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A CONSISTENT TEST OF SIGNIFICANCE

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Abstract

This paper presents a test of significance consistent under nonparametric alternatives. Under the null hypothesis, a regressor has no effect on the regression model. Our statistic does not require to estimate the model on the alternative hypothesis, which is left unspecified. Hence, no smoothing techniques are required. The statistic is a weighted empirical process which resembles the Cramèr-von Mises. The asymptotic test is consistent under Pitman's alternatives converging to the null at a rate $n^{-1/2}$. A Monte-Carlo experiment illustrates the performance of the test in small samples. We also include two applications involving biomedical and acid rain data.

Key Words
Test of Significance; Empirical Process; Cramèr-von Mises; Nonparametric Alternatives; Pitman's Departures.

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1 Introduction

The article proposes a test of constancy of regression curves, which is consistent in the direction of nonparametric alternatives. When the regression function is parametrized, standard F-tests can be implemented, and they will be consistent in the direction of the parametrized alternative. However, F-tests may be inconsistent under certain alternatives when the underlying model is misspecified.

The problem of consistent testing a fit of a parametric model in the direction of non-parametric alternatives attached the attention of many authors. Some references on this subject are Yanagimoto and Yanagimoto (1987), Cleveland and Devlin (1988), Cox et al. (1988), Eubank and Spiegelman (1990), Kozek (1991), and Härdle and Mammem (1993). All the above mentioned tests are based on the distance between a parametric and a nonparametric fit. Raz (1990) proposed a randomization test for no effect based on the residual sum of squares of smooth regression estimates. The estimation of the model on the alternative requires the choice of the amount of smoothing employed, and the performance of the test will usually depend on such a choice.

The test proposed in this paper avoids estimation of the model under the alternative. It is based on a weighted empirical process, which has been used before by Hong-zhi and Bing (1991) and by Delgado (1993) in other contexts.

The rest of the paper is organized as follows. In next section we present the statistic, which resembles in spirit the Cramèr-von Mises (Cramèr 1928 and von-Mises 1931). In fact they share the same asymptotic null distribution. Section 3 presents results of a small Monte-Carlo experiment which illustrates the performance of our test in practice. Section 4 provides two applications. First, we apply the statistic to test mean independence between lymphocyte concentrations and inmunological status in men with human inmunodeficiency virus infection. In the second application, we test the hypothesis of equal trend functions of rain sulfate concentration (adjusted by amount of rainfall) in two cities.

2 Test statistic

We assume that data are recorded in the form $\{(x_i,Y_i),\ i=1,2,..,n\}$, where Y_i is a real valued response and x_i is a scalar design variable which becomes dense in the observational interval as the sample size increases. The x_i come from the interval [0,1], or any other bounded interval. We want to test the hypothesis of no relationship between the response and the design variable. Formally, we assume that the data are structured according to the model

$$Y_i = m(x_i) + \varepsilon_i, \quad i = 1, 2, ..n, \tag{1}$$

where ε_i , $i \geq 1$, denote zero mean independent errors which do not depend on x_i, and $Var(\varepsilon_i) = \sigma^2$. The regression function $m(\cdot)$ is of unknown functional form and it is assumed to be bounded and continuous on [0,1]. Let the design be such that, for each n and $i = 1, 2, ..., n$, x_i is the i/n quantile of a distribution with density $r(\cdot)$. Our test is based on the functional $S(t) = \int_0^t (m(u) - \mu)\, r(u)\, du$, where $\mu = \int_0^1 m(u)\, r(u)\, du$. Thus, the hypothesis of no relationship can be characterized as

$$H_0 : S(t) = 0 \quad \textit{all } t \in [0,1], \quad \textit{versus} \quad H_1 : S(t) \neq 0 \quad \textit{some } t \in [0,1]. \tag{2}$$

The null hypothesis entails that $m(x) = \mu$ for all x in any subinterval of [0,1] where $r(x)$ does not vanish. Under the alternative hypothesis, there exists a subinterval of $[0, 1]$ where $m(x)$ varies with x and $r(x)$ does not vanish.

Thought the discussion here is centered in the fixed design case, our test can also be applied to a regression model with random regressors assuming independence between regression errors and explanatory variables.

A natural estimate of $S(t)$ is given by

$$S_n(t) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})\ 1(x_i \leq t) \equiv \frac{1}{n}\sum_{i=1}^{[nt]}(Y_i - \bar{Y}),$$

where $1(A)$ is the indicator function of the event A . Note that $S_n(x_k) = n^{-1}\sum_{i=1}^{k}(Y_i - \bar{Y})$. Then, the behavior of $S_n(x_k)$ under H_0 does not depend on the assumptions made on the design points. We propose the test statistic

$$T_n = \sum_{i=1}^{n} S_n(x_i)^2/\hat{\sigma}_n^2, \tag{3}$$

where $\hat{\sigma}_n^2 = (2n)^{-1}\sum_{i=1}^{n-1}(Y_{i+1} - Y_i)^2$ is a strongly consistent estimate of σ^2 both under H_0 and H_1 (note that $m(\cdot)$ is continuous). The usual variance estimate $s_n^2 = n^{-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ is also strongly consistent under H_0 but under H_1, $s_n^2 \to \sigma^2 + \int_0^1 m(u)^2\ r(u)\ du > \sigma^2$ with probability 1 as $n \to \infty$. Therefore, a test statistic scaling by $\hat{\sigma}_n^2$ will always be more powerful than a test statistic scaling by s_n^2. The scale factor $\hat{\sigma}_n^2$ has also been used before by Rice (1984), Hall and Hart (1990), King et al. (1991) and Delgado (1993).

By Donsker's invariance principle applied to the partial sums $n^{-1/2}\sum_{i=1}^{[nt]}\varepsilon_i/\sigma$ and the continuous mapping theorem

$$n^{1/2}S_n(t)/\sigma \to W(t) \tag{4}$$

weakly in distribution as $n \to \infty$, where $W(t)$ is a Brownian Bridge. Then, since $\hat{\sigma}_n^2$ is strongly consistent, by (4) and applying the continuous mapping theorem,

$$T_n \to T = \int_0^1 W(t)^2\ dt \tag{5}$$

in distribution as $n \to \infty$ under H_0.

Interestingly, the statistic T_n has the same asymptotic null distribution than the Cramèr-von Mises statistic. The distribution of T has been tabulated by Anderson and Darling (1952).

Under H_1, with t fixed, we have

$$S_n(t) \rightarrow S(t) \tag{6}$$

with probability 1 as $n \rightarrow \infty$. Thus, T_n diverges to infinity under H_1.

Define T_α such that $\Pr(T \geq T_\alpha) = \alpha$, then (5) and (6) imply that

$$\lim_{n\rightarrow\infty} \Pr(T_n > T_\alpha) = \alpha \textit{ under } H_0 \quad \textit{and} \quad \lim_{n\rightarrow\infty} \Pr(T_n > c) = 1 \textit{ all fixed } c \textit{ under } H_1.$$

The null hypothesis H_0 will be rejected at the level α of significance when the observed T_n dominates T_α. Some critical values are $T_{0.1} = 0.34730$, $T_{0.05} = 0.46136$ and $T_{0.01} = 0.74346$.

Consider local alternatives

$$H_{1n} : m(x) = \mu + n^{-1/2} c\, h(x) \quad \textit{for each } x \in [0,1], \tag{7}$$

where c is a fixed constant and $h(\cdot)$ is a continuous and bounded function. Under H_{1n}

$$T_n \rightarrow \int_0^1 \left(c\,\sigma^{-1} \int_0^t h(u)\, r(u)\, du + W(t) \right)^2 dt \tag{8}$$

in distribution as $n \rightarrow \infty$. Then T_n diverges to ∞ as $c \rightarrow \infty$ under H_{1n}. Hence, the test is asymptotically powerful under alternatives converging to the null at a rate $n^{-1/2}$.

3 Monte-Carlo

In all the experiments observations are generated according to the model

$$Y_i = m(x_i) + \varepsilon_i, \quad i = 1, 2, ..., n,$$

where ε_i are generated independent and identically distributed with different distributions and the design variable $x_i = i/n$. We consider the same regression models and errors distributions as Raz (1990) for the sake of comparison:

Null : $Y_i = 4 + \varepsilon_i$.

Linear : $Y_i = 2.001\ x_i + \varepsilon_i$.

Quadratic : $Y_i = 9.513\ (x_i - \bar{x})^2 + \varepsilon_i$.

Sinusoidal : $Y_i = 1.155\ \sin(4(i-1)\pi/n) + \varepsilon_i$.

The errors are generated according to a normal, $\sinh^{-1}$-normal and lognormal, each with zero mean an unit variance.

We report percentage of rejections under the null and the different alternatives of our test and the standard t-ratio.

TABLE I ABOUT HERE

The empirical and nominal levels of the t-ratio test are very close for all sample sizes. For the smallest sample sizes ($n = 15, 30$) our test overreject H_0, but for the greatest sample sizes both tests behave similarly. Under the linear alternative, our test rejects more for the smallest sample sizes, as it also happened under H_0. Under the quadratic alternative the empirical power of the t-ratio test is even smaller than its nominal level, while our test shows to be very powerful. Under the sinusoidal alternative both tests are powerful but ours behaves better.

TABLE II ABOUT HERE

Other tests based on nonparametric estimates of the regression model on the alternative hypothesis, typically depend on the choice of the amount of smoothing. The reported empirical powers in Raz (1990) variate a lot depending on the amount of smoothing employed, though this variation decreases when higher order kernels (of order 4) are employed. Our test is at least as powerful as the test proposed by Raz (1990) for the majority of smoothing numbers chosen.

The level distortions of the asymptotic test, when the sample size is very small, can be corrected by approximating the exact critical values by a bootstrap or implementing a random permutation test. A bootstrap approximation to the exact critical values can be based on the statistic

$$T_n^* = \sum_{i=1}^{n} [n^{-1} \sum_{j=1}^{i} (Y_j^* - \overline{Y^*})]^2,$$

where $\{Y_1^*, Y_2^*, ..., Y_n^*\}$ is a random sample with replacement from $\{Y_1, Y_2, ..., Y_n\}$ and $\overline{Y^*} = \sum_{i=1}^{n} Y_i^*$. By repeated resampling compute $\hat{T}_{n\alpha}$ such that $\Pr\left\{T_n \geq \hat{T}_{n\alpha} \mid \mathcal{X}\right\} = \alpha$, where $\mathcal{X}=\{(x_i, Y_i), i = 1, ..., n\}$. The bootstrap test consists of rejecting H_0 when the observed T_n exceeds $\hat{T}_{n\alpha}$. Note that $E\left(Y_i^* - \overline{Y^*} \mid \mathcal{X}\right) = 0$ and $\lim_{n\to\infty} \Pr\left(T_n^* \geq x \mid \mathcal{X}\right) = \Pr\left(T \geq x\right)$. In fact, it can be proved, using similar arguments than Hall and Hart (1990) that the bootstrap error level is of order n^{-2} rather than n^{-1} corresponding to the asymptotic test. Also, while T_α approximate the exact critical values with an error of order $n^{-\frac{3}{2}}$, the error of the asymptotic test is of order n^{-1}.

TABLE III ABOUT HERE

Table III reports the proportion of rejections using the bootstrap version of our test. It exhibits extraordinary level accuracy for all distributions and sample sizes. The empirical power is comparable to that of the t-ratio test under the linear alternative. Under the quadratic and sinusoidal alternative, the bootstrap test is also very powerful.

4 Empirical Applications.

We apply the proposed test to data on cell counts of lymphocyte concentrations in a sample of 58 men aged 34-36 with a positive test for the human immunodeficiency virus (HIV) antibody. Details on the data set are given in Lang et al. (1987). This data set was also used by Raz (1990) for illustrating his test procedure. The explanatory variable is immunological status explained by a skin test score measuring skin reactions on seven antigens. Cell counts of Len 2a lymphocytes are used as response variables. Under the null hypothesis, lymphocyte counts are unrelated to the skin score test. Figure I presents two regression curve estimates using kernel smoothers and a bandwidth close to that chosen by Raz (1990). One of the regression curve estimates do not use the last six observations in order to avoid boundary effects.

FIGURE I ABOUT HERE

The two curves seem quite horizontal, suggesting that there is not clear relationship between the response and the design variable. The few extreme observations can determine the slope of the curve. Raz (1990) obtained a P-value of 0.045, while $T_n = 0.1983$ (P-value$\approx$ 0.27). Thus, the null hypothesis can not be rejected using our statistic at any reasonable significance level.

The test is also applied to detect a difference between the shapes of two regression functions. That is, we have data consisting of observations $\{(Y_i, Z_i, x_i), \quad i = 1, 2, .., n\}$ and structured according to the model

$$Y_i = f(x_i) + \varepsilon_{yi} \quad , \quad Z_i = g(x_i) + \varepsilon_{zi}. \tag{9}$$

We want to test the hypothesis that $f(\cdot)$ and $g(\cdot)$ differ by a simple shift. This hypothesis is tested by means of the statistic T_n applied to the response variable $Y_i - Z_i$.

The test is applied to data on concentration of sulfate in North Caroline rain. This data have been used before by Hall and Hart (1990), and consist of weekly measurements of rainfall amount and concentration of sulfate in the rain over the period 1979 to 1983 on two towns, Coweeta and Lewiston. It is compared the natural logarithm of acid concentration adjusted for the covariate 'amount of rainfall' as a function of time in the two towns. There are several weeks where data is not available for both locations and this is why the number of observations do not correspond to the weeks observed. In fact, we have 189 available observations for comparing the two regression curves, among the 260 weeks in the study. Hall and Hart (1990) did not find evidence of autocorrelation of the error terms based on residuals computed from nonparametric kernel regression estimates.

FIGURE II ABOUT HERE

Hall and Hart (1990) obtained a P-value of 0.097 concluding that the shape of the two regression curves are different. We arrive to the same conclusion, using the whole data set (189 available observations). The value obtained for our test statistic is $T_n = 1.1175$, (P-value$\approx$ 0.001) which implies rejection of H_0 at any reasonable significance level. Figure II presents a plot of the data with the regression curves estimated by Nadaraya-Watson kernel method (Nadaraya 1964, Watson 1964) and with a bandwidth number close to that used by Hall and Hart (1990). For this bandwidth choice, the shape of the regression curves is sinusoidal, but this shape is not so evident when we use other bandwidth numbers. The regression curve corresponding to Coweeta change its shape around the week 150th. We applied the test for the first 150 weeks (109 available observations), the corresponding statistic is $T_n = 0.043$ (P-value$\approx$ 0.9), so we are unable to reject the null hypothesis that the two regression functions have identical shape when using the first 150 weeks.

We also applied our statistic to each town in order to test the presence of a trend in the regression curves. Using the whole sample, $T_n = 2.0514$ (P-value$\approx$ 0) in Coweeta and $T_n = 0.1042$ (P-value$\approx$ 0.43) in Lewiston. Thus, we reject the null hypothesis that there is not a trend in the regression curve in Coweeta but we are unable to reject such hypothesis in Lewiston. However, using observations from the week 19 to the week 194 (152 observations), in order to avoid the outliers at the beginning and at the end of the observational period, we obtain $T_n = 0.3406$ (P-value$\approx$ 0.1).

The bootstrap P-values were always very close to their asymptotic counterparts in all applications.

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Table I. Proportion of rejections under H_0 in 5000 replications.

NULL	Error	α	n=15	n=30	n=50	n=100
T-Ratio	Normal	0.1 0.05 0.01	0.098 0.049 0.009	0.095 0.048 0.009	0.105 0.052 0.011	0.110 0.058 0.012
	Lgnml	0.1 0.05 0.01	0.089 0.041 0.007	0.098 0.042 0.006	0.098 0.046 0.006	0.108 0.054 0.008
	$Sinhip^{-1}$	0.1 0.05 0.01	0.096 0.043 0.006	0.096 0.045 0.007	0.101 0.051 0.010	0.106 0.052 0.010
Nonparamet.	Normal	0.1 0.05 0.01	0.164 0.099 0.039	0.131 0.070 0.020	0.123 0.065 0.014	0.117 0.061 0.014
	Lgnml	0.1 0.05 0.01	0.157 0.097 0.037	0.128 0.070 0.018	0.113 0.052 0.013	0.110 0.054 0.009
	$sinhip^{-1}$	0.1 0.05 0.01	0.158 0.097 0.032	0.130 0.070 0.017	0.118 0.063 0.011	0.110 0.053 0.010

Table II. Proportion of rejections under H_1 for three alternative regression models in 5000 replications.

Model	Error	α	T-RATIO n= 15	T-RATIO n= 30	T-RATIO n= 50	T-RATIO n= 100	NON PARAMETRIC n= 15	NON PARAMETRIC n= 30	NON PARAMETRIC n= 50	NON PARAMETRIC n= 100
LINEAR	Normal	0.1	0.674	0.920	0.994	1.000	0.727	0.928	0.993	1.000
		0.05	0.538	0.859	0.985	1.000	0.631	0.867	0.981	1.000
		0.01	0.258	0.654	0.931	0.999	0.422	0.716	0.927	0.999
	Lgnml	0.1	0.802	0.910	0.971	0.995	0.827	0.916	0.969	0.995
		0.05	0.728	0.863	0.954	0.992	0.774	0.878	0.954	0.991
		0.01	0.545	0.758	0.906	0.982	0.657	0.792	0.907	0.981
	$Sinhip^{-1}$	0.1	0.757	0.916	0.982	0.999	0.787	0.923	0.979	0.999
		0.05	0.653	0.865	0.967	0.998	0.718	0.879	0.964	0.998
		0.01	0.413	0.719	0.916	0.991	0.563	0.766	0.915	0.991
QUADRATIC	Normal	0.1	0.043	0.040	0.045	0.052	0.469	0.761	0.943	0.999
		0.05	0.015	0.014	0.016	0.018	0.313	0.585	0.850	0.997
		0.01	0.002	0.001	0.001	0.001	0.116	0.254	0.514	0.958
	Lgnml	0.1	0.004	0.013	0.019	0.029	0.671	0.820	0.914	0.979
		0.05	0.000	0.002	0.003	0.008	0.548	0.726	0.851	0.962
		0.01	0.000	0.000	0.000	0.000	0.324	0.511	0.668	0.898
	$Sinhip^{-1}$	0.1	0.032	0.033	0.035	0.043	0.589	0.806	0.928	0.993
		0.05	0.011	0.010	0.013	0.015	0.431	0.683	0.861	0.985
		0.01	0.000	0.000	0.001	0.001	0.193	0.391	0.612	0.924
SINUSOID.	Normal	0.1	0.162	0.350	0.566	0.862	0.352	0.658	0.880	0.997
		0.05	0.079	0.212	0.413	0.750	0.237	0.503	0.765	0.985
		0.01	0.015	0.054	0.168	0.473	0.089	0.258	0.476	0.887
	Lgnml	0.1	0.169	0.442	0.647	0.868	0.511	0.774	0.880	0.974
		0.05	0.066	0.272	0.507	0.784	0.370	0.677	0.815	0.950
		0.01	0.006	0.054	0.221	0.558	0.158	0.468	0.647	0.863
	$Sinhip^{-1}$	0.1	0.175	0.393	0.605	0.868	0.418	0.735	0.880	0.985
		0.05	0.074	0.243	0.462	0.776	0.298	0.606	0.796	0.965
		0.01	0.009	0.057	0.187	0.518	0.117	0.359	0.565	0.883

Table III: Proportion of rejections of the bootstrap test under H_0 and H_1 in 5000 replications (with 2000 bootstrap samples).

		NULL			LINEAR			QUADRATIC			SINUSOIDAL		
n	α	Normal	Lgnml	$Sinh^{-1}n$	Normal	Lgnml	$Sinh^{-1}n$	Normal	Lgnml	$Sinh^{-1}n$	Normal	Lgnml	$Sinh^{-1}n$
15	0.10	0.099	0.098	0.099	0.631	0.776	0.719	0.312	0.548	0.436	0.238	0.369	0.295
	0.05	0.051	0.051	0.047	0.486	0.698	0.616	0.156	0.391	0.256	0.121	0.210	0.157
	0.01	0.012	0.009	0.010	0.214	0.498	0.351	0.026	0.125	0.049	0.026	0.036	0.029
30	0.10	0.098	0.101	0.096	0.901	0.899	0.902	0.677	0.777	0.746	0.579	0.730	0.671
	0.05	0.047	0.049	0.050	0.825	0.852	0.848	0.455	0.657	0.591	0.414	0.613	0.509
	0.01	0.009	0.008	0.008	0.595	0.733	0.687	0.132	0.388	0.247	0.153	0.355	0.243
50	0.10	0.102	0.092	0.098	0.992	0.966	0.977	0.921	0.899	0.913	0.846	0.862	0.858
	0.05	0.050	0.044	0.050	0.975	0.949	0.958	0.797	0.823	0.824	0.703	0.787	0.758
	0.01	0.008	0.008	0.007	0.888	0.891	0.891	0.385	0.607	0.521	0.382	0.582	0.490
100	0.10	0.107	0.102	0.099	1.000	0.995	0.999	0.999	0.978	0.992	0.996	0.971	0.983
	0.05	0.053	0.050	0.050	1.000	0.992	0.998	0.997	0.960	0.984	0.981	0.946	0.962
	0.01	0.011	0.008	0.009	0.999	0.980	0.990	0.936	0.891	0.909	0.860	0.851	0.862

Figure I. HIV infection data set with kernel smoothers.

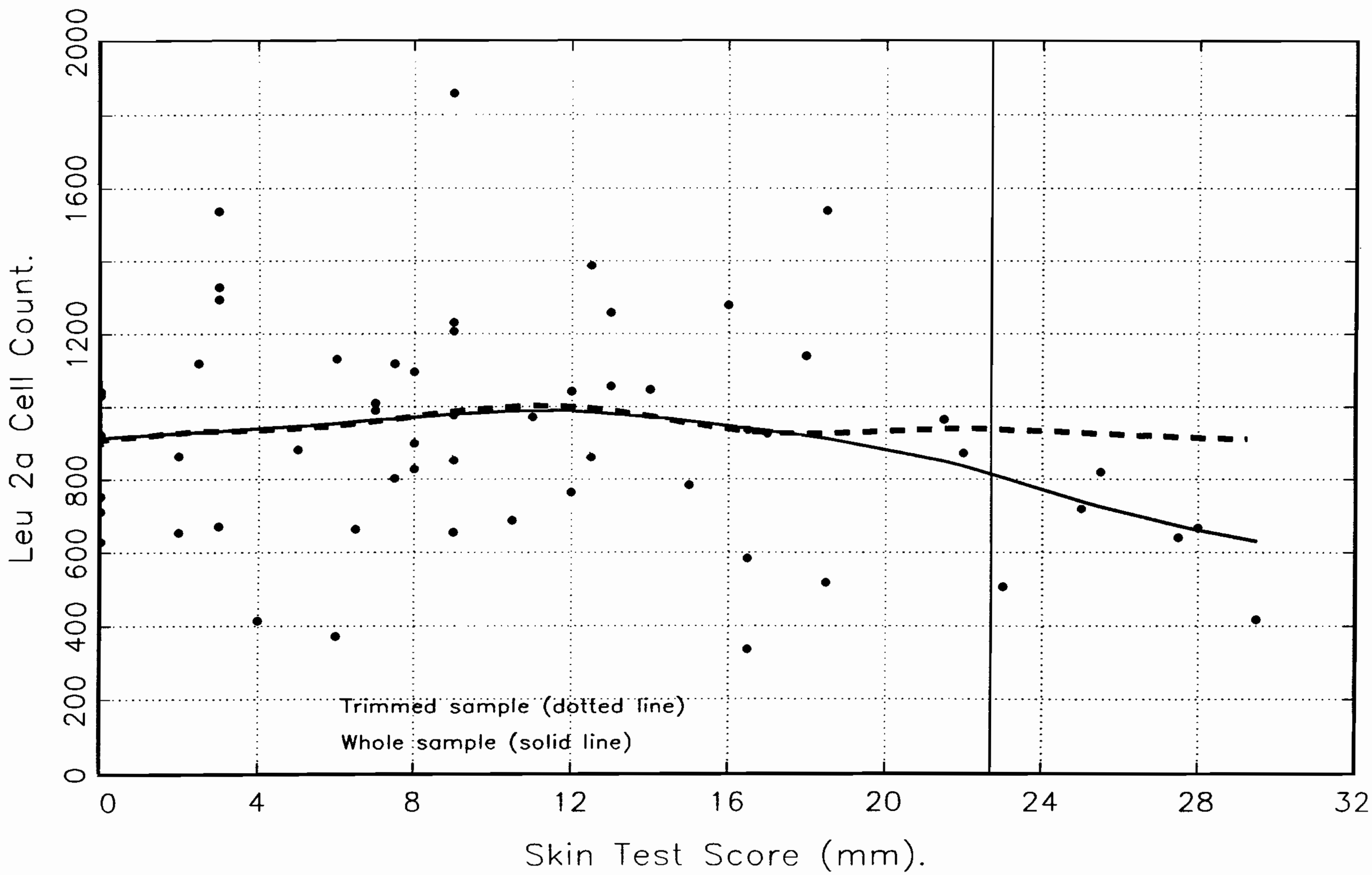

Figure II. Acid rain data with kernel smoothers for the two cities.