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A FIXED-WIDTH INTERVAL FOR α/β IN REGRESSION

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Abstract

Consider the regression model $y_i = \alpha + \beta x_i + e_i$ and the problem of constructing a confidence interval for α/β with $|\alpha| > \alpha_*$) and $\beta \in (0, \beta^*)$ where $\alpha_* > 0$ and $\beta^* > 0$. Uniformity down to $\beta = 0$ is a major difficulty. In fact any procedure based on a fixed sample size, will have either infinite expected width or zero confidence (Gleser and Hwang 1987), confidence being the infimum of the coverage probability. Sequential sampling is used to construct fixed length intervals of the form

$$(\hat{\alpha}_\tau/\hat{\beta}_\tau - h,\ \hat{\alpha}_\tau/\hat{\beta}_\tau + h)$$

where τ is an integer valued stopping time, $\hat{\alpha}_\tau$ and $\hat{\beta}_\tau$ are the least squares estimators for α and β based on τ-observations and h is the half-width of the interval. Stopping times τ_h are derived so that these intervals have coverage probabilities converging to a set value γ as $h \to 0$. This convergence is uniform down to $\beta = 0$. Furthermore the predictors x_i may be chosen adaptively.

Key Words
Brownian motion; sequential estimation; Strassen.

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A Fixed-Width Interval for α/β in Regression

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Consider the regression model $y_i = \alpha + \beta x_i + e_i$ and the problem of constructing a confidence interval for α/β with $|\alpha| > \alpha_*$ and $\beta \in (0, \beta^*)$ where $\alpha_* > 0$ and $\beta^* > 0$. Uniformity down to $\beta = 0$ is a major difficulty. In fact any procedure based on a fixed sample size, will have either infinite expected width or zero confidence (Gleser and Hwang 1987), confidence being the infimum of the coverage probability. Sequential sampling is used to construct fixed length intervals of the form

$$(\hat{\alpha}_\tau/\hat{\beta}_\tau - h, \hat{\alpha}_\tau/\hat{\beta}_\tau + h)$$

where τ is an integer valued stopping time, $\hat{\alpha}_\tau$ and $\hat{\beta}_\tau$ are the least squares estimators for α and β based on τ-observations and h is the half-width of the interval. Stopping times τ_h are derived so that these intervals have coverage probabilities converging to a set value γ as $h \to 0$. This convergence is uniform down to $\beta = 0$. Furthermore the predictors x_i may be chosen adaptively.

1 An interval for α/β

Fixed-width, asymptotic confidence intervals are set for α/β, from the model

$$Y_i = \alpha + x_i\beta + e_i. \tag{1}$$

Intervals are of the form

$$(\hat{\alpha}_\tau/\hat{\beta}_\tau - h, \hat{\alpha}_\tau/\hat{\beta}_\tau + h), \tag{2}$$

where τ is an integer valued stopping time, $\hat{\alpha}_\tau/\hat{\beta}_\tau$ is the ratio of the least squares estimators based on τ observations and h is the half-length. Stopping times τ_a are derived so that these confidence intervals have coverage probabilities converging to the desired coverage probability $\gamma \in (0,1)$ as $h \to 0$ or as $a \to \infty$ where

$$a = \sqrt{-\Phi^{-1}\left(\frac{1-\gamma}{2}\right)/h} \tag{3}$$

and Φ is the c.d.f. of a standard normal. This coverage is uniform over the set

$$\Theta = \Theta(a) = \{(\alpha, \beta) \in \mathbb{R}^2 \,|\, |\alpha| > \alpha^* a^{-\frac{1}{4}} \text{ and } \beta \in (0, \beta^* a^{\frac{1}{4}})\}$$

where $\alpha^* > 0$ and $\beta^* > 0$ are constants set by the experimenter.

Furthermore, the predictors, x_i, may be chosen adaptively. That is, x_i may be a function of $(x_{i-1}, y_{i-1}, ..., x_1, y_1)$. In particular, x_i may be a function of $\hat{\alpha}_{i-1}$ and $\hat{\beta}_{i-1}$ and depend implicitly on the parameters α and β.

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The problem considered here is a generalization of setting fixed-width confidence intervals for $1/\beta$ from the model $y_i = x_i\beta + \epsilon_i$, see Coleman (1994).

Assume the following assumption on the errors.

(E) The errors, e_i, are i.i.d. with $\mathbb{E}e_i = 0$, $\mathbb{E}e_i^2 = \sigma^2$ and for some $p > 1$, $\mathbb{E}|e_i|^{2p} < \infty$.

The least squares estimators for β, α and σ^2 are

$$\hat{\beta}_n = s_n^{-1}\sum_{i=1}^{n}(x_i - \overline{x}_n)y_i,\ \hat{\alpha}_n = \overline{y}_n - \overline{x}_n\hat{\beta}_n \text{ and } \hat{\sigma}_n^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n}(y_i - \overline{y}_n)^2 - s_n\hat{\beta}_n^2\right] + s_n^{-\frac{1}{4}},$$

where

$$\overline{x}_n = \frac{1}{n}\sum_{i=1}^{n}x_i,\ \overline{y}_n = \frac{1}{n}\sum_{i=1}^{n}y_i,\ t_n = \sum_{i=1}^{n}x_i^2, \text{ and } s_n = \sum_{i=1}^{n}(x_i - \overline{x}_n)^2.$$

The estimator for σ^2 is modified by adding $s_n^{-\frac{1}{4}}$ to prevent early stopping.

To motivate the stopping time assume that

$$\frac{1}{\hat{\sigma}_n}\begin{pmatrix}\sqrt{s_n}(\hat{\beta}_n - \beta)\\ \sqrt{n}[(\hat{\alpha}_n + \hat{\beta}_n\overline{x}_n) - (\alpha + \beta\overline{x}_n)]\end{pmatrix} \Rightarrow N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right).$$

Then by a Taylor's series expansion

$$\begin{aligned}
&\frac{\sqrt{s_n}}{\hat{\sigma}_n}\left(\frac{\hat{\alpha}_n}{\hat{\beta}_n} - \frac{\alpha}{\beta}\right)\\
\approx\ & \frac{1}{\beta}\frac{\sqrt{s_n}}{\sqrt{n}}\frac{\sqrt{n}}{\hat{\sigma}_n}[(\hat{\alpha}_n + \hat{\beta}_n\overline{x}_n) - (\alpha + \beta\overline{x}_n)] - \frac{1}{\beta}\left(\overline{x}_n + \frac{\alpha}{\beta}\right)\frac{\sqrt{s_n}}{\hat{\sigma}_n}(\hat{\beta}_n - \beta)\\
\Rightarrow\ & N\left(0, \frac{1}{\beta^2}(x^* - x_*^2) + \frac{1}{\beta^2}\left(x_* + \frac{\alpha}{\beta}\right)^2\right)\\
=\ & N\left(0, \frac{1}{\beta^4}q\right),
\end{aligned}$$

where $x^* = x^*(\alpha, \beta)$ and $x_* = x_*(\alpha, \beta)$ are such that

$$n^{-1}s_n \rightarrow_p x^* - x_*^2 \text{ and } \overline{x}_n \rightarrow_p x_*$$

and

$$q = \left(x^* - x_*^2\right)\beta^2 + (x_*\beta + \alpha)^2 = x^*\beta^2 + 2\alpha\beta x_* + \alpha^2. \tag{4}$$

Hence

$$\mathbb{P}\left(\left|\frac{\hat{\alpha}_n}{\hat{\beta}_n} - \frac{\alpha}{\beta}\right| \le h\right) \approx 1 - 2\Phi\left(\frac{-h\beta^2\sqrt{s_n}}{\sigma\sqrt{q}}\right).$$

This coverage should be at least γ, the desired coverage probability. Replace α, β, σ and q with their estimators to obtain

$$1 - 2\Phi\left(\frac{-\hat{\beta}_n^2 h\sqrt{s_n}}{\hat{\sigma}_n\sqrt{\hat{q}_n}}\right) \ge \gamma_c,$$

where

$$\hat{q}_n = \left(\frac{s_n}{n}\right)\hat{\beta}_n^2 + \left(\overline{x}_n\hat{\beta}_n + \hat{\alpha}_n\right)^2 = \frac{t_n}{n}\hat{\beta}_n^2 + 2\hat{\alpha}_n\hat{\beta}_n\overline{x}_n + \hat{\alpha}_n^2. \tag{5}$$

Then

$$\frac{h\hat{\beta}_n^2\sqrt{s_n}}{\hat{\sigma}_n\sqrt{\hat{q}_n}} \geq -\Phi^{-1}\left(\frac{1-\gamma_c}{2}\right) \text{ and } \hat{\beta}_n^2\sqrt{s_n} \geq a^2\hat{\sigma}_n\sqrt{\hat{q}_n}$$

where a is defined in (3). Hence

$$\left|\sum_{i=1}^{n}(x_i - \overline{x}_n)y_i,\right| \geq a s_n^{\frac{3}{4}}\hat{\sigma}_n^{\frac{1}{2}}\hat{q}_n^{\frac{1}{4}}.$$

Based on these calculations it's natural to consider the stopping times

$$\tau = \tau(a) = \inf\left\{n | n \geq 3, s_n \geq s^o \text{ and } \sum_{i=1}^{n}(x_i - \overline{x}_n)y_i \geq a s_n^{\frac{3}{4}}\sqrt{\hat{\sigma}_n}\left(\hat{q}_n + s_n^{-\frac{1}{4}}\right)^{\frac{1}{4}}\right\} \tag{6}$$

where $s^o > 0$ is a constant set by the experimenter. The term $s_n^{-\frac{1}{4}}$ is added to $\hat{q}_n$ to prevent early stopping, see Lemma 8. Theorem 1 below shows that this choice of τ produces fixed-width, asymptotic confidence intervals of the form described in (2).

Let $\lfloor z \rfloor$ be the largest integer less than or equal to z. Let $M > 0$ and $m > 0$ denote constants that do not depend on α or β. Let $f(a) = O(g(a))$ denote the existence of $M > 0$ such that

$$\limsup_{a\to\infty}\left|\frac{f(a)}{g(a)}\right| \leq M.$$

Assume the following assumptions on the predictors:

(P1) $x_i = x_i((x_{i-1}, y_{i-1}), ..., (x_1, y_1), v_i)$ where v_i are independent random variables such that $\{v_i\}$ is independent of $\{\epsilon_j\}$,
(P2) $\exists k \geq p$ such that $\sup_\Theta \sum_{i=1}^{a}\mathbb{E}|x_i|^{2k} = O(a)$,
(P3) $\sup_\Theta \mathbb{E}\sup_{3\leq n\leq a}|\overline{x}_n\sum_{i=1}^{n}e_i|^{2p} = O(a^p)$,
(P4) for $\epsilon > 0$ and $\phi > \frac{1}{2}$, $\sup_\Theta \epsilon^{2p}\mathbb{P}\left(\sup_{n>a} n^{-\phi}|\overline{x}_n\sum_{i=1}^{n}e_i| > \epsilon\right) = O\left(a^{-(\phi-\frac{1}{2})2p}\right)$,
(P5) $\exists z_o > 0$ such that $\sup_\Theta \mathbb{P}\left(\sup_{n>a} n s_n^{-1} \geq z_o\right) = O(a^{-\frac{k}{2}})$,
(P6) $x^* = x^*(\alpha,\beta)$ and $x_* = x_*(\alpha,\beta)$ are such that $x^* - x_*^2 > m_x$ and $\max\{x^*, |x_*|\} < M_x$ where $M_x > 0$ and $m_x > 0$ are constants,
(P7) for $\epsilon > 0$, $\lim_{a\to\infty}\sup_\Theta \epsilon^{2p}\mathbb{P}\left(\sup_{n>a}|\overline{x}_n - x_*| > \epsilon\right) = 0$,
(P8) for $\epsilon > 0$, $\lim_{a\to\infty}\sup_\Theta \epsilon^{2p}\mathbb{P}\left(\sup_{n>a}|n^{-1}t_n - x^*| > \epsilon\right) = 0$.

If the predictors are deterministic the assumptions simplify to

(P2) $\exists k \geq p$ such that $\limsup_{n\to\infty} n^{-1}\sum_{i=1}^{n}|x_i|^{2k} < \infty$,
(P5) $\exists z_o > 0$ such that $\limsup_{n\to\infty} n s_n^{-1} < z_o$,
(P6) $x^* - x_*^2 > m_x$ for $m_x > 0$,
(P7) $\lim_{n\to\infty}\overline{x}_n = x_*$,
(P8) $\lim_{n\to\infty} n^{-1}t_n = x^*$.

If the predictors x_i are independent, identitically distributed such that $\{x_i\}$ is independent of $\{e_i\}$, $\mathbb{E}x_1^2 > 0$ and $\mathbb{E}|x_1|^{2k} < \infty$ for some $k \geq p$ assumptions are satisfied. This is a special case of an adaptive procedure proposed in Section 4.

For the remainder of this paper assume (P1) through (P8) and (E).

Define the sigma-field

$$\mathcal{X}_n = \sigma\{e_n, ..., e_1, x_{n+1}, ..._1\}. \tag{7}$$

The main result is stated in the next theorem. Recall, both Θ and τ depend on a.

Theorem 1 *For* $\gamma \in (0,1)$,

$$\lim_{a\to\infty} \sup_{\Theta} \left| \mathbb{P}\left(\left| \frac{\hat{\alpha}_\tau}{\hat{\beta}_\tau} - \frac{\alpha}{\beta} \right| \leq h \right) - \gamma \right| = 0.$$

For $p' < 4kp/(4k+5p)$,

$$\lim_{a\to\infty} \sup_{\Theta} \left|\frac{\beta}{\alpha}\right|^{p'} \mathbb{E} \left| \frac{\hat{\alpha}_\tau}{\hat{\beta}_\tau} - \frac{\alpha}{\beta} \right|^{p'} = 0.$$

The proof of Theorem 1 requires some properties of the stopping time. At stopping

$$\sum_{i=1}^{\tau}(x_i - \overline{x}_\tau)y_i = \sum_{i=1}^{\tau}(x_i - \overline{x}_n)e_i + s_\tau\beta \geq a s_\tau^{\frac{3}{4}} \hat{\sigma}_\tau^{\frac{1}{2}} \left(\hat{q}_\tau + s_\tau^{-\frac{1}{4}} \right)^{\frac{1}{4}}. \tag{8}$$

Assuming no excess over the boundary, set $\sum_{i=1}^{\tau}(x_i - \overline{x}_n)e_i = 0 = s_\tau^{-\frac{1}{4}}$, solving for s_τ yields

$$s_\tau \approx (a/\beta)^4 \hat{\sigma}_\tau^2 \hat{q}_\tau.$$

Hence uniformity for β down to zero is obtained by sampling until s_n is sufficiently large. Let

$$s_\tau^* = \frac{\beta^4 s_\tau}{a^4 \sigma^2 q}.$$

Let $d > 0$ such that

$$d < k^2/(k+2) \text{ for } k \leq 2 \text{ and } d < \min(k/2, p) \text{ for } k > 2. \tag{9}$$

Theorem 2 *For* $\epsilon_o > 0$,

$$\lim_{a\to\infty} \sup_{\Theta} \sup_{\epsilon > \epsilon_o} \epsilon^d \mathbb{P}\left(s_\tau^* \geq 1 + \epsilon\right) = 0.$$

For $\epsilon \in (0,1)$,

$$\lim_{a\to\infty} \sup_{\Theta} \mathbb{P}\left(s_\tau^* \leq 1 - \epsilon\right) = 0.$$

Furthermore,

$$\lim_{a\to\infty} \sup_{\Theta} \mathbb{E}\left|s_\tau^* - 1\right|^d = 0.$$

The rate ϵ^d in the first assertion of Theorem 2, leads directly to the expectation in the third assertion. Theorem 1 is proved in Section 3 and Theorem 2 is proved in Section 2. In Section 4 an adaptive procedure is proposed .

2 Results for the Stopping Time τ

Lemmas 1, 2 and 3 are inequalities which are used throughout this paper. Lemma 1, adapted from Brunk and Chung, uses Burkholder's inequality to make sharp bounds for martingale differences, see Corollary 2, pg. 397 and Theorem 3, pg. 345 of Chow and Teicher (1988).

Lemma 1 *Let $d_i = d_i(\alpha, \beta)$ be martingale differences, $S_n = \sum_{i=1}^n d_i, k > 1$, $\phi > \frac{1}{2}$ and*

$$\sup_{\Theta} \sum_{i=1}^{a} \mathbb{E}|d_i|^k = O(a). \tag{10}$$

Then

$$\sup_{\Theta} \mathbb{E} \sup_{1 \le n \le a} |S_n|^k = O\left(a^{\frac{k}{2}}\right)$$

and for $\epsilon > 0$,

$$\sup_{\Theta} \epsilon^k \mathbb{P}\left(\sup_{n>a} n^{-\phi}|S_n| > \epsilon\right) = O\left(a^{-(\phi-\frac{1}{2})k}\right).$$

For a proof see Lemma 3, Coleman (1994).

Lemma 2 provides bounds for the quantity,

$$\sum_{i=1}^{n}(x_i - \overline{x}_n)y_i - s_n\beta = s_n(\hat{\beta}_n - \beta) = \sum_{i=1}^{n}(x_i - \overline{x}_n)e_i.$$

Lemma 2

$$\sup_{\Theta} \mathbb{E} \sup_{3 \le n \le a} \left|\sum_{i=1}^{n}(x_i - \overline{x}_n)e_i\right|^{2p} = O(a^p).$$

For $\phi > \frac{1}{2}$ and $\epsilon > 0$,

$$\sup_{\Theta} \epsilon^{2p} \mathbb{P}\left(\sup_{n>a} n^{-\phi}\left|\sum_{i=1}^{n}(x_i - \overline{x}_n)e_i\right| > \epsilon\right) = O\left(a^{-(\phi-\frac{1}{2})2p}\right).$$

Proof. The sums, $\sum_{i=1}^n x_i e_i$, with the filtration $\mathcal{X}_n$, is a martingale then by (P2),

$$\sup_{\Theta} \sum_{i=1}^{a} \mathbb{E}|x_i e_i|^{2p} = \mathbb{E}|e_1|^{2p} \sup_{\Theta} \sum_{i=1}^{a} \mathbb{E}|x_i|^{2p} = O(a).$$

The lemma follows by Lemma 1, (P3) and (P4). □

The following lemma states some easy albeit essential properties of q.

Lemma 3

$$\inf_{\Theta} q/\alpha^2 \ge m_x/M_x, \quad \inf_{\Theta} q/\beta^2 \ge m_x \text{ and } \inf_{\Theta} q \ge a^{-\frac{1}{2}}\alpha_*^2(m_x/M_x).$$

Proof. By (P6)

$$\inf_{\Theta} q/\alpha^2 = \inf_{\Theta} x^*(\beta/\alpha)^2 + 2x_*(\beta/\alpha) + 1 \geq \inf_{\Theta}(x^* - x_*^2)/x^* \geq m_x/M_x,$$

$$\inf_{\Theta} q/\beta^2 \geq \inf_{\Theta}(x^* - x_*) \geq m_x \textit{ and } \inf_{\Theta} q \geq \inf_{\Theta} \alpha^2(q/\alpha^2) \geq a^{-\frac{1}{2}}\alpha_*^2(m_x/M_x).$$

□

Consider the first assertion of Theorem 2. For d defined in (9) choose $\delta > 0$ such that

$$\delta < \min\{1, 2k/(k+2)\}, \ d < k\delta/2, \text{ and } d < p\delta.$$

For $\epsilon > 0$, define the stopping time $n^* = n^*(a, \delta, \alpha, \beta, \epsilon)$ by

$$n^* = \inf\left\{n \geq 3 | s_n \geq (a/\beta)^4\sigma^2 q\,(1+\epsilon)^\delta\right\}. \tag{11}$$

Define the set

$$\mathcal{A}_a = \left\{a < n^* \leq n^o, (a/\beta)^4\sigma^2 q\,(1+\epsilon)^\delta \leq s_{n^*} < (a/\beta)^4\sigma^2 q\,(1+\epsilon)\right\},$$

where

$$n^o = \lfloor 2z_o(a/\beta)^4\sigma^2 q(1+\epsilon)^\delta \rfloor \tag{12}$$

and z_o is defined in (P5). Hence on the set $\mathcal{A}_a$,

$$\left\{s_\tau \geq (a/\beta)^4\sigma^2 q(1+\epsilon)\right\} \subseteq \left\{\sum_{i=1}^{n^*}(x_i - \overline{x}_{n^*})e_i + s_n\beta < a s_{n^*}^{\frac{3}{4}}\hat{\sigma}_{n^*}^{\frac{1}{2}}(\hat{q}_{n^*} + s_{n^*}^{-\frac{1}{4}})^{\frac{1}{4}}\right\} \tag{13}$$

Lemma 4 states the $\mathbb{P}(\mathcal{A}^c)$ tends to zero, Lemmas 5 and 6 show that $\hat{\sigma}_{n^*}$ and $\hat{q}_{n_*}$ converge to σ and q. Lemma 7 proves the first assertion of Theorem 2.

Lemma 4 *For* $\epsilon > 0$,

$$\lim_{a\to\infty} \sup_{\Theta} \epsilon^d \mathbb{P}\,(\mathcal{A}_a^c) = 0.$$

Proof. Since $s_n \leq t_n$ by Holder's inequality, Jensen's inequality and (P2)

$$\begin{aligned}
\sup_{\Theta} \mathbb{E} s_{\lfloor a \rfloor}^k &\leq \sup_{\Theta} \mathbb{E}\left(\sum_{i=1}^{\lfloor a \rfloor} x_i^2\right)^k \\
&\leq \sup_{\Theta} \mathbb{E}\left(\lfloor a \rfloor^{\frac{k-1}{k}}\left(\sum_{i=1}^{\lfloor a \rfloor}|x_i|^{2k}\right)^{\frac{1}{k}}\right)^k \\
&= \lfloor a \rfloor^{k-1}\sup_{\Theta}\sum_{i=1}^{\lfloor a \rfloor}\mathbb{E}|x_i|^{2k} \\
&= O\left(a^k\right).
\end{aligned} \tag{14}$$

Since s_n is nondecreasing in n,

$$\{a \geq n^*\} \subseteq \left\{s_{\lfloor a \rfloor} \geq s_{n^*}\right\} \subseteq \left\{s_{\lfloor a \rfloor} \geq (a/\beta)^4 \sigma^2 q(1+\epsilon)^\delta\right\}$$

then by (14) the first probability is

$$\sup_\Theta \mathbb{P}(a \geq n^*) \leq \sup_\Theta \left(\frac{\beta^4}{a^4\sigma^2 q(1+\epsilon)^\delta}\right)^k \mathbb{E} s_{\lfloor a \rfloor}^k = (1+\epsilon)^{-\delta k} O\left(a^{-\frac{5}{2}k}\right).$$

Since n^* is a stopping time,

$$\{n^* > n^o\} \subseteq \left\{(a/\beta)^4\sigma^2 q(1+\epsilon)^\delta \geq s_{n^o}\right\} \subseteq \left\{\frac{n^o}{s_{n^o}} \geq \frac{\beta^4 n^o}{a^4\sigma^2 q}(1+\epsilon)^\delta\right\} \subseteq \left\{\frac{n^o}{s_{n^o}} \geq z_o\right\}$$

then by (P5) the second probability is

$$\sup_\Theta \mathbb{P}(n^* > n^o) \leq \sup_\Theta \mathbb{P}\left(\frac{n^o}{s_{n^o}} \geq z_o\right) = \sup_\Theta O\left((n^o)^{-2p}\right) = (1+\epsilon)^{-\frac{\delta k}{2}} O\left(a^{-\frac{7}{4}k}\right).$$

Since $s_{n^*-1} < (a/\beta)^4\sigma^2 q(1+\epsilon)^\delta$ then for $n^* \leq n^o$

$$\begin{aligned} s_{n^*} - (a/\beta)^4\sigma^2 q(1+\epsilon) &\leq s_{n^*} - s_{n^*-1} - (a/\beta)^4\sigma^2 q\left[(1+\epsilon) - (1+\epsilon)^\delta\right] \\ &\leq (x_{n^*} - \overline{x}_{n^*-1})^2 - (a/\beta)^4\sigma^2 q(1-\delta)\epsilon \\ &\leq 4 \sup_{1 \leq n \leq n^o} x_n^2 - (a^4/\beta^2) 4m\epsilon \end{aligned}$$

where $m > 0$ is such that $\inf_{\theta\in\Theta} \sigma^2(q/\beta^2)(1-\delta) \geq 4m$. Then by (P2),

$$\begin{aligned} \sup_\Theta \mathbb{P}\left(s_{n^*} \geq \frac{a^4}{\beta^4}\sigma^2 q(1+\epsilon), n^* \leq n^o\right) &\leq \sup_\Theta \left(\frac{\beta^2}{a^4 m\epsilon}\right)^k \sum_{n=1}^{n^o} \mathbb{E}|x_n|^{2k} \\ &= \epsilon^{-(k-\delta)} O\left(a^{-\frac{7}{2}(k-1)}\right). \end{aligned}$$

Since $\delta < 2k/(k+2)$ then $k - \delta \geq k\delta/2 \geq d$. The result follows by comparing these rates. □

Lemma 5 *For* $\epsilon_o > 0$,

$$\lim_{a\to 0} \sup_\Theta \sup_{\epsilon > \epsilon_o} \epsilon^d \mathbb{P}\left(\hat{\sigma}_{n^*}^2 > (1+\epsilon)^{\frac{\delta}{2}} \sigma^2, \mathcal{A}_a\right) = 0.$$

For $\epsilon \in (0,1)$,

$$\lim_{a\to 0} \sup_\Theta \mathbb{P}\left(\inf_{n>a} \hat{\sigma}_n^2 < (1-\epsilon)\sigma^2\right) = 0.$$

A similar statement is proved in Lemma 5, Coleman (1994).

Lemma 6 *For* $\epsilon_o > 0$,

$$\lim_{a\to\infty} \sup_\Theta \sup_{\epsilon>\epsilon_o} \epsilon^d \mathbb{P}\left(\hat{q}_{n^*} + s_{n^*}^{-\frac{1}{4}} \geq (1+\epsilon)^{\frac{d}{2}} q, \mathcal{A}_a\right) = 0.$$

For $\epsilon \in (0,1)$,

$$\lim_{a\to\infty} \sup_\Theta \mathbb{P}\left(\inf_{n>a} \hat{q}_n \leq (1-\epsilon) q\right) = 0.$$

Proof. Some algebra yields,

$$\begin{aligned}\hat{q}_n - q &= \frac{s_n}{n}(\hat{\beta}_n - \beta)^2 + 2\beta\frac{s_n}{n}(\hat{\beta}_n - \beta) + \beta^2\left(\frac{t_n}{n} - x^*\right)\\ &+ \left[2\alpha\bar{e}_n + 2\beta\bar{x}_n\bar{e}_n + \bar{e}_n^2\right] + \left[2\beta\alpha(\bar{x}_n - x_*)\right].\end{aligned}$$

On the set $\mathcal{A}_a$, $2z_o(s_{n^*}/n^*) \geq 1$, $s_{n^*}^{-\frac{1}{4}} \leq \left[(a/\beta)^4\sigma^2 q(1+\epsilon)^\delta\right]^{-\frac{1}{4}} = O\left(a^{-\frac{7}{8}}\right)$and $n^* > a$, by Lemma 3 there exist $M > 0$ such that

$$\begin{aligned}&\frac{1}{q}\left(|\hat{q}_{n^*} - q| + s_{n^*}^{-\frac{1}{4}}\right)\\ &\leq \sup_{n>a}\left[\frac{2z_o}{a^{\frac{1}{2}}q}\frac{s_n^2}{n^{\frac{7}{4}}}\left|\hat{\beta}_n - \beta\right|^2 + \frac{2\beta}{a^{\frac{1}{4}}q}\frac{s_n}{n^{\frac{7}{8}}}\left|\hat{\beta}_n - \beta\right| + \frac{\beta^2}{q}\left|\frac{t_n}{n} - x^*\right| + \frac{2|\alpha|}{a^{\frac{1}{4}}q}n^{\frac{1}{8}}|\bar{e}_n|\right.\\ &\left.+\frac{2\beta}{a^{\frac{1}{4}}q}n^{\frac{1}{8}}|\bar{x}_n\bar{e}_n| + \frac{1}{a^{\frac{1}{2}}q}\left|n^{\frac{1}{8}}\bar{e}_n\right|^2 + \frac{2\beta|\alpha|}{q}(\bar{x}_n - x_*) + \frac{1}{q}s_{n^*}^{-\frac{1}{4}}\right]\\ &\leq M\sup_{n>a}\left[\frac{s_n^2}{n^{\frac{7}{4}}}\left|\hat{\beta}_n - \beta\right|^2 + \frac{s_n}{n^{\frac{7}{8}}}\left|\hat{\beta}_n - \beta\right| + \left|\frac{t_n}{n} - x^*\right| + n^{\frac{1}{8}}|\bar{e}_n|\right.\\ &\left.+n^{\frac{1}{8}}|\bar{x}_n\bar{e}_n| + \left(n^{\frac{1}{8}}|\bar{e}_n|\right)^2 + |\bar{x}_n - x_*| + O(a)^{-\frac{3}{8}}\right].\end{aligned}$$

Let $m = m(\epsilon_o, \delta) > 0$ be such that $(1+\epsilon)^{\frac{\delta}{2}} - 1 \geq 8m\epsilon^{\frac{\delta}{2}}$ for all $\epsilon > \epsilon_o$. Hence

$$\mathbb{P}\left(\hat{q}_{n^*} + s_{n^*}^{-\frac{1}{4}} \geq (1+\epsilon)^{\frac{\delta}{2}} q, \mathcal{A}_a\right) \leq \mathbb{P}\left(\frac{1}{q}\left(|\hat{q}_{n^*} - q| + s_{n^*}^{-\frac{1}{4}}\right) \geq 8m\epsilon^{\frac{\delta}{2}}, \mathcal{A}_a\right)$$

can be bounded by the sum of eight probabilities. The first probability is

$$\lim_{a\to\infty}\sup_{\Theta}\sup_{\epsilon>\epsilon_o}\epsilon^{\frac{p\delta}{2}}\mathbb{P}\left(\sup_{n>a}\frac{s_n^2}{n^{\frac{7}{4}}}|\hat{\beta}_n - \beta|^2 \geq \frac{m}{M}\epsilon^{\frac{\delta}{2}}\right) = 0. \tag{15}$$

The bound follows from Lemma 2. Lemma 2 applies to the second probability, (P8) applies to the third, Lemma 1 applies to the fourth and sixth, (P4) applies to the fifth, (P7) applies the seventh and the eighth is zero for a sufficiently large. The second assertion is proved in a similar manner. □

Lemma 7 *For $\epsilon_o > 0$*

$$\lim_{a\to\infty}\sup_{\Theta}\sup_{\epsilon>\epsilon_o}\epsilon^d\mathbb{P}\left(s_\tau^* \geq 1+\epsilon\right) = 0.$$

Proof. Fix $\epsilon_o > 0$. Define the event

$$\mathcal{B}_a = \left\{\hat{\sigma}_{n^*}^{\frac{1}{2}}\left(\hat{q}_{n^*} + s_{n^*}^{-\frac{1}{4}}\right)^{\frac{1}{4}} < \left(1+\frac{\epsilon}{2}\right)^{\frac{\delta}{4}}\sigma^{\frac{1}{2}}q^{\frac{1}{4}}, \mathcal{A}_a\right\}.$$

By Lemmas 4, 5 and 6,

$$\lim_{a\to\infty}\sup_{\Theta}\sup_{\epsilon>\epsilon_o}\epsilon^d\mathbb{P}\left(\mathcal{B}_a^c\right) = 0.$$

Choose $M = M(\epsilon_o, \delta) > 0$ such that for all $\epsilon > \epsilon_o$,

$$(1+\epsilon)^{\frac{\delta}{4}}\left[\left(1+\frac{\epsilon}{2}\right)^{\frac{\delta}{4}} - (1+\epsilon)^{\frac{\delta}{4}}\right] \geq -M\epsilon^{\frac{\delta}{2}}.$$

On the set $\mathcal{B}_a$, $(a/\beta)^4\sigma^2 q(1+\epsilon)^\delta \leq s_{n^*} < (a/\beta)^4\sigma^2 q(1+\epsilon)$ and

$$a s_{n^*}^{\frac{3}{4}}\hat{\sigma}_{n^*}^{\frac{1}{2}}\left(\hat{q}_{n^*} + s_{n^*}^{-\frac{1}{4}}\right)^{\frac{1}{4}} - s_{n^*}\beta \leq s_{n^*}^{\frac{3}{4}}\left[a\sigma^{\frac{1}{2}}q^{\frac{1}{4}}\left(1+\frac{\epsilon}{2}\right)^{\frac{\delta}{4}} - s_{n^*}^{\frac{1}{4}}\beta\right] < 0.$$

This quantity is maximized by replacing s_{n^*} with it's lower bound. Hence

$$\begin{aligned} a s_{n^*}^{\frac{3}{4}}\hat{\sigma}_{n^*}^{\frac{1}{2}}\left(\hat{q}_{n^*} + s_{n^*}^{-\frac{1}{4}}\right)^{\frac{1}{4}} - s_{n^*}\beta &\leq \frac{a^4}{\beta^3}\sigma^2 q(1+\epsilon)^{\frac{3\delta}{4}}\left[\left(1+\frac{\epsilon}{2}\right)^{\frac{\delta}{4}} - (1+\epsilon)^{\frac{\delta}{4}}\right] \\ &\leq -a^2 m(n^o)^{\frac{1}{2}} M\epsilon^{\frac{\delta}{2}} \end{aligned}$$

where $m > 0$ is such that $m \leq \inf_\Theta \sigma\sqrt{q}/(\beta\sqrt{2z_o})$ and n^o was defined in (12). By (13) and the first assertion of Lemma 2.1,

$$\begin{aligned} &\sup_\Theta \mathbb{P}\left(s_\tau \geq \frac{a^4}{\beta^4}\sigma^2 q(1+\epsilon), \mathcal{B}_a\right) \\ &\leq \sup_\Theta \mathbb{P}\left(\sum_{i=1}^{n^*}(x_i - \overline{x}_{n^*})e_i + s_{n^*}\beta < a s_{n^*}^{\frac{3}{4}}\hat{\sigma}_{n^*}^{\frac{1}{2}}\left(\hat{q}_{n^*} + s_{n^*}^{-\frac{1}{4}}\right)^{\frac{1}{4}}, \mathcal{B}_a\right) \\ &\leq \sup_\Theta \mathbb{P}\left(\sup_{1\leq n\leq n^o}(n^o)^{-\frac{1}{2}}\left|\sum_{i=1}^{n}(x_i - \overline{x}_n)e_i\right| > a^2 m M\epsilon^{\frac{\delta}{2}}\right) \\ &= \epsilon^{-p\delta}O\left(a^{-3p}\right). \end{aligned}$$

□

Consider the second assertion of Theorem 2. For $\epsilon \in (0,1)$, define

$$n_* = \sup\left\{n | s_n \leq (a/\beta)^4\sigma^2 q(1-\epsilon)\right\}.$$

Then

$$\{s_\tau^* \leq 1-\epsilon\} = \left\{s^o \leq s_\tau \leq s_{\lfloor a\rfloor}\right\}\bigcup\left\{s_{\lfloor a\rfloor} < s_\tau \leq s_{n_*}\right\}. \tag{16}$$

Lemma 8

$$\sup_\Theta \mathbb{P}\left(s^o \leq s_\tau \leq s_{\lfloor a\rfloor}\right) = O(a^{-p}).$$

Proof. Let $B = (2\beta^*)^{-\frac{8}{3}}$ and define the set

$$\mathcal{D}_a = \left\{s_{\lfloor a\rfloor} \leq a^2 B\right\}.$$

Then by (14),

$$\sup_\Theta \mathbb{P}\left(\mathcal{D}_a^c\right) \leq \sup_\Theta \left(a^2 B\right)^{-k}\mathbb{E}s_{\lfloor a\rfloor}^k = O\left(a^{-k}\right). \tag{17}$$

On the set $\mathcal{D}_a \cap \{s_n \geq s^o\}$ with $0 < \beta < \beta^* a^{\frac{1}{4}}$, and a sufficiently large,

$$\sup_{3\leq n\leq a} s_n\beta - a s_n^{\frac{5}{8}} \leq \sup_{s^o\leq s_n\leq a^2 B} s_n^{\frac{5}{8}}\left(s_n^{\frac{3}{8}} a^{\frac{1}{4}}\beta^* - a\right) \leq (s^o)^{\frac{5}{8}}\left(\frac{a}{2} - a\right) = -(s^o)^{\frac{5}{8}}\frac{a}{2}.$$

Since

$$a s_n^{\frac{3}{4}}\hat{\sigma}_n^{\frac{1}{2}}\left(\hat{q}_n + s_n^{-\frac{1}{4}}\right)^{\frac{1}{4}} \geq a s_n^{\frac{3}{4}}\left(s_n^{-\frac{1}{4}} s_n^{-\frac{1}{4}}\right)^{\frac{1}{4}} = a s_n^{\frac{5}{8}},$$

$$\begin{aligned}
\sup_\Theta \mathbb{P}\left(s_3^o \leq s_\tau \leq s_{\lfloor a\rfloor}, \mathcal{D}_a\right) &\leq \sup_\Theta \mathbb{P}\left(\sup_{3\leq n\leq a}\sum_{i=1}^n (x_i - \overline{x}_n)e_i + s_n\beta - a s_n^{\frac{5}{8}} \geq 0, s_n \geq s^o, \mathcal{D}_a\right) \\
&\leq \sup_\Theta \mathbb{P}\left(\sup_{3\leq n\leq a}\sum_{i=1}^n (x_i - \overline{x}_n)e_i \geq (s^o)^{\frac{5}{8}}\frac{a}{2}\right) \\
&= O\left(a^{-p}\right).
\end{aligned}$$

□

Two preliminaries are needed before analyzing the second set in (16). The first is to approximate the sum, $\sigma^{-1}\sum_{i=1}^n (x_i - \overline{x}_n)e_i$, with a Brownian motion. This requires the martingale, $\sigma^{-1}\sum_{i=1}^n (x_i - x_*)e_i$, the sum

$$r_n = \sigma^{-2}\sum_{i=1}^n \mathbb{E}\left((x_i - x_*)^2 e_i^2 | \mathcal{X}_{i-1}\right) = \sum_{i=1}^n (x_i - x_*)^2 = s_n + n(\overline{x}_n - x_*)^2 \tag{18}$$

and the following strong approximation result for martingales, adapted from Theorem 4.4, Strassen (1965).

Theorem 3 *Let $\Theta \subseteq \mathbb{R}^k$ for k a positive integer, $\theta \in \Theta$ and $\Theta_a \subseteq \Theta$ such that $\Theta_{a'} \subseteq \Theta_a$ for all $a' \leq a$. Assume e_i satisfy (E), $d_i = d_i(\theta)$ are such that d_i is independent of $\{e_j, j \geq i\}$ and*

$$\sup_\Theta \sum_{i=1}^a \mathbb{E}|d_i|^{2k} = O(a).$$

Then, without loss of generality, there exist Brownian motions $W(t) = W_\theta(t)$ such that for $\gamma > \frac{1}{4}$, $\frac{1}{4} < \gamma' < \gamma$, $\gamma' \leq (6k + p - 2)/4p$ and $\epsilon > 0$,

$$\sup_\Theta \epsilon^{2p}\mathbb{P}\left(\sup_{n>a} n^{-\gamma}\left|\sigma^{-1}\sum_{i=1}^n d_i e_i - W\left(\sum_{i=1}^n d_i^2\right)\right| > \epsilon\right) = O\left(a^{-(2\gamma' - \frac{1}{2})p}\right).$$

Proof. See Theorem 3, Coleman (1994). □

Here as in Strassen, the phrase, without loss of generality, means that there exist a probability space with a Brownian motion and random variables equal in distribution to the original random variables such that the relation is satisfied.

Lemma 9

$$\sup_\Theta \mathbb{E}\sup_{3\leq n\leq a}\left|\sum_{i=1}^n (x_i - x_*)e_i\right|^{2p} = O(a^p).$$

For $\phi > \frac{1}{2}$ and $\epsilon > 0$,

$$\sup_{\Theta} \epsilon^{2p} \mathbb{P}\left(\sup_{n>a} n^{-\phi} \left| \sum_{i=1}^{n} (x_i - x_*) e_i \right| > \epsilon \right) = O\left(a^{-(\phi - \frac{1}{2})2p} \right).$$

There exist Brownian motions $W(r) = W_\theta(r)$ such that for $\epsilon > 0$,

$$\sup_{\Theta} \mathbb{P}\left(\sup_{n>a} s_n^{-\frac{3}{4}} \left| \sum_{i=1}^{n} (x_i - \overline{x}_n) e_i - W(r_n) \right| > \epsilon \right) = O\left(a^{-\frac{p}{2}} \right).$$

Proof. The sums $\sum_{i=1}^{n} (x_i - x_*) e_i$ with the filtration $\mathcal{X}_n$ is a martingale and

$$\sup_{\Theta} \sum_{i=1}^{a} \mathbb{E} \left| (x_i - x_*) e_i \right|^{2p} \leq 2^{2p} \mathbb{E} |e_i|^{2p} \sup_{\Theta} \sum_{i=1}^{\lfloor a \rfloor} \left(\mathbb{E} |x_i|^{2p} + M_x^{2p} \right) = O(a).$$

The first and second assertions follow from Lemma 1. Since

$$\begin{aligned} &\sup_{n>a} s_n^{-\frac{3}{4}} \left| \sum_{i=1}^{n} (x_i - \overline{x}_n) e_i - W(r_n) \right| \\ &\leq \sup_{n>a} \left(n^{\frac{3}{4}} s_n^{-\frac{3}{4}} \right) \left(|\overline{x}_n - x_*| n^{-\frac{3}{4}} \left| \sum_{i=1}^{n} e_i \right| + n^{-\frac{3}{4}} \left| \sum_{i=1}^{n} (x_i - x_*) e_i - W(r_n) \right| \right), \end{aligned}$$

the third assertion follows by (P5), (P7), Lemma 1 and Theorem 3. □

The second preliminary result is the following lemma.

Lemma 10 *Let $W(r)$ be a standard Brownian motion, $c = c(a, \alpha, \beta) > 0$, such that* $\inf_{\Theta} ac \to \infty$ *as $a \to \infty$,*

$$a' = \frac{ac}{s^o} \left(1 - \frac{1}{\sqrt{acs^o}} \right) \text{ and } \tau_W = \inf \left\{ r | r \geq s^o \text{ and } W(r) + r\mu \geq acr^{\frac{3}{4}} \right\}.$$

Then for $ac > e^4$ and $0 < \mu \leq a'$,

$$\mathbb{P}\left(\tau_W \leq \left(\frac{ac}{\mu} - \sqrt{\frac{1}{\mu}} \right)^4 \right) \leq 11 \left(1 - \Phi\left(\sqrt{acs^o} \right) \right) + 4ac\phi\left(\sqrt{ac} - 1 \right)$$

where Φ and ϕ are the distribution and density functions of a $N(0,1)$ random variable.

After rescaling for c this lemma is the second result in Proposition 2.3 of Keener and Woodroofe(1992).

Lemma 11 *For $\epsilon \in (0,1)$,*

$$\sup_{\Theta} \mathbb{P}\left(s_{\lfloor a \rfloor} < s_\tau \leq s_{n_*} \right) = O(a^{-\frac{p}{2}}).$$

Proof. Fix $\epsilon \in (0,1)$. Define the set

$$\mathcal{E}_a = \left\{ \inf_{n>a} \frac{s_n}{r_n} \geq \left(1-\frac{\epsilon}{4}\right)^{\frac{1}{12}} \right\}$$

then

$$\mathcal{E}_a^c = \left\{ \sup_{n>a} \frac{r_n}{s_n} > \left(1-\frac{\epsilon}{4}\right)^{-\frac{1}{12}} \right\} \subseteq \left\{ \sup_{n>a} \frac{n}{s_n} > z_o \right\} \bigcup \left\{ \sup_{n>a} (\overline{x}_n - x_*)^2 > \frac{\epsilon}{48 z_o} \right\}.$$

By (P5) and (P7),

$$\lim_{a\to\infty} \sup_{\Theta} \mathbb{P}\left(\mathcal{E}_a^c\right) = 0. \tag{19}$$

Define the set

$$\mathcal{F}_a = \left\{ \inf_{n>a} \left[a s_n^{\frac{3}{4}} \sigma^{-1} \hat{\sigma}_n^{\frac{1}{2}} \hat{q}_n^{\frac{1}{4}} - \Delta_n \right] - \left[\left(1-\frac{\epsilon}{2}\right)^{\frac{1}{4}} a r_n^{\frac{3}{4}} \sigma^{-\frac{1}{2}} q^{\frac{1}{4}} \right] \geq 0, \mathcal{E}_a \right\}.$$

where

$$\Delta_n = \left| \sigma^{-1} \sum_{i=1}^{n} (x_i - \overline{x}_n) e_i - W(r_n) \right|.$$

For a sufficiently large and all (α,β) in Θ,

$$a^{-1}\epsilon < \left[\left(1-\frac{\epsilon}{4}\right)^{\frac{1}{4}} - \left(1-\frac{\epsilon}{2}\right)^{\frac{1}{4}} \right] \sigma^{\frac{1}{2}} q^{\frac{1}{4}}.$$

Then, multiplying by $a^{-1} r_n^{-\frac{3}{4}} \sigma$,

$$\begin{aligned}
\mathcal{F}_a^c &= \left\{ \inf_{n>a} \left(\frac{s_n}{r_n}\right)^{\frac{3}{4}} \hat{\sigma}_n^{\frac{1}{2}} \hat{q}_n^{\frac{1}{4}} - \left(1-\frac{\epsilon}{2}\right)^{\frac{1}{4}} \sigma^{\frac{1}{2}} q^{\frac{1}{4}} - a^{-1}\sigma r_n^{-\frac{3}{4}} \Delta_n < 0, \mathcal{E}_a \right\} \bigcup \mathcal{E}_a^c \\
&\subseteq \left\{ \inf_{n>a} \left[\left(1-\frac{\epsilon}{4}\right)^{\frac{1}{16}} \hat{\sigma}_n^{\frac{1}{2}} \hat{q}_n^{\frac{1}{4}} - \left(1-\frac{\epsilon}{2}\right)^{\frac{1}{4}} \sigma^{\frac{1}{2}} q^{\frac{1}{4}} - \frac{\epsilon}{a} \right] + \frac{1}{a}\left[\epsilon - \sigma r_n^{-\frac{3}{4}} \Delta_n \right] < 0 \right\} \bigcup \mathcal{E}_a^c \\
&\subseteq \left\{ \inf_{n>a} \hat{\sigma}_n^{\frac{1}{2}} \hat{q}_n^{\frac{1}{4}} < \left(1-\frac{\epsilon}{2}\right)^{\frac{3}{16}} \sigma^{\frac{1}{2}} q^{\frac{1}{4}} \right\} \bigcup \left\{ \sup_{n>a} s_n^{-\frac{3}{4}} \Delta_n > \frac{\epsilon}{\sigma} \right\} \bigcup \mathcal{E}_a^c.
\end{aligned}$$

By Lemmas 5, 6 and 9 and (19)

$$\lim_{a\to\infty} \sup_{\Theta} \mathbb{P}\left(\mathcal{F}_a^c\right) = 0. \tag{20}$$

On the set $\mathcal{F}_a \bigcap \{n^* > a\}$ and for a sufficiently large, define R as

$$\begin{aligned}
r_{n_*} &\leq s_{n_*} \left(1-\frac{\epsilon}{4}\right)^{-\frac{1}{12}} \leq (a/\beta)^4 \sigma^2 q (1-\epsilon) \left(1-\frac{\epsilon}{4}\right)^{-\frac{1}{12}} \\
&\leq (a/\beta)^4 \sigma^2 q \left(1-\frac{\epsilon}{2}\right) \leq \left(\frac{a\sqrt{\sigma} q^{\frac{1}{4}} (1-\frac{\epsilon}{2})^{\frac{1}{4}}}{\beta} - \sqrt{\frac{\sigma}{\beta}} \right)^4 = R.
\end{aligned}$$

Then

$$
\begin{aligned}
&\left\{s_{\lfloor a\rfloor} < s_\tau \le s_{n_*}, \mathcal{F}_a\right\} \\
\subseteq\ & \left\{\sum_{i=1}^{n}(x_i - \overline{x}_n)e_i + s_n\beta \ge a s_n^{\frac{3}{4}}\hat{\sigma}_n^{\frac{1}{2}}(\hat{q}_n + s_n^{-\frac{1}{4}})^{\frac{1}{4}}, \text{for some } n \in (a, n_*], \mathcal{F}_a\right\} \\
\subseteq\ & \left\{W(r_n) + r_n\frac{\beta}{\sigma} \ge \frac{a}{\sigma}s_n^{\frac{3}{4}}\hat{\sigma}_n^{\frac{1}{2}}\hat{q}_n^{\frac{1}{4}} - \Delta_n \text{ for some } n \in (a, n_*], \mathcal{F}_a\right\} \\
\subseteq\ & \left\{W(r_n) + r_n\frac{\beta}{\sigma} \ge a r_n^{\frac{3}{4}}\sigma^{-\frac{1}{2}}q^{\frac{1}{4}}\left(1 - \frac{\epsilon}{2}\right)^{\frac{1}{4}} \text{ for some } n \in (a, n_*], \mathcal{F}_a\right\} \\
\subseteq\ & \left\{W(r) + r\frac{\beta}{\sigma} \ge a r^{\frac{3}{4}}\sigma^{-\frac{1}{2}}q^{\frac{1}{4}}\left(1 - \frac{\epsilon}{2}\right)^{\frac{1}{4}} \text{ for some } r \in [s^o, R]\right\} \\
\subseteq\ & \{\tau_W \le R\} \qquad (21)
\end{aligned}
$$

where τ_W is the stopping time in Lemma 10 with

$$c = \sigma^{-\frac{1}{2}}q^{\frac{1}{4}}\left(1 - \frac{\epsilon}{2}\right)^{\frac{1}{4}} \text{ and } \mu = \frac{\beta}{\sigma} < \frac{\beta^*}{\sigma}a^{\frac{1}{4}}.$$

Hence by Lemma 10,

$$\lim_{a\to 0}\sup_{0<\beta<\beta_a^*}\mathbb{P}\left(s_{\lfloor a\rfloor} < s_\tau \le s_{n_*}, \mathcal{F}_a\right) \le \lim_{a\to 0}\sup_{0<\beta<\beta_a^*}\mathbb{P}(\tau_W \le R) = 0 \qquad (22)$$

The result follows by (19) and (22). □

Proof of Theorem 2. Lemmas 7, 8 and 11 imply the first two assertions of Theorem 2. For the final assertion of Theorem 2, let $d' < d$. Then for $\epsilon \in (0,1)$,

$$
\begin{aligned}
\sup_\Theta \mathbb{E}\,|s_\tau^* - 1|^{d'} &= \sup_\Theta\Big[\mathbb{E}\left(|s_\tau^* - 1|^{d'}; s_\tau^* \le 1-\epsilon\right) + \mathbb{E}\left(|s_\tau^* - 1|^{d'}; |s_\tau^* - 1| \le \epsilon\right) \\
&\quad + \mathbb{E}\left(|s_\tau^* - 1|^{d'}; 1+\epsilon \le s_\tau^* \le 2\right) + \mathbb{E}\left(|s_\tau^* - 1|^{d'}; s_\tau^* \ge 2\right)\Big] \\
&\le \sup_\Theta\left[\mathbb{P}(s_\tau^* < 1-\epsilon) + \epsilon^{d'} + \mathbb{P}(s_\tau^* \ge 1+\epsilon) + \sum_{n=1}^{\infty}\mathbb{P}\left((s_\tau^* - 1)^{d'} \ge n\right)\right] \\
&\le \epsilon + \epsilon^{d'} + \epsilon + o(1)\sum_{n=1}^{\infty} n^{-\frac{d}{d'}} \\
&\le 4\epsilon
\end{aligned}
$$

for a sufficiently large. Since ϵ was arbitrary the result follows.

3 Proof of the Main Result, Theorem 1

In this section let $n^o = \lfloor 4z_o(a/\beta)^4\sigma^2 q\rfloor$. It was previously defined in (12). For $\epsilon \in (0,1)$, define the set

$$\mathcal{G}_a = \mathcal{G}_a(\epsilon) = \{|s_\tau^* - 1| < \epsilon \text{ and } a < \tau \le n^o\}.$$

Lemma 12

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}(\mathcal{G}_a^c)=0.$$

Proof. Consider

$$\mathcal{G}_a^c=\{|s_\tau^*-1|\geq\epsilon\}\bigcup\{3\leq\tau\leq a\}\bigcup\{\tau>n^o,s_\tau^*<1+\epsilon\}.$$

By Theorem 2, the probability of the first set tends to zero, uniformly over Θ. By Lemma 8, the probability of the second set is

$$\sup_{\Theta}\mathbb{P}(3\leq\tau\leq a)=\sup_{\Theta}\mathbb{P}\left(s^o\leq s_\tau\leq s_{\lfloor a\rfloor}\right) \quad=\quad O(a^{-p}). \tag{23}$$

Since

$$\{n^o<\tau,s_\tau^*<1+\epsilon\}\subseteq\{s_{n^o}\leq s_\tau,s_\tau^*<1+\epsilon\}\subseteq\{s_{n^o}\leq(a/\beta)^4\sigma^2q(1+\epsilon)\}\subseteq\{n_os_{n_o}^{-1}\geq z_o\}$$

By (P5), the probability of the third set is

$$\sup_{\Theta}\mathbb{P}(n^o<\tau,s_\tau^*<1+\epsilon)\quad\leq\quad\sup_{\Theta}\mathbb{P}\left(\frac{n^o}{s_{n^o}}\geq z_o\right)=\sup_{\Theta}O\left((n^o)^{-\frac{k}{2}}\right)=O\left(a^{-\frac{7}{4}k}\right).$$

□

Let

$$N_a=\lfloor\frac{a^4\sigma^2q}{\beta^4(x^*-x_*^2)}\rfloor \text{ and } N_a'=\frac{a^4\sigma^2q}{\beta^4(x^*-x_*^2)}.$$

By Lemma 3, $\inf_\Theta N_a\to\infty$ as $a\to\infty$.

Lemma 13 *For* $\delta\in(0,1)$,

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{\tau}{N_a}-1\right|>\delta\right)=0,$$

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{\sum_{i=1}^{\tau}(x_i-x_*)e_i}{\sigma\sqrt{\tau}}-\frac{\sum_{i=1}^{N_a}(x_i-x_*)e_i}{\sigma\sqrt{N_a}}\right|>\delta\right)=0$$

and

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{\sum_{i=1}^{\tau}e_i}{\sigma\sqrt{\tau}}-\frac{\sum_{i=1}^{N_a}e_i}{\sigma\sqrt{N_a}}\right|>\delta\right)=0.$$

Proof. Given $\delta\in(0,1)$, choose $\epsilon\in(0,1)$ such that on the set $\mathcal{G}_a$,

$$\left|\frac{1}{s_\tau^*}-1\right|<\delta.$$

On the set $\mathcal{G}_a$,

$$\begin{aligned}
&(x^* - x_*^2)\left[\left|\frac{N_a'}{\tau} - 1\right| - 2\delta\right] \\
&= \left|\frac{1}{s_\tau^*}\frac{s_\tau}{\tau} - (x^* - x_*^2)\right| - 2\delta(x^* - x_*^2) \\
&\le \left|\frac{1}{s_\tau^*} - 1\right|\frac{s_\tau}{\tau} - \delta(x^* - x_*^2) + \left|\frac{s_\tau}{\tau} - (x^* - x_*^2)\right| - \delta(x^* - x_*^2) \\
&\le (1+\delta)\left|\frac{s_\tau}{\tau} - (x^* - x_*^2)\right| - \delta(x^* - x_*^2) \\
&\le (1+\delta)\sup_{n>a}\left|\frac{s_n}{n} - (x^* - x_*^2)\right| - \delta m_x
\end{aligned}$$

then by (P7) and (P8)

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{N_a'}{\tau} - 1\right| > 2\delta, \mathcal{G}_a\right) \le \lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\sup_{n>a}\left|\frac{s_n}{n} - (x^* - x_*^2)\right| > \frac{\delta m_x}{1+\delta}\right) = 0.$$

The first assertion follows easily. Anscombe's Theorem, Anscombe (1952), applies to the second and third assertions, see Section 1.3 of Woodroofe (1982). The first condition of Anscombe's Theorem is the first assertion of this theorem, the second is uniform continuity in probability, (U.C.I.P.). For the second assertion this is, for all $\delta > 0$ there exists $\lambda > 0$ such that

$$\sup_{\Theta}\mathbb{P}\left(\max_{1\le k\le \lambda N_a}\left|\frac{\sum_{i=1}^{N_a}(x_i - x_*)e_i}{\sigma\sqrt{N_a}} - \frac{\sum_{i=1}^{N_a+k}(x_i - x_*)e_i}{\sigma\sqrt{N_a+k}}\right| > \delta\right) < \delta.$$

Now,

$$\begin{aligned}
&\left|\frac{\sum_{i=1}^{N_a+k}(x_i - x_*)e_i}{\sigma\sqrt{N_a+k}} - \frac{\sum_{i=1}^{N_a}(x_i - x_*)e_i}{\sigma\sqrt{N_a}}\right| \\
&\le \left|\frac{\sum_{i=1}^{N_a+k}(x_i - x_*)e_i}{\sigma\sqrt{N_a+k}} - \frac{\sum_{i=1}^{N_a}(x_i - x_*)e_i}{\sigma\sqrt{N_a+k}}\right| + \left|\frac{\sum_{i=1}^{N_a}(x_i - x_*)e_i}{\sigma\sqrt{N_a+k}} - \frac{\sum_{i=1}^{N_a}(x_i - x_*)e_i}{\sigma\sqrt{N_a}}\right| \\
&\le \left|\frac{\sum_{i=N_a+1}^{N_a+k}(x_i - x_*)e_i}{\sigma\sqrt{N_a}}\right| + \left|\frac{\sqrt{N_a}}{\sqrt{N_a+k}} - 1\right|\left|\frac{\sum_{i=1}^{N_a}(x_i - x_*)e_i}{\sigma\sqrt{N_a}}\right|.
\end{aligned}$$

Set $\lambda = \delta^4$. By Lemma 9,

$$\sup_{\Theta}\mathbb{P}\left(\max_{1\le k\le \delta^4 N_a}\frac{1}{\sigma\sqrt{N_a}}\left|\sum_{i=N_a+1}^{N_a+k}(x_i - x_*)e_i\right| > \frac{\delta}{2}\right) \le \sup_{\Theta}\left(\frac{2}{\sigma\delta\sqrt{N_a}}\right)^{2p}\left(\delta^4 N_a\right)^p = \delta^{2p}O(1).$$

Since

$$\max_{1\le k\le \delta^4 N_a}\left|\frac{\sqrt{N_a}}{\sqrt{N_a+k}} - 1\right| = 1 - (1+\delta^4)^{-\frac{1}{2}} \le \frac{1}{2}\delta^4$$

then by Lemma 9 the second probability is

$$\sup_{\Theta} \mathbb{P}\left(\max_{1\le k\le \delta^4 N_a}\left|\frac{\sqrt{N_a}}{\sqrt{N_a+k}}-1\right|\left|\frac{\sum_{i=1}^{N_a}(x_i-x_*)e_i}{\sigma\sqrt{N_a}}\right|>\frac{\delta}{2}\right) \le \sup_{\Theta}\mathbb{P}\left(\left|\frac{\sum_{i=1}^{N_a}(x_i-x_*)e_i}{\sigma\sqrt{N_a}}\right|>\frac{1}{\delta^3}\right) = \delta^{10p}O(1).$$

This proves the second assertion of the lemma. The proof of the third assertion is similar.

Lemma 14 *For $\delta > 0$,*

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{a^4\sigma^2 q}{\beta^3\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)y_i}-1\right|>\delta\right)=0,$$

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{\alpha+\beta\overline{x}_\tau}{q^{\frac{1}{2}}}\sqrt{\frac{\tau}{s_\tau}}-\frac{\alpha+\beta x_*}{q^{\frac{1}{2}}}(x^*-x_*^2)^{-\frac{1}{2}}\right|>\delta\right)=0$$

and

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{\beta}{q^{\frac{1}{2}}}\sqrt{\frac{s_\tau}{\tau}}+\frac{(\overline{x}_\tau-x_*)(\alpha+\beta\overline{x}_\tau)}{q^{\frac{1}{2}}}\sqrt{\frac{\tau}{s_\tau}}-\frac{\beta}{q^{\frac{1}{2}}}(x^*-x_*^2)^{\frac{1}{2}}\right|>\delta\right)=0.$$

Proof. Given $\delta > 0$, choose $\epsilon < \delta/2$ in the set $\mathcal{G}_a$. Since $\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)y_i = \sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)e_i+\beta s_\tau$,

$$\begin{aligned}
&\sup_{\Theta}\mathbb{P}\left(\left|\frac{\beta^3\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)y_i}{a^4\sigma^2 q}-1\right|>\delta\right)\\
\le\ &\sup_{\Theta}\left[\mathbb{P}\left(\frac{\beta^3}{a^4\sigma^2 q}\left|\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)e_i\right|>\frac{\delta}{2},\mathcal{G}_a\right)+\mathbb{P}\left(|s_\tau^*-1|>\frac{\delta}{2},\mathcal{G}_a\right)+\mathbb{P}\left(\mathcal{G}_a^c\right)\right]\\
\le\ &\sup_{\Theta}\mathbb{P}\left(\sup_{1\le n\le n^o}\left|\sum_{i=1}^{n}(x_i-\overline{x}_n)e_i\right|>\frac{a^2\sigma\sqrt{q}\delta}{2\beta\sqrt{4z_o}}\sqrt{n^o}\right)+\frac{\delta}{2}\\
=\ &\delta
\end{aligned}$$

for a sufficiently large. The last two assertions follow easily from the following bounds. By Theorem 2 and Lemma 13,

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{\tau}{s_\tau}(x^*-x_*^2)-1\right|>\delta\right) = \lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|\frac{\tau}{N_a'}\frac{1}{s_\tau^*}-1\right|>\delta\right)=0.$$

and by (P7) and (23),

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(|\overline{x}_\tau-x_*|>\delta\right) = \lim_{a\to\infty}\sup_{\Theta}\left[\mathbb{P}\left(\sup_{n>a}|\overline{x}_n-x_*|>\delta\right)+\mathbb{P}\left(\tau\le a\right)\right]=0.$$

□

Lemma 15 *There exist Brownian motions, $W(z) = W_\theta(z)$, such that for $\delta > 0$,*

$$\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\left|a^2\left(\frac{\hat{\alpha}_\tau}{\hat{\beta}_\tau}-\frac{\alpha}{\beta}\right)-\frac{W(N_a)}{\sqrt{N_a}}\right|>\delta\right)=0.$$

Note that $W(N_a)/\sqrt{N_a} \sim N(0,1)$.

Proof. Note that,

$$
\begin{aligned}
a^2\left(\frac{\hat{\alpha}_\tau}{\hat{\beta}_\tau}-\frac{\alpha}{\beta}\right) &= a^2\left(\frac{\alpha+\beta\overline{x}_\tau+\overline{e}_\tau}{\hat{\beta}_\tau}-\overline{x}_\tau-\frac{\alpha}{\beta}\right)\\
&= \frac{a^2}{\hat{\beta}_\tau}\left[\overline{e}_\tau-\left(\frac{\alpha}{\beta}+\overline{x}_\tau\right)\left(\hat{\beta}_\tau-\beta\right)\right]\\
&= \frac{a^2 s_\tau}{\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)y_i}\left[\frac{1}{\tau}\sum_{i=1}^{\tau}e_i-\left(\frac{\alpha}{\beta}+\overline{x}_\tau\right)\frac{1}{s_\tau}\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)e_i\right]\\
&= \frac{a^4\sigma^2 q}{\beta^3\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)y_i}\frac{\beta^2 s_\tau^{\frac{1}{2}}}{a^2\sigma q^{\frac{1}{2}}}\\
&\quad \left[\frac{\beta}{q^{\frac{1}{2}}}\left(\frac{s_\tau}{\tau}\right)^{\frac{1}{2}}\frac{\sum_{i=1}^{\tau}e_i}{\sigma\sqrt{\tau}}-\frac{\alpha+\beta\overline{x}_\tau}{q^{\frac{1}{2}}}\left(\frac{\tau}{s_\tau}\right)^{\frac{1}{2}}\frac{\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)e_i}{\sigma\sqrt{\tau}}\right]\\
&= b_\tau\left[d_\tau\frac{\sum_{i=1}^{\tau}e_i}{\sigma\sqrt{\tau}}-c_\tau\frac{\sum_{i=1}^{\tau}(x_i-x_*)e_i}{\sigma\sqrt{\tau}}\right], \qquad (24)
\end{aligned}
$$

where

$$
b_\tau=\frac{a^4\sigma^2 q}{\beta^3\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)y_i}(s_\tau^*)^{\frac{1}{2}},\ c_\tau=\frac{\alpha+\beta\overline{x}_\tau}{q^{\frac{1}{2}}}\sqrt{\frac{\tau}{s_\tau}}
$$

and

$$
d_\tau=\frac{\beta}{q^{\frac{1}{2}}}\sqrt{\frac{s_\tau}{\tau}}+\frac{(\overline{x}_\tau-x_*)(\alpha+\beta\overline{x}_\tau)}{q^{\frac{1}{2}}}\sqrt{\frac{\tau}{s_\tau}}.
$$

Hence

$$
\begin{aligned}
&\left|a^2\left(\frac{\hat{\alpha}_\tau}{\hat{\beta}_\tau}-\frac{\alpha}{\beta}\right)-\frac{W(N_a)}{\sqrt{N_a}}\right|\\
\le\ & \left|b_\tau\left[d_\tau\frac{\sum_{i=1}^{\tau}e_i}{\sigma\sqrt{\tau}}-c_\tau\frac{\sum_{i=1}^{\tau}(x_i-x_*)e_i}{\sigma\sqrt{\tau}}\right]-\left[d_o\frac{\sum_{i=1}^{\tau}e_i}{\sigma\sqrt{\tau}}-c_o\frac{\sum_{i=1}^{\tau}(x_i-x_*)e_i}{\sigma\sqrt{\tau}}\right]\right|\\
&+\left|\left[d_o\frac{\sum_{i=1}^{\tau}e_i}{\sigma\sqrt{\tau}}-c_o\frac{\sum_{i=1}^{\tau}(x_i-x_*)e_i}{\sigma\sqrt{\tau}}\right]-\left[d_o\frac{\sum_{i=1}^{N_a}e_i}{\sigma\sqrt{N_a}}-c_o\frac{\sum_{i=1}^{N_a}(x_i-x_*)e_i}{\sigma\sqrt{N_a}}\right]\right|\\
&+\left|\left[d_o\frac{\sum_{i=1}^{N_a}e_i}{\sigma\sqrt{N_a}}-c_o\frac{\sum_{i=1}^{N_a}(x_i-x_*)e_i}{\sigma\sqrt{N_a}}\right]-\frac{W(z_{N_a})}{\sqrt{N_a}}\right|\\
&+\left|\frac{W(z_{N_a})}{\sqrt{N_a}}-\frac{W(N_a)}{\sqrt{N_a}}\right|, \qquad (25)
\end{aligned}
$$

where

$$
z_{N_a}=c_o^2 s_{N_a}+2c_o d_o N_a(\overline{x}_{N_a}-x_*)^2+d_o^2 N_a,
$$

$$
c_o=\frac{\alpha+\beta x_*}{q^{\frac{1}{2}}}(x^*-x_*)^{-\frac{1}{2}} \text{ and } d_o=\frac{\beta}{q^{\frac{1}{2}}}(x^*-x_*^2)^{\frac{1}{2}}.
$$

By (P6) and Lemma 2,

$$
|c_o|\le m_x^{-\frac{1}{2}} \text{ and } d_o\le 1. \qquad (26)
$$

It's sufficient to show the four terms in (25) converge in probability to zero uniformly in Θ. Consider the first term in (25). By Theorem 3,

$$\frac{\sum_{i=1}^{N_a}(x_i - x_*)e_i}{\sigma\sqrt{N_a}} \text{ and } \frac{\sum_{i=1}^{N_a} e_i}{\sigma\sqrt{N_a}}$$

are stochastically bounded, uniformly over Θ. Hence by Lemma 13,

$$\frac{\sum_{i=1}^{\tau}(x_i - x_*)e_i}{\sigma\sqrt{\tau}} \text{ and } \frac{\sum_{i=1}^{\tau} e_i}{\sigma\sqrt{\tau}}$$

are stochastically bounded, uniformly over Θ. By Lemma 14 and Theorem 2, $b_\tau \to_p 1$, $c_\tau \to_p c_o$ and $d_\tau \to_p d_o$, uniformly over Θ. Hence the first probability tends to zero uniformly over Θ.

By Lemma 13 and (26) the second term in (25) converges to zero, uniformly over Θ.

Consider the third term in (25). The sums $\sum_{i=1}^{n}(d_o - c_o(x_i - x_*))\sigma^{-1}e_i$ with the filtration $\mathcal{X}_n$, is a martingale. By (P2), (P5) and (26)

$$\sup_\Theta \sum_{i=1}^{a} \mathbb{E}|(d_o - c_o(x_i - x_*))|^{2p} = O(a).$$

By Theorem 3, with $\gamma = \frac{1}{2}$ and $\gamma' = \frac{3}{8}$,

$$\begin{aligned}
&\sup_\Theta \mathbb{P}\left(\left|\left[d_o\frac{\sum_{i=1}^{N_a} e_i}{\sigma\sqrt{N_a}} + c_o\frac{\sum_{i=1}^{N_a}(x_i - x_*)e_i}{\sigma\sqrt{N_a}}\right] - \frac{W(z_{N_a})}{\sqrt{N_a}}\right| > \delta\right) \\
&\le \sup_\Theta \mathbb{P}\left(\sup_{n>a} n^{-\frac{1}{2}}\left|\sigma^{-1}\sum_{i=1}^{n}[d_o - c_o(x_i - x_*)]\,e_i - W(z_n)\right| > \delta\right) \\
&= O\left(a^{-\frac{p}{4}}\right). \qquad (27)
\end{aligned}$$

Anscombe's Theorem applies to the fourth probability in (25). Two assumptions must be verified. By (P7), (P8) and (26)

$$\begin{aligned}
&\lim_{a\to\infty} \sup_\Theta \mathbb{P}\left(\left|\frac{z_{N_a}}{N_a} - 1\right| > \delta^4\right) \\
&\le \lim_{a\to\infty} \sup_\Theta \left[\mathbb{P}\left(c_o^2\left|\frac{s_{N_a}}{N_a} - (x^* - x_*^2)\right| > \frac{\delta^4}{3}\right)\right. \\
&\quad \left. + \mathbb{P}\left(\left|d_o^2 - c_o^2(x^* - x_*^2) - 1\right| > \frac{\delta^4}{3}\right) + \mathbb{P}\left(2|c_o|d_o\,|\overline{x}_{N_a} - x_*| > \frac{\delta^4}{3}\right)\right] \\
&= 0.
\end{aligned}$$

Here (P7) and (P8) apply to the first probability, the second probability is zero and (P8) applies to the third probability. The second assumption is U.C.I.P., by Levy's Inequality

$$\begin{aligned}
&\sup_\Theta \mathbb{P}\left(\sup_{|z-N_a|<\delta^4 N_a}\left|\frac{W(z)}{\sqrt{N_a}} - \frac{W(N_a)}{\sqrt{N_a}}\right| > \delta\right) \\
&\le 4\sup_\Theta \mathbb{P}\left(\left|W((1-\delta^4)N_a) - W((1+\delta^4)N_a)\right| > \sqrt{N_a}\delta\right) \\
&= 8\delta^2.
\end{aligned}$$

□

Lemma 13 implies the first assertion of Theorem 1.

The proof of the second assertion of Theorem 1 requires the following lemma.

Lemma 16

$$\sup_{\Theta} \mathbb{E}\left|\frac{\beta}{\hat{\beta}_\tau} - 1\right|^{2p'} = O\left(a^{-p}\right), \ \sup_{\Theta} \mathbb{E}|\overline{x}_\tau - x_*|^{2p'} = O(1), \text{ and } \sup_{\Theta} \mathbb{E}\left|\overline{e}_\tau\right|^{p'} = O\left(a^{-\frac{3}{8}p'}\right)$$

Proof. Claim for $k' < k$,

$$\sup_{\Theta} \mathbb{E}\left(\left(\frac{\tau}{s_\tau}\right)^{\frac{k'}{4}}; \tau > a\right) \leq \sup_{\Theta} \mathbb{E}\left(\sup_{n>a}\left(\frac{n}{s'_n}\right)^{\frac{k'}{4}}; s_n \geq s^o\right) = O\left(1\right). \tag{28}$$

By $(P5)$ and for $z \geq z_o$ and $m \geq 1$

$$\sup_{\Theta} \mathbb{P}\left(\sup_{a^m < n \leq a^{m+1}} n s_n^{-1} \geq z, s_n \geq s^o\right) = O\left(a^{-\frac{km}{2}}\right).$$

If $z > a^{m+1}(s^o)^{-1}$ then the probability is zero. If $z \leq a^{m+1}(s^o)^{-1}$ then $(s^o)^{-\frac{k}{4}} z^{-\frac{k}{4}} \geq a^{-\frac{k}{4}(m+1)}$. Hence

$$\sup_{\Theta} \mathbb{P}\left(\sup_{a^m < n \leq a^{m+1}} n s_n^{-1} > z, s_n \geq s^o\right) = z^{-\frac{k}{4}} O\left(a^{-\frac{k}{4}(m-1)}\right).$$

Summing m over the positive integers yields

$$\sup_{\Theta} \mathbb{P}\left(\sup_{n>a} n s_n^{-1} > z, s_n \geq s^o\right) = z^{-\frac{k}{4}} O\left(\frac{1}{\log(a)}\right)$$

and (28) follows. Since $p' < 4kp/(5p + 4k)$ then

$$k > \frac{5p'p}{4(p-p')} = \frac{5p'}{4} + \frac{5(p')^2}{4(p-p')}.$$

Choose p'' such that $p' < p'' < p$ and

$$k > \frac{5p'p''}{4(p''-p')}.$$

At stopping,

$$\sum_{i=1}^{\tau}(x_i - \overline{x}_n)y_i \geq a s_\tau^{\frac{3}{4}} \sqrt{\hat{\sigma}_\tau}\left(\hat{q}_\tau + s_\tau^{-\frac{1}{4}}\right)^{\frac{1}{4}} \geq a s_\tau^{\frac{5}{8}}.$$

Therefore

$$\left|\frac{\beta}{\hat{\beta}_\tau} - 1\right| = \left|\frac{\beta s_\tau - \sum_{i=1}^{\tau}(x_i - \overline{x}_n)y_i}{\sum_{i=1}^{\tau}(x_i - \overline{x}_n)y_i}\right| \leq \left|\frac{\sum_{i=1}^{\tau}(x_i - \overline{x}_\tau)e_i}{a s_\tau^{\frac{5}{8}}}\right|.$$

By Hölder's inequality,

$$
\begin{aligned}
&\sup_{\Theta} \mathbb{E}\left|\frac{\beta}{\hat{\beta}_\tau}-1\right|^{2p'} \\
\le\;& \sup_{\Theta}\left\{\mathbb{E}\left(\left|\frac{\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)e_i}{a s_\tau^{\frac{5}{8}}}\right|^{2p'};3\le\tau\le a\right)+\mathbb{E}\left(\left|\frac{\sum_{i=1}^{\tau}(x_i-\overline{x}_\tau)e_i}{a s_\tau^{\frac{5}{8}}}\right|^{2p'};\tau>a\right)\right\} \\
\le\;& \sup_{\Theta} a^{-2p'}\left\{(s^o)^{-\frac{5}{4}p'}\mathbb{E}\left(\sup_{3\le n\le a}\left|\sum_{i=1}^{n}(x_i-\overline{x}_n)e_i\right|^{2p'};3\le\tau\le a\right)\right. \\
&\left.+\,\mathbb{E}\left(\sup_{n>a}\left|n^{-\frac{5}{8}}\sum_{i=1}^{n}(x_i-\overline{x}_n)e_i\right|^{2p'}\left(\frac{\tau}{s_\tau}\right)^{\frac{5}{4}p'};\tau>a\right)\right\} \\
\le\;& \sup_{\Theta} a^{-2p'}\left\{(s^o)^{-\frac{5}{4}p'}\left[\mathbb{E}\sup_{3\le n\le a}\left|\sum_{i=1}^{n}(x_i-\overline{x}_n)e_i\right|^{2p}\right]^{\frac{p'}{p}}\left[\mathbb{P}\left(3\le\tau\le a\right)\right]^{\frac{p-p'}{p}}\right. \\
&\left.+\left[\mathbb{E}\sup_{n>a}\left|n^{-\frac{5}{8}}\sum_{i=1}^{n}(x_i-\overline{x}_n)e_i\right|^{2p''}\right]^{\frac{p'}{p''}}\left[\mathbb{E}\left(\frac{\tau}{s_\tau}\right)^{\frac{5p'p''}{4(p''-p')}};\tau>a\right]^{\frac{p''-p'}{p''}}\right\} \\
\le\;& a^{-2p'}\left[\left[O\left(a^{p}\right)\right]^{\frac{p'}{p}}\left[O\left(a^{-p}\right)\right]^{\frac{p-p'}{p}}+o(1)O(1)\right] \\
=\;& O\left(a^{-\min\{p,2p'\}}\right).
\end{aligned}
$$

The first and second expectations are bounded in Lemma 2, the probability is calculated in (23), the third expectation is finite by (28).

Consider the second expectation. By (P2) and (P6),

$$
\sup_{\Theta}\mathbb{E}\left(\sup_{3\le n\le a}|\overline{x}_n-x_*|\right)^{2k}\le 2^{2k}\left(\sup_{\Theta}\mathbb{E}\sum_{n=1}^{a}|x_n|^{2k}+M_x^{2k}\right)=O(a).
$$

By (23) and (P7),

$$
\begin{aligned}
\sup_{\Theta}\mathbb{E}|\overline{x}_\tau-x_*|^{2p'} &\le \sup_{\Theta}\left[\mathbb{E}\left(\sup_{3\le n\le a}|\overline{x}_n-x_*|^{2p'};3\le\tau\le a\right)+\mathbb{E}\sup_{n>a}|\overline{x}_n-x_*|^{2p'}\right] \\
&\le \sup_{\Theta}\left[\mathbb{E}\left(\sup_{3\le n\le a}|\overline{x}_n-x_*|^{2k}\right)\right]^{\frac{p'}{k}}\left[\mathbb{P}\left(3\le\tau\le a\right)\right]^{\frac{k-p'}{k}}+O(1) \\
&= [O(a)]^{\frac{p'}{k}}\left[O(a^{-p})\right]^{\frac{k-p'}{k}}+O(1) \\
&= O(1).
\end{aligned}
$$

In a similar manner

$$
\sup_{\Theta}\mathbb{E}|\overline{e}_\tau|^{p'} \le \sup_{\Theta}\left[\mathbb{E}\left(\sup_{3\le n\le a}|\overline{e}_n|^{2p}\right)\right]^{\frac{p'}{2p}}\left[\mathbb{P}\left(3\le\tau\le a\right)\right]^{\frac{2p-p'}{2p}}+\left[\mathbb{E}\sup_{n>a}|\overline{e}_n|^{p}\right]^{\frac{p'}{p}}
$$

$$= O(1)\left[O(a^{-p})\right]^{\frac{2p-p'}{2p}} + O(a^{-\frac{3}{8}p'}) = O(a^{-\frac{3}{8}p'}).$$

The probability is bounded in (23) and the expectations are bounded by the Marcinkiewicz-Zygmund inequality, see Theorems 3 and 4 pages 369-370 of Chow and Teicher(1988). □

For the second assertion of Theorem 1, consider

$$\begin{aligned}
&\left|\frac{\beta}{\alpha}\left(\frac{\hat{\alpha}_T}{\hat{\beta}_T} - \frac{\alpha}{\beta}\right)\right|^{p'} \\
&= \left|\frac{\beta}{\alpha}\left(\frac{\alpha + \beta\bar{x}_T + \bar{e}_T}{\hat{\beta}_T} - \bar{x}_T - \frac{\alpha}{\beta}\right)\right|^{p'} \\
&= \left|\frac{\beta}{\hat{\beta}_T} + \frac{\beta}{\alpha}\frac{\beta}{\hat{\beta}_T}\bar{x}_T + \frac{\beta}{\hat{\beta}_T}\frac{\bar{e}_T}{\alpha} - \frac{\beta}{\alpha}\bar{x}_T - 1\right|^{p'} \\
&= \left|\left(1 + \frac{\beta}{\alpha}x_*\right)\left(\frac{\beta}{\hat{\beta}_T} - 1\right) + \frac{\beta}{\alpha}(\bar{x}_T - x_*)\left(\frac{\beta}{\hat{\beta}_T} - 1\right) + \frac{\bar{e}_T}{\alpha}\left(\frac{\beta}{\hat{\beta}_T} - 1\right) + \frac{\bar{e}_T}{\alpha}\right|^{p'} \\
&\leq 4^{p'}\left[\left|\left(1 + \frac{\beta}{\alpha}x_*\right)\left(\frac{\beta}{\hat{\beta}_T} - 1\right)\right|^{p'} + \left|\frac{\beta}{\alpha}(\bar{x}_T - x_*)\left(\frac{\beta}{\hat{\beta}_T} - 1\right)\right|^{p'} \left|\frac{\bar{e}_T}{\alpha}\left(\frac{\beta}{\hat{\beta}_T} - 1\right)\right|^{p'} + \left|\frac{\bar{e}_T}{\alpha}\right|^{p'}\right].
\end{aligned}$$

By Schwarz' inequality,

$$\begin{aligned}
&\sup_{\Theta}\mathbb{E}\left|\frac{\beta}{\alpha}\left(\frac{\hat{\alpha}_T}{\hat{\beta}_T} - \frac{\alpha}{\beta}\right)\right|^{p'} \\
&\leq 4^{p'}\sup_{\Theta}\left\{\left|1 + \frac{\beta}{\alpha}x_*\right|^{p'}\left[\mathbb{E}\left|\frac{\beta}{\hat{\beta}_T} - 1\right|^{2p'}\right]^{\frac{1}{2}}\left|\frac{\beta}{\alpha}\right|^{p'}\left[\mathbb{E}\left|\bar{x}_T - x_*\right|^{2p'}\right]^{\frac{1}{2}}\left[\mathbb{E}\left|\frac{\beta}{\hat{\beta}_T} - 1\right|^{2p'}\right]^{\frac{1}{2}}\right. \\
&\qquad\left. + \left|\frac{1}{\alpha}\right|^{p'}\left[\mathbb{E}\left|\bar{e}_T\right|^{2p'}\right]^{\frac{1}{2}}\left[\mathbb{E}\left|\frac{\beta}{\hat{\beta}_T} - 1\right|^{2p'}\right]^{\frac{1}{2}} + \left|\frac{1}{\alpha}\right|^{p'}\mathbb{E}\left|\bar{e}_T\right|^{p'}\right\} \\
&= O\left(a^{\frac{p'}{2}}\right)\left[O\left(a^{-\min\{p,2p'\}}\right)\right]^{\frac{1}{2}} + O\left(a^{\frac{p'}{2}}\right)O(1)\left[O\left(a^{-\min\{p,2p'\}}\right)\right]^{\frac{1}{2}} \\
&\qquad + O\left(a^{\frac{p'}{4}}\right)o(1)\left[O\left(a^{-\min\{p,2p'\}}\right)\right]^{\frac{1}{2}} + O\left(a^{\frac{p'}{4}}\right)O\left(a^{-\frac{3}{8}p'}\right) \\
&\to 0.
\end{aligned}$$

□

4 An Applicable Procedure

In this section, a class of procedures is shown to satisfy assumptions (P1) through (P8). Set constants M, m such that $0 < m < M < \infty$. Let

$$x_i = l_i + u_i v_i$$

where for $i = 1$, 2 and 3 the experimenter chooses constants l_i and u_i such that $|l_i| \leq M$ and $m \leq u_i \leq M$, for $i \geq 4$, $l_i = l(\hat{\alpha}_{i-1}, \hat{\beta}_{i-1})$ such that for $(a,b) \in \mathbb{R}$, $|l(a,b)| \leq M$ and for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\sup_{|a-a'|\leq\delta, |b-b'|\leq\delta} |l(a,b) - l(a',b')| \leq \epsilon, \tag{29}$$

$u_i = u(\hat{\sigma}^2_{i-1})$ such that for $s \geq 0$, $m \leq u(s) \leq M$ and for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\sup_{|s-s'|\leq\delta} |u(s) - u(s')| \leq \epsilon$$

and for $i \geq 1$, v_i are i.i.d. r.v.s such that $\{v_i\}$ is independent of $\{e_j\}$, $\mathbb{E}v_1 = 0$, $\mathbb{E}v_1^2 = 1$ and for some $k \geq p$, $\mathbb{E}|v_1|^k < \infty$.

Assumptions (P1) and (P2) are easily verified. Consider (P3), since

$$|\overline{x}_n \sum_{i=1}^n e_i| \leq M|\sum_{i=1}^n e_i| + M|\overline{v}_n||\sum_{i=1}^n e_i|$$

then by independence assumption (P3) is

$$\sup_\Theta \mathbb{E} \sup_{3\leq n\leq a} |\overline{x}_n \sum_{i=1}^n e_i|^{2p} \leq M\mathbb{E} \sup_{3\leq n\leq a} |\sum_{i=1}^n e_i|^{2p} + M\mathbb{E} \sup_{3\leq n\leq a} |\overline{v}_n|^{2p} \mathbb{E} \sup_{3\leq n\leq a} |\sum_{i=1}^n e_i|^{2p} = O(a^p).$$

The first and third expectations are $O(a^p)$ by the first assertion of Lemma 1. The second expectation is $O(1)$ by Marcinkiewicz-Zygmund inequality, see Theorem 3, pg. 369, Chow and Teicher (1988). For (P4), let ϕ' be such that $\frac{1}{2} < \phi' < \min\{\phi, \frac{3}{4}\}$. Then by Markov's inequality and independence

$$\begin{aligned}
&\sup_{\epsilon>\epsilon_o} \left(\frac{\epsilon}{2M}\right)^{2p} \mathbb{P}\left(\sup_{n>a} n^{-\phi}|\overline{x}_n \sum_{i=1}^n e_i| > \epsilon\right) \\
\leq\ & \sup_{\epsilon>\epsilon_o} \left(\frac{\epsilon}{2M}\right)^{2p} \mathbb{P}\left(\sup_{n>a} n^{-\phi}|\sum_{i=1}^n e_i| > \frac{\epsilon}{2M}\right) \\
&+ a^{-(1-\phi')2p}\mathbb{E}\left(\sup_{n>a}|n^{-\phi'}\sum_{i=1}^n v_i|^{2p}\right) a^{-(\phi-\phi')2p}\mathbb{E}\left(\sup_{n>a}|n^{-\phi'}\sum_{i=1}^n e_i|^{2p}\right) \\
=\ & O(a^{-(\phi-\frac{1}{2})2p})
\end{aligned}$$

The probability is bounded by the second assertion of Lemma 1 and the expectations are $O(1)$ by Marcinkiewicz-Zygmund inequality, see Theorem 4, pg. 370, Chow and Teicher (1988). For (P5), define $w_1 = s_1 = 0$ and $\mathcal{W}_1 = \sigma(\emptyset, \Omega)$, and for $n \geq 2$,

$$w_n = s_n - s_{n-1} = \frac{n-1}{n}(l_n + u_n v_n - \overline{x}_{n-1})^2 \text{ and } \mathcal{W}_n = \sigma(e_1, ..., e_{n-1}, v_2, ..., v_n).$$

Then w_n is measurable $\mathcal{W}_n$, v_n is independent of $\mathcal{W}_{n-1}$ and l_n, u_n and $\overline{x}_{n-1}$ are measurable $\mathcal{W}_{n-1}$. The sums,

$$\sum_{i=1}^n [w_i - \mathbb{E}(w_i|\mathcal{W}_{i-1})] = \sum_{i=1}^n \left[w_i - \frac{i-1}{i}\left[u_i^2 + (l_i - \overline{x}_{i-1})^2\right]\right]$$

with the filtration $\mathcal{W}_n$ is a martingale with

$$\sum_{i=1}^{n} \mathbb{E}\left|w_i - \mathbb{E}\left(w_i|\mathcal{W}_{i-1}\right)\right|^k = O(n).$$

For $n \geq 2$,

$$\frac{1}{n}\sum_{i=2}^{n} \mathbb{E}\left(w_i|\mathcal{W}_{i-1}\right) \;\geq\; \frac{1}{n}\sum_{i=2}^{n} \frac{i-1}{i}\left[m^2 + (l_i - \overline{x}_{i-1})^2\right] \geq \frac{m^2}{4}.$$

For $a \geq 1$ and $z_o > 4/m^2$,

$$\begin{aligned}
\sup_{\Theta} \mathbb{P}\left(\sup_{n>a} n s_n^{-1} \geq z_o\right) &\leq \sup_{\Theta} \mathbb{P}\left(\sup_{n>a} -\frac{1}{n}\sum_{i=2}^{n} w_i \geq -\frac{1}{z_o}\right) \\
&\leq \sup_{\Theta} \mathbb{P}\left(\sup_{n>a} \frac{1}{n}\left|\sum_{i=2}^{n}\left[w_i - \mathbb{E}\left(w_i|\mathcal{W}_{i-1}\right)\right]\right| \geq \frac{m^2}{4} - \frac{1}{z_o}\right) \\
&\leq O\left(a^{-\frac{k}{2}}\right).
\end{aligned}$$

Let

$$x_* = l(\alpha,\beta) \text{ and } x^* = u(\sigma^2) + x_*^2.$$

Then $\max(x_*, x^*) \leq u(\sigma^2) + M^2 + M$ and $x^* - x_*^2 \geq u(\sigma^2)$ and (P6) is satisfied. By (P5) and the second assertion of Lemma 2.1, which requires (P2) and (P4),

$$\begin{aligned}
&\lim_{a\to\infty} \sup_{\Theta} \mathbb{P}\left(\sup_{n>\delta a}\left|\hat{\beta}_n - \beta\right| \geq \delta\right) \\
&\leq \lim_{a\to\infty} \sup_{\Theta}\left[\mathbb{P}\left(\sup_{n>\delta a} n^{-1}\left|\sum_{i=1}^{n}(x_i - \overline{x}_n)e_i\right| \geq \frac{\delta}{z_o}\right) + \mathbb{P}\left(\sup_{n>\delta a}\frac{n}{s_n} \geq z_o\right)\right] \\
&= 0,
\end{aligned} \tag{30}$$

for $\delta > 0$. Similarly,

$$\lim_{a\to\infty} \sup_{\Theta} \mathbb{P}\left(\sup_{n>\delta a} |\hat{\alpha}_n - \alpha| \geq \delta\right) = 0, \tag{31}$$

for $\delta > 0$. Fix $\epsilon > 0$ and choose $\delta > 0$ such that $2\delta M < \epsilon/4$ and

$$\sup_{|a-\alpha|<\delta, |b-\beta|<\delta} |l(a,b) - l(\alpha,\beta)| < \frac{\epsilon}{4}.$$

Since $|l(a,b) - l(a',b')| < 2M$, by (30) and (31).

$$\begin{aligned}
&\lim_{a\to\infty} \sup_{\Theta} \mathbb{P}\left(\sup_{n>a} |x_* - \overline{l}_n| > \epsilon\right) \qquad (32) \\
&\leq \lim_{a\to\infty} \sup_{\Theta}\left[\mathbb{P}\left(\sup_{n>a} n^{-1}\sum_{i=1}^{n\delta} |l(\alpha,\beta) - l(\hat{\alpha}_{i-1}, \hat{\beta}_{i-1})| > \frac{\epsilon}{2}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}\left(\sup_{n>a} n^{-1}\sum_{i>n\delta}^{n} |l(\alpha,\beta) - l(\hat{\alpha}_{i-1},\hat{\beta}_{i-1})| > \frac{\epsilon}{2}\right)\Bigg] \\
\leq\ & \lim_{a\to\infty}\sup_{\Theta}\left[\mathbb{P}\left(\sup_{n>\delta a}\left|\hat{\beta}_n - \beta\right| \geq \delta\right) + \mathbb{P}\left(\sup_{n>\delta a}|\hat{\alpha}_n - \alpha| \geq \delta\right)\right] \\
=\ & 0.
\end{aligned}
\tag{33}
$$

Here $l(\hat{\alpha}_{i-1},\hat{\beta}_{i-1}) = l_i$ for $i = 1, 2$ and 3. By Lemma 1 and (33),

$$
\begin{aligned}
&\lim_{a\to\infty}\sup_{\Theta}\mathbb{P}\left(\sup_{n>a}|\overline{x}_n - x_*| > \epsilon\right) \\
\leq\ & \lim_{a\to\infty}\sup_{\Theta}\left[\mathbb{P}\left(\sup_{n>a}|\overline{u}_n - x_*| > \frac{\epsilon}{2}\right) + \mathbb{P}\left(\sup_{n>a}|\overline{v}_n| > \frac{\epsilon}{2M}\right)\right] \\
=\ & 0.
\end{aligned}
$$

Since $|l(a,b)| < M$, uniformity for $\epsilon \geq 0$ is immediate. Verification of (P8) is similar.

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