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A MULTIVARIATE KOLMOGOROV-SMIRNOV TEST OF GOODNESS OF FIT

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Abstract

This paper presents a distribution free multivariate Kolmogorov-Smirnov goodness of fit test. The test uses an statistic which is built using Rosenblatt's transformation and an algorithm is developed to compute it in the bivariate case. An approximate test, that can be easily computed in any dimension, is also presented. The power of these multivariate tests is studied in a simulation study.

Key words:

Empirical distribution function. Kolmogorov-Smirnov statistics. Rosenblatt's transformation.

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1. INTRODUCTION

Goodness of fit tests have been mostly developed for univariate distributions and, except for the case of multivariate normality, very few references can be found in the literature about multivariate goodness of fit tests. (See Krishnaiak, 1980, Kotz and Johnson, 1983 and D'Agostino and Stephens, 1986).

In principle, the chi-square test can be applied for testing any multivariate distribution but it is unknown what is the best way to choose the cell limits and what is the best statistic to be used. Moore and Stubblebine (1981) suggested choosing as cell boundaries the concentric hyperellipses centered at the sample mean and with shape determined by the inverse of the covariance matrix, and used the Rao and Robson (1974) modification of the chi-squared statistic. However, much work need to be done on the properties of this test.

The two most important classes of tests of goodness of fit based on the empirical distribution function of a random sample, the Kolmogorov-Smirnov statistic and the Cramer-von Mises group, have not been extended to the multivariate case. The problem is that the probability distribution of these multivariate statistics are not distribution free as in the univariate case. Rosenblatt (1956) proposed a simple transformation of an absolutely continuous p -variate distribution into the uniform distribution on the p -dimensional hypercube and suggested using this transformation to build multivariate goodness of fit tests. The distribution function of the Cramer-von Mises statistic in the multivariate case has been studied by a number of authors (see Kotz and Johnson, 1985, pp: 35-39) but a general multivariate test of goodness of fit based on this statistic that can be readily applied has not yet been developed.

Mardia (1970) seems to be the first to have published a practical multivariate test of goodness of fit for the multinormal distribution using multivariate measures of skewness and kurtosis. Malkovich and Afifi (1973) used the univariate normality of all linear combinations to derive three tests of normality. These tests search for the linear combination with the largest measures of skewness, kurtosis and the negative of the Shapiro and Wilk statistics, respectively. Cox and Small (1978) suggested finding the linear combination of the variables maximizing the curvature when regressed on the other. Other procedures to check for multivariate normality have been proposed by Andrews et al. (1973), Dahiya and

Gurland (1973), Hensler et al. (1977), Csörgö (1986), Mudholkar et al. (1992) and Ghosh and Ruymgaart (1992).

In this paper we present two multivariate goodness of fit test. In section 2 we present a multivariate goodness of fit statistics which is distribution free and reduces to the Kolmogorov-Smirnov statistic in the univariate case. The computation of the proposed statistic is a problem in itself, and in this section we develop another statistic that can be easily computed for any dimension. Section 3 presents a procedure to compute the test statistics in the bivariate case. In section 4 we present some simulation results for the exact and the approximated Kolmogorov-Smirnov statistics. Finally, section 5 includes some concluding remarks.

2. THE MULTIVARIATE KOLMOGOROV-SMIRNOV STATISTIC

Given a sample $\mathbf{x}_1, \dots, \mathbf{x}_n$ of i.i.d. random variables with distribution function F , consider the problem of testing $H_0 : F = F_0$ versus $H_1 : F \neq F_0$, where F_0 is some specified distribution function. In the univariate case, H_0 can be tested using the Kolmogorov-Smirnov statistic

$$D_n = \sup_{x \in \mathbb{R}} | F_n(x) - F(x) |,$$

where F_n is the empirical distribution function of the sample. It is also well known that this statistic is distribution free and it can be expressed as

$$D_n = \sup_{0 \leq u \leq 1} | G_n(u) - u |, \quad (2.1)$$

where $G_n(u)$ is the empirical distribution function of the uniform 0-1 transformed sample $u_i = F_0(y_i)$, for $i = 1, \dots, n$.

The distribution free property of the Kolmogorov-Smirnov statistic is derived from the result that any continuous random variable X with distribution function F can be transformed to a uniform random variable Y by the transformation $Y = F(X)$. A similar result holds for a continuous multivariate random variable \mathbf{X} , as it is shown in the following theorem, due to Rosenblatt (1952).

THEOREM 1. *Let $\mathbf{X} = (X_1, \dots, X_p)$ be a random vector with joint density*

$$f(\mathbf{x}_1, \dots, \mathbf{x}_p) = f_1(x_1)f_2(x_2 | x_1) \cdots f_p(x_p | x_1, \dots, x_{p-1}),$$

and define the transformation $\mathbf{Y} = T(\mathbf{X})$, by

$$\begin{aligned} Y_1 &= F_1(X_1) \\ Y_i &= F_i(X_i | X_1, \dots, X_{i-1}) \quad i = 2, \dots, p. \end{aligned} \quad (2.2)$$

Then Y_1, \dots, Y_p are i.i.d. uniform 0-1.

The probability distribution function of the statistic

$$\sup_{\mathbf{x} \in \mathbf{R}^p} | F(\mathbf{x}_1, \dots, \mathbf{x}_p) - F_n(\mathbf{x}_1, \dots, \mathbf{x}_p) |,$$

where F_n is the empirical distribution function, is not distribution free. However, as suggested by Rosenblatt (1952) we could use the transformation defined in theorem 1 to test whether the values (y_1, \dots, y_n) are a sample from a uniform distribution on the p -dimensional hypercube. The natural extension of the statistic (2.1) to the multivariate case is

$$D_n = \sup_{\mathbf{y}} | G_n(\mathbf{y}) - y_1 \cdots y_p | \quad (2.3)$$

where G_n is the empirical distribution function of the transformed sample $\mathbf{y} = T(\mathbf{x})$.

Unfortunately, and unlike in the univariate case, the computation of (2.3) is very involved, as it is shown in the next section, in which we present an algorithm for the case $p = 2$. Although this algorithm could be extended to the $p > 2$ case, the computation difficulties appear to be considerable. A much simpler to compute statistic, \tilde{D}_n , can be defined by taking the supremum on the set of transformed sample points A ,

$$\tilde{D}_n = \sup_{\mathbf{y} \in A} | G_n(\mathbf{y}) - y_1 \cdots y_p |,$$

which, by the same argument above, is also distribution free. When n is large, \tilde{D}_n will be close to D_n , as it is shown in the simulation results reported in Section 4.

By theorem 1, the multivariate Kolmogorov-Smirnov statistic D_n and its approximation \tilde{D}_n are distribution free in the class of continuous multivariate distributions. The percentiles of the distribution of D_n can be computed for the bivariate case by Monte Carlo simulation and table 1 presents the percentiles of this statistic in the standard case in which F_0 is completely specified by H_0 . Table 2 presents the percentiles in the particular case of testing

normality and when the parameter are estimated by the sample mean and the sample covariance matrix, that is the multivariate generalization of the Kolmogorov-Smirnov-Lilliefords statistic.

(Table 1 and 2 around here)

Table 3 presents the percentiles for the distribution of the approximated Kolmogorov-Smirnov statistic in the bivariate case.

(Table 3 around here)

3. AN ALGORITHM TO COMPUTE THE BIVARIATE KOLMOGOROV-SMIRNOV STATISTIC

In a one-dimensional sample the empirical distribution changes only in the observed points, and the univariate Kolmogorov-Smirnov statistic is obtained by evaluating the distance between the empirical and theoretical distribution functions in these points. Nevertheless, when the dimension p is larger than one, the empirical distribution function jumps on an infinite number of points. Here we develop a procedure for calculating the Kolmogorov-Smirnov statistic (2.3) in the two-dimensional case by evaluating it on a finite set.

Since theorem 1 holds we may assume that $u_1 = (x_1, y_1), \dots, u_n = (x_n, y_n)$ is a random sample from two independent uniform 0-1 distributions. In this context, the pair (x_j, y_i) is called *an intersection point* if $x_i < x_j$ and $y_i > y_j$. For $u = (x, y)$ we define the *superior distance* $D_n^+(u) = (G_n(u) - G(u))$ and the *inferior distance* $D_n^-(u) = (G(u) - G_n(u))$, where G is the distribution function of two independent uniform random variables on $(0, 1)$ and G_n is the empirical distribution function. Also, the left empirical distribution function in u is defined as $G_n(u^-) = \lim_{\epsilon \rightarrow 0} G_n(x - \epsilon, y - \epsilon)$. The proof is based on the behavior of the lateral Kolmogorov-Smirnov statistics $D_n^+ = \sup_u D_n^+(u)$ and $D_n^- = \sup_u D_n^-(u)$.

LEMMA 1. *If $x_0 = y_0 = 0$, then $D_n^+ = \max_{v \in I} D_n^+(v)$, where*

$$I = \{(x_j, y_i) \mid x_i \leq x_j, y_i \geq y_j; i, j = 0, 1, \dots, n\}.$$

Elements in the set I are: the pair $(0, 0)$, the observed points and the intersection points.

Proof. For each $u = (x, y)$ in the unit square, let x^u and y^u be

$$\begin{aligned}x^u &= x_{k_u} = \max_{i=0,1,\dots,n} \{x_i \mid x_i \leq x, y_i \leq y\} \\y^u &= y_{p_u} = \max_{i=0,1,\dots,n} \{y_i \mid y_i \leq y, x_i \leq x\}.\end{aligned}$$

The relationship between the coordinates is given by $x_{p_u} \leq \max \{x_i \leq x \mid y_i \leq y\} = x_{k_u}$ and $y_{k_u} \leq \max \{y_i \leq y \mid x_i \leq x\} = y_{p_u}$. Hence $(x^u, y^u) \in I$. By the definition of (x^u, y^u) , it is immediate that $G_n(x, y) = G_n(x^u, y) = G_n(x, y^u)$ and the set $\{x_i \mid x_i \in (x^u, x], y_i \in (y^u, y]\}$ is empty. Then

$$G_n(x^u, y^u) = G_n(x^u, y) + G_n(x, y^u) - G_n(x, y) = G_n(x, y)$$

and, therefore, for each u

$$D_n^+(u) = G_n(x, y) - G(x, y) \leq G_n(x^u, y^u) - G(x^u, y^u) \leq \max_{v \in I} D_n^+(v).$$

Hence the lemma follows.

LEMMA 2. *If $x_0 = 0, y_0 = 1, x_{n+1} = 1$ and $y_{n+1} = 0$, then $D_n^- = \max_{v \in P} (G(v) - G_n(v^-))$, where*

$$P = \{(x_j, y_i) \mid x_j > x_i, y_j < y_i; i, j = 0, 1, \dots, n+1\}.$$

Elements in set P are: the pair $(1, 1)$, the intersection points and the projections of the observed points on the right and on the top unit square borders.

Proof. For each $u = (x, y)$ in the unit square, let x^u and y^u be

$$\begin{aligned}x^u &= x_{k_u} = \min_{i=0,1,\dots,n+1} \{x_i \mid x_i > x\} \\y^u &= y_{p_u} = \min_{i=0,1,\dots,n+1} \{y_i \mid y_i > y, x_i \leq x\}.\end{aligned}$$

Obviously, $x_{k_u} > x \geq x_{p_u}$ and $y_{p_u} > y$. Hence (x^u, y^u) is in the set

$$\begin{aligned}Q &= \{(x_j, y_i) \mid x_j > x_i; i, j = 0, 1, \dots, n+1\} \\&= P \cup \{(x_j, y_i) \mid x_j > x_i, y_j > y_i; i, j = 1, \dots, n\}.\end{aligned}$$

Since F is continuous and increasing, the inferior distance is bounded by

$$\begin{aligned}D_n^-(u) &= G(u) - G_n(u) < G(x^u, y^u) - G_n(u) = \\&= G(x^u, y^u) - \lim_{\epsilon \rightarrow 0} G_n(x^u - \epsilon, y^u - \epsilon) + \lim_{\epsilon \rightarrow 0} G_n(x^u - \epsilon, y^u - \epsilon) - G_n(u) \leq \\&\leq \max_{v \in Q} (G(v) - G_n(v^-)) + \lim_{\epsilon \rightarrow 0} G_n(x^u - \epsilon, y^u - \epsilon) - G_n(u).\end{aligned}\tag{3.4}$$

Because of the definition of (x^u, y^u) , $G_n(u) = \lim_{\epsilon \rightarrow 0} G_n(x^u - \epsilon, y) = \lim_{\epsilon \rightarrow 0} G_n(x, y^u - \epsilon)$ and the set $\{(x_i, y_i) \mid x_i \in (x, x^u), y_i \in (y, y^u)\}$ is empty. The left empirical distribution function verifies

$$\lim_{\epsilon \rightarrow 0} G_n(x^u - \epsilon, y^u - \epsilon) = \lim_{\epsilon \rightarrow 0} G_n(x^u, y - \epsilon) + \lim_{\epsilon \rightarrow 0} G_n(x, y^u - \epsilon) - G_n(u) = G_n(u). \quad (3.5)$$

In addition, if $(x^u, y^u) \in Q - P$, we define $x^w = \min \{x_i \mid x_i > x^u, y_i < y^u\}$. Then the pair (x^w, y^u) is in P , $G(x^u, y^u) < G(x^w, y^u)$ and the set $\{(x_i, y_i) \mid x_i \in [x^u, x^w), y_i \in (-\infty, y^u)\}$ is empty. Hence $\lim_{\epsilon \rightarrow 0} G_n(x^w - \epsilon, y^u - \epsilon) = \lim_{\epsilon \rightarrow 0} G_n(x^u - \epsilon, y^u - \epsilon)$ and

$$G(x^u, y^u) - \lim_{\epsilon \rightarrow 0} G_n(x^u - \epsilon, y^u - \epsilon) < \max_{v \in P} (G(v) - G_n(v^-)). \quad (3.6)$$

By (3.4), (3.5) and (3.6), $\max_{v \in P} (G(v) - G_n(v^-))$ is a superior bound for $D_n^-(u)$.

Finally, let $u_0 = (x_0, y_0)$ be given by $u_0 = \arg \max_{v \in P} (G(v) - G_n(v^-))$, then

$$\begin{aligned} \max_{v \in P} (G(v) - G_n(v^-)) &= \lim_{\epsilon \rightarrow 0} (G(x_0 - \epsilon, y_0 - \epsilon) - G_n(x_0 - \epsilon, y_0 - \epsilon)) = \\ &= \lim_{\epsilon \rightarrow 0} D_n^-(x_0 - \epsilon, y_0 - \epsilon). \end{aligned}$$

Hence $D_n^- = \max_{v \in P} (G(v) - G_n(v^-)) = \sup_u D_n^-(u)$ and the lemma follows.

THEOREM 2. *If $p = 2$, the Kolmogorov-Smirnov statistic (2.3) is*

$$D_n = \max_{u \in I, v \in P} \{G_n(u) - G(u), G(v) - G_n(v^-)\}.$$

The Kolmogorov-Smirnov statistics may be expressed as $D_n = \max \{D_n^+, D_n^-\}$ and the proof is straightforward by lemmas 1 and 2.

As a consequence of theorem 2, D_n may be obtained by evaluating the distance in a finite amount of points which ranks from $3n$ to $3n + \binom{n}{2}$ depending on the sample configuration. The theorem leads to the following procedure to compute the Kolmogorov-Smirnov statistic (2.3):

1. Compute the maximum distance in the observed points, $D_n^1 = \max_{i=1, \dots, n} D_n^+(u_i)$.
2. Compute the maximum and minimum distances in the intersection points, $D_n^2 = \max_{i, j=1, \dots, n} \{D_n^+(x_j, y_i) \mid x_j > x_i, y_j < y_i\}$ and $D_n^3 = 2/n - \min_{i, j=1, \dots, n} \{D_n^+(x_j, y_i) \mid x_j > x_i, y_j < y_i\}$.

3. Compute the maximum distance among the projections of the observed points on the right unit square border, $D_n^4 = 1/n - \min_{i=1, \dots, n} D_n^+(1, y_i)$.
4. Compute the maximum distance among the projections of the observed points on the top unit square border, $D_n^5 = 1/n - \min_{i=1, \dots, n} D_n^+(x_i, 1)$.
5. Compute the maximum $D_n = \max\{D_n^1, D_n^2, D_n^3, D_n^4, D_n^5\}$.

4. SOME SIMULATION RESULTS

The power of the exact and the approximate multivariate Kolmogorov-Smirnov statistics when used as a normality test and as a general multivariate goodness of fit test have been investigated. In the first case, the null hypothesis is bivariate normal with mean $\mu = 0$ and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

The alternative distribution is

$$(1 - \epsilon)N(\mathbf{0}, \Sigma) + \epsilon N(\mu, \Sigma)$$

for several values of ϵ and μ . Table 4 shows the power of the normality test. As we may have expected, the power increases with n and is larger for the exact test than for the approximate one. However, for moderately large n ($n \geq 50$ say) the power of the approximate test is very close to that of the exact one. Table 4 shows that both tests are very powerful when n is large and $\epsilon \geq 0.2$.

(Table 4 around here)

Table 5 shows the power of these statistics when the null distribution is Morgenstern (see Morgenstern, 1956) with parameters $\alpha = 0.5$. Similar results were also found for other values of α . We have chosen this distribution because it may have fixed marginal distribution allowing different degrees of dependency. The joint density function for the uniform marginal case is

$$f(x_1, x_2) = 1 + \alpha(2x_1 - 1)(2x_2 - 1) \quad 0 \leq x_1, x_2 \leq 1 \quad -1 \leq \alpha \leq 1$$

and it is straightforward to show that for this distribution

$$F(x_1) = x_1 \quad 0 \leq x_1 \leq 1$$

$$F(x_2 | x_1) = (1 - \alpha(2x_1 - 1))x_2 + \alpha(2x_1 - 1)x_2^2 \quad 0 \leq x_2 \leq 1.$$

The alternative distributions are independent Beta distributions with several combination of shape parameters, to allow for different degrees of asymmetry. Table 5 shows that again, as one could expect, for n small ($n = 10$) the power is very low unless the degree of kurtosis or asymmetry is high. The difference between the power of the exact and approximate test is negligible for large n ($n \leq 50$).

(Table 5 around here)

5. CONCLUDING REMARKS

As in the univariate case, the multivariate Kolmogorov-Smirnov test presented in this paper may provide a general and flexible goodness of fit test, specially for situations when specific test are yet to be developed. The main problem in the application of the test is the computation of the statistic in the case $p > 2$. An extension of the computing algorithm developed in this paper may be possible, but still the numerical complications seem considerable. However, our simulation results show that the approximate Kolmogorov-Smirnov test statistics introduced in this paper, that is trivial to compute, seems to be a promising alternative with a very small loss of power when n is moderately large.

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Table 1: *Monte Carlo approximation to the percentiles of the Kolmogorov-Smirnov statistic distribution, with 2,000 replications.*

n	0.25	0.2	0.15	0.1	0.05	0.01
10	0.4076	0.4244	0.4432	0.4668	0.5022	0.5731
11	0.3863	0.4022	0.4190	0.4408	0.4796	0.5548
12	0.3739	0.3882	0.4041	0.4275	0.4629	0.5229
13	0.3576	0.3716	0.3883	0.4082	0.4378	0.5015
14	0.3491	0.3623	0.3753	0.3988	0.4246	0.4874
15	0.3383	0.3516	0.3644	0.3871	0.4116	0.4657
16	0.3222	0.3347	0.3517	0.3699	0.4018	0.4637
17	0.3177	0.3291	0.3435	0.3628	0.3896	0.4415
18	0.3120	0.3216	0.3349	0.3526	0.3819	0.4321
19	0.3016	0.3137	0.3271	0.3435	0.3718	0.4237
20	0.2923	0.3041	0.3161	0.3329	0.3585	0.4088
25	0.2643	0.2733	0.2849	0.3019	0.3251	0.3759
30	0.2410	0.2498	0.2602	0.2744	0.2964	0.3501
35	0.2238	0.2330	0.2441	0.2601	0.2831	0.3221
50	0.1901	0.1981	0.2063	0.2174	0.2330	0.2763
100	0.1360	0.1414	0.1473	0.1559	0.1686	0.1934

Table 2: *Monte Carlo approximation to the Kolmogorov-Smirnov Lilliefords statistic distribution, with 2,000 replications.*

n	0.25	0.2	0.15	0.1	0.05	0.01
10	0.3067	0.3155	0.3264	0.3416	0.3609	0.4040
11	0.2935	0.3031	0.3140	0.3285	0.3539	0.3895
12	0.2818	0.2892	0.2990	0.3096	0.3274	0.3725
13	0.2713	0.2796	0.2889	0.3004	0.3227	0.3650
14	0.2635	0.2713	0.2803	0.2946	0.3110	0.3473
15	0.2583	0.2654	0.2749	0.2870	0.2997	0.3333
16	0.2489	0.2561	0.2646	0.2760	0.2922	0.3338
17	0.2400	0.2477	0.2567	0.2685	0.2888	0.3175
18	0.2384	0.2455	0.2536	0.2637	0.2805	0.3077
19	0.2307	0.2379	0.2467	0.2579	0.2759	0.3063
20	0.2253	0.2318	0.2410	0.2504	0.2673	0.3022
25	0.2026	0.2088	0.2163	0.2254	0.2407	0.2773
30	0.1862	0.1916	0.1981	0.2080	0.2204	0.2490
35	0.1737	0.1798	0.1862	0.1932	0.2069	0.2325
50	0.1469	0.1508	0.1556	0.1625	0.1737	0.1910
100	0.1020	0.1048	0.1081	0.1125	0.1187	0.1329

Table 3: *Monte Carlo approximation to the percentiles of the approximated Kolmogorov-Smirnov statistic distribution, with 2,000 replications.*

n	0.25	0.2	0.15	0.1	0.05	0.01
10	0.3044	0.3209	0.3386	0.3654	0.4056	0.4758
11	0.2914	0.3085	0.3264	0.3511	0.3896	0.4568
12	0.2854	0.2998	0.3176	0.3452	0.3785	0.4394
13	0.2732	0.2874	0.3066	0.3266	0.3589	0.4278
14	0.2641	0.2770	0.2903	0.3117	0.3440	0.4194
15	0.2631	0.2764	0.2907	0.3105	0.3494	0.4221
16	0.2483	0.2640	0.2772	0.2970	0.3266	0.3838
17	0.2441	0.2572	0.2748	0.2964	0.3277	0.3875
18	0.2377	0.2484	0.2613	0.2812	0.3083	0.3690
19	0.2351	0.2455	0.2569	0.2764	0.3058	0.3631
20	0.2303	0.2401	0.2505	0.2702	0.2978	0.3510
25	0.2072	0.2168	0.2284	0.2438	0.2715	0.3261
30	0.1919	0.2026	0.2138	0.2303	0.2513	0.2878
35	0.1798	0.1883	0.1992	0.2147	0.2365	0.2786
50	0.1511	0.1586	0.1670	0.1797	0.1993	0.2396
100	0.1134	0.1180	0.1249	0.1325	0.1449	0.1708

Table 4: Empirical power of the Kolmogorov-Smirnov (KS) and approximated Kolmogorov-Smirnov (AKS) test with size 0.1. The null hypothesis is a $N(0, \Sigma)$ and the samples are generated from a normal mixture $(1 - \epsilon)N(0, \Sigma) + \epsilon N(\mu, \Sigma)$.

		$n = 15$		$n = 25$		$n = 50$		$n = 100$	
		KS	AKS	KS	AKS	KS	AKS	KS	AKS
$\mu = (3, 3)'$	$\epsilon = 0.1$	0.15	0.13	0.18	0.16	0.27	0.25	0.44	0.41
	$\epsilon = 0.2$	0.30	0.26	0.45	0.41	0.72	0.70	0.96	0.95
	$\epsilon = 0.4$	0.75	0.73	0.93	0.92	1.00	1.00	1.00	1.00
$\mu = (3, -1)'$	$\epsilon = 0.1$	0.16	0.11	0.21	0.14	0.30	0.21	0.50	0.36
	$\epsilon = 0.2$	0.34	0.20	0.51	0.33	0.77	0.60	0.97	0.91
	$\epsilon = 0.4$	0.80	0.59	0.95	0.83	1.00	0.99	1.00	1.00

Table 5: Empirical power of the Kolmogorov-Smirnov (KS) and the approximated Kolmogorov-Smirnov (AKS) test of size α . The null hypothesis is a Morgenstern with parameter 0.5 and uniform marginals. The samples are generated from two independent Beta(a,b).

	n	$\alpha = 0.1$		$\alpha = 0.05$		$\alpha = 0.01$	
		KS	AKS	KS	AKS	KS	AKS
Beta(10,10)	10	0.948	0.424	0.842	0.310	0.372	0.157
	20	1.000	0.777	0.999	0.692	0.994	0.505
	50	1.000	0.999	1.000	0.996	1.000	0.975
	100	1.000	1.000	1.000	1.000	1.000	1.000
Beta(3,3)	10	0.241	0.177	0.129	0.101	0.020	0.029
	20	0.640	0.385	0.459	0.260	0.175	0.097
	50	0.992	0.897	0.968	0.793	0.756	0.502
	100	1.000	0.999	1.000	0.997	0.999	0.973
Beta(3,2)	10	0.167	0.145	0.079	0.059	0.008	0.007
	20	0.518	0.424	0.332	0.244	0.089	0.053
	50	0.983	0.969	0.943	0.878	0.574	0.471
	100	1.000	1.000	1.000	1.000	0.998	0.992
Beta(0.5,1)	10	0.579	0.575	0.439	0.422	0.220	0.211
	20	0.842	0.816	0.753	0.713	0.508	0.451
	50	0.994	0.993	0.989	0.985	0.911	0.898
	100	1.000	1.000	1.000	1.000	0.999	0.999
Beta(0.5,0.5)	10	0.308	0.247	0.212	0.153	0.077	0.053
	20	0.516	0.403	0.388	0.284	0.186	0.111
	50	0.850	0.792	0.759	0.651	0.456	0.358
	100	0.994	0.987	0.976	0.960	0.887	0.832