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"EQUILIBRIA WITH SOCIAL SECURITY"

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Abstract

We model pay-as-you-go (PAYG) social security systems as the outcome of majority voting within a standard OLG model with production and an exogenous population growth rate. At each point in time individuals work, save, consume and invest by taking the social security policy as given. The latter consists of a tax on current wages transferred to elderly people. When they vote, individuals have to make two choices: If they want to keep the commitment made by the previous generation by paying the elderly the promised amount of benefits, and which amount they want paid to themselves next period. We show that when the growth rate of population is high enough compared to the productivity of capital there exists an equilibrium where PAYG pensions are voted into existence and maintained. PAYG systems are kept even when everybody knows that they will surely be abandoned, and that some generation will pay and not be paid back. We characterize the steady state and dynamic properties of these equilibria and study their welfare properties. Equilibria achieved by voting are typically inefficient; however, they may be so due to overaccumulation, as well as, in other cases, due to under accumulation. On the other hand, the efficient steady states turn out to be dynamically unstable: so we are presenting an unpleasant alternative for policy making.

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1. Introduction.

We are aging, and this may be more than just our personal and unavoidable experience. Indeed around the world, and particularly in the most advanced countries, the average citizen is becoming older. A sharp increase in the population's proportion of elderly and retired individuals has occurred in the last twenty years. An even more drastic shift in the same direction will take place in the near future should the current demographic trends continue.

This has caused the political debate to move from the previous concentration on expanding the social security system, to concerns about the future viability of the system itself. Substantial reforms have already been implemented in some countries (Chile, Argentina) or are explicitly being proposed or even starting to be implemented elsewhere (Italy, Spain, Sweden). It is very likely that in the next twenty years similar phenomena will spread around the globe and that we will be facing the necessity to introduce drastic modifications in the way in which our social security systems are organized and run.

Evaluating the social and economic impact of these predictions requires more than an examination of the demographic assumptions from which they are derived. It also requires an understanding of the process of decision making through which social security systems are managed.

Pension systems are defined benefit systems which are at best partially funded, with the unfunded "pay-as-you-go" (PAYG) portion growing over time. Most often even in a system that began as partially funded the current receipts from contributions and the income from previous investments are no longer adequate to cover current benefits and the contribution rate is subject to periodical increments. These changes have not occurred by chance, but have been the outcome of an historical evolution in which the political conflicts between different interest groups and the intervention of the state in the provision of other public services have played a major role. Indeed the current system of social welfare and public goods provision (pensions, health services, educational services) appears as a unity in which when one part is modified or taken away other parts have also to be changed. As suggested by Becker and Murphy (1988), in modern societies the welfare system is better interpreted as the outcome of a rather sophisticated intergenerational agreement that, through the channel of the state and of the political system, solves a bargaining problem which originates from within the family and which involves at the same time the

education of young generations, the care of the elderly and the provision of assistance to individuals unable to work.

In the literature one can find a variety of different explanations for why social security systems have been introduced (*e.g.* Becker and Murphy (1988), Diamond (1977), Kotlikoff (1987), Lazear (1979), Merton (1983), Sala-i-Martin (1992)). They seem to concentrate around the idea that public pension systems are efficiency enhancing, either because of the overaccumulation that would occur without them, or because they provide for efficient risk-sharing in presence of incomplete annuities markets and adverse selection, or because they are a way around the lack of efficient intergenerational credit markets, or because of the presence of pervasive negative externalities, etcetera.

While we find each of these motivations compelling, we believe a clearer theoretical understanding of the nature of social security systems requires an explicit consideration of their redistributive features and of the fact that they are sustained by the consensus of a majority of the population. This is particularly true for pay-as-you-go (PAYG) systems (as opposed to Capital-Reserve (CR)), which have now become the rule in almost all advanced countries. We are far from being the first to claim that PAYG social security systems redistribute income between generations. This was made clear by Edgar Browning a while ago with two important articles, Browning (1973, 1975). He also pointed out that, because of the potential for intergenerational redistribution, median voter models predict that public social security plans will tend to be larger than required by economic efficiency. A number of contributions clarifying and extending Browning's initial intuition have followed since (see Verbon (1987) for a relatively recent update).

The temporal credibility problem of a PAYG pension system in the context of an OLG model with majority voting rule has been analysed in Sjoblom (1985). He considers a stationary environment, without capital accumulation or production, and shows that in the one-shot game there is no Nash equilibrium with SoSeSy. In the repeated game he shows that there is a subgame perfect Nash equilibrium at which each player may achieve a pre-selected level of utility. This is shown in a deterministic setting by *a priori* restricting the set of beliefs adopted by the players. A median voter model of the Social Security System is also in Boadway and Wildasin (1989).

A related approach is that of Tabellini (1991) who studies the intertemporal sustainability of public debt in a two-period model with intergenerational altruism. He shows that, even without a commitment technology, a majority voting mechanism will lead a

coalition formed by the first generation and the wealthiest portion of the second to approve full repayment of the debt even if the latter ex-ante redistributes income from the second to the first generation.

More recently Esteban and Sákovics (1993) have looked at institutions that can transfer income or wealth over time and across generations in the context of an OLG model. They model the creation of “institutions” by means of a fixed cost and consider both non-cooperative and cooperative games among generations. The basic intuition is similar to the one we use here, that is to say that “trust” can be built over time and maintained in order to achieve superior outcomes. On the other hand they do not model the voting mechanism explicitly and are not interested in the dynamic interactions between the transfer system and the process of capital accumulation. Their analysis is nevertheless relevant to our own investigation as they characterize the efficiency properties of such institutions and the increase in efficiency achievable as set-up costs are reduced.

At least three aspects of the problem seem to have remained in the background: a) the set of conditions under which a PAYG social security scheme remains viable in an intertemporal context, when agents are selfish and are called to vote upon it in each period b) the dynamic properties of such a system in the presence of an explicitly modelled accumulation of productive capital and a stochastic population growth rate; c) the impact that changes in the exogenous growth rate of population and of technological progress may have on this dynamics. The theoretical framework presented here should allow for a careful examination of these issues. We show that a PAYG public pension plan is a subgame perfect equilibrium of a majority voting game in an OLG model with production and capital accumulation when the growth rate of population and the initial stock of capital satisfy a certain set of restrictions [†]. We characterize the properties of such systems for the general case and then give a detailed characterization of our model economy for selected functional forms of the utility function, the production functions and the stochastic process of the growth rate of population.

For reasons of analytical simplicity we restrict our analysis to the case in which only the growth rate of the population is uncertain, but it would be relatively straightforward to duplicate the same results in the presence of random technology shocks affecting the marginal productivity of labor.

[†] In Boldrin and Rustichini (1995b) this analysis is extended to the case of heterogeneity among agents within each generations.

Mention should be made here of the recent and independent contribution of Cooley and Soares (1994) addressing the existence and viability of a PAYG Social Security System from a point of view which is quite similar to ours. Indeed also Cooley and Soares adopt the overlapping generations framework to show that there exist majority voting equilibria in which Social Security is implemented by means of a majority voting mechanism. The basic intuition is apparently the same we introduce here, but they concentrate more on quantifying the welfare gains/losses generated by a social security tax by means of an explicit parameterization and of numerical simulations of the model. On the other hand they restrict their attention only to the case in which a constant Social Security tax is the equilibrium outcome and do not discuss the dynamic evolution of the political equilibrium and its impact on the accumulation of the aggregate stock of capital. Also they consider only in passing the issue which is instead central to our analysis, i.e. the sustainability of a PAYG system in the long run and the form in which it could collapse in front of a decrease in the growth rate of the population. The two exercises therefore appear as complementary rather than substitute.

We should add that a number of interesting properties of the equilibria we describe seem to be consistent with the basic features of existing social security arrangements. We show that altruistic considerations (as for example in Verbon (1987) or Hansson and Stuart (1989)) are not needed to explain the intertemporal persistence of PAYG systems. We also show that for such a system to be an equilibrium it must entail a windfall for the generation of old people alive at the time of its introduction: they always receive a transfer even if they had never made a contribution. In general the equilibrium PAYG system is not efficient, in the sense that it is not identical to the one that would be implemented by a benevolent government maximizing the utility of the average generation. Furthermore the equilibrium level of pension payments is linked to the real wages and it tends to get larger as income per capita grows, therefore behaving as a superior good.

Some other questions we are able to address are:

- i) Under which conditions are the current social security systems sustainable? What is the most likely evolution of social security contributions and payments?
- ii) What can be predicted about the future flows of government payments and taxes given a certain evolution of the demographic and income distribution variables? Should we expect a smooth phasing out of the public social security system in front of decreasing population growth rates or a sudden collapse?

- iii) In which direction will these movements affect capital accumulation and economic growth?
- iv) How will this affect the welfare of different generations, in the present and the future?

The rest of the paper is organized as follows. In the next section we introduce the basic economic model, and its competitive equilibrium. In Section 3 we introduce a simple voting model and characterize the sub-game perfect equilibria of the underlying game. Then we introduce the full social security game, and the definition of equilibrium. In the following sections we study the basic properties of the equilibria of this game, and its welfare properties. In Section 4 we study the steady state equilibria, in Section 5 the dynamics of the equilibrium paths. Section 6 is devoted to the welfare analysis. We begin by summarizing the dynamic of capital accumulation when a benevolent social planner pick taxes and transfers in order to maximize lifetime utility of a representative agent at steady state, and then study the dynamic properties the paths where the rate of return between the investment in physical capital and the investment in pensions are equalized. This provides the benchmark against which the welfare properties of the political equilibria are discussed. In Section 7 we introduce a stochastic population growth rate to address the issue of the stability of the pay-as-you-go system. Section 8 concludes.

2. The Economy Without Voting.

We begin with a brief illustration of the commonly adopted model of social security and capital accumulation, as described for instance in Blanchard and Fischer (1989, chapt.3). In this environment agents are homogenous within each generation and do not have the power to vote on the level of taxes and transfers, so that fiscal policies are exogenously given. Within this simplified framework one can establish some baseline results against which the predictions of a model with voting can be contrasted. As we plan to concentrate our analysis on PAYG systems we will skip the description of the fully funded mechanism.

Consider an OLG model with production and agents living for two periods, with no labor endowment during the second one. Denote with $d_t = w_t \tau_t$ the per-capita contribution of young people and with $b_t = (1 + n_t)d_t$ the benefits received by an elderly person during the same period. $(1 + n_t)$ is the ratio between the young and the old people at time t . The representative young agent in period t solves

$$\begin{aligned} \max \quad & u(c_t) + \delta u(c_{t+1}) \\ \text{subject to: } & c_t + s_t \leq w_t(1 - \tau_t), \end{aligned} \tag{2.1}$$

$$c_{t+1} \leq s_t \pi_{t+1} + w_{t+1}(1 + n_{t+1})\tau_{t+1},$$

$$\text{and : } 0 \leq s_t$$

The *equilibrium saving function* is the function of $(k_t, \tau_t, \tau_{t+1})$ that solves the maximisation problem of the agent born at time t , when he takes as given wages and profits in the second period given by the capital accumulation determined by the savings of his own generation. When $s_t > 0$ the following first order conditions for interior solutions define the optimal saving of an agent:

$$u'(w_t(1 - \tau_t) - s_t) = \delta u'(s_t \pi_{t+1} + w_{t+1}(1 + n_{t+1})\tau_{t+1})\pi_{t+1} \quad (2.2)$$

and the equilibrium saving function is the value of s such that (2.2) is satisfied. For non interior solutions, obvious modifications are necessary. The production side is summarized by the three equilibrium conditions:

$$w_t = w(k_t) = f(k_t) - k_t f'(k_t)$$

$$\pi_t = \pi(k_t) = f'(k_t)$$

$$k_{t+1} = s_t / (1 + n_{t+1})$$

We write c_t^s to denote the consumption at time t of an agent born at time s . For a given sequence (n_t, τ_t) of population growth rates and tax rates, a *competitive equilibrium* is a sequence $(w_t, \pi_t, c_t^t, c_{t+1}^t)$ of wages, profits, and consumption pairs of the agents such that (1) the consumption solves the maximization problem (2.1) above, and (2) wages, profits and capital stocks satisfy the three equilibrium conditions just given.

When $n_t = n$ and $\tau_t = \tau$ for every t , we may define as usual a *steady state equilibrium* as an equilibrium sequence of *constant* wages, profits, and consumption pairs.

3. The Economy with Voting.

Consider now the same framework as before but assume that the social security tax τ_t is not a fixed sequence, but is instead chosen by the citizens through majority voting. In each period both generations vote on whether to have a Social Security System (SoSeSy) and at what level it should be implemented.

More precisely assume the political decision making process operates in the following form. If at time t a SoSeSy does not exist voters will decide to either remain without one

or introduce it. In the second case they have to decide what is the per-capita transfer (if any) to be paid to the currently old generation, $d_t = \tau_t w_t$, and at the same time make a "proposal" as to the amount of money a currently young individual will receive next period $b_{t+1} = (1 + n_{t+1})d_{t+1}$. Instead if a SoSeSy is already in place the citizens will be asked if they like to disband it or not. Keeping it entails paying the promised amount to the old folks and setting a promise for the payments the currently young are entitled to claim next period.

Individuals are homogeneous within a given generation. It is obvious then, that the elderly will always favor keeping the SoSeSy as the latter is financed by taxes on current wage income, while the young workers face an intertemporal trade off. The young know that if the SoSeSy is approved at time t and maintained thereafter, they have the equilibrium value of the problem:

$$\begin{aligned} \max \quad & u(c_t) + \delta u(c_{t+1}) & (3.1) \\ \text{subject to : } & c_t + s_t \leq w_t - d_t \\ & \text{and : } c_{t+1} \leq \pi_{t+1} s_t + b_{t+1} \end{aligned}$$

where d_t is given and b_{t+1} has to be chosen. If instead the SoSeSy is introduced at time t but then rejected during the following period they receive total utility

$$\begin{aligned} \max \quad & u(c_t) + \delta u(c_{t+1}) & (3.2) \\ \text{subject to : } & c_t + s_t \leq w_t - d_t \\ & \text{and : } c_{t+1} \leq \pi_{t+1} s_t \end{aligned}$$

On the other hand, if the SoSeSy is rejected at time t but introduced at time $t + 1$ then they have the equilibrium value of

$$\max u(c_t) + \delta u(c_{t+1}) \quad (3.3)$$

$$\text{subject to : } c_t + s_t \leq w_t$$

$$\text{and : } c_{t+1} \leq \pi_{t+1}s_t + b_{t+1}$$

Finally, if the SoSeSy is rejected at both time t and time $t + 1$ then they will obtain the lifetime utility

$$\max u(c_t) + \delta u(c_{t+1}) \quad (3.4)$$

$$\text{subject to : } c_t + s_t \leq w_t$$

$$\text{and : } c_{t+1} \leq \pi_{t+1}s_t$$

The key point is that while the value associated to (3.3) dominates the one associated to (3.4) which in turn dominates the one given by (3.2) there is no ways in which, a-priori, one may rank the payoffs associated to (3.1) and (3.4) In general this will depend on the population growth rate n_t , on the present and future stocks of capital (k_t, k_{t+1}) and on the promised level of benefits for the currently retired, b_t .

This means that, under certain circumstances, the young agents may find it utility maximizing to introduce the SoSeSy and to change their saving-consumption behavior accordingly. Notice though that, in order to decide what their vote should be, the young need not only to be able to find a level of τ_{t+1} that makes them better off under the first program, but also be sure that such a promise will be accepted by their descendants next period.

An equilibrium with Social Security is now described by a sequence of capital stocks plus a sequence of $\{\tau_t\}_{t=0}^{\infty}$ such that, at the equilibrium of the economy where the τ 's are taken as given, the value for the young generation in period t is at least the same as in the equilibrium without Social Security, for all t 's. A formal definition is given in Section 3.2.

The intuition for why a SoSeSy may be voted into existence is that the young can take advantage of the system of transfers by exploiting the growth of the population and the process of capital accumulation that increases future labor productivity. They give up a fraction of their salary, but they get back in the second period a fraction of the future salary times the population growth factor $(1 + n_{t+1})$.

We should note also that the same argument could be applied if labor-augmenting technological progress were introduced in the model. In that case one would have to look

at the combined impact of population growth and increases in labor efficiency units in order to evaluate the tradeoff. To ease the exposition we use only the symbol n_t and keep calling it the growth rate of the population.

The strategy of introducing a SoSeSy to reap benefits from future generations will be successful, if the sequence of future payments and transfers implicit in it does not make any of the future median voters worse off. In other words, one has to show that the proposed sequence of (b_t, d_t) is an equilibrium of a game with an infinite number of players. We will do this in the following subsections.

3.1 A Simple Game.

We begin with a simplified model, which will serve the purpose of illustrating the main idea.

Consider the following game. There are countably many players who move sequentially. (In the more complex social security game each player will in fact be an entire generation of agents.) At time t the designated player can choose to give out of his pocket to the previous player an amount d_t . If such payment is actually made, the player receiving the transfer will in fact be paid the amount $(1 + n_t)d_t$ rather than simply d_t ; the logic through which the precise value of d_t is selected will be determined later. The action of each player is perfectly observable by all those following him. As the amounts d_t and n_t are given, the action available to each player is just

$$a_t \in \{\mathcal{Y}, \mathcal{N}\}$$

where \mathcal{Y} stands for *yes* and \mathcal{N} stands for *no*. A history h_{t-1} of the game at time t , when it is player's t turn to move, is:

$$h_{t-1} = (a_1, a_2, \dots, a_{t-1})$$

so that a strategy for player t is a map

$$\sigma_t : (a_1, a_2, \dots, a_{t-1}) \mapsto \{\mathcal{Y}, \mathcal{N}\}.$$

Identify \mathcal{Y} with 1 and \mathcal{N} with 0. The payoff for player t is determined by the value function $V_t : \{0, 1\} \times \{0, 1\} \mapsto \mathfrak{R}$ defined by

$$V_t(a_t, a_{t+1}) \equiv \max u(c_t) + \delta u(c_{t+1}) \tag{3.5}$$

subject to : $c_t + s_t \leq w_t - d_t a_t$

and : $c_{t+1} \leq \pi_{t+1} s_t + b_{t+1} a_{t+1}$

For the time being we only need to assume that the sequences $\{d_t, n_t\}_{t=0}^{\infty}$ are such that

$$V_t(1, 1) \geq V_t(0, 0) \quad \text{for all } t. \quad (3.6)$$

Then one has the following result:

Proposition 1 For both $\sigma_0 = 0$ and $\sigma_0 = 1$ the strategies:

$$\sigma_t = \min\{a_1, a_2, \dots, a_{t-1}\}$$

for all $t \geq 1$ are a subgame perfect equilibrium (SPE).

Proof: By definition of SPE, we have to prove that for any subgame given by a history h_{t-1} , the strategy profile $(\sigma_t, \sigma_{t+1}, \dots)$ restricted to the history h_{t-1} is a Nash equilibrium. We have two possible cases.

Case 1: $\min\{a_1, \dots, a_{t-1}\} = 0$; then $\min\{a_1, \dots, a_{t-1}, a_t\} = 0$ for any a_t ; so $\sigma_{t+1}(h_{t-1}, a_t) = 0$, and so (since $V_t(0, 0) > V_t(1, 0)$), the best choice of player t is 0, which is equal to $\min\{a_1, \dots, a_{t-1}\}$.

Case 2: $\min\{a_1, \dots, a_{t-1}\} = 1$; then $\min\{a_1, \dots, a_{t-1}, a_t\} = a_t$ for all a_t ; then $\sigma_{t+1}(h_{t-1}, a_t) = a_t$. Now from the assumption that $V_t(1, 1) \geq V_t(0, 0)$ the best choice for t is 1, and the claim follows in this case too. **Q.E.D.**

Remark 1 Of course if $\sigma_0 = 1$ then the equilibrium outcome is $(1, 1, \dots)$; and if $\sigma_0 = 0$ then the equilibrium outcome is $(0, 0, \dots)$.

Remark 2 Consider the "relenting strategies":

$$\sigma_0 = 1; \quad \sigma_t(a_1, \dots, a_{t-1}) = a_{t-1} \quad \text{for } t \geq 1.$$

in which deviations of generations further than the immediate predecessor are forgiven. It is easy to show that they are not an SPE.

The last remark shows that when a SoSeSy is in place (i.e. a pair (b_t, d_t) has been approved and paid during the previous period) a one-period default is enough to destroy the credibility of the system for the very long future. Let us now see how the equilibrium outcomes look like.

Proposition 2 For any equilibrium outcome (a_1^*, a_2^*, \dots) of the game, if $a_t^* = 0$ for some t , then $a_i^* = 0$ for all $i < t$.

Proof: By backward induction, it is enough to show that $a_{t-1}^* = 0$. In fact, if $a_{t-1}^* = 1$, then player $t - 1$ will obtain $V_{t-1}(1, 0)$, while he could insure for himself $V_{t-1}(0, 0) > V_{t-1}(1, 0)$ as payoff. **Q.E.D.**

Proposition 3 *The only equilibrium outcomes are of the form:*

$$(0, 0, \dots, 0, 1, 1, \dots).$$

Also let T be the first period with $a_t = 1$. Then for any $T \geq 0$ there is a corresponding equilibrium of this form.

Proof: Consider the equilibrium outcome $(a_1^*, \dots, a_t^*, \dots)$, and let $i \equiv \min\{t : a_t = 1\}$ as usual, $i = \infty$ if the set is empty. For a finite i , $a_i = 1$ and $a_{i-1} = 0$; then if $a_k = 0$ for some $k > i$ we have a contradiction with the previous proposition. The rest is obvious. **Q.E.D.**

The fact that the SoSeSy, once established, cannot collapse can also be changed by introducing a stochastic element. This will be clearer in Section 7. Let us see how the intuition developed in this simple game can be transferred to the full blown Social Security problem.

3.2 The Social Security Game.

In the simple game that we have analysed so far the amount of the transfers b_t and d_t was exogenously given. To obtain a complete model of the social security game we have to express these quantities as tax receipts from labor income, on the one hand, and social security transfers on the other and then derive a political decision rule through which their amounts may get determined. This will require a richer description of a history of the game.

Indeed, a full description of the history of the game at time t would require the list of all alternative proposals presented at the voting, as well as the winning outcome. In addition, since the competitive equilibrium is itself part of the game, we should add a complete record of all consumption and saving decisions. A major simplification is achieved if no deviation are allowed at the stage in which the competitive equilibrium is determined. Once we do this, then the strategic aspects of the model are confined to the choice, by voting, of the tax rates. Let us be more precise.

In the game that we are going to define, a history h_{t-1} at time t is again a sequence:

$$h_{t-1} = (a_1, a_2, \dots, a_{t-1})$$

where for each $s \in (1, \dots, t-1)$, $a_s \equiv (a_s^1, a_s^2) \in \{\mathcal{Y}, \mathcal{N}\} \times [0, 1]$. Take now any infinite history $h \equiv (a_1, \dots, a_t, \dots)$. For any pair (a_{t-1}, a_t) in the history the *effective tax rate* (τ_t^h) at time t is defined to be $\tau_t^h = a_{t-1}^2$ if $a_t^1 = Y$, and equal to 0 otherwise.

The competitive equilibrium of the economy given the sequence (n_t, τ_t) is well defined, and so is the lifetime utility of each agent in the economy. We have most of the elements necessary for a well defined game: action sets and payoffs; if we take each generation of agents as a single player, (that we call for convenience the *generation player*), we have a completely specified game. Formally we say:

Definition 1 *The generation player game of the economy is the extensive form game where:*

- (1) *players are indexed by $t = (1, 2, \dots)$;*
- (2) *the action set of each player is $\{\mathcal{Y}, \mathcal{N}\} \times [0, 1]$;*
- (3) *for every history $(a_1, a_2, \dots, a_t, \dots)$ of actions, the payoff to each player is equal to the lifetime utility of the representative agent born at time t in the competitive equilibrium of the economy with $(n_t, \tau_t), (n_{t+1}, \tau_{t+1})$.*

Now that we have defined the game, we may proceed with the definition of the equilibrium.

Definition 2 *For a given sequence (n_t) of population growth rates, a political equilibrium is a sequence $\{\tau_t, w_t, \pi_t, c_t^i, c_{t+1}^i\}_{i=0}^\infty$ such that:*

- (1) *the sequence $\{w_t, \pi_t, c_t^i, c_{t+1}^i\}_{i=0}^\infty$ is a competitive equilibrium given $\{n_t, \tau_t\}_{i=0}^\infty$;*
- (2) *there exists a sequence of strategies $\{\sigma_t\}_{i=0}^\infty$ for the generation player game which is a subgame perfect equilibrium, and such that $\{\tau_t\}_{i=0}^\infty$ is the sequence of effective tax rates associated with the equilibrium history.*

In the next sections we study the equilibria that we have just defined.

4. Steady State Equilibria

Some equilibrium outcomes are particularly simple to study; as usual, steady states are first among them. We define a *political equilibrium steady state* to be the political equilibrium where all quantities are constant in time; in the following k_τ will denote the steady state value of the *per capita* capital stock, in a competitive equilibrium where the sequence of tax rates is fixed to τ .

In this section we deal only with steady states: so the subscript to k will denote the tax rate, and not the time period. For instance k_0 means the steady state for $\tau = 0$, and not the first period capital stock. Also $V(k_\tau, \tau)$ denotes the lifetime value of utility for the representative generation, at the steady state value of the capital stock associated to a stationary tax rate $\tau > 0$.

Suppose now that, with $k = k_\tau$, the social security system is voted down. The sequence of tax rates is then set to zero, and the sequence of values of capital stocks in the competitive equilibrium will in general diverge from k_τ . The generation of agents born in that period will achieve a level of utility different from $V(k_\tau, \tau)$, and dependent only on the value of k_τ ; let $v(k_\tau)$ denote this value. Note that clearly

$$(k_0, 0) = v(k_0).$$

Which values of steady state can be supported as political equilibria? Clearly a necessary condition is that

$$V(k_\tau, \tau) \geq v(k_\tau)$$

since otherwise the social security system would be immediately voted down. But it is easy to see that the condition is also sufficient:

Proposition 4 *The steady state value (k_τ) can be supported as the outcome of a political equilibrium if and only if $V(k_\tau, \tau) \geq v(k_\tau)$.*

Proof: The sequence of strategies $\{\sigma_t\}_{t=0}^\infty$ where, for every history h_t , $\sigma_t(h_t) = (\mathcal{Y}, \tau)$ if for every $s < t$, $a_s = (\mathcal{Y}, \tau)$, and equal to (\mathcal{N}, τ) otherwise satisfies our definition of political equilibrium, and supports the steady state. **Q.E.D.**

Let us see the steady state equilibria of some simple model.

Example 1. Let first $u(c) = \log(c)$ and $f(k) = k^\alpha$. The steady state capital stock is easily found to be

$$k_\tau^{(1-\alpha)} = \frac{\alpha\delta(1-\alpha)(1-\tau)}{(1+n)[\alpha(1+\delta) + (1-\alpha)\tau]},$$

a decreasing function of τ . One can also compute that

$$V(k_\tau, \tau) = \log\left(\frac{s_\tau}{\delta} + \frac{w_\tau(1+n)\tau}{\pi_\tau\delta}\right) + \delta \log(s_\tau\pi_\tau + w_\tau(1+n)\tau)$$

where (w_τ, π_τ) are the wages and the profit rate at the steady state value of the capital stock, and $s_\tau = (1+n)k_\tau$ is the equilibrium saving *per capita*. The alternative for the

generation of young voters when the economy is at k_τ is to vote the system down, expect no transfer from future generations, and get a lifetime utility given by:

$$v(k_\tau) = \log\left(\frac{s_\tau^v}{\delta}\right) + \delta \log\left(s_\tau^v \pi_\tau^v\right)$$

where s_τ^v is the competitive equilibrium saving in the economy with initial capital stock k_τ and zero tax transfers for all periods; the precise value is found to be $s_\tau^v = (1+n)k_0^{(1-\alpha)}k_\tau^\alpha$. So the difference between the two values, denoted $D(\tau)$, is:

$$D(\tau) = (1 + \alpha\delta)(1 + \alpha) \log\left(\frac{k_\tau}{k_0}\right)^{(1-\alpha)} + (1 + \delta) \log\left(\frac{\alpha + (1 - \alpha)\tau}{\alpha}\right)$$

which some additional computation shows to be equal to

$$(1 + \alpha\delta)(1 + \alpha) \log\left[\frac{(1 - \tau)}{(\alpha(1 + \delta) + (1 - \alpha)\tau)}\right] + (1 + \delta) \log\left(\frac{\alpha + (1 - \alpha)\tau}{\alpha}\right) + C$$

where C is a constant independent of τ . Simple analysis of the function D shows that it is zero at zero (as we knew), concave, and with

$$\lim_{\tau \rightarrow 1} D(\tau) = -\infty.$$

Finally, the sign of $D'(0)$ (the derivative of D at $\tau = 0$) is the same as the sign of

$$[(1 + \delta)^2(1 - \alpha) - (1 + \alpha\delta)^2(1 + \alpha)].$$

This last relation gives the key condition on the parameters for the existence of a steady state with positive tax transfers. In particular, two cases are possible.

In the first case, when $D'(0)$ is positive, there exists a largest value of the steady state which can be supported as a political equilibrium. This happens for instance when α is small enough, or, for some α 's (but not all, as a careful study of the sign of $D'(0)$ will reveal), when agents are patient enough. In this first case any smaller capital stock can also be supported, in correspondence with any lower value of the tax rate, because $D(\tau)$ is positive in this interval.

On the other hand, when $D'(0)$ is negative, then the value of keeping the social security system is always lower than the value of voting it down, and no steady state with positive transfers can be supported.

Example 2. For future reference we also characterize the steady state political equilibria for the case of linear utility and Cobb Douglas production function: $u(c) = c$ and $f(k) = k^\alpha$. In this case we find that

$$V(k_\tau, \tau) = \delta(1 - \tau)^\alpha(1 + n)^{1-\alpha}[\alpha + (1 - \alpha)\tau]w(k_\tau)^\alpha$$

while

$$v(k_\tau) = \delta\alpha(1 + n)^{1-\alpha}w(k_\tau)^\alpha$$

This time we take the ratio of the two functions, defined $R(\tau)$, and observe that $R(0) = 1$, $R(1) = 0$, $R'(0) = -\alpha^2 - \alpha + 1$, and that the derivative with respect to τ of $\log(R)$ is decreasing.

The structure of the set of steady states in the present case is very similar to the one of the previous example. There are no steady states when the capital share is large (more precisely, when $\alpha \geq \frac{5^{1/2}-1}{2} \equiv \alpha^*$); on the other hand when the capital share is small there is an entire interval of tax rates, $[0, \tau_p]$ say, and correspondingly of steady states, which can be supported as political equilibria.

5. The Dynamics of the Political Equilibrium.

We now move to the harder task of determining the equilibrium paths out of an initial condition on the capital stock that is not necessarily a steady state. Our aim is to determine how the capital accumulation evolves when the decision on tax rates is determined by a voting mechanism. Given the difficulty of providing a complete characterization for the general case we will still proceed by means of examples.

5.1 Logarithmic utility and linear production.

Let $u(c) = \log(c)$ and $f(k) = ak + b$. This example is extremely simple and also somewhat paradoxical (for instance, one of possible steady state capital stocks at the political equilibrium is zero, and there is no interesting dynamics in the capital stock), but it should help to clarify the intuition. The maximization problem of the young agent gives the following saving function

$$s_t = \max\left\{b \frac{a\delta(1 - \tau_t) - (1 + n)\tau_{t+1}}{a(1 + \delta)}, 0\right\}.$$

Since the saving function does not depend on the current stock of capital, this is also the equilibrium saving function. Now take the difference between the lifetime utility for the

representative agent if the social security is operating in his life, at the rates τ_t and τ_{t+1} , and the utility if there are zero transfers in both periods. This difference is obviously independent of k ; it is equal to

$$(1 + \delta) \log \left[(1 - \tau_t) + \frac{1+n}{a} \tau_{t+1} \right].$$

when $a\delta(1 - \tau_t) > (1 + n)\tau_{t+1}$, and savings are positive; and is equal to

$$\log(1 - \tau_t) + \delta \log \tau_{t+1} + C$$

when the savings are zero. Here C is a constant that depends only on the parameters of the model ($C = \delta \log(a\delta) - \delta \log(1 + n)(1 + \delta) \log(1 + \delta)$, for the picky reader). Now consider the function $\tau_{t+1} = \phi(\tau_t)$ defined by:

$$\begin{aligned} \tau_{t+1} &= \frac{a}{1+n} \tau_t, & \text{when } \tau_t < \frac{\delta}{1+\delta} \\ \tau_{t+1} &= \exp\left(\delta^{-1}(C - \log(1 - \tau_t))\right), & \text{when } \tau_t \geq \frac{\delta}{1+\delta} \end{aligned}$$

where C is the constant defined previously. This is the next period value of the tax rate necessary, for a given tax rate τ_t today, to make the social security system at least as good as the zero transfers system. It is a continuous, increasing function, and as τ_t tends to 1, τ_{t+1} tends to $+\infty$.

Now we can distinguish two cases.

- a) If $a > 1 + n$ the graph of the function is all above the diagonal, for strictly positive τ 's. So for any initial positive value of τ to make the system survive it is necessary to make the next period tax rate even higher, and in finite time this rate goes above 1. Hence, there is no equilibrium with Social Security. This is quite intuitive as the direct investment in capital always dominates the rate of return on the security system.
- b) Take now $a < 1 + n$. There is a unique positive fixed point τ_p , such that $\tau_{t+1} \leq \tau_t$ if $\tau_t \leq \tau_p$, and *viceversa* for the other values. The rate τ_p is the maximum rate that can be supported as political equilibrium. In this case we can give a complete description of all the equilibrium paths:

Proposition 5 Define the correspondence from $[0, \tau_p]$ to intervals of $[0, 1]$:

$$\Phi(\tau) = [\phi(\tau), \tau_p].$$

Then a sequence of capital stocks and taxes (k_t, τ_t) is the outcome of a political equilibrium if and only if it is the solution of the system of difference inclusions:

$$k_{t+1} = \max\left\{b \frac{a\delta(1 - \tau_t) - (1 + n)\tau_{t+1}}{a(1 + \delta)}, 0\right\}; \quad \tau_{t+1} \in \Phi(\tau_t),$$

for every $t \geq 1$.

Of course it is crucial that the solution be defined, and contained in $[0, 1]$ for every period. Take now the initial tax rate as given, with $\tau_0 \in [0, \tau_p]$. Here are two important equilibrium paths. In the first equilibrium $\tau_t^1 \equiv \tau_p$ for $t \geq 0$, while the capital stock $k_t^1 \equiv 0$. In the second the tax rate is defined by $\tau_{t+1}^2 = \phi(\tau_t^2)$ for $t \geq 0$ and some initial condition $\tau_0^2 < \tau_p$, and correspondingly a capital stock sequence $\{k_t^2\}_{t=0}^{\infty}$ converging to a steady state value $\frac{b\delta}{(1+\delta)(1+n)}$. These two sequences describe the entire set of equilibria; in fact they are extreme points of this set, *i.e.* any equilibrium path satisfies $\tau_t \in [\tau_t^2, \tau_t^1]$, $k_t \in [k_t^1, k_t^2]$.

In the next example the dynamics of the capital stock are more interesting.

5.2 Linear utility and Cobb-Douglas production.

Let $u(c) = c$ and $f(k) = k^\alpha$. For a given pair of taxes (τ_t, τ_{t+1}) and a current capital stock k_t the optimal saving of a representative agent is determined by solving

$$\max_{s \geq 0} (1 - \tau_t)w_t - s + \delta[\pi_{t+1} \cdot s + (1 + n)w_{t+1}\tau_{t+1}] \quad (5.1)$$

The solution to this problem is $(1 - \tau_t)w_t$, or the interval $[0, (1 - \tau_t)w_t]$, or 0, if $(-1 + \delta\pi_{t+1})$ is positive, zero, or negative respectively. Since $\delta\pi(k_{t+1}) \geq 1$ if and only if $k_{t+1} \leq (\alpha\delta)^{1/(1-\alpha)}$ the equilibrium saving function is

$$s_t = s^*(k_t, \tau_t) = \min\left\{(1 - \tau_t)(1 - \alpha)k_t^\alpha, (1 + n)(\alpha\delta)^{1/(1-\alpha)}\right\} \quad (5.2)$$

which is independent of τ_{t+1} . For a given value of $\tau_t = \tau$ the equation (5.2) has a unique non zero stable steady state. When $\tau_t = 0$ for all t , the evolution of the per-capita stock of capital reduces to

$$k_{t+1} = \min\left\{\frac{(1 - \alpha)k_t^\alpha}{(1 + n)}, (\alpha\delta)^{1/(1-\alpha)}\right\}$$

which, if the population has a constant growth rate, converges to the steady state $k^* = \min\left\{(1 - \alpha)/(1 + n), \alpha\delta\right\}^{1/(1-\alpha)}$. The first of the values over which the minimum is

taken satisfies the Golden Rule condition only when $\alpha = 1/2$, whereas the second requires $\delta(1+n) = 1$.

To determine the equilibrium strategies and the consequent evolution of the tax rate, let us compute the value for the representative member of the young generation of turning down the SoSeSy and going alone. As a function of the current stock of capital k and of the expected growth rate of the population n this is

$$v(k_t) = (1 - \alpha)k_t^\alpha - s^*(k_t, 0) + \delta\alpha(1+n)^{1-\alpha}s^*(k_t, 0)^\alpha. \quad (5.3)$$

On the other hand the value of keeping the SoSeSy as a function of the same variables, of the current tax rate τ_t and of the proposed tax rate τ_{t+1} is

$$V(k_t, \tau_t, \tau_{t+1}) = (1 - \tau_t)(1 - \alpha)k_t^\alpha - s^*(k_t, \tau_t) + \delta(1+n)^{1-\alpha}s^*(k_t, \tau_t)^\alpha(\alpha + (1 - \alpha)\tau_{t+1}). \quad (5.4)$$

To organize the analysis it is convenient to distinguish three separate regions in the capital stock-tax rate space $\mathfrak{R}_+ \times [0, 1]$:

$$\mathbf{R1} \equiv \{(k, \tau) : (\alpha\delta)^{\frac{1}{1-\alpha}}(1+n) \leq (1 - \tau)w(k)\};$$

$$\mathbf{R2} \equiv \{(k, \tau) : (1 - \tau)w(k) \leq (\alpha\delta)^{\frac{1}{1-\alpha}}(1+n) \leq w(k)\};$$

$$\mathbf{R3} \equiv \{(k, \tau) : w(k) \leq (\alpha\delta)^{\frac{1}{1-\alpha}}(1+n)\}.$$

Note that the boundary between the second and the third region is a line $\{(k^3, \tau) : \tau \in [0, 1]\}$. The equilibrium saving in the regions **R2** and **R3** is equal to $(1 - \tau)w(k)$, and to $(\alpha\delta)^{\frac{1}{1-\alpha}}(1+n)$ in **R1**. From the form of the equilibrium saving function we have that two cases are possible for the steady state when the tax rate is identically zero: it is either equal to $(\frac{1-\alpha}{1+n})^{\frac{1}{1-\alpha}}$ or it is equal to $(\alpha\delta)^{\frac{1}{1-\alpha}}$. We concentrate on the first, more interesting, case and assume in the rest that:

$$\frac{1 - \alpha}{1 + n} \leq \alpha\delta.$$

Let us immediately note three implications of this assumption: First, $(\alpha\delta)^{\frac{1}{1-\alpha}}(1+n) \leq k^3$. Second, k^3 is larger than the steady state of the economy with zero taxes, and therefore also larger than the steady state of the economy with any tax rate. Finally from the form of the equilibrium saving function we have that for any initial condition and any sequence of tax rates, $k_t \leq k^3$ for any t after the initial period: in other words, the third region is invariant for any competitive equilibrium path beginning within it.

Since in the third region the equilibrium saving function is equal to $(1 - \tau)w(k_t)$, and it is easily found that the inequality $V(k_t, \tau_t, \tau_{t+1}) \geq v(k_t)$ is equivalent to

$$\alpha + (1 - \alpha)\tau_{t+1} \geq \alpha(1 - \tau_t)^{-\alpha} \quad (5.5)$$

An interesting property of (5.5), which will be used repeatedly in the future, is that it does not involve the stock of capital. Rewriting (5.5) in explicit form gives

$$\tau_{t+1} \geq \frac{\alpha}{1 - \alpha} \left[\frac{1}{(1 - \tau_t)^\alpha} - 1 \right] = \phi(\tau_t) \quad (5.5bis)$$

If the restriction $0 \leq \alpha < (\sqrt{5} - 1)/2$ is satisfied the convex function ϕ starts at the origin and has a unique interior fixed point τ_p which is unstable under repeated iterations. For larger values of α only the degenerate equilibrium without SoSeSy exists. The analysis is now very similar to the previous case. For any τ_t larger than τ_p the next period transfers necessary to make the social security system acceptable to the next generation are (recall that the third region is invariant) equal to $\phi(\tau_t)$ or larger, and iteration of this argument leads to a tax rate larger than 1 in finite time, hence the system cannot be supported. On the other hand as long as the tax rate remains in the region $[0, \tau_p]$ the social security system can be supported. In fact we can completely characterize the set of equilibrium paths. Again we define the correspondence

$$\Phi(\tau) \equiv [\phi(\tau), \tau_p]$$

and then:

Proposition 6 *A sequence (k_t, τ_t) is an equilibrium outcome of the generation player game if and only if it is a solution of the system of difference inclusions:*

$$k_{t+1} = \frac{s^*(k_t, \tau_t)}{1 + n}; \quad \tau_{t+1} \in \Phi(\tau_t),$$

for every $t \geq 1$.

The proof of this statement is immediate.

We have proved earlier that the third region is invariant, and therefore the comparison between the two values V and v only depends on the tax rates, as formulated in equation (5.5) above. Equilibrium strategies are now easy to define. In this case too we may concentrate on a few, more interesting equilibria.

In the first equilibrium the tax rate is immediately set equal to the highest possible value compatible with the existence of a social security system. When the initial capital stock is less or equal than k^3 , this rate is equal to τ_p . Then the equilibrium path has the form (k_t, τ_p) , where the sequence k_t converges to the steady state k_{τ_p} . When the initial capital stock is higher than k^3 , the highest possible tax rate turns out to be lower than τ_p . In fact there is a curve in the capital-tax rate space, where the tax rate is decreasing with the capital stock, that describes the equilibrium tax rate. Details are in Appendix 2, but the intuition is clear: when the capital stock is high the value of the alternative of dropping the social security system is too high, and the generation who introduces the social security system has to accept lower transfers to make the system viable. After one period, however, the equilibrium path enters the third region, and the time sequence is by now familiar.

In the second equilibrium we consider the sequence of tax rates defined by:

$$\tau_{t+1} = \phi(\tau_t).$$

For simplicity we only consider the case $k_0 \leq k^3$. The tax rate converges to zero, and the capital stock converges to the steady state value $\left(\frac{1-\alpha}{1+n}\right)^{\frac{1}{1-\alpha}}$ of the economy with zero transfers. This equilibrium corresponds to the slow disappearance of the social security system.

6. The Welfare Analysis of the Political Equilibria

6.1 The Planner's Choice

In the tradition of Diamond (1965) and Samuelson (1975) we consider the tax rates which are efficient at steady state. To be precise we define an *efficient steady state tax rate* as the tax rate which gives, at the associated value of steady state for the capital stock, the highest utility per capita to each generation. Note that the social planner is free to choose the initial endowment of capital stock: a possible interpretation of this is that the economy is running from the infinite past into the infinite future.

For the first model we consider we have already analyzed the steady state political equilibria in section 4, example 2. Let $u(c) = c$, and $f(k) = k^\alpha$. In this case:

Proposition 7 *The efficient steady state tax rate is:*

$$\tau_e = \frac{1 - 2\alpha}{1 - \alpha}$$

with corresponding steady state capital stock

$$k = \left(\frac{\alpha}{(1+n)} \right)^{1/(1-\alpha)}$$

if $\alpha \leq \frac{1}{2}$ and $\delta(1+n) > 1$. The efficient steady state tax rate is equal to 0 otherwise, with steady state capital stock:

$$k = \left(\frac{(1-\alpha)}{(1+n)} \right)^{1/(1-\alpha)}$$

The main lines of the proof of this proposition can be found in the Appendix 1. The reader can draw his conclusions on the efficiency of the political equilibrium by himself. We do it later, in section 6.3, after we deal with the dynamic aspect of the issue.

6.2 The Dynamic of Taxes and Capital

One can give an explanation for the inefficiency of the competitive equilibrium without social security by observing that the rates of return of the voluntary savings s_t and of the forced saving achieved through the social security system, $\tau_t w_t$, are not equalized. In fact, a necessary condition for efficiency is precisely that these two rates are equalised. This condition is not sufficient, as one can see from example 1 below: there may be more than one steady state in which the condition is satisfied. This poses the next question.

From a normative point of view, we may consider a social planner who, in the initial period where a capital stock k_0 is given, chooses the initial tax rate, and then imposes the equality among rates, perhaps through some market mechanism. In this interpretation it is important to determine the long run properties of the competitive equilibrium paths that are determined in this way: in particular, if the efficient steady states are stable. They will turn out not to be. In our framework the equality of rates requires

$$\frac{(1+n_{t+1})w_{t+1}\tau_{t+1}}{\tau_t w_t} = \pi_{t+1} \quad (6.1)$$

whenever the tax rates τ_t, τ_{t+1} are different from zero. At a steady state with positive tax transfers, this is the *Golden Rule*.

Replacing the factor market equilibrium conditions in (6.1) and adding the capital accumulation equation yields a two-dimensional dynamical system in implicit form

$$k_{t+1} = \frac{s[w(k_t), \pi(k_{t+1}), \tau_t, \tau_{t+1}]}{(1 + n_{t+1})} \quad (6.2)$$

$$\tau_{t+1} = \frac{\tau_t \cdot w(k_t) \cdot \pi(k_{t+1})}{(1 + n_{t+1})w(k_{t+1})}$$

which applies for all quadruples $\{(k_t, \tau_t), (k_{t+1}, \tau_{t+1})\}$. Fixing a constant growth rate of the population $n_t = n$ one can compute the unique steady state of (6.2) with positive taxes which is given by the pair (k^*, τ^*) satisfying:

$$k^* = g^{-1}(1 + n), \quad (g \equiv f') \quad (6.3a)$$

$$u'(w(k^*)(1 - \tau^*) - k^*(1 + n)) = \delta u'(k^*(1 + n)f'(k^*) + w(k^*)(1 + n)\tau^*)f'(k^*) \quad (6.3b)$$

This is obviously the steady state at which the rate of return on capital $f'(k^*) - 1$ equals the rate of return on social security taxes which also equals the rate of growth of the population n . It is immediate to verify that a low population growth rate induces a higher steady state level of per-capita capital stock $k^*(n)$.

On the other hand its effect on the efficient size of the social security transfer is ambiguous. Since the equation that gives equality of rates of return is satisfied for any value of capital when the tax rate is zero, there may be other steady states, of the form $(k^*, 0)$. For future comparison we characterize the behavior of the system for some specific pairs of utility and production functions.

Example 1 (Continue). As in example 1 of Section 4 let $u(c) = \log c$, and $f(k) = k^\alpha$. We get:

$$s = \frac{\delta \pi(1 - \tau)w - \tau'(1 + n)w'}{\pi(1 + \delta)}$$

where a "prime" denotes next period variable. By substituting the equilibrium values for w , w' and π we get the following expression for the dynamical system (6.2)

$$k_{t+1} = \frac{\alpha \delta (1 - \alpha)(1 - \tau_t)k_t^\alpha}{\alpha(1 + \delta)(1 + n) + (1 - \alpha)(1 + n)\tau_{t+1}}$$

$$\tau_{t+1} = \frac{\alpha \tau_t k_t^\alpha}{(1 + n)k_{t+1}}$$

There are two possible steady states:

$$(k_1^*, \tau_1^*) = \left(\left(\frac{\alpha}{(1+n)} \right)^{1/(1-\alpha)}, \frac{\delta}{(1+\delta)} - \frac{\alpha}{(1-\alpha)} \right);$$

and

$$(k_2^*, \tau_2^*) = \left(\left(\frac{\delta(1-\alpha)}{(1+\delta)(1+n)} \right)^{\frac{1}{1-\alpha}}, 0 \right).$$

when $\frac{\delta}{(1+\delta)} - \frac{\alpha}{(1-\alpha)}$ is positive. In this case the first pair is unstable, and the second stable.

Example 2 (Continue). The situation is very similar in the next example, which uses the utility and production functions considered in example 2 of section 4: $u(c) = c$, and $f(k) = k^\alpha$. The saving function is

$$s_t = s^*(k_t, \tau_t) = \min \{ (1-\tau_t)(1-\alpha)k_t^\alpha, (1+n_{t+1})(\alpha\delta)^{1/(1-\alpha)} \} \quad (6.4)$$

Together with the condition that equalizes the return on private saving to the return on social security payments, (6.4) yields a dynamical system for (τ_t, k_t)

$$k_{t+1} = \min \left\{ \frac{(1-\tau_t)(1-\alpha)k_t^\alpha}{(1+n)}, (\alpha\delta)^{1/(1-\alpha)} \right\}$$

$$\tau_{t+1} = \max \left\{ \frac{\alpha\tau_t}{(1-\alpha)(1-\tau_t)}, \frac{\alpha\tau_t k_t^\alpha}{(1+n)(\alpha\delta)^{1/(1-\alpha)}} \right\}$$

Setting aside the hairline case in which $\delta(1+n) = 1$, when $\alpha < 1/2$ the latter has two stationary states:

$$(k_1^*, \tau_1^*) = \left(\left(\frac{\alpha}{1+n} \right)^{1/(1-\alpha)}, \frac{1-2\alpha}{1-\alpha} \right)$$

and

$$(k_2^*, \tau_2^*) = \left(\left(\frac{1-\alpha}{1+n} \right)^{1/(1-\alpha)}, 0 \right).$$

When $\delta(1+n) > (1-\alpha)/\alpha$ the largest of the two steady states is smaller than $(\alpha\delta)^{1/(1-\alpha)}$, and so eventually the behavior around the two stationary states is governed by the system

$$k_{t+1} = \frac{(1-\tau_t)(1-\alpha)k_t^\alpha}{(1+n)}$$

$$\tau_{t+1} = \frac{\alpha\tau_t}{(1-\alpha)(1-\tau_t)}$$

The two eigenvalues defining the linear approximation near each of the steady states are respectively

$$\begin{aligned}\lambda_1(k_1^*, \tau_1^*) &= \alpha, & \lambda_2(k_1^*, \tau_1^*) &= \frac{1 - \alpha}{\alpha} \\ \lambda_1(k_2^*, \tau_2^*) &= \alpha, & \lambda_2(k_2^*, \tau_2^*) &= \frac{\alpha}{1 - \alpha}\end{aligned}$$

Since $\alpha < 1/2$ the first steady state is a saddle and the second a sink. The stable manifold of (τ_1^*, k_1^*) is then a vertical line in the (τ, k) space in correspondence to the value $\tau = \tau_1^*$. As we have seen, for this value of the parameters the efficient steady state taxes are positive, and equal to τ_2^* . So the efficient steady state is the unstable steady state. For all initial conditions (τ_0, k_0) such that $\tau_0 < \tau_1^*$ the equilibrium path converges to the stationary state (τ_2^*, k_2^*) without SoSeSy. When $\tau_0 = \tau_1^*$ it converges to the stationary state (τ_1^*, k_1^*) . There is no equilibrium that begins with $\tau_0 > \tau_1^*$, since the tax rate becomes larger than 1 in a finite number of periods. In the other case of $\alpha > 1/2$ only (k_2^*, τ_2^*) is an admissible steady state, hence the only efficient steady state, which is now unstable.

6.3 The political equilibria

In this section we show, by means of a simple example, that the political equilibria are typically inefficient, but may be so either because there is too much capital accumulation at equilibrium, or, an alternative possibility with different parameters, because there is too little capital accumulation.

The model we consider is the the one labelled so far as Example 2: linear utility and Cobb Douglas production function (see sections 4 and 6.2). The critical parameters turn out to be the capital share, α and the discount factor δ times the population growth rate $(1 + n)$. The reason for the inefficiency of the competitive equilibrium without social security transfers is unambiguous and well known: without transfers there is an overaccumulation of capital at equilibrium. This is obvious from the equation determining the value of the steady state capital, which is decreasing in τ ; so when it is efficient to have positive transfers it is so because the equilibrium capital will be reduced.

It is easy first of all to determine the relative position of the efficient tax rate with respect to the interval $[0, \tau_p]$. If we substitute the value $\frac{1-2\alpha}{1-\alpha}$ into the equation that determines τ_p we have that $\tau_p > \tau_e$ if and only if $\alpha < 1/2$; so that over the entire parameter region,

$$\tau_p \geq \tau_e.$$

Some easy calculus shows that τ_p is an increasing function of α , and that $\lim_{\alpha \rightarrow 0} \tau_p = 1$. There are therefore three completely different possibilities:

1. When $\alpha \in [\alpha^*, 1]$ the efficient level of tax rate is zero, but also the only tax rate that can be supported as political equilibrium is zero.
2. For all the values of $\alpha \in [1/2, \alpha^*]$ (and any δ), and for values of α and δ such that $\alpha < 1/2$ and $\delta(1+n) < 1$ there are many levels of tax rate that can be supported in a political equilibrium. In the same region the efficient level of tax rate is zero. So in this region any political equilibrium steady state with positive transfers is unambiguously inefficient, with a steady state level of capital stock too low.
3. The political equilibria are still inefficient, but for the opposite reason, in the rest of the parameter space. When $\alpha \leq 1/2$ and $\delta(1+n) > 1$ then the maximum tax rate that can be supported at equilibrium is higher than the efficient rate, and so the political equilibria can be inefficient because there is too much or too little capital accumulation.

7. Stochastic Population Growth

In the recent historical experience of many countries the immediate cause for the crisis of the social security system seems to be the long run fall in the population growth rate coupled with a relatively low growth rate of labor productivity. Since this is not an unexpected event, but rather quite foreseeable, and since it seems to lead to either the dismissal of the special pay-as-you-go version of the system or to a sharp reduction of the benefits paid to retired individuals, it is natural to ask if such an institution could be supported as the equilibrium result of rational decision making of voters, and if so, what predictions we can derive from this explanation. The model that we present now will provide a first answer to these questions. As in the rest of the paper we restrict attention to changes in the growth rate of population, the extension to exogenous labor-augmenting technological progress being immediate.

7.1 Logarithmic utility, linear production and stochastic population growth.

We reconsider here one of our examples (Example 1), modified to make the growth rate of population a stochastic process. More precisely we assume that there exists a sequence of growth rates $\{n(j)\}_{j=0}^{\infty}$ satisfying the restrictions $n(j+1) < n(j)$, for all j ,

and $\lim_{j \rightarrow \infty} n(j) = 0$, and a transition probability

$$\Pr (n_{t+1} = n(j); n_t = n(j)) = 1 - p,$$

$$\Pr (n_{t+1} = n(j + 1); n_t = n(j)) = p;$$

for all j , with $0 < p < 1$. The definition of equilibrium with a stochastic growth rate of the population is very similar to the one we have adopted for the models with a deterministic dynamic of the population growth rate. We refer the reader to Appendix 3. for a formal statement. We let

$$a > 1,$$

so that $a > 1 + n_t$, eventually almost surely. The strategy profiles of the political equilibrium are defined as follows. Let n_t be $n(j)$, then

$$\sigma_t(h_t) = (\mathcal{Y}, \tau(n(j)), \tau(n(j + 1)))$$

if for every $s < t$, $a_s = (\mathcal{Y}, \tau(n(i)), \tau(n(i + 1gr)))$, where $n_s = n(i)$. On the other hand the strategy of the player t sets

$$\sigma_t(h_t) = (\mathcal{N}, \tau(n(j)), \tau(n(j + 1)))$$

in all other cases. The proof of this statement can be found in Appendix 1 as part of the proof of the following characterization of the political equilibrium.

Proposition 8 *If $a > 1$, then for any political equilibrium of the generation player game, eventually almost surely $\tau_t = 0$ for every t : that is the social security system is terminated almost surely*

The proof is in Appendix 1.

Of course the proposition is interesting only if there are equilibria which give as an outcome tax rates which are not identically zero. This requires that at least in the initial periods the population growth rate is sufficiently high to make the return from the savings invested in the security system larger than the return from private investment. To avoid uninteresting situations we assume, in addition, that

$$a < (1 - p)(1 + n_0).$$

The proof of proposition 10 will make it clear why on the right hand side we do not have the expected growth rate.

We now prove that equilibria with positive tax rates and transfers in the initial periods exist, even if it is known that the system of social security will be eventually dismissed almost surely. As usual, the construction of an equilibrium rests on the comparison between the value of keeping the social security system, and the value of dropping it. In the model we are discussing the second value is a constant, v :

$$v = (1 + \delta) \log\left(\frac{b}{a(1 + \delta)}\right) + \delta \log(a\delta).$$

First we prove a simple proposition which provides the constructive methods through which equilibria are found:

Proposition 9 *There is a sequence of tax rates $\{\tau(j)\}_{j=0}^{\infty}$ such that*

$$\max_{s \geq 0} \left\{ \log[(1 - \tau(j))b - s] + \delta(1 - p) \log[as + (1 + n(j))\tau(j)b] + \right. \\ \left. + \delta p \log[as + (1 + n(j + 1))\tau(j + 1)b] \right\} = v \quad (7.1)$$

for all j 's, and $\tau(j) > 0$ for some j .

The proof is in Appendix 1.

It is based on backward induction and encompasses the following steps:

1. First, for any $n(j)$ we determine the value of τ (call it $\underline{\tau}(j)$) which satisfies the following condition: "If $\underline{\tau}(j)$ is to be paid to the old generation, then the program $(a, \underline{\tau}(j), 0)$ wins the elections." In other words, when $\underline{\tau}(j)$ is the promised tax rate a realization of the growth rate $n(j + 1)$ will trigger the collapse of the SoSeSy. Notice that for some $n(j)$'s such a tax rate may not exist in $[0, 1]$.
2. Let $n(j)$ be a growth rate for which a $\underline{\tau}(j)$ exists and call it the *least growth rate*. Taking $n(j)$ as given one can use (7.1) and proceed backward to determine the sequence of tax rates that yield an equality in each period.
3. Finally let the first young generation choose the initial value $\tau(0)$. This is accomplished by selecting the best among all the sequences of $\tau(j)$ which were computed in the previous step. Notice that each such sequence (and hence each initial $\tau(0)$) corresponds to a different least growth rate.

The equilibrium we construct has a stationary nature: to each possible population growth rate is associated a tax rate, which is positive as long as the population growth is higher than a critical level, and then drops to zero. At that point the system collapses.

When the population growth rate reaches the level which is immediately next to the critical rate the generation of young voters still prefers to keep the system going, even if they know that with positive probability they will pay and then will not be paid back by the next generation. Since the rate n_t is falling over time, it is not immediately clear how the equilibrium tax rates behave over time. We have:

Proposition 10 *The sequence of tax rates described in proposition 9 above is the outcome of a political equilibrium. On this equilibrium the sequence of tax rates is decreasing, that is*

$$\tau(j + 1) < \tau(j)$$

for all j .

8. Conclusions.

We have shown that a PAYG Social Security System may be supported as the sub-game perfect equilibrium of an infinite horizon game in which economic agents choose the contribution and benefit rates by majority voting in every period and competitive markets determine saving and consumption levels. No altruistic motivations are needed in our model to explain the existence of PAYG pension plans.

A majority voting equilibrium may lead to the establishment of social security transfers from the young to the old even in circumstances in which the competitive equilibrium would otherwise be converging to a consumption efficient steady-state. We conclude from this fact that when social security policies are determined through voting, one may expect them to be typically inefficient.

An interesting property of our model is the following. If a society faces an uncertain but asymptotically decreasing growth rate of the labor force the majority voting equilibrium will lead to the disappearance of the PAYG SoSeSy within a finite number of periods. The actual elimination of the SoSeSy will be voted in at the time in which a certain lower bound on the growth rate of the labor force is realized. Nevertheless we also show that in the meanwhile, i.e. until the least growth rate is not realized, it is perfectly rational for the median voter to maintain alive a SoSeSy. In this circumstances the majority voting system leads to a non-increasing sequence of taxes and benefits rates.

The model also suggests a number of interesting questions worth investigating. One would be interested in classifying the efficiency properties of the political equilibria and

the set of "constitutional restrictions" (if any) that might guarantee the maximization of one or another type of social welfare function by means of the majority voting system.

One would also like to verify the extent to which the political equilibrium with PAYG social security is modified by the introduction of income heterogeneity within each generation. A first step in this direction can be found in Tabellini (1990) who uses a static, two-period version of an OLG economy and shows that social security can be supported in a majority voting equilibrium by a coalition of old people and the poorest among the young ones. On the other hand he does not examine the dynamic implications of this extension of the redistributive features of a PAYG system to the intragenerational level and the impact this may have on the set of intertemporal equilibria. By providing further incentives to redistribute income, such a modification would certainly lead to an increase in the size of the SoSeSy as well as in the number of circumstances in which it may be adopted. On the other hand the reduction in saving this would cause may be large enough to bring about a collapse of the system more often than in the previous circumstances.

Finally one is interested in characterizing a more sophisticated type of redistributive political equilibrium in which people have to simultaneously vote upon a tax to finance SoSeSy and a tax to finance the public education system. This is considered in Boldrin and Rustichini (1995a).

9. Appendix

9.1 Appendix 1.

Proof of proposition 7. We first determine the competitive equilibrium steady state as a function of the parameters, for a fixed tax rate. The steady state value is given by $(\alpha\delta)^{\frac{1}{1-\alpha}}$ if:

$$\frac{(1-\alpha)(1-\tau)}{1+n} \geq \alpha\delta,$$

and

$$k_\tau = \left(\frac{(1-\alpha)(1-\tau)}{1+n} \right)^{\frac{1}{1-\alpha}}.$$

otherwise. The value τ_0 is determined by taking equality in the first inequality above and is equal to $\tau_0 = \max\left\{\frac{1-\alpha[1+\delta(1+n)]}{1-\alpha}, 0\right\}$. We now denote for convenience $k_{\tau_0} \equiv (\alpha\delta)^{\frac{1}{1-\alpha}}$, substitute the value of the steady state capital in the utility function of the representative agent, and obtain a function of τ which is defined piecewise, and we denote by $U(\tau)$. In fact when $\tau \leq \tau_0$ the function is equal to $(1-\alpha)k_{\tau_0}^\alpha [1-\tau + \delta(1+n)\tau]$; while when $\tau \geq \tau_0$ the function $U(\tau)$ is equal to $\delta[(1-\tau)w(k_\tau)\pi(k_\tau) + w(k_\tau)(1+n)\tau]$. Clearly the only thing we need to do is to compute the derivative of $U(\tau)$ in the interval $[\tau_0, 1]$. Algebra, and the use of the equation $\frac{dk_\tau}{d\tau} = -\frac{k_\tau^\alpha}{1+n}$ gives that this derivative has the same sign as $\frac{1-2\alpha}{\alpha} - \tau$. Our claim then follows. **Q.E.D.**

Proof of proposition 8. A necessary condition for a sequence of tax rates τ_t to be the outcome of a political equilibrium is that the value of keeping the system is larger in each period, and every realization of the (n_t) process, than the value of dropping it. Recall that the second and third components of $\sigma_t(h_t)$, denoted $\sigma_t^i(h_t)$, $i = 2, 3$, are the tax rates that the generation of young voters is setting for future periods in the event that the population growth rate does not change, or respectively does change. We denote these two rates τ^1 and τ^2 respectively. In our case, since the equilibrium saving function equals the solution of the maximization problem of the representative young consumer, this inequality is equivalent to:

$$\max_{s \geq 0} \log(b(1-\tau_t) + \delta(1-p)\log(as + b(1+n(j))\sigma^2(h_t) +$$

$$\delta p \log(as + b(1+n(j+1))\sigma^3(h_t)) \geq v;$$

where $n_t = n(j)$.

To prove our claim it is enough to prove that there is no solution to this infinite system of inequalities. Since the left hand side, keeping everything else fixed, is increasing in the population growth rate, and the rate itself reaches almost surely in finite time any value lower than the initial, it is enough to show that there is no solution to the same system when the population growth rate is fixed to a value n such that $a > 1 + n$. So it is enough to prove that the system

$$\max_{s \geq 0} \log(b(1 - \tau_t) + \delta(1 - p) \log(as + b(1 + n)\tau_{t+1}^1 + \delta p \log(as + b(1 + n)\tau_{t+1}^2) \geq v$$

does not have a solution extending in the infinite future. The latter defines implicitly a stochastic difference inclusion in the following way. Let first $\Phi(\tau_t)$ be the set of $(\tau^1, \tau^2) \in [0, 1]^2$ such that the inequality is satisfied. This Φ plays here a role similar to the correspondence of the same name defined in section 5.

The stochastic difference inclusion is defined now as: $\tau_{t+1} \in \Psi(\tau_t)$, where

$$\Psi(\tau_t) = \{\tau^1 : (\tau^1, \tau^2) \in \Phi(\tau_t), \text{ for some } \tau^2\}, \text{ if } n_{t+1} = n_t;$$

and

$$\Psi(\tau_t) = \{\tau^2 : (\tau^1, \tau^2) \in \Phi(\tau_t), \text{ for some } \tau^1\}, \text{ if } n_{t+1} < n_t.$$

We prove that for any path τ_t which is not identically zero the correspondence $\Phi(\tau_t)$ is empty valued in finite time, almost surely. The proof will extend to the present case the argument that proves a similar statement for the deterministic case, discussed in section 5.

Let us list the properties of Φ that will be used in the sequel: 1. The set $\Phi(\tau)$ is convex: this follows from the fact that the function $\tau \rightarrow \log(as + b(1 + n)\tau)$ is concave. Also it is easy to show by implicit differentiation that at the point of intersection between the diagonal in $[0, 1]^2$ and the boundary of this set the supporting line to the set has normal vector proportional to $(1 - p, p)$. 2. Let ϕ be the function defined in section 5.1. We know that its graph lies above the diagonal. Also, from the convexity of the image of Φ :

$$\Phi(\tau) \subseteq \{(\tau^1, \tau^2) : (1 - p)\tau^1 + p\tau^2 \geq 0\} + (\phi(\tau), \phi(\tau)).$$

Addition of sets is defined as usual, element by element. Now from the fact that $\phi(\tau) > 1$ for τ large enough (precisely, for $\tau > 1 - \epsilon^C$, where C is the constant defined in section 5.1), our claim follows. **Q.E.D.**

Proof of proposition 9. Define

$$\Psi_1(\tau, n) \equiv \max_s \left\{ \log[(1 - \tau)b - s] + \delta(1 - p) \log[as + (1 - n)\tau b] + \delta p \log as \right\}$$

This is the lifetime utility achievable by a young generation starting with $n_t = n$ and $\tau_t = \tau$ and expecting to receive τ if $n_{t+1} = n$ and 0 if $n_{t+1} < n$. Notice that the equilibrium saving level $s^*(\tau)$ is positive if $(1 - \tau)b > 0$ and that $s^*(0) = \frac{\delta b}{1 + \delta}$. Of course $\Psi_1(0, n) = v$ for any n ; but also the function Ψ_1 satisfies $\lim_{\tau \rightarrow 1} \Psi_1(1, n) = -\infty$ and

$$\frac{\partial \Psi_1}{\partial \tau}(0, n) = \frac{b\delta}{s^*(0)} \left(\frac{(1 - p)(1 + n)}{a} - 1 \right) > 0$$

if and only if $\frac{a}{1+n} < 1 - p$. Our assumptions imply that this is true for a non-empty but finite set of population growth rates in the sequence $n(j)$. Now if we take any such $n(j)$ to be the value of n in the function Ψ_1 , we conclude that a $\underline{\tau}(j) > 0$ exists for any j such that $\Psi_1(\underline{\tau}(j), n(j)) = v$.

We move now to the backward induction construction of the tax rates. Take $n(j)$ and $\underline{\tau}(j)$ as defined in the previous step. Recall that this implies that when $n(j + 1)$ is realized the equilibrium tax is zero and the system collapses. For $i = 1, 2, \dots$ assume that the value of $\tau(j - i)$ which solves

$$\max_s \left\{ \log[(1 - \tau(j - i))b - s] + \delta(1 - p) \log[as + (1 + n(j - i))\tau(j - i)b] + \right. \\ \left. + \delta p \log[as + (1 + n(j - i + 1))\underline{\tau}(j - i + 1)b] \right\} = v$$

has been found. We look for the next value $\tau(j - i - 1)$ to be associated to $n(j - i - 1)$ by solving

$$\Psi_2(\tau, n(j - i - 1)) - v = 0$$

where

$$\Psi_2(\tau, n) \equiv \max_s \left\{ \log[(1 - \tau)b - s] + \delta(1 - p) \log[as + (1 + n)\tau b] + \right. \\ \left. + \delta p \log[as + (1 + n(j - i))\underline{\tau}(j - i)b] \right\} = v$$

Note that

$$\Psi_2(1, n) = -\infty$$

and

$$\Psi_2(0, n) > \max_s \log[b - s] + \delta \log as = v$$

hence a solution $\tau(j - i - 1)$ always exists. Repeating this procedure for all $i = 1, \dots, j - 1$ and all $n(j)$ generates a the required sequence of non-zero tax rates. **Q.E.D.**

Proof of proposition 10. As usual given the equality between the value of keeping the social security system and the value of dropping it, a sequence of equilibrium strategies is easy to define, and is given in the main text.

We turn to the real issue, of proving that the equilibrium sequence of tax rates is decreasing over time; in fact, we prove the statement for any sequence $(\tau(j))$ satisfying the equality stated in proposition 9. First we fix some notation: j_0 is the index for the least growth rate to which is associated a positive tax rate; so $\tau(j_0) > 0$, and $\tau(j) = 0$ for any $j \geq j_0 + 1$. We also denote the function $\Psi(s; n(j), \tau(j), n(j+1), \tau(j+1))$ as:

$$\log[(1 - \tau(j))b - s] + \delta(1 - p) \log[as + (1 + n(j))\tau(j)] + \delta p \log[as + (1 + n(j+1))\tau(j+1)b].$$

For future use, we now note an obvious but important property of Ψ . The function $(s, \tau) \rightarrow \Psi(s; n(j), \tau, n(j+1), \tau(j+1))$ is concave, since it is the composition of concave and linear functions. Therefore the function: $\tau \rightarrow \max_{s \geq 0} \Psi(s; n(j), \tau, n(j+1), \tau(j+1))$ is also concave.

We now proceed with the proof of our main claim. The proof is by induction on the index j . We begin with the inequality

$$\tau(j_0) \leq \tau(j_0 - 1).$$

Note first that for any s and τ the following inequality is immediate from the definition of Ψ : (recall that $\tau(j_0 + 1) = 0$):

$$\Psi(s; n(j_0), \tau, n(j_0 + 1), \tau(j_0 + 1)) \leq \Psi(s; n(j_0 - 1), \tau, n(j_0), \tau(j_0)).$$

Hence the same inequality is preserved by taking $\max_{s \geq 0}$ on both sides: this operation gives us two functions of τ , Ψ_{j_0} and Ψ_{j_0-1} . say, with the first function less or equal to the function on the second, pointwise over $[0, 1]$. We have already seen that these two functions are equal to v for at least one value of τ (in fact, we have already substituted one of these values, $\tau(j_0)$, in the corresponding Ψ). From the fact that they are concave we now know that this value is unique; and from the inequality between the two functions we know that the value for the function Ψ_{j_0} , $\tau(j_0)$ is less or equal to the value of the function Ψ_{j_0-1} at $\tau(j_0 - 1)$.

Now for the other tax rates. By the induction hypothesis we have the inequality $\tau(j+1) \leq \tau(j)$, and we now claim that $\tau(j) \leq \tau(j-1)$. The induction hypothesis, together with the inequality on growth rates of population, gives the inequality:

$$\Psi(s; n(j), \tau, n(j+1), \tau(j+1)) \leq \Psi(s; n(j-1), \tau, n(j), \tau(j))$$

for any τ . The argument now is identical to the one given in the first step, and we conclude our proof. **Q.E.D.**

9.2 Appendix 2.

Recall that under our assumption on the parameters, from any initial capital stock larger than k^3 the competitive equilibrium sequence, irrespective of the tax rate, has value less than k^3 after one period; so we are looking for the highest tax rate that makes the function V larger than v , conditional on the next period tax rate being equal to τ_p . Let us now begin to describe the first region. Here the inequality $V(k, \tau, \tau_p) \geq v(k)$ is found to be equivalent to

$$\delta(1+n)(\alpha\delta)^{\frac{\alpha}{1-\alpha}}\tau_p \geq \tau k^\alpha.$$

The equality determines a function from τ to k ; it is immediate that this function is decreasing in τ . The same remarks we just made hold in the case of the second region. Here the inequality between V and v , again conditional on the next period tax rate being equal to τ_p , is equivalent to:

$$\delta(1+n)[\alpha + (1-\alpha)\tau_p] \geq (1-\alpha)^{1-\alpha}k^{\alpha(1-\alpha)}(1-\tau)^{-\alpha}.$$

Calculus shows that the equality determines a function from tax rate to capital stock, decreasing, and with value at τ_p equal to k^3 , as it should.

9.3 Appendix 3.

Here we provide the relevant definitions for the model with stochastic population growth rates. They follow closely the lines of the definitions for the deterministic model.

A *tax process* is a measurable function from the history of the economy to $[0, 1]$.

Definition 3 For a given pair of processes of population growth rates and tax rates, a **competitive equilibrium** is a process $(w_t, \pi_t, c_t^l, c_{t+1}^l)$ such that, almost surely:

- (1) the equilibrium conditions for the firms in section 2 above are satisfied;
- (2) the saving process (s_t) maximizes the expected utility of the representative young agent, conditional on the history;
- (3) markets clear.

On the basis of this, we say that:

Definition 4 A generation player game is the extensive form game where

- (1) players are indexed by $t \in (1, 2, \dots)$;
- (2) the action set of each player is $\{\mathcal{Y}, \mathcal{N}\} \times [0, 1]^2$;
- (3) for every history of the population growth rates and actions, the payoff to the generation player t is the expected utility at the corresponding competitive equilibrium conditional on that history.

The element (Y, τ_1, τ_2) of the action set is to be interpreted as follows: the generation of young players accepts to pay the tax rate set in the previous period, and sets a transfer rate (to themselves) of τ_1 in the event that the population growth rate stays constant, and τ_2 otherwise.

We also remark that histories are now a list of past actions of the generation players, and of the outcome of the population growth rate in the past; to be precise by a history at time t we mean an element of the form $(n_1, a_1, \dots, a_{t-1}, n_t)$. Finally we say that:

Definition 5 For a given stochastic process of the population, (n_t) , a **political equilibrium** is a process $(\tau_t, w_t, \pi_t, c_t^l, c_{t+1}^l)$ such that:

- (1) the process $(w_t, \pi_t, c_t^l, c_{t+1}^l)$ is a competitive equilibrium for the given population and tax processes;
- (2) there exists a sequence (σ_t) of strategies of the generation player game which is sub-game perfect equilibrium and gives as outcome the process (τ_t) .

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