Power comparison among tests for fractional unit roots

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Abstract

This article compares the asymptotic power properties of the Wald, the Lagrange Multiplier and the Likelihood Ratio test for fractional unit roots. The paper shows that there is an asymptotic inequality between the three tests that holds under fixed alternatives.

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Several tests for the null hypothesis of unit root against the alternative of long memory have recently been proposed. Robinson (1991, 1994) proposed a Lagrange Multiplier (LM) test that was further analyzed by Tanaka (1999), whereas Lobato and Velasco (2007, LV, hereinafter) proposed an efficient Wald type test. Although both tests are locally asymptotically optimal, in the simulations in LV, it appears that the Wald test presents more empirical power than the LM test. Since LV provided no further explanation, the aim of this note is to present some analysis to help explain the apparent power superiority of the efficient Wald test over the LM test. Note that this apparent empirical power superiority also shows in the inefficient tests considered by Dolado, Gonzalo and Mayoral (2002) and Lobato and Velasco (2006).

Let y_t denote a fractionally integrated process whose true order of integration is d, denoted as $I(d)$,

$$\Delta^d y_t 1\{t>0\} \quad \varepsilon_t, \qquad t \quad 1,2,\ldots, \tag{1}$$

where ε_t are independent and identically distributed random variables with zero mean and finite variance, and $1\{\}$ denotes the indicator function. The fractional difference operator $\Delta^d = (1 \quad L)^d$ is defined in terms of the lag operator L by the formal expansion,

$$\Delta^\alpha : \quad \sum_{i \quad 0}^{\infty} \pi_i(\alpha) L^i,$$

for any real α; where for $\alpha \neq 1, 2, \ldots$,

$$\pi_i(\alpha) \quad \frac{\Gamma(i \quad \alpha)}{\Gamma(i+1)\Gamma(\quad \alpha)},$$

and Γ is the Gamma function, with $\Gamma(0)/\Gamma(0)=1$, so the first coefficients are $\pi_0(\alpha)=1$ and $\pi_1(\alpha)= \quad \alpha$: We consider testing the null hypothesis

$$H_0 : d \quad 1,$$

versus either a simple alternative

$$H_A : d \quad d_A < 1,$$

or a composite alternative

$$H_1 : d < 1.$$

Robinson's (1991, 1994) LM test statistic is

$$\mathrm{LM}^{1/2} \quad T^{1/2} \left(\frac{\pi^2}{6}\right)^{-1/2} \sum_{j \quad 1}^{T-1} j^{-1} \hat{\rho}_{\Delta y}(j), \tag{2}$$

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where $\hat{\rho}_{\Delta y}(j)$ denotes the sample autocorrelation of order j of Δy_t. This test is asymptotically locally optimal in a Gaussian framework.

LV proposed to test the null hypothesis by testing the significance of the coefficient of $z_{t\quad 1}(d)$ in the regression model

$$\Delta y_t = \phi z_{t-1}(d) + u_t, \qquad t = 1, \ldots, T, \tag{3}$$

where

$$z_{t-1}(d) = \frac{(\Delta^{d-1} \quad 1)\Delta y_t}{(1 \quad d)}.$$

by means of a left sided test based on the t ratio test statistic, denoted by $t(d)$. In practice, a consistent estimator of d should be plug in.

Both tests are asymptotically equivalent in a local alternative framework, but this theoretical result does not help to un derstand the apparent power superiority of the efficient Wald test over the LM test in practice. In this note, we compare both tests in a fixed alternative framework. For completeness, we also introduce the likelihood ratio (LR) test, and establish an inequality that recalls the well known inequality among the three classical tests.

The LR test can be derived under Gaussianity assumptions from model (1), or it can be motivated by the fact that in model (3), the null and alternative hypotheses are also simple hypotheses when testing H_0 against H_A, in particular, the null corresponds to $\phi \quad 0$ and the alternative to $\phi \quad\quad 1$. In both cases, for the simple alternative case the LR statistic is

$$\mathrm{LR} = \quad 2T\log\left(\frac{\sum(\Delta^{d_A} y_t)^2}{\sum(\Delta y_t)^2}\right)^{1/2}.$$

In the composite alternative H_1 case, d_A should be replaced by the maximum likelihood estimator of d.

It can be shown that, under the appropriate assumptions, the three tests share the same asymptotic null distribution, are locally asymptotically equivalent, and are consistent. Hence, in order to compare them, we are going to use a fixed alternative framework and compute the probabilistic limit of the properly normalized test statistics. Without loss of generality, we are going to assume that ε_t has unit variance.

First, consider the LM test statistic in Eq. (2). Since Δy_t $\Delta^{1\quad d}\varepsilon_t$, $d<1$; its autocorrelations are given by

$$\rho_{\Delta y}(j) = \frac{\Gamma(2 \quad d)}{\Gamma(d \quad 1)} \frac{\Gamma(j+d \quad 1)}{\Gamma(j+2 \quad d)},$$

and

$$\operatorname*{plim}_{T\to\infty}\left(\frac{\mathrm{LM}^{1/2}}{T^{1/2}}\right)^2 = \frac{6}{\pi^2}\left(\sum_{j\quad 1}^{\infty}\frac{\rho_{\Delta y}(j)}{j}\right)^2$$

$$= \frac{6}{\pi^2}\frac{\Gamma(2 \quad d)^2}{\Gamma(d \quad 1)^2}\left(\sum_{j\quad 1}^{\infty}\frac{\Gamma(j+d \quad 1)}{j\Gamma(j+2 \quad d)}\right)^2. \tag{4}$$

Second, using the basic relation of least squares regression theory, the efficient Wald test can be expressed as

$$W = t(d)^2 = T\frac{R^2(d)}{1 \quad R^2(d)},$$

where $R^2(d)$ denotes the squared sample correlation between Δy_t and $z_{t\quad 1}(d)$. Therefore, we get that

$$\operatorname*{plim}_{T\to\infty}\frac{W}{T} = \frac{\varrho^2}{1 \quad \varrho^2},$$

where ϱ denotes the population correlation between reg ressand and regressor in model (3). Note that the regressand is $\Delta y_t \quad \Delta^{1\quad d}\varepsilon_t$ whereas the regressor is $(1 \quad d)^{\quad 1}(1 \quad \Delta^{d\quad 1})\Delta y_t$ $(1 \quad d)^{\quad 1}(\Delta^{1\quad d} \quad 1)\varepsilon_t$. It is simple to show that ϱ^2 can be written as

$$\varrho^2 = \frac{\sum_{j\quad 1}^{\infty}\pi_j(1 \quad d)^2}{1+\sum_{j\quad 1}^{\infty}\pi_j(1 \quad d)^2},$$

and so

$$\operatorname*{plim}_{T\to\infty}\frac{W}{T} = \sum_{j\quad 1}^{\infty}\pi_j(1 \quad d)^2 = \frac{\Gamma(3 \quad 2d)}{\Gamma(2 \quad d)^2} \quad 1. \tag{5}$$

Finally, since $T^{\quad 1}\sum(\Delta^d y_t)^2 \to_p \sigma_\varepsilon^2 \quad 1$, it can be shown that under H_A,

$$\operatorname*{plim}_{T\to\infty}\frac{\mathrm{LR}}{T} = \log\left(1+\sum_{j\quad 1}^{\infty}\pi_j(1 \quad d)^2\right)$$

$$= \log\left(\frac{\Gamma(3 \quad 2d)}{\Gamma(2 \quad d)^2}\right). \tag{6}$$

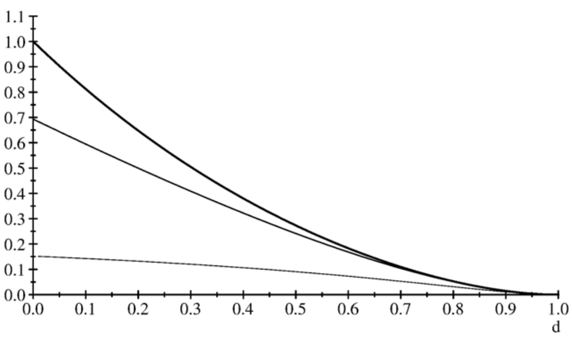

Fig. 1. Efficiency of the LM-thin line-, LR-medium line-, and W-thick line- tests under fixed alternatives. Respectively, these drifts are given in Eqs. (4), (5) and (6), as a function of d.

A comparison of expressions (5) and (6) for the case $d = 1 - c/\sqrt{T}$ shows that the LR and the W test are locally asymptotically equivalent, since for this case

$$\frac{\log\left(1+\sum_{j=1}^{\infty}\pi_j(1-d)^2\right)}{\sum_{j=1}^{\infty}\pi_j(1-d)^2}\to 1,$$

as $d\to 1$.

Fig. 1 plots the drifts of the three tests given in Eqs. (4), (5), and (6) for values of d between 0 and 1. In spite of the asymptotically locally equivalence among the three tests, this Figure shows that for values of d distant from 1, the $\text{plim}_{T\to\infty}$ (LM/T) is much lower than the $\text{plim}_{T\to\infty}$ (W/T). Note that the inequalities (derived under the alternative hypothesis)

$$\underset{T\to\infty}{\text{plim}}\frac{W}{T}\geq\underset{T\to\infty}{\text{plim}}\frac{\text{LR}}{T}\geq\underset{T\to\infty}{\text{plim}}\frac{\text{LM}}{T},$$

can also be written as

$$P(W\geq\text{LR}\geq\text{LM})\to 1 \text{ as } T\to\infty,$$

which recalls the well known inequalities between the three classical tests. This analysis supports the simulations results in LV, and suggests that, when some reliable estimator of the true d is available, the Wald test proposed in LV may be preferable to the optimal LM test of Robinson because of the local character of this LM test.

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