, if bidders in the first group have some uncertainty about whether bidders in the second group are active or not, condition (a) in Proposition 4 cannot be satisfied. This means that there are only two ways by which the necessary conditions of Proposition 4 can be satisfied. The first one is when the auctioneer knows when bidders of G_2 are active. Then, if bidders G_2 are active, the auctioneer can convey this information to the bidders and eliminate the heterogeneity in the sense of Proposition 4. The other case is when the auctioneer knows when bidders of G_2 are inactive. Then, if bidders G_2 are inactive the auctioneer knows that all the active bidders hold the same beliefs about the number of active bidders. He can convey this information to the bidders so that condition (a) is satisfied and compute the active bidders beliefs in the sense of condition (b). Hence, entry fees can be optimal only if the auctioneer has very precise information about the number of active bidders.

5.2 Auctioneer Incentives to Reveal False Information

Proposition 4 states necessary conditions for entry fees to be optimal. It is easy to show that these conditions are sufficient if the auctioneer's announcements of his private information are credible, i.e. if the auctioneer can commit to a given report function. Note, however, that the information revealed by the auctioneer affects the bidders' behaviour. And hence, the auctioneer could have incentives to mislead the bidders with false information.

We illustrate with a simple example that entry fees can be sub-optimal even if the conditions of Proposition 4 are met. The reason is that it could be that the auctioneer has no way of conveying credibly his private information.

We assume that there are two potential bidders, bidder 1 and bidder 2. Each of them is active with an independent probability, say ρ_1 and ρ_2 respectively, where $\rho_1 \neq \rho_2$. Moreover, suppose they do not receive any private signal S_i. We also assume that the auctioneer knows the number of active bidders, i.e. $S_0 = N$. If the auctioneer cannot convey his private information, condition (a) of Proposition 4 implies that entry fees are sub-optimal. However, if the auctioneer reports the number of active bidders, this is uses a reporting function $\Pi(S_0, Z) \equiv S_0$, the conditions of Proposition 4 are satisfied.

We consider the case in which the auctioneer reveals the number of active bidders. We shall show that:

Lemma 1. *Suppose that the bidders think that the auctioneer reports truthfully. Suppose also that the auctioneer use a positive entry fee when he announces that there is one bidder active. Then, the auctioneer increases his expected payoffs if he announces that there is one bidder active when in reality there are two bidders active.*

Proof. For the sake of simplicity we restrict the proof to the case in which the auctioneer uses a second price auction and he can only choose the reserve price and the entry fee. If bidders belief his announcement of the number of active bidders, the optimal auction can be implemented with a combination of reserve price and entry fee (r_n, e_n), where n is the announced number of active bidders, only if they satisfy (see, for instance the arguments in the proof of Proposition 2):

$$(v^* - r_n)F(v^*)^{n-1} - e_n = 0, \text{ for } n = 1, 2. \tag{2}$$

In this case, it is an equilibrium for each active bidder to enter the auction if and only if her type is greater than v^*, in both cases: when the auctioneer reports one active bidder and the combination of reserve price and entry fee is (r_1, e_1), and when the auctioneer reports two active bidders and the combination of reserve price and entry fee is (r_2, e_2).

Consequently, the auctioneer's expected revenue when there are two active bidders equals:

$$2\int_{v^*}^{1} v(1-F(v))f(v)dv + 2(1-F(v^*))(r_nF(v^*)+e_n), \text{ for } n=1,2,$$

if the auctioneer reports that there is one active bidder and announces (e_1, r_1), or if he reports that there are two active bidders and announces (e_2, r_2).

The first term is the expected revenue from the bids. This is the expected value of the second highest valuation conditional on being greater than v^* times the probability that the second highest valuation is greater than v^*. The second term is the expected revenue from the reserve price and the entry fee. The probability that only one bidder has valuation above v^* times the reserve price plus the probability that a bidder enters the auction by the number of bidders, two, times the entry fee.

Note that from equation (2), $F(v^*)r_2 + e_2 = F(v^*)v^* = F(v^*)v^* + F(v^*)e_1 < F(v^*)v^* + e_1$, hence $r_1F(v^*) + e_1 > r_2F(v^*) + e_2$ when $e_1 > 0$. This means that the auctioneer gets strictly higher revenue reporting that there is only one active bidder and announcing (r_1, e_1) than reporting that there are two active bidders and announcing (r_2, e_2), when there are two active bidders and $e_1 > 0$. ■

The intuition is clear. Bidders are less willing to pay an entry fee if they expect to meet much competition in the auction. Hence, the auctioneer has incentives to mislead the bidders making them believe that there will face less competition than they will do.

6 Concluding Remarks

In this paper we have shown that when there is uncertainty about the number of bidders the auctioneer can achieve his maximum expected utility with an appropriate auction (e.g. a second price auction) with a reserve price. We have also shown that entry fees are, however, sub-optimal mainly due to two reasons: heterogeneity in bidder's beliefs about the number of active bidders; and, auctioneer's uncertainty about the bidders' beliefs and hence, the optimal entry fee, if any. Under these two conditions, an entry fee will act distorting the entry with respect to the bidders' entry decisions when the auction only has an optimal reserve price.

The problem of uncertainty about the number of bidders seems to be crucial in real life auctions. We have provided a quite general model to analyse exogenous uncertainty, however it is still missing a general model that considers not only exogenous uncertainty but also endogenous uncertainty. Models in which endogenous uncertainty about the number of bidders play an important role as Levin and Smith (1994) restrict to the case in which all bidders are symmetric, and hence hold homogenous beliefs. We have shown in this paper that this restriction can play an important role.

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