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Three Essays on Game Theory

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To my parents

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CHAPTER 1

Introduction

The main text of this thesis is divided into three chapters. The three papers are contributions to the literature on equilibrium refinements in non-cooperative game theory. Each chapter can be read independently of the rest.

Chapter 2 characterizes the class of finite extensive forms for which the sets of Subgame Perfect and Sequential equilibrium strategy profiles coincide for any possible payoff function. In addition, it identifies the class of finite extensive forms for which the outcomes induced by these two solution concepts coincide, and study the implications of our results for perfect Bayesian equilibrium.

Chapter 3 shows that in games with population uncertainty some perfect equilibria are in dominated strategies. It is proved that every Poisson game has at least one perfect equilibrium in undominated strategies.

Chapter 4 shows that the set of probability distributions over networks induced by Nash equilibria of the network formation game proposed by Myerson (1991) is finite for a generic assignment of payoffs to networks. The same result can be extended to several variations of the game found in the literature.

CHAPTER 2

Conditions for Equivalence Between Sequentiality and Subgame Perfection¹

2.1. Introduction

Analysis of backward induction in finite extensive form games provides useful insights for a wide range of economic problems. The basic idea of backward induction is that each player uses a best reply to the other players' strategies, not only at the initial node of the tree, but also at any other information set.

To capture this type of rationality Selten (1965) defined the subgame perfect equilibrium concept. While subgame perfection has some important applications, it does not always eliminate irrational behavior at every information set. In order to solve this problem, Selten (1975) introduced the more restrictive notion of "trembling-hand" perfection.

Sequential equilibrium, due to Kreps and Wilson (1982), requires that every player maximizes her expected payoff at every information set, according to some consistent beliefs. They showed that "trembling-hand" perfection implies sequentiality, which in turn implies subgame perfection. They also proved that for generic payoffs, almost all sequential equilibrium strategies are "trembling-hand" perfect, a result that was strengthened by Blume and Zame (1994) who proved that for a fixed extensive form and generic payoffs it is the case that the two concepts coincide.

¹This chapter is based on Gonzalez Pimienta and Litan (2005).

Although it is a weaker concept than Selten's perfection, Kohlberg and Mertens (1986) note that "sequential equilibrium seems to be the direct generalization [of backward induction] to games of imperfect information". It fulfills all the properties that characterize subgame perfection (backward induction) in games of perfect information. This is no longer true with different concepts like perfect or proper equilibrium.²

In this paper we find the maximal set of finite extensive forms (extensive games without any payoff assignment) for which sequential and subgame perfect equilibrium yield the same set of equilibrium strategies, for every possible payoff function (Proposition 2.1). It can be characterized as the set of extensive forms, such that for any behavior strategy profile every information set is reached with positive probability conditional on the smallest subgame that contains it. Whenever the extensive form does not have this structure, payoffs can be assigned such that the set of subgame perfect equilibria does not coincide with the set of sequential equilibria.

However, it may still happen that the set of equilibrium outcomes of both concepts coincides for any possible assignment of the payoff function. Thus, we also identify the maximal set of finite extensive forms for which subgame perfect and sequential equilibrium always yield the same equilibrium outcomes (Proposition 2.2).

In many applications of extensive games with incomplete information, the so called "perfect Bayesian equilibrium" is used. It places no restrictions at all on beliefs off the equilibrium path of every subgame. Hence, it implies subgame perfection and it is implied by sequential equilibrium. We obtain as corollaries that our equivalence conditions remain true if we substitute sequential for perfect Bayesian.

²See Kohlberg and Mertens (1986) for details.

Notice that, unlike related results on equivalence between refinements of Nash equilibrium, where the object of analysis is the payoff space (e.g. Kreps and Wilson (1982), Blume and Zame (1994)), we find conditions on the game form. Our results characterize the information structures where applying sequential rationality does not make a relevant difference with respect to subgame perfection. We consider them as tools for economic modelling. They allow us to know if, for the extensive game under study, subgame perfect and sequential equilibrium are always equivalent, either in equilibrium strategies or in equilibrium outcomes.

The paper is organized as follows: in Section 2.2 we briefly introduce the main notation and terminology of extensive form games. This closely follows van Damme (1991). Section 2.3 contains definitions. Results are formally stated and proved in Section 2.4. In Section 2.5 we give some examples where our results can be applied.

2.2. Notation and Terminology

The analysis is restricted to finite extensive form games with perfect recall. Since our characterization is based on the structural properties of extensive games, we cannot dispose of a complete formal description of extensive form games. However, and in consideration with those readers who are already familiar with extensive games, we relegate such a long discussion to the appendix and only offer in Figure 2.1 a brief list with very terse explanations of the symbols that we require.

We need the following definitions before moving to the next section.

If $x \in X$, let \mathbb{P}_x^b denote the probability distribution on Z if the game is started at x and the players play according to the strategy profile b . Given a system of beliefs μ , a strategy profile b and an information set u , we define the probability distribution $\mathbb{P}_u^{b,\mu}$ on Z as $\mathbb{P}_u^{b,\mu} = \sum_{x \in u} \mu(x) \mathbb{P}_x^b$.

Notation	Terminology	Comments
Ξ	Extensive form	Extensive game without payoff assignment
T	Set of nodes in Ξ	Typical elements $x, y \in T$
\leq	Precedence relation on T	\leq partially orders T
U_i	Player i 's information sets	Typical elements $u, v, w \in U_i$
C_u	Choices available at u	Typical elements $c, d, e \in C_u$
Z	Set of final nodes	$\{z \in T : \nexists x \in T \text{ s.t. } z < x\}$
X	Set of decision nodes	$X = T \setminus Z$
r_i	Player i 's payoff function	$r_i : Z \rightarrow \mathbb{R}, r = (r_1, \dots, r_n)$
Γ	n -player extensive game	$\Gamma = (\Xi, r)$
b_i	Player i 's behavioral strategy	$b_i \in B_i, b = (b_1, \dots, b_n)$
\mathbb{P}^b	Probability measure on Z	Induced by b
$R_i(b)$	Player i 's expected utility at b	$\sum_{z \in Z} \mathbb{P}^b(z) r_i(z)$
$Z(A)$	Final nodes coming after A	$A \subseteq T$
$\mathbb{P}^b(A)$	Probability of $A \subseteq T$	$\mathbb{P}^b(Z(A))$
Ξ_y	Subform starting at y	Subgame without payoff assignment
Γ_y	Subgame starting at y	$\Gamma_y = (\Xi_y, \hat{r})$
μ	System of beliefs	$\mu(\cdot) \geq 0, \sum_{x \in u} \mu(x) = 1, \forall u$

FIGURE 2.1. Notation and terminology of finite extensive games with perfect recall

These probability distributions allow us to compute expected utilities at parts of the extensive game other than the initial node, already considered in $R_i(b)$. Define $R_{ix}(b) = \sum_{z \in Z} \mathbb{P}_x^b(z) r_i(z)$ as player i 's expected payoff at node x . In a similar fashion, $R_{iu}(b) = \sum_{z \in Z} \mathbb{P}^b(z|u) r_i(z) = \sum_{x \in u} \mathbb{P}^b(x|u) R_{ix}(b)$ is player i 's expected payoff at every information set u such that $\mathbb{P}^b(u) >$

0. Furthermore, under the system of beliefs μ , $R_{iu}^\mu(b) = \sum_{z \in Z} \mathbb{P}_u^{b, \mu}(z) r_i(z)$ denotes player i 's expected payoff at the information set u .

2.3. Definitions

We use the substitution notation $b \setminus b'_i$ to denote the strategy profile in which all players play according to b , except player i who plays b'_i . The strategy b_i is said to be a best reply against b if it is the case that $b_i \in \arg \max_{b'_i \in B_i} R_i(b \setminus b'_i)$. If $\mathbb{P}^b(u) > 0$, we say that the strategy b_i is a best reply against b at the information set $u \in U_i$ if it maximizes $R_{iu}(b \setminus b'_i)$ over the domain where it is well defined.

The strategy b_i is a best reply against (b, μ) at the information set $u \in U_i$ if $b_i \in \arg \max_{b'_i \in B_i} R_{iu}^\mu(b \setminus b'_i)$. If b_i prescribes a best reply against (b, μ) at every information set $u \in U_i$, we say that b_i is a sequential best reply against (b, μ) . The strategy profile b is a sequential best reply against (b, μ) if it prescribes a sequential best reply against (b, μ) for every player.

With this terminology at hand we define several equilibrium concepts.

DEFINITION 2.1 (Nash Equilibrium). A strategy profile $b \in B$ is a Nash equilibrium of Γ if every player is playing a best reply against b .

We denote by $\text{NE}(\Gamma)$ the set of Nash equilibria of Γ . Subgame perfection refines the Nash equilibrium concept by requiring a Nash equilibrium in every subgame. Formally,

DEFINITION 2.2 (Subgame Perfect Equilibrium). A strategy profile b is a subgame perfect equilibrium of Γ if, for every subgame Γ_y of Γ , the restriction b_y constitutes a Nash equilibrium of Γ_y .

We denote by $\text{SPE}(\Gamma)$ the set of subgame perfect equilibria of Γ . We write $\text{SPEO}(\Gamma) = \{\mathbb{P}^b : b \in \text{SPE}(\Gamma)\}$ for the set of subgame perfect equilibrium outcomes, and $\text{SPEP}(\Gamma) = \{R(b) : b \in \text{SPE}(\Gamma)\}$ for the set of subgame perfect equilibrium payoffs, where $R(b) = (R_1(b), \dots, R_n(b))$.

Sequential rationality is a refinement of subgame perfection. Every player must maximize at every information set according to her beliefs about how the game has evolved so far. If b is a completely mixed strategy profile, beliefs are perfectly defined by Bayes' rule. Otherwise, beliefs should meet a consistency requirement. A sequential equilibrium is an assessment that satisfies such a consistency requirement together with an optimality requirement. This is formalized by the next two definitions.

DEFINITION 2.3 (Consistent Assessment). An assessment (b, μ) is consistent if there exists a sequence $\{(b_t, \mu_t)\}_t$, where b_t is a completely mixed strategy profile and $\mu_t(x) = \mathbb{P}^{b_t}(x|u)$ for $x \in u$, such that $\lim_{t \rightarrow \infty} (b_t, \mu_t) = (b, \mu)$.

DEFINITION 2.4 (Sequential Equilibrium). A sequential equilibrium of Γ is a consistent assessment (b, μ) such that b is a sequential best reply against (b, μ) .

If Γ is an extensive game, we denote by $\text{SQE}(\Gamma)$ the set of strategies b such that (b, μ) is a sequential equilibrium of Γ , for some μ . Moreover, $\text{SQEO}(\Gamma) = \{\mathbb{P}^b : b \in \text{SQE}(\Gamma)\}$ denotes the set of sequential equilibrium outcomes and $\text{SQEP}(\Gamma) = \{R(b) : b \in \text{SQE}(\Gamma)\}$ the set of sequential equilibrium payoffs. Recall that $\text{SQE}(\Gamma) \subseteq \text{SPE}(\Gamma)$ for any game Γ .

We now introduce some new definitions that are needed for the results.

DEFINITION 2.5 (Minimal Subform of an Information Set). Given an information set u , the minimal subform that contains u , to be denoted $\Xi(u)$, is the subform Ξ_y that contains u and does not properly include any other subform that contains u .

We say that $\Gamma_y = (\Xi_y, \hat{r})$ is the minimal subgame that contains u if Ξ_y is the minimal subform that contains u .

In a given extensive form there are information sets that are always reached with positive probability. When this does not happen we say that the information set is avoidable, formally:

DEFINITION 2.6 (Avoidable information set). An information set u is avoidable in the extensive form Ξ if $\mathbb{P}^b(u) = 0$, for some $b \in B$. Likewise, we say that the information set u is avoidable in the subform Ξ_y if $\mathbb{P}_y^b(u) = 0$, for some $b \in B$.

For reasons that will become clear in the next section, we are interested in identifying extensive games where no information set is avoidable in its minimal subform. To get an idea about the set of extensive forms that we have in mind consider Figures 2.2 and 2.3. In the former, no information set is avoidable in the extensive form. While in the latter, no information set is avoidable in its minimal subform.

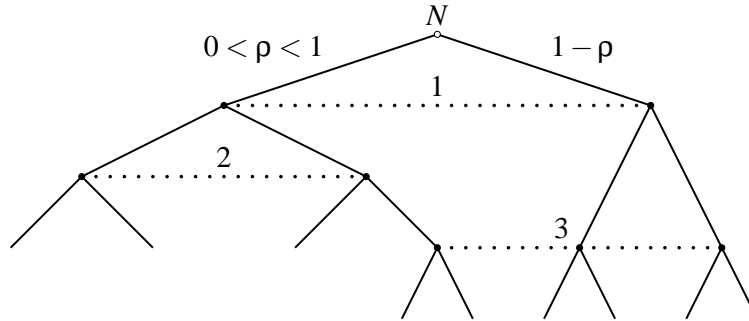


FIGURE 2.2. Extensive form where no information set is avoidable.

Conversely, consider Figure 2.4. Player 2's information set is avoidable in the extensive form (also in its minimal subform since the entire game is the only proper subgame) because player 1 can decide not to let her move.

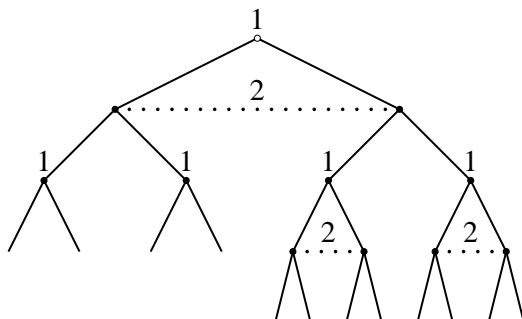


FIGURE 2.3. Extensive form where no information set is avoidable in its minimal subform.

2.4. Results

The three “best reply” concepts introduced in Section 2.3 relate to each other, as it is shown in the first two statements of the next lemma. The third assertion of the same lemma shows that maximizing behavior at an information set is independent of the subgame of reference.

LEMMA 2.1. *Fix a game $\Gamma = (\Xi, r)$. The following assertions hold:*

- (1) *Given a strategy profile b , if $u \in U_i$ is such that $\mathbb{P}^b(u) > 0$ and b_i is a best reply against b , then b_i is a best reply against b at the information set u .*
- (2) *Given a consistent assessment (b, μ) , if $u \in U_i$ is such that $\mathbb{P}^b(u) > 0$ and b_i is a best reply against b at the information set u , then b_i is a best reply against (b, μ) at the information set u .*
- (3) *If Γ_y is the minimal subgame that contains u and (b_y, μ_y) is the restriction of some assessment (b, μ) to Γ_y , then b_i is a best reply against (b, μ) at the information set u in the game Γ if and only if $b_{y,i}$ is a best reply against (b_y, μ_y) at the information set u in the game Γ_y .*

PROOF. Part 1 is known.³ Proofs for 2 and 3 are trivial. \square

In the next proposition we identify the set of extensive forms where sequential equilibrium has no additional bite over subgame perfection. The latter concept allows for the play of non-credible threats at information sets that might never be reached conditional on its minimal subgame. However, if we restrict attention to extensive form games where no information set is avoidable in its minimal subform, we can use the previous lemma to show that sequential and subgame perfect equilibrium coincide.

It turns out that not only is this particular restriction sufficient but also necessary for the equivalence, in the following sense: we can always find a payoff assignment so that the sets of subgame perfect and sequential equilibrium differ when the restriction fails to hold. The construction of such payoff assignment is based on, first, taking one information set that is avoidable in its minimal subform out of one subgame perfect equilibrium path and, second, making one of the available actions at this avoidable information set a strictly dominated action. Take for instance the game contained in Figure 2.4. If player 1 moves *Out* she gives player 2 the possibility of taking the strictly dominated move *H*, which forms a subgame perfect equilibrium which is not sequential.

PROPOSITION 2.1. *Let Ξ be an extensive form such that no information set u is avoidable in $\Xi(u)$. Then for any possible payoff vector r , the game $\Gamma = (\Xi, r)$ is such that $\text{SPE}(\Gamma) = \text{SQE}(\Gamma)$. Conversely, if Ξ is an extensive form with an information set u that is avoidable in $\Xi(u)$, then we can find a payoff vector r such that for the game $\Gamma = (\Xi, r)$, $\text{SPE}(\Gamma) \neq \text{SQE}(\Gamma)$.*

³For instance, see van Damme (1991), Theorem 6.2.1.

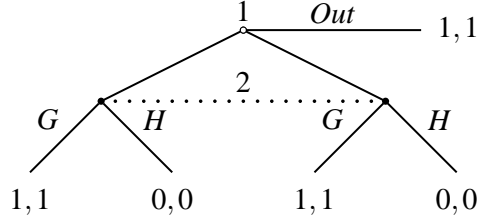


FIGURE 2.4. Example of the use of the algorithm contained in the proof of Proposition 2.1 to generate a game where $\text{SPE}(\Gamma) \neq \text{SQE}(\Gamma)$.

PROOF. Let us prove the first part of the proposition. We only have to show that $\text{SPE}(\Gamma) \subseteq \text{SQE}(\Gamma)$. Consider $b \in \text{SPE}(\Gamma)$ and construct a consistent assessment (b, μ) .⁴ We have to prove that the set

$$(1) \quad \tilde{U}(b, \mu) = \bigcup_{i=1}^n \left\{ u \in U_i : b_i \notin \arg \max_{\tilde{b}_i \in B_i} R_{iu}^\mu(b \setminus \tilde{b}_i) \right\}$$

is empty. Assume to the contrary that $\tilde{U}(b, \mu) \neq \emptyset$, and consider $u \in \tilde{U}(b, \mu)$. Let Γ_y be the minimal subgame that contains u and let j be the player moving at u . By lemma 2.1.3, $b_{y,j}$ is not a best reply against (b_y, μ_y) at u in the game Γ_y . Part 2 implies either that $\mathbb{P}_y^b(u) = 0$ or that $b_{y,j}$ is not a best reply against b_y at u . If the latter was true, part 1 would anyway imply that $\mathbb{P}_y^b(u) = 0$. However, u is not avoidable in Ξ_y . This provides the contradiction.

⁴A general method to define consistent assessments (b, μ) for any given $b \in B$, in an extensive form, is the following: take a sequence of completely mixed strategy profile $\{b_t\}_t \rightarrow b$ and for each t , construct $\mu^t(x) = \mathbb{P}^{b_t}(x|u) \in [0, 1]$, $\forall x \in u$, for all information sets u . Call $k = |X \setminus P_0|$. The set $[0, 1]^k$ is compact and since $\mu^t \in [0, 1]^k, \forall t$, there exists a subsequence of $\{t\}$, call it $\{t_j\}$, such that $\{\mu^{t_j}\}_{t_j}$ converges in $[0, 1]^k$. Define beliefs as $\mu = \lim_{j \rightarrow \infty} \mu^{t_j}$.

Let us now prove the second part of the proposition. Suppose $u \in U_i$ is an information set that is avoidable in $\Xi(u)$ and let $c \in C_u$ be an arbitrary choice available at u . Assign the following payoffs:

$$(2) \quad \begin{cases} r_i(z) = 0 & \forall i \text{ if } z \in Z(c) \\ r_i(z) = 1 & \forall i \text{ elsewhere.} \end{cases}$$

Clearly any strategy $b_i = b_i \setminus c$ cannot be part of a sequential equilibrium since playing a different choice at u gives player i strictly higher expected payoff at that information set.

We now have to show that there exists a subgame perfect equilibrium b such that $b_i = b_i \setminus c$. By assumption there exists b' such that $\mathbb{P}_y^{b'}(u) = 0$ in the minimal subgame Γ_y that contains u . The equality $\mathbb{P}_y^b(u) = 0$ also holds for $b = b' \setminus c$. The strategy profile b_y is a Nash equilibrium of Γ_y since nobody can obtain a payoff larger than one. By the same argument, b induces a Nash equilibrium in every subgame, hence it is a subgame perfect equilibrium. This completes the proof. \square

We use the extensive form of Selten's horse game (Figures 2.5 and 2.6) to show that the algorithm (used in the proof of the second part of Proposition 2.1) does not depend either on the particular avoidable information set, or on the particular choice that is taken to construct the payoffs. Information set u in the algorithm corresponds to player 2's (player 3's) information set in Figure 2.5 (Figure 2.6), and choice $c \in C_u$ in the algorithm corresponds to choice B (choice R) in Figure 2.5 (Figure 2.6).

Notice that the payoff assignment in the previous proof yields a difference in equilibrium strategies but not in equilibrium payoffs. The reason is that we cannot always achieve difference in equilibrium outcomes (therefore, neither in equilibrium payoffs). Figure 2.7 contains an extensive form where the second information set of player 1 is avoidable in its minimal

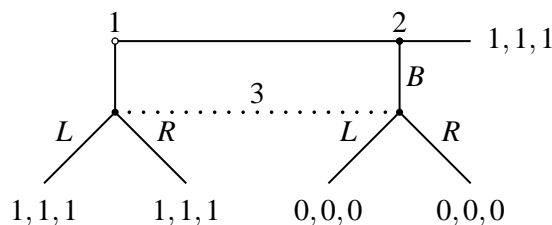


FIGURE 2.5. Selten's horse. An example of the use of the algorithm contained in the proof of proposition 2.1 to generate a game where $\text{SPE}(\Gamma) \neq \text{SQE}(\Gamma)$.

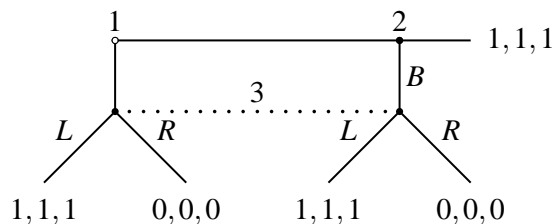


FIGURE 2.6. Selten's horse. A different use of the algorithm contained in Proposition 2.1.

subform, and nevertheless, the sets of sequential and subgame perfect equilibrium outcomes always coincide, regardless of what the payoffs assigned to final nodes are. Proposition 2.2 provides a sufficient and necessary condition for the sets of equilibrium outcomes (also, of equilibrium payoffs) to be equal for any conceivable payoff function.

Before that, we need to be able to identify which players can avoid a given information set. Let u be an information set and let $\Xi_y = \Xi(u)$. Construct the set of strategies $B(u) = \{b \in B : \mathbb{P}_y^b(u) > 0\}$.

DEFINITION 2.7. We say that the information set u can be avoided in $\Xi(u)$ by player i if there exists a strategy profile $b \in B(u)$, and a choice $c \in C_v$, with $v \in U_i$, such that $\mathbb{P}_y^{b \setminus c}(u) = 0$.

Remember that for an information set u that is avoidable in $\Xi(u) = \Xi_y$ there must be a strategy profile b such that $\mathbb{P}_y^b = 0$ (Definition 2.6). If a player, say player i , is able to unilaterally modify a strategy profile b' for which $\mathbb{P}_y^{b'} > 0$, by changing only one of her choices, and hereby construct one b for which $\mathbb{P}_y^b = 0$, then we say that the information set u can be avoided in $\Xi(u)$ by player i . Therefore, associated with any information set, there is a (possibly empty) list of players who can avoid it in its minimal subform. Figure 2.7 is an example of an extensive form where for every information set such a list is either empty or contains only the owner of the information set. When this happens, sequential equilibrium has no additional bite over subgame perfection regarding equilibrium outcomes. The reason is that subgame perfection allows a player to choose actions suboptimally, but given the particular structure of the game form, it can only happen at information sets already avoided by her own previous behavior, and choices at such information sets do not affect the outcome of the game.

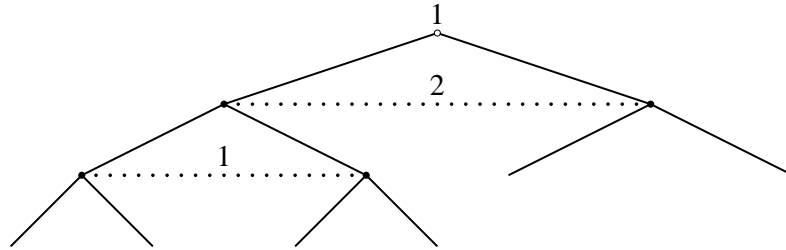


FIGURE 2.7. The second information set of player 1 can only be avoided by player 1. Proposition 2.2 implies that $\text{SPEP}(\Gamma) = \text{SQEP}(\Gamma)$.

This condition is also necessary for equivalence in equilibrium outcomes in the following sense: if player i can avoid the information set u in its minimal subform, and if j is the owner of the information set u , there exists a payoff assignment so that player j can “non-credibly” threaten player i (something ruled out by sequential equilibrium but not by subgame perfection) bringing about the difference in equilibrium outcomes.

The following lemma is useful for the proof of Proposition 2.2.

LEMMA 2.2. *Let Ξ be an extensive form such that, whenever an information set u is avoidable in $\Xi(u)$, it can only be avoided in $\Xi(u)$ by its owner. Let (b, μ) and (b', μ') be two consistent assessments. If b and b' are such that $\mathbb{P}_y^b = \mathbb{P}_y^{b'}$ for every subform Ξ_y , then $\mu = \mu'$.*

PROOF. Let (b, μ) and (b', μ') be two consistent assessments such that $\mathbb{P}_y^b = \mathbb{P}_y^{b'}$ for every subform Ξ_y . Note that b' can be obtained from b by changing behavior at information sets that are reached with zero probability within their minimal subform. Hence, without loss of generality, let b and b' differ only at one such information set, say $u \in U_i$, and let $\Xi_y = \Xi(u)$. The shift from b to b' may cause a change in beliefs only at information sets that come after u and are in the same minimal subform Ξ_y . Let $v \in U_j$ be one of those information sets.

If $j = i$, perfect recall and consistency imply that there is no change in beliefs at the information set v . If $j \neq i$ there are two possible cases, either $\mathbb{P}_y^b(v) > 0$ or $\mathbb{P}_y^b(v) = 0$. In the first case the beliefs at v are uniquely defined, therefore, $\mu(x) = \mu'(x), \forall x \in v$ and moreover, $\mu(x) = \mu'(x) = 0, \forall x \in v$ such that $u < x$. In the second case, since the information set v can only be avoided by player j in $\Xi(u)$ there exists a choice $c \in C_w$ of player j such that $\mathbb{P}_y^{b \setminus c}(v) > 0$, otherwise player i would also be able to avoid the information set u in $\Xi(u)$. Let $b'' = b \setminus c$ and $b''' = b' \setminus c$, then by the discussion of the first case, $\mu''(x) = \mu'''(x), \forall x \in v$, furthermore, perfect recall and consistency

imply $\mu''(x) = \mu(x)$ and $\mu'''(x) = \mu'(x), \forall x \in v$, which in turn implies $\mu(x) = \mu'(x), \forall x \in v$. \square

We are now ready to state and prove our second equivalence result.

PROPOSITION 2.2. *Let Ξ be an extensive form such that, whenever an information set u is avoidable in $\Xi(u)$, it can only be avoided in $\Xi(u)$ by its owner. Then for any possible payoff vector r , the game $\Gamma = (\Xi, r)$ is such that $\text{SPEO}(\Gamma) = \text{SQEO}(\Gamma)$. Conversely, if Ξ is an extensive form with an information set u that can be avoided in $\Xi(u)$ by a different player than its owner, then we can find a payoff vector r such that for the game $\Gamma = (\Xi, r)$, $\text{SPEP}(\Gamma) \neq \text{SQEP}(\Gamma)$.*

PROOF. Let us prove the first part of the proposition. We need to prove that $\forall b \in \text{SPE}(\Gamma), \mathbb{P}^b \in \text{SQEO}(\Gamma)$. Take an arbitrary $b \in \text{SPE}(\Gamma)$ and construct some consistent beliefs μ .

If the set $\tilde{U}(b, \mu) = \bigcup_{i=1}^n \left\{ u \in U_i : b_i \notin \arg \max_{\tilde{b}_i \in B_i} R_{iu}^\mu(b \setminus \tilde{b}_i) \right\}$ is empty, then $b \in \text{SQE}(\Gamma)$ and $\mathbb{P}^b \in \text{SQEO}(\Gamma)$. Otherwise, we need to find a sequential equilibrium (b^*, μ^*) such that $\mathbb{P}^{b^*} = \mathbb{P}^b$.

Step 1: Take an information set $u \in \tilde{U}(b, \mu)$. Let i be the player that moves at this information set, and let $\Gamma_y = (\Xi(u), \hat{r})$. As in the proof of proposition 2.1, notice that by Lemma 2.1, u should be such that $\mathbb{P}_y^b(u) = 0$, hence it is avoidable in its minimal subform. By assumption, u can only be avoided by player i .

Step 2: Let b' be the strategy profile b modified so that player i plays a best reply against (b, μ) at the information set u . Construct a consistent assessment (b', μ') . Notice that $\mathbb{P}^{b'} = \mathbb{P}^b$ and, in particular, $\mathbb{P}_y^{b'} = \mathbb{P}_y^b$. By Lemma 2.2, μ and μ' assign the same probability distribution on every information set.

Step 3: We now prove that $b' \in \text{SPE}(\Gamma)$. For this we need $b'_y \in \text{NE}(\Gamma_y)$. Given the strategy profile b'_y in the subgame Γ_y , player i cannot profitably deviate because this would mean that she was also able to profitably deviate when b_y was played in the subgame Γ_y , which contradicts $b_y \in \text{NE}(\Gamma_y)$.

Suppose now that there exists a player $j \neq i$ who has a profitable deviation $b''_{y,j}$ from $b'_{y,j}$ in the subgame Γ_y . The hypothesis on the extensive form Ξ implies $\mathbb{P}_y^{b \setminus b'_{y,j}} = \mathbb{P}_y^{b' \setminus b''_{y,j}}$, which further implies that $b''_{y,j}$ should have also been a profitable deviation from b_y . However, this is impossible since $b_y \in \text{NE}(\Gamma_y)$.

Step 4: By step 2, $|\tilde{U}(b', \mu')| = |\tilde{U}(b, \mu)| - 1$. If $|\tilde{U}(b', \mu')| \neq 0$, apply the same type of transformation to b' . Suppose that the cardinality of $\tilde{U}(b, \mu)$ is q , then in the q th transformation we will obtain a consistent assessment $(b^{(q)}, \mu^{(q)})$ such that $b^{(q)} \in \text{SPE}(\Gamma)$, $\mathbb{P}^b = \mathbb{P}^{b^{(q)}}$, and $\tilde{U}(b^{(q)}, \mu^{(q)}) = \emptyset$. Observe that, $b^{(q)} \in \text{SPE}(\Gamma)$ and $\tilde{U}(b^{(q)}, \mu^{(q)}) = \emptyset$ imply $b^{(q)} \in \text{SQE}(\Gamma)$. Therefore $(b^{(q)}, \mu^{(q)})$ is the sequential equilibrium (b^*, μ^*) we were looking for.

Let us now prove the second part of the proposition. For notational convenience, it is proved for games without proper subgames, however, the argument extends immediately to the general case.

Given a node $x \in T$, the set $\text{Path}(x) = \{c \in \bigcup_u C_u : c < x\}$ of choices is called path to x .

Suppose that u is an information set that can be avoided in Ξ by a player, say player j , different from the player moving at it, say player i . Note that there must exist an $x \in u$ and a choice $c \in C_v$, where $v \in U_j$, such that if $b = b \setminus \text{Path}(x)$, then $\mathbb{P}^{b \setminus c}(u) = 0$ is true.

Let $f \in C_u$ be an arbitrary choice available to player i at u . Assign the following payoffs:

$$(3) \quad \begin{cases} r_j(z) = 0 & \text{if } z \in Z(c) \\ r_i(z) = r_j(z) = 0 & \text{if } z \in Z(f) \\ r_i(z) = r_j(z) = 1 & \text{if } z \in Z(u) \setminus Z(f). \end{cases}$$

Let $d \in \text{Path}(x)$ with $d \notin C_v$, assign payoffs to the terminal nodes, whenever allowed by 3, in the following fashion:

$$(4) \quad r_k(z) > r_k(z') \text{ where } z \in Z(d) \text{ and } z' \in Z(C_w \setminus \{d\}).$$

Player k above is the player who has choice d available at the information set w . Give zero to every player everywhere else.

In words, player j moves with positive probability in the game. She has two choices, either moving towards the information set u and letting player i decide, or moving away from the information set u . If she moves away she gets zero for sure. If she lets player i decide, player i can either make both get zero by choosing f , or make both get one by choosing something else. Due to 4, no player will disturb this description of the playing of the game.

This game has a Nash equilibrium in which player i moves f and player j obtains a payoff equal to zero by moving c . However, in every sequential equilibrium of this game, player i does not choose f and, as a consequence, player j takes the action contained in $\text{Path}(x) \cap C_v$. Therefore, in every sequential equilibrium, players i and j obtain a payoff strictly larger than zero.⁵ This completes the proof. \square

For a very simple application of the previous algorithm, consider the extensive game of Figure 2.4 and substitute the payoff vector following move *Out* of player 1, with the payoff vector $(0, 0)$. Again, the first player

⁵Equilibrium payoffs are not necessarily equal to one due to eventual moves of Nature.

moving *Out* and the second player taking the strictly dominated move *H*, is a subgame perfect equilibrium that yields an equilibrium payoff vector equal to $(0,0)$. However, in any sequential equilibrium, player 2 moves *G* and player 1 does not move *Out*, which makes $(1,1)$ the only sequential equilibrium payoff vector.

REMARK 2.1. Notice that, in the set of extensive forms under study in the last proposition, beliefs are always uniquely defined for any given strategy profile (consider $b' = b$ in Lemma 2.2). One may incorrectly think that it is the uniqueness of the beliefs that is behind the equivalence. Consider a modification of the game form in Figure 2.7 so that the second information set of player 1 is controlled by a new player 3. This modified extensive form has a unique system of consistent beliefs for any given strategy profile but, as seen in Proposition 2.2, the set of equilibrium outcomes is not the same for both concepts for every possible payoff vector.

2.4.1. Perfect Bayesian equilibrium. These results can be helpful in applied work. But many applied economists use Perfect Bayesian Equilibrium in extensive games with incomplete information. This motivates us to analyze the relationship between this concept and our previous findings. The formal definition that we use is:

DEFINITION 2.8. An assessment (b, μ) is a perfect Bayesian equilibrium of the extensive game Γ if it satisfies the following conditions:

- (1) For every information set u if $\mathbb{P}_y^b(u) > 0$, then $\mu(x) = \mathbb{P}_y^b(x|u)$, where $\Xi_y = \Xi(u)$, for all $x \in u$;
- (2) b is a sequential best reply against (b, μ) .⁶

⁶This is the weakest and the most used version. See Fudenberg and Tirole (1991) for related definitions.

Let $\text{PBE}(\Gamma)$ be the set of strategies that together with some system of beliefs make up a perfect Bayesian equilibrium. Let $\text{PBEP}(\Gamma)$ and $\text{PBEO}(\Gamma)$ be the sets of, respectively, perfect Bayesian equilibrium payoffs and perfect Bayesian equilibrium outcomes.

A quick inspection of the definition reveals that perfect Bayesian equilibrium implies subgame perfection and that it is implied by sequential equilibrium. This observation by itself proves that the sufficiency parts of Propositions 2.1 and 2.2 hold if we replace $\text{SQE}(\Gamma)$ with $\text{PBE}(\Gamma)$ and $\text{SQEO}(\Gamma)$ with $\text{PBEO}(\Gamma)$.

As for the necessity part of both propositions, the algorithms proposed are also valid to construct subgame perfect equilibria (subgame perfect equilibrium payoffs) that are not perfect Bayesian (perfect Bayesian equilibrium payoffs). Note that the irrational move prohibited to a player having consistent beliefs is also forbidden to a player that has any conceivable beliefs.

In other words, the conditions for equivalence between subgame perfection and perfect Bayesian equilibrium parallel those between subgame perfection and sequentiality. Formally:

COROLLARY 2.1. *If Ξ is an extensive form such that no information set u can be avoided in $\Xi(u)$, then for any possible payoff vector r , the game $\Gamma = (\Xi, r)$ is such that $\text{SPE}(\Gamma) = \text{PBE}(\Gamma)$. If Ξ is an extensive form with an information set u that can be avoided in $\Xi(u)$, then we can find a payoff vector r such that for the game $\Gamma = (\Xi, r)$, $\text{SPE}(\Gamma) \neq \text{PBE}(\Gamma)$.*

The analogous result regarding equilibrium outcomes and equilibrium payoffs is:

COROLLARY 2.2. *Let Ξ be an extensive form such that, whenever an information set u can be avoided in $\Xi(u)$, it can only be avoided in $\Xi(u)$ by its owner, then for any possible payoff vector r , the game $\Gamma = (\Xi, r)$ is such*

that $\text{SPEO}(\Gamma) = \text{PBEO}(\Gamma)$. If Ξ is an extensive form with an information set u that can be avoided in $\Xi(u)$ by a different player than its owner, then we can find a payoff vector r such that for the game $\Gamma = (\Xi, r)$, $\text{SPEP}(\Gamma) \neq \text{PBEP}(\Gamma)$.

2.5. Examples

These results can be applied to many games considered in the economic literature. It allows us to identify in a straightforward way the finite extensive form games of imperfect information for which subgame perfect equilibria are still conforming with backward induction expressed in a sequential equilibrium.

Besley and Coate (1997) proposed an economic model of representative democracy. The political process is a three-stage game. In stage 1, each citizen decides whether or not to become a candidate for public office. At the second stage, voting takes place over the list of candidates. At stage 3 the candidate with the most votes chooses the policy. Besley and Coate solved this model using subgame perfection and found multiple subgame perfect equilibria with very different outcomes in terms of number of candidates. This may suggest that some refinement might give sharper predictions. However, given the structure of the game that they considered, it follows immediately from the results of the previous section that all subgame perfect equilibria in their model are also sequential. Thus, no additional insights would be obtained by requiring this particular refinement.

The information structure of Besley and Coate's model is a particular case of the more general framework offered by Fudenberg and Levine (1983). They characterized the information structure of finite-horizon multistage games as "almost" perfect, since in each period players simultaneously choose actions, Nature never moves and there is no uncertainty at the end of each stage. As they noticed, sequential equilibrium does not refine

subgame perfection in this class of games. This can also be obtained as an implication of Proposition 2.1 in the present paper.

In their version of the Diamond and Dybvig (1983) model, Adão and Temzelides (1998) discussed both the issue of potential banking instability as well as that of the decentralization of the optimal deposit contract. They addressed the first question in a model with a “social planner” bank. The bank offers the efficient contract as a deposit contract in the initial period. In the first stage agents sequentially choose whether to deposit in the bank or to remain in autarky. In the second stage, those agents who were selected by Nature to be patient, simultaneously choose whether to misrepresent their preferences and withdraw, or report truthfully and wait. The reduced normal form of the game has two symmetric Nash equilibria in pure strategies. The first one has all agents choosing depositing in the bank and reporting faithfully, the second one has all agents choosing autarky. The fact that both equilibria are sequential is presented in their Proposition 2. Because of the game form they used, our Proposition 2.1 also implies their result.

In the implementation theory framework, Moore and Repullo (1988) present the strength of subgame perfect implementation. If a choice function is implementable in subgame perfect equilibria by a given mechanism, the strategy space is finite, and no information set is avoidable in its minimal subform in the extensive form of the mechanism, then our work establishes the implementability in sequential equilibrium. (See, for instance, the example they study in Section 5, pp. 1213-1215.)

More examples can be found in Game Theory textbooks, like those of Fudenberg and Tirole (1996), Myerson (1991) and Osborne and Rubinstein (1994). Notice that whenever subgame perfect and sequential equilibrium differ for an extensive game, there are information sets that are avoidable

in its minimal subform. As examples consider Figures 8.4 and 8.5 in Fudenberg and Tirole (1996), Figures from 4.8 to 4.11 in Myerson (1991) and Figures 225.1 and 230.1 in Osborne and Rubinstein (1994).

2.6. Appendix: Notation and Terminology

2.6.1. Extensive form. An n -player extensive form is a sextuple $\Xi = (T, \leq, P, U, C, p)$, where T is the finite set of nodes and \leq is a partial order on T , representing precedence. We use the notation $x < y$ to say that node y comes after node x . The immediate predecessor of x is $A(x) = \max\{y : y < x\}$, and the set of immediate successors of x is $S(x) = \{y : x \in A(y)\}$. The pair (T, \leq) is a tree with a unique root α : for any $x \in T, x \neq \alpha$, there exists a unique sequence $\alpha = x_0, x_1, \dots, x_n = x$ with $x_i \in S(x_{i-1}), 1 \leq i \leq n$. The set of endpoints is $Z = \{x : S(x) = \emptyset\}$ and $X = T \setminus Z$ is the set of decision points. We write $Z(x) = \{y \in Z : x < y\}$ to denote the set of terminal successors of x , and if E is an arbitrary set of nodes we write $Z(E) = \{z \in Z(x) : x \in E\}$.

2.6.2. Player partition. The player partition, P , is a partition of X into sets P_0, P_1, \dots, P_n , where P_i is the set of decision points of player i and P_0 stands for the set of nodes where chance moves. The probability assignment p specifies for every $x \in P_0$ a completely mixed probability distribution p_x on $S(x)$.

2.6.3. Information partition. The information partition U is an n -tuple (U_1, \dots, U_n) , where U_i is a partition of P_i into information sets of player i , such that (i) if $u \in U_i, x, y \in u$ and $x \leq z$ for $z \in X$, then we cannot have $z < y$, and (ii) if $u \in U_i, x, y \in u$, then $|S(x)| = |S(y)|$. Therefore, if u is an information set and $x \in X$, it makes sense to write $u < x$. Also, if $u \in U_i$, we often refer to player i as the owner of the information set u .

2.6.4. Choice partition. If $u \in U_i$, the set C_u is the set of choices available for i at u . A choice $c \in C_u$ is a collection of $|u|$ nodes with one, and

only one, element of $S(x)$ for each $x \in u$. If player i chooses $c \in C_u$ at the information set $u \in U_i$ when she is actually at $x \in u$, then the next node reached by the game is the element of $S(x)$ contained in c . The entire collection $C = \{C_u : u \in \bigcup_{i=1}^n U_i\}$ is called the choice partition. We assume throughout that $|C_u| > 1$ for every information set u .

2.6.5. Extensive form game. We define a finite n -person extensive form game as a pair $\Gamma = (\Xi, r)$, where Ξ is an n -player extensive form and r , the payoff function, is an n -tuple (r_1, \dots, r_n) , where r_i is a real valued function with domain Z . We assume throughout that the extensive form Ξ satisfies perfect recall, i.e. for all $i \in \{1, \dots, n\}$, $u, v \in U_i$, $c \in C_u$ and $x, y \in v$, we have $c < x$ if and only if $c < y$. Therefore, we can say that choice c comes before the information set v (to be denoted $c < v$) and that the information set u comes before the information set v (to be denoted $u < v$).

2.6.6. Behavior strategies, beliefs and assessments. A behavior strategy b_i of player i is a sequence of functions $(b_i^u)_{u \in U_i}$ such that $b_i^u : C_u \rightarrow \mathbb{R}_+$ and $\sum_{c \in C_u} b_i^u(c) = 1, \forall u$. The set B_i represents the set of behavior strategies available to player i . A behavior strategy profile is an element of $B = \prod_{i=1}^n B_i$. As common in extensive form games, we restrict attention to behavior strategies.⁷ Throughout, we simply refer to them as strategies. If $b_i \in B_i$ and $c \in C_u$ with $u \in U_i$, then $b_i \setminus c$ denotes the strategy b_i changed so that c is taken with probability one at u . If $b \in B$ and $b'_i \in B_i$ then $b \setminus b'_i$ is the strategy profile $(b_1, \dots, b_{i-1}, b'_i, b_{i+1}, \dots, b_n)$. If c is a choice of player i then $b \setminus c = b \setminus b'_i$, where $b'_i = b_i \setminus c$.

A system of beliefs μ is a function $\mu : X \setminus P_0 \rightarrow [0, 1]$ with $\sum_{x \in u} \mu(x) = 1, \forall u$. An assessment (b, μ) is a strategy profile together with a system of beliefs.

⁷We can do this without loss of generality due to perfect recall and Kuhn's Theorem, see Kuhn (1953).

2.6.7. Subforms and subgames. Let $\hat{T} \subset T$ be a subset of nodes such that (i) $\exists y \in \hat{T}$ with $y < x$, $\forall x \in \hat{T}$, $x \neq y$, (ii) if $x \in \hat{T}$ then $S(x) \subset \hat{T}$, and (iii) if $x \in \hat{T}$ and $x \in u$ then $u \subset \hat{T}$. Then we say that $\Xi_y = (\hat{T}, \hat{\leq}, \hat{P}, \hat{U}, \hat{C}, \hat{p})$ is a subform of Ξ starting at y , where $(\hat{\leq}, \hat{P}, \hat{U}, \hat{C}, \hat{p})$ are defined from Ξ in \hat{T} by restriction. A subgame is a pair $\Gamma_y = (\Xi_y, \hat{r})$, where \hat{r} is the restriction of r to the endpoints of Ξ_y . We denote by b_y the restriction of $b \in B$ to the subform Ξ_y (to the subgame Γ_y). The restriction of a system of beliefs μ to the subform Ξ_y (to the subgame Γ_y) is denoted by μ_y .

CHAPTER 3

Undominated (and) Perfect Equilibria in Poisson Games¹

3.1. Introduction

Models of population uncertainty have been introduced by Myerson (1998, 2000) and Milchtaich (2004), in order to describe situations in which players do not know the number of opponents. Among these games, a special attention has been reserved to Poisson games, where the number of players is a Poisson random variable with a given mean and where the players' types are independent identically distributed random variables. The properties of the Poisson distribution make Poisson games an extremely convenient subclass of games. They are characterized by the properties of *independent actions* (for every possible strategy profile the number of players who take different actions are independent random variables) and *environmental equivalence* (a player assesses the same probability for the type profile of the others as an external observer does for the type profile of the whole game, where a type profile is a vector that lists how many players there are of each type).

Myerson (1998) extends the definition of Nash equilibrium and acknowledges its existence. The existing literature on equilibrium refinements in noncooperative game theory warns that we should be cautious about the strategic stability of the Nash equilibrium concept. If this concern is well founded, we can ask which Nash equilibria are self-enforcing in this setting.

The following example serves us to both introducing Poisson games to the reader and illustrating the nature of the question. A player is sitting at

¹This chapter is based on De Sinopoli and Gonzalez Pimienta (2007)

home and faces two possible alternatives, either she goes out to some social event, or she stays home. She does not know how many players are facing this same disjunctive, but she knows that this number is a Poisson random variable with parameter n . If she goes out and meets somebody she receives a payoff equal to 1. If she meets nobody or decides to stay home, she gets a payoff equal to 0. Every player faces this same two options and has the same preferences.

The strategy “everybody stays home” is a Nash equilibrium of the described game. However, we cannot consider it a good equilibrium since players use a dominated strategy. It is not difficult to come up with similar examples with patently implausible Nash equilibria.²

Recall that in conventional normal form games (from now on just normal form games), a modest refinement like perfection only selects undominated strategies. This is the case in the previous example. However, in Poisson games this is not true in general. We can go further, straightforward extensions of proper and strictly perfect equilibrium do not satisfy undominance either and, in addition, not every game has a strictly perfect equilibrium.

On the other hand, as it happens in normal form games, not every undominated equilibrium is perfect. The same arguments that in normal form games suggest that we should dispose of some of the undominated equilibria that are not perfect are valid here. The difference being that, as argued above, some perfect equilibria may be dominated.

We define undominated perfect equilibria for Poisson games as strategy combinations that are limits of sequences of undominated equilibria of perturbed Poisson games. We prove that every Poisson game has at least one

²For instance, Myerson (2002), analyzing voting contexts, considers only Nash equilibria in which weakly dominated actions have been eliminated for all the types.

undominated perfect equilibrium and that the set of undominated perfect equilibria is exactly the set of perfect equilibria which are also undominated.

Our analysis is focused on Poisson games. However, we must point out that none of the implications that we derive relies on the specific shape of the Poisson distribution. Only some payoffs and thresholds used in some examples would have to be recomputed if we want to translate them into a framework with a different underlying probability distribution.

This paper is organized as follows: In the next section we formally define Poisson games, strategies and Nash equilibria. We closely follow the description of Poisson games made by Myerson (1998). The third section is devoted to examine the properties of undominated strategies in Poisson games, where we show that there exist important asymmetries with respect to normal form games. The fourth section studies the perfect equilibrium concept and some of its possible variations. We define the concept of undominated perfect equilibrium for Poisson games in Section 3.5, where some of its properties are also proven.

3.2. Preliminaries

Recall that a Poisson random variable is a discrete probability distribution that takes only one parameter. The probability that a Poisson random variable of parameter n takes the value k , being k a nonnegative integer, is

$$f(k;n) = e^{-n} \frac{n^k}{k!}.$$

A Poisson game Γ is a five-tuple (n, T, r, C, u) . The number of players in the game is a Poisson random variable with parameter $n > 0$. The set T represents the set of possible types of players, we assume it to be a nonempty finite set.

As usual, if A is a finite set, $\Delta(A)$ represents the set of probability distributions over A . Given the event that a player is in the game, she is of type $t \in T$ with probability $r(t)$. This information is contained in the vector $r \in \Delta(T)$. The *decomposition property* of the Poisson distribution implies that for each type t in T , the number of players of the game whose type is t is a Poisson random variable with parameter $nr(t)$. These random variables together are mutually independent and form a vector, called the *type profile*, which lists the number of players in the game who have each type.

For any finite set S , we denote as $Z(S)$ the set of elements $w \in \mathbb{R}^S$ such that $w(s)$ is a nonnegative integer for all $s \in S$. Using this notation, the set $Z(T)$ denotes the set of possible values for the type profile in the game.

The set C is the set of available choices or pure actions that a player may take. We assume that it is common to all players regardless of their type and that it is a finite set containing at least two different alternatives. The set $\Delta(C)$ is the set of mixed actions. Henceforth, we refer to mixed actions simply as actions.

The utility to each player depends on her type, on the action that she chooses and on the number of players, not counting herself, who choose each possible action. A vector that lists these numbers of players for each possible element of C is called an *action profile* and belongs to the set $Z(C)$. We assume that preferences of a player of type t can be summarized with a bounded function $u_t : C \times Z(C) \rightarrow \mathbb{R}$, i.e. $u_t(b, x)$ is the payoff that a player of type t receives if she takes the pure action b and the number of players who choose action c is $x(c)$, for all $c \in C$. Furthermore, let $u = (u_t)_{t \in T}$.

In games with population uncertainty, as Myerson (1998, p. 377) argues, "... players' perceptions about each others' strategic behavior cannot be formulated as a strategy profile that assigns a randomized strategy to each specific individual of the game, because a player is not aware of the

specific identities of all the other players”. Notice that two players of the same type do not have any other known characteristic by which others can assess different conducts. The conclusion of the previous reasoning is that a strategy σ is an element of $(\Delta(C))^T$, i.e. a mapping from the set of types to the set of possible actions.³

This symmetry assumption is a fundamental part of the description of the game. Notice that it is not made for convenience, on the contrary, symmetry is a critical assumption of a model of population uncertainty for it to be meaningful and well constructed.

If players play according to the strategy σ , $\sigma_t(c)$ is the probability that a player of type t chooses the pure action c . The decomposition property of the Poisson distribution implies that the number of players of type $t \in T$ who choose the pure action c is a Poisson distribution with parameter $nr(t)\sigma_t(c)$. The *aggregation property* of the Poisson distribution implies that any sum of independent Poisson random variables is also a Poisson random variable. It follows that the total number of players who take the pure action c is a Poisson distribution with parameter $n\tau(c)$, where $\tau(c) = \sum_{t \in T} r(t)\sigma_t(c)$.

A player of type t who plays the pure action $b \in C$ while all other players are expected to play according to σ has expected utility equal to

$$U_t(b, \sigma) = \sum_{x \in Z(C)} \mathbb{P}(x|\sigma) u_t(b, x)$$

where,

$$\mathbb{P}(x|\sigma) = \prod_{c \in C} e^{-n\tau(c)} \frac{(n\tau(c))^{x(c)}}{x(c)!}$$

³One may wonder how the game might be affected if the subdivision of types was finer, thus, allowing a larger variety of different behaviors. Myerson (1998) proves that, for Poisson games, utility-irrelevant subdivisions of types cannot substantially change the set of Nash equilibria (Theorem 4, page 386).

and her expected utility from playing action $\theta \in \Delta(C)$ is

$$U_t(\theta, \sigma) = \sum_{b \in C} \theta(b) U_t(b, \sigma).$$

The set of best responses for a player of type t against a strategy σ is the set of actions that maximizes her expected utility given that the rest of the players, including those whose type is t , behave as prescribed by σ . The set $\text{PBR}_t(\sigma) = \{c \in C : c \in \arg \max_{b \in C} U_t(b, \sigma)\}$ is the set of pure best responses against σ for a player of type t . The set of mixed best responses against σ for a player of type t is the set of actions $\text{BR}_t(\sigma) = \Delta(\text{PBR}_t(\sigma))$.

DEFINITION 3.1. The strategy σ^* is a Nash equilibrium if $\sigma_t^* \in \text{BR}_t(\sigma^*)$ for all t .

Standard fixed-point arguments show that every Poisson Game has at least one Nash equilibrium, see Myerson (1998).

3.3. Dominated Strategies

The admissibility principle, which in normal form games stipulates that no player must choose a dominated strategy, translates into the current framework imposing that no player should choose a dominated action.

DEFINITION 3.2. The action $\theta \in \Delta(C)$ is dominated for a player of type t if there exists an alternative action θ' such that $U_t(\theta, \sigma) \leq U_t(\theta', \sigma)$, for every possible strategy σ and $U_t(\theta, \sigma') < U_t(\theta', \sigma')$ for at least one σ' .

Although contained in a voting framework, Myerson (2002) offers a weaker definition of dominated action. Under such definition the (pure) action c is dominated for a player of type t if there exists an alternative (pure) action b such that $u_t(c, x) \leq u_t(b, x)$ for every $x \in Z(C)$ and with strict inequality for at least one x' . However, we prefer the former since it is equivalent to the definition of dominated strategy for normal form games.

In games with population uncertainty dominated strategies are defined in the following way:

DEFINITION 3.3. A strategy σ is dominated if there is some type t for which σ_t is a dominated action.

We can use this formal apparatus to revisit the example discussed in the introduction. Let a stand for “going out” and b for “staying home”:

EXAMPLE 3.1. Let Γ be a Poisson game with $n > 0$, only one possible type, set of available choices $C = \{a, b\}$, and utility function:

$$u(a, x) = \begin{cases} 1 & \text{if } x(a) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$u(b, x) = 0 \quad \forall x \in Z(C).$$

Since this Poisson game has only one possible type, we can identify the set of strategies with the set of actions. There are two equilibria, a and b . We have already argued that the equilibrium strategy b is unsatisfactory. Notice that b is a dominated action, even when we consider the weaker definition given by Myerson (2002), which makes b a dominated strategy.

The example highlights that the Nash equilibrium concept is inadequate for Poisson games since it allows for equilibrium points where players use dominated actions (strategies).

In normal form games it is well known that a dominated strategy is never a best response against a completely mixed strategy of the opponents. This property implies, for instance, that a perfect equilibrium only selects undominated strategies. Ideally, we would like to establish an analogy between the properties of (un)dominated strategies in normal form games and (un)dominated actions in Poisson games. In the remainder of this section

we examine which are the differences and similarities between the two settings with regard to (un)dominated strategies.

The following straightforward result is true in both cases, although it has to be stated in terms of strategies for normal form games. (Henceforth we skip this last clarification when comparing actions of Poisson games with strategies of normal form games.)

LEMMA 3.1. *If a pure action is dominated then every mixed action that gives positive probability to that pure action is also dominated.*

This implies that a strategy that prescribes that some type plays an action which gives positive probability to a dominated pure action is dominated. On the other hand, as so happens in normal form games, a dominated mixed action does not necessarily give positive weight to a dominated pure action. We illustrate this in the following example.

EXAMPLE 3.2. Consider a Poisson game with an expected number of players such that $n > \ln 2$, only one possible type, three available choices in the set $C = \{a, b, c\}$, and utility function:

$$u(a, x) = \begin{cases} 10 & \text{if } x(a) \geq x(b) \\ 0 & \text{otherwise} \end{cases} \quad u(b, x) = \begin{cases} 10 & \text{if } x(a) < x(b) \\ 0 & \text{otherwise} \end{cases}$$

$$u(c, x) = 6 \quad \forall x \in Z(C).$$

The pure action a is not dominated. It is the unique best response against the strategy a . The pure action b is not dominated either. In particular, notice that it is not dominated by a , given the assumption that $n > \ln 2$, whose unique purpose is to make sufficiently small the probability that the number of players who turn up in the game is equal to zero. As for the pure action c , it does better than a against the strategy b and better than b against the strategy a .

The mixed action $\theta = 1/2a + 1/2b$ is dominated by the pure action $\theta' = c$. To see this note that given a strategy, we can assign probability p to the event $x(a) \geq x(b)$ and probability $1 - p$ to the event $x(a) < x(b)$. We can compute the expected utility of playing action $\theta = 1/2a + 1/2b$ as $1/2(1 - p)10 + 1/2p10 = 5$.

Therefore, we have proved:

LEMMA 3.2. *An action that does not give positive probability to a dominated pure action may be dominated.*

It is also true that a pure strategy may only be dominated by a mixed strategy. Modify the utility function of the previous example so that $u(x, c) = 4$ for all x in $Z(C)$, and raise the lower bound of n to $\ln(5/2)$. In this modified game, the pure action c is dominated by neither a nor b , but it is dominated by the action $\theta = 1/2a + 1/2b$.

In normal form games, the process of discerning which strategies are dominated is simplified by the fact that it suffices to consider only pure strategies of the opponents. As the next example illustrates, this is not enough in Poisson games.

EXAMPLE 3.3. Let Γ be a Poisson game with expected number of players equal to n , only one possible type, set of choices equal to $C = \{a, b, c\}$, and utility function:

$$u(a, x) = \begin{cases} 1 & \text{if } x(a) = x(b) > 0 \\ 0 & \text{otherwise} \end{cases} \quad u(b, x) = \begin{cases} 1 & \text{if } x(a) = x(b) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$u(c, x) = 0 \quad \forall x \in Z(C).$$

The pure action c does strictly worse than the pure actions a and b if and only if the strategy σ gives strictly positive probability to both a and b .

Nevertheless, to compute expected payoffs and, therefore, to identify dominated actions for one player of some type, it suffices to consider that every other player plays the same action, regardless of her type. This is so because from the strategy $\sigma \in (\Delta(C))^T$ we can define a global action $\tau \in \Delta(C)$ given by $\tau(c) = \sum_{t \in T} r(t)\sigma_t(c)$, which implies the same probability distribution over the set of action profiles $Z(C)$.

An important fact about undominated strategies in normal form games is that a strategy is undominated if and only if it is a best response against some element contained in the interior of the simplex of the set of pure strategy combinations of the opponents. As mentioned above this implies that a perfect equilibrium only selects undominated strategies. Our previous circumspection suggests that things may work differently in the present framework. As it turns out, no result similar to this is true for Poisson games.

If A is a finite set, let $\Delta^0(A)$ stand for the set of probability distributions over A that give positive probability to every element of A .

LEMMA 3.3. *An undominated action may be a best response against no element of $\Delta^0(C^T)$.*

PROOF. Consider a Poisson game with expected number of players $n = 1$,⁴ only one possible type so that $\Delta^0(C^T) = (\Delta^0(C))^T$, set of available choices equal to $C = \{a, b, c\}$ and utility function:

⁴The set of examples in the paper is designed to be as clear and simple as possible. This is the reason why we many times fix the expected number of players to be $n = 1$ or $n = 2$. This contrasts with the fact that Poisson games fit more naturally to a situation where the expected number of players is large. At the expense of computational simplicity, similar examples can be constructed that put no restrictions on the Poisson parameter n .

$$\begin{aligned}
 u(a,x) &= \begin{cases} 5e & \text{if } x(a) = 1 \text{ and } x(c) = 0 \\ 5e & \text{if } x(c) > 0 \\ 0 & \text{otherwise} \end{cases} \\
 u(b,x) &= \begin{cases} 5e & \text{if } x(b) = 1 \text{ and } x(c) = 0 \\ 5e & \text{if } x(c) > 0 \\ 0 & \text{otherwise} \end{cases} \\
 u(c,x) &= 4 \quad \forall x \in Z(C).
 \end{aligned}$$

If a player expects every other potential player to behave according to the strategy $\sigma = b$, the action c gives her a larger payoff than the action a . In turn, if she expects every other potential player to behave according to the strategy $\sigma = a$, the action c gives her a larger payoff than the action b . To see that no mixed action between a and b dominates c , consider that $\sigma = a$, then the following inequalities hold:

$$U(b, \sigma) = 0 < U(c, \sigma) = 4 < U(a, \sigma) = 5.$$

From here it follows that under the strategy $\sigma = a$, the action c does strictly better than the action $\theta = \lambda a + (1 - \lambda)b$ for $\lambda \in [0, 4/5)$. If $\sigma = b$,

$$U(a, \sigma) = 0 < U(c, \sigma) = 4 < U(b, \sigma) = 5,$$

in which case the action c does better than the action $\theta = \lambda a + (1 - \lambda)b$ for $\lambda \in (1/5, 1]$. Therefore, no mixed action between a and b does always at least as good as the action c for every possible strategy σ .

It remains to prove that the action c is never a best response to any strategy σ . Consider first the case where σ randomizes only between a and b . Note that to minimize the maximum payoff obtained by playing either a

or b we need $\sigma = 1/2a + 1/2b$. However, in such a case

$$4 = U(c, \sigma) < U(a, \sigma) = U(b, \sigma) = \frac{5}{2}\sqrt{e}.$$

Finally, the action c is never a best response against any completely mixed strategy because any weight that the strategy σ puts in the choice c increases the expected payoff from both the actions a and b . \square

The next lemma completes the previous one. In Poisson games a dominated action can be a best response even if every other player uses a completely mixed action.

LEMMA 3.4. *A dominated action may be a best response against a completely mixed strategy.*

PROOF. Consider the following example:

EXAMPLE 3.4. Let Γ be a Poisson game with expected number of players equal to $n = 2$, only one possible type, set of choices $C = \{a, b\}$, and utility function

$$u(a, x) = e^{-2} \quad \forall x$$

$$u(b, x) = \begin{cases} 1 & \text{if } x(a) = x(b) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that e^{-2} is the probability that $x(a) = x(b) = 1$ under the strategy $\sigma = 1/2a + 1/2b$. Also notice that the action b is dominated by the action a , the former only does as good as the latter against the strategy $\sigma = 1/2a + 1/2b$, and does strictly worse for any other strategy $\sigma' \neq \sigma$. However, it is a best response against $\sigma \in \Delta^0(C)$. \square

As we mentioned above, in normal form games undominated strategies are characterized by the existence of a probability distribution in the interior

of the simplex of the set of pure strategy combinations of the opponents, against which the undominated strategy is a best response. This property gives a means of proposing equilibrium concepts that ensure that no player chooses a dominated strategy.

In normal form games the admissibility requirement is taken care of by perfection. Every perfect equilibrium selects only undominated strategies and, moreover, perfect equilibrium conditions do not admit just every equilibrium in undominated strategies, but only a subset of them.

Mertens (2004) links undominance and perfection through the concept of admissibility. He defines 3 possible concepts of admissible best response:

- (α) θ is an admissible best response against σ if there exists a sequence of completely mixed σ^k converging to σ such that θ is a best response against each (σ^k).
- (β) θ is an admissible best response against σ if θ is a best response against σ and there exist completely mixed σ' such that θ is a best response against σ' .
- (γ) θ is an admissible best response against σ if θ is a best response against σ and no other best response θ' is at least as good against every σ' and better against some.

The third concept corresponds to the usual concept of admissibility, i.e. undominance, while the first one is a characterization of perfect equilibria. In normal form games, the first concept is strictly stronger than the second, which in turn is strictly stronger than the third.

Lemmas 3.3 and 3.4 cast doubt upon the fact that the previous relationship holds for Poisson games (apart from the fact that the second concept is clearly weaker than the first). We are interested in finding out if there is any connection between α and γ in the present setting. Once we know this, we

will be able to propose a definition of a strong version of admissibility for Poisson games.

This is done in Section 3.5. Before that we have to extend the perfect equilibrium concept to Poisson games and look into its properties.

3.4. Perfection

Three equivalent definitions of perfect equilibrium have been proposed for normal form games. One based on perturbed games (Selten, 1975), a second one based on the item α of the previous list (also Selten, 1975) and a last one based on ε -perfect equilibria (Myerson, 1978). Below we provide the three corresponding definitions for Poisson games and prove their equivalence, so that we always have the most advantageous definition available.

The leading definition that we use is the one based on perturbed games

DEFINITION 3.4. Let Γ be a Poisson Game, for every $t \in T$, let η_t and $\Sigma_t(\eta_t)$ be defined by:

$$\eta_t \in \mathbb{R}^C \text{ with } \eta_t(c) > 0 \text{ for all } c \in C \text{ and } \sum_{c \in C} \eta_t(c) < 1$$

$$\Sigma_t(\eta_t) = \{\theta \in \Delta(C) : \theta(c) \geq \eta_t(c) \text{ for all } c \in C\}.$$

Furthermore, let $\eta = (\eta_t)_t$. The perturbed Poisson game (Γ, η) is the Poisson game (n, T, r, C, u) where players of type t are restricted to play only actions in $\Sigma_t(\eta_t)$, for every t .

In the perturbed Poisson game (Γ, η) , an action $\theta \in \Sigma_t(\eta_t)$ is a best reply against $\sigma \in \Sigma(\eta) = \prod_{t \in T} \Sigma_t(\eta_t)$ for a player of type t if every pure action c that is not a best response in Γ against σ for a player of type t is played with minimum probability, that is to say, $\sigma_t(c) = \eta_t(c)$. A strategy $\sigma \in \Sigma(\eta)$ is an equilibrium of the perturbed Poisson game (Γ, η) if for every type t , σ_t is a best response to σ in (Γ, η) . Kakutani fixed point theorem implies that:

LEMMA 3.5. *Every perturbed Poisson game has an equilibrium.*

Perturbed games lead to the following definition of perfection:

DEFINITION 3.5. A strategy σ is a perfect equilibrium if it is the limit point of a sequence $\{\sigma^\eta\}_{\eta \rightarrow 0}$, where σ^η is an equilibrium of the perturbed game (Γ, η) , for all η .

Since every perturbed Poisson game has an equilibrium and since this equilibrium is contained in the compact set $(\Delta(C))^T$, every Poisson game has a perfect equilibrium.⁵ By continuity of the utility function, every perfect equilibrium is also a Nash equilibrium.

As we mentioned earlier, another possible definition of perfect equilibrium uses ε -perfect equilibria. A completely mixed strategy σ^ε is an ε -perfect equilibrium if it satisfies:

$$U_t(c, \sigma^\varepsilon) < U_t(d, \sigma^\varepsilon), \text{ then } \sigma_t^\varepsilon(c) \leq \varepsilon \text{ for all } t \in T.$$

What follows is an adaption to Poisson games of some results and proofs of the book of van Damme (1991, pp. 26–29) for perfect equilibrium in normal form games. Although this is rather straightforward, we include it here to maintain the paper self-contained. The next lemma lists the two remaining concepts of perfect equilibrium and proves their equivalence.

LEMMA 3.6. *Let Γ be a Poisson game, and let $\sigma \in (\Delta(C))^T$. The following assertions are equivalent:*

- (1) σ is a perfect equilibrium of Γ ,
- (2) σ is a limit point of a sequence $\{\sigma^\varepsilon\}_{\varepsilon \rightarrow 0}$, where σ^ε is an ε -perfect equilibrium of Γ , for all ε , and

⁵Take any sequence of $\eta \rightarrow 0$, and for each η , an equilibrium σ^η of (Γ, η) . The sequence $\{\sigma^\eta\}_{\eta \rightarrow 0}$ has a convergent subsequence whose limit point is a perfect equilibrium.

(3) σ is a limit point of a sequence $\{\sigma^\varepsilon\}_{\varepsilon \rightarrow 0}$ of completely mixed strategy combinations with the property that, for all t , σ_t is a best response against each element σ^ε in this sequence.

PROOF. (1)→(2): Let σ be a limit point of a sequence $\{\sigma^\eta\}_{\eta \rightarrow 0}$, where σ^η is equilibrium of $\Gamma(\eta)$ for all η . Define $\varepsilon(\eta) \in \mathbb{R}_{++}$ by

$$\varepsilon(\eta) = \max_{t,c} \eta_t(c).$$

Then σ^η is an $\varepsilon(\eta)$ -perfect equilibrium for Γ .

(2)→(3): Let $\{\sigma^\varepsilon\}_{\varepsilon \rightarrow 0}$ be a sequence of ε -perfect equilibria with limit σ . By continuity, every element of the carrier of σ , which from now on we denote as $C(\sigma)$, is a best response against $\sigma(\varepsilon)$ for ε close enough to zero.

(3)→(1): Let $\{\sigma^\varepsilon\}_{\varepsilon \rightarrow 0}$ be a sequence as in (3) with limit σ . Define η^ε by:

$$\eta_t^\varepsilon(c) = \begin{cases} \sigma_t^\varepsilon(c) & \text{if } c \notin C(\sigma_t) \\ \varepsilon & \text{otherwise} \end{cases} \quad \text{for all } t, c.$$

For ε small enough σ^ε is equilibrium of the perturbed Poisson game $(\Gamma, \eta^\varepsilon)$, which establishes (1). \square

EXAMPLE 3.4 (Continued). We already saw that the action b is dominated by the action a and that both are best responses against $\sigma = 1/2a + 1/2b$. By Lemma 3.1, the action $\theta = 1/2a + 1/2b$ is also dominated by a . Nevertheless it is a best response against the strategy σ . Consequently, the dominated strategy σ is a perfect equilibrium.

The next example is more illustrative in showing how the perfect equilibrium concept fails to select only undominated strategies in Poisson games.

EXAMPLE 3.5. Consider the Poisson game $\Gamma = \{n, T, r, C, u\}$, with expected number of players $n = 2$, set of types $T = \{1, 2\}$, with equal probability for each type $r(1) = r(2) = 1/2$, set of choices $C = \{a, b\}$, and utility function:

$$u_1(a, x) = \begin{cases} 1 & \text{if } x(b) = 1 \\ 0 & \text{otherwise} \end{cases} \quad u_2(a, x) = e^{-1} \quad \forall x \in Z(C)$$

$$u_1(b, x) = e^{-1} \quad \forall x \in Z(C) \quad u_2(b, x) = \begin{cases} 1 & \text{if } x(a) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The number of players of type 1 is a Poisson random variable with expected value equal to 1. The same is true for type 2. Notice also that e^{-1} coincides with the probability that a Poisson random variable of parameter 1 is equal to 1. The action a is dominated for players of type 1, while action b is dominated for players of type 2. We claim that the strategy $\sigma = (\sigma_1, \sigma_2) = (a, b)$ is a perfect equilibrium. Take the sequence of ε -perfect equilibria $\sigma_1^\varepsilon = (1 - \varepsilon)a + \varepsilon b$, $\sigma_2^\varepsilon = \varepsilon a + (1 - \varepsilon)b$. For every ε , $U_t(a, \sigma^\varepsilon) = U_t(b, \sigma^\varepsilon)$, and the sequence $\{\sigma^\varepsilon\}_{\varepsilon \rightarrow 0}$ converges to σ .

Each one of this last two examples actually proves the next proposition:

PROPOSITION 3.1. *A Perfect equilibrium can be dominated.*

Hence, the doubts that we have raised at the end of the previous section are justified. In Poisson games, the relationship between α and γ of the possible concepts of admissible best response listed by Mertens is different from the one that holds in normal form games.

In the last example, the undominated equilibrium $\sigma = (\sigma_1, \sigma_2) = (b, a)$ is also perfect. The next question that we must answer is whether or not undominance implies perfection. Proposition 3.2 shows that in this case things work as they do in normal form games.

PROPOSITION 3.2. *An undominated equilibrium may not be perfect.*

PROOF. Consider a Poisson game Γ , with expected number of players equal to n , two possible types with equal probabilities, i.e. $T = \{1, 2\}$ and $r(1) = r(2) = 1/2$, set of available choices $C = \{a, b, c\}$ and utility function:⁶

$$\begin{aligned} u_1(a, x) &= x(a) + x(b) & u_2(a, x) &= x(a) \\ u_1(b, x) &= |x(a) + x(b) - x(c)| & u_2(b, x) &= 0 \quad \forall x \in Z(C) \\ u_1(c, x) &= 0 \quad \forall x \in Z(C) & u_2(c, x) &= 0 \quad \forall x \in Z(C). \end{aligned}$$

The game has a continuum of undominated equilibria $(\lambda a + (1 - \lambda)b, a)$, for λ taking values in the closed interval $[0, 1]$. Note, in particular, that the action b is not dominated for players of type 1 since it does better than the action a against the strategy $\sigma = (\sigma_1, \sigma_2) = (c, c)$. However, the strategy $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2) = (a, a)$ is the unique perfect equilibrium of the game. \square

The example used in the proof of the last proposition depicts that there may be unreasonable equilibria in undominated strategies. Consider the strategy $\sigma' = (\lambda a + (1 - \lambda)b, a)$ with $\lambda \in [0, 1)$. It is difficult to justify that a player of type 1 will stick to the prescribed strategy. A rational player should not risk his equilibrium payoff, even more when there is no possible expected benefit from such behavior. Suppose there was an unexpected deviation from σ' toward c , placing weight in the action b would pay off to players of type 1 if and only if such a deviation was drastic and it would hurt otherwise.

⁶Notice that the utility functions that we use in this example, and in some of the following ones are not bounded, as we assumed in the general description of Poisson games made in Section 3.2. The main features of all the examples discussed are preserved if we put an upper bound on utilities, that is to say, if utilities are given by $\tilde{u}_t(y, x) = \min\{u_t(y, x), K\}$, where K is a sufficiently large number with respect to n . However, we maintain the unbounded functions for the sake of simplicity.

Since perfection does not imply undominance and undominance does not imply perfection, we would like to have available an equilibrium concept that implies both. At this early stage, we do not want to go very far apart from the perfect equilibrium concept. We notice, nevertheless, that the equilibrium discussed in Example 3.5 is also proper, for a straightforward extension of this concept to Poisson games,⁷ since every player has only two possible choices.⁸ Strictly perfect equilibrium, does not help either. As argued above, the strategy $\sigma = 1/2a + 1/2b$ is an equilibrium of the

⁷A completely mixed strategy σ^ε is an ε -proper equilibrium if it satisfies:

$$U_t(c, \sigma^\varepsilon) < U_t(d, \sigma^\varepsilon), \text{ then } \sigma_t^\varepsilon(c) \leq \varepsilon \sigma_t^\varepsilon(d) \text{ for all } t \in T.$$

A strategy σ is proper if it is a limit point of a sequence $\{\sigma^\varepsilon\}_{\varepsilon \rightarrow 0}$, where σ^ε is an ε -proper equilibrium of Γ , for all ε .

⁸As it should be expected, not every proper equilibrium is perfect. Consider the Poisson game $\Gamma = \{n, T, r, C, u\}$, with expected number of players $n = 2$, two possible types that are equally probable, i.e. $T = \{1, 2\}$ and $r(1) = r(2) = 1/2$, set of choices $C = \{a, b, c, d\}$ and utility function:

$$\begin{aligned} u_1(x, a) &= 0 \quad \forall x & u_2(x, a) &= \begin{cases} 1 & \text{if } x(b) = 1 \\ 0 & \text{otherwise} \end{cases} \\ u_1(x, b) &= x(d) - x(c) & u_2(x, b) &= e^{-1} \quad \forall x \\ u_1(x, c) &= -1 \quad \forall x & u_2(x, c) &= -1 \quad \forall x \\ u_1(x, d) &= -2 \quad \forall x & u_2(x, d) &= -2 \quad \forall x. \end{aligned}$$

The action a is dominated for players of type 2 by action b . The strategy $\sigma = (\sigma_1, \sigma_2) = (b, a)$ is perfect. To see this consider the sequence of ε -perfect equilibria:

$$\begin{aligned} \sigma_1^\varepsilon &= \frac{1}{3}\varepsilon a + (1 - \varepsilon)b + \frac{1}{3}\varepsilon c + \frac{1}{3}\varepsilon d \\ \sigma_2^\varepsilon &= (1 - \varepsilon - 2\varepsilon^2)a + \varepsilon b + \varepsilon^2 c + \varepsilon^2 d \end{aligned}$$

For every type, action d is always strictly worse than action c , hence, in any ε -proper equilibrium, the former is played with strictly less probability than the latter. Therefore, a player of type 1 plays the action b with a probability less than ε times the probability that she gives to a . Hence, in no proper equilibrium she plays b with positive probability.

Poisson game described in Example 3.4. Notice that this equilibrium uses completely mixed strategies, and consequently, it is a strictly perfect equilibrium (again, using a straightforward extension of the concept to Poisson games).⁹

Examples 3.4 and 3.5 suggest that we may also demand robustness against perturbations other than trembles. (In Example 3.4, the payoff e^{-2} coincides with the probability that $x(a) = x(b) = 1$ under the strategy $\sigma = 1/2a + 1/2b$. In example 3.5 the payoff e^{-1} coincides with the probability that $x(a) = 1$, also that $x(b) = 1$, under the strategy $\sigma = (\sigma_1, \sigma_2) = (a, b)$.) Specifically, perturbations in the Poisson parameter n seem like the natural candidate as the model is of population uncertainty. Let us study the following equilibrium concept.

DEFINITION 3.6. The strategy σ is a perfect* equilibrium of the Poisson game $\Gamma = (n, T, r, C, u)$ if there exists a $\xi > 0$ such that σ is a perfect equilibrium of the Poisson game $\tilde{\Gamma} = (\tilde{n}, T, r, C, u)$ for all $\tilde{n} \in (n - \xi, n + \xi)$.

A perfect* equilibrium is a perfect equilibrium, not only of the original game, but also of every game that is obtained by small perturbations in the expected number of players. Notice that we cannot rely exclusively on perturbations in the expected numbers of players. One can easily construct

⁹In addition strictly perfect equilibrium does not satisfy existence. To see this, consider a Poisson game with expected number of players $n > 0$, only one possible type, four different choices $C = \{a, b, c, d\}$ and utility function:

$$u(a, x) = 1 + x(c)$$

$$u(b, x) = 1 + x(d)$$

$$u(c, x) = 0 \quad \forall x$$

$$u(d, x) = 0 \quad \forall x.$$

Notice that there is no equilibrium that is “robust” to every possible tremble.

examples that do not pose any restriction in the expected number of players with unreasonable Nash equilibria. See for instance Example 3.1.

Let us analyze why the perfect* equilibrium concept is not adequate by means of the following example.

EXAMPLE 3.6. Consider the family of Poisson games with expected number of players equal to $n > \frac{4}{7}$,¹⁰ with only one type, set of choices $C = \{a, b\}$, and utility function:

$$u(a, x) = x(b)$$

$$u(b, x) = \begin{cases} 1 & \text{if } x(a) = x(b) = 0 \\ 2x(a) & \text{otherwise.} \end{cases}$$

Every game has a unique equilibrium and it depends on n .¹¹ Consequently, it does not have a perfect* equilibrium.

This example prompts us to discard the previous equilibrium concept and reveals that demanding stability against variations in the Poisson parameter n forces to tolerate, at least, smooth variations of the equilibrium strategy if we want to retain existence. Therefore, if σ is a perfect equilibrium of Γ , we may want any game that only differs from Γ in that it has a slightly different number of expected players to have a perfect equilibrium that is not far away from σ .

As the next example shows, this relaxation would bring back dominated equilibria.

EXAMPLE 3.7. Let Γ be a Poisson game with expected number of players equal to $n = 6$, two different types $T = \{1, 2\}$ with $r(1) = 2/3$ and

¹⁰It is enough that n is such that $\ln n > -n$.

¹¹The unique equilibrium is $\sigma = \alpha a + (1 - \alpha)b$, where $\alpha = (1 - \frac{1}{ne^n}) / (3 - \frac{2}{e^n})$.

$r(2) = 1/3$, set of available choices $C = \{a, b, c, d\}$, and utility function:

$$\begin{aligned}
 u_1(h, x) &= 0 \quad \forall x \in Z(C), \forall h \in C \\
 u_2(a, x) &= \begin{cases} 1 & \text{if } x(c) = x(d) = 1 \\ 0 & \text{otherwise} \end{cases} \\
 u_2(b, x) &= e^{-2} \quad \forall x \in Z(C) \\
 u_2(h, x) &= -1 \quad \forall x \in Z(C), h = c, d.
 \end{aligned}$$

Notice first that the number of players with type 1 is a Poisson random variable of parameter 4. The strategy $\sigma = (\sigma_1, \sigma_2) = (1/4a + 1/4b + 1/4c + 1/4d, a)$ implies that the event $x(c) = x(d) = 1$ occurs with probability e^{-2} . The strategy σ is a perfect equilibrium where players of type 2 play dominated strategies. Take g to be a small number. The Poisson game $\Gamma^g = \{n + g, T, r, C, u\}$ has a dominated perfect equilibrium very close to σ where players of type 1 play action $(1/4 + \kappa, 1/4 + \kappa, 1/4 - \kappa, 1/4 - \kappa)$, for $\kappa = g/(24 + 4g)$, and players of type 2 play action a . On the other hand, the Poisson game $\Gamma^g = \{n - g, T, r, C, u\}$ also has a dominated perfect equilibrium very close to σ , where players of type 1 play action $(1/4 - \kappa', 1/4 - \kappa', 1/4 + \kappa', 1/4 + \kappa')$, for $\kappa' = g/(24 - 4g)$, and players of type 2 play action a .

So far we have provided a number of results and examples that show that some equilibrium concepts proposed for normal form games do not retain either admissibility or existence when extended to Poisson games. In the next section we propose an equilibrium concept that shows that, in this setting, these properties are not incompatible.

3.5. Undominated Perfect Equilibria

The same arguments that in normal form games compel to dispose of the undominated equilibria that are not perfect are also well founded here. Perfection is a weak requirement, it asks for stability against one single perturbation, not against every possible perturbation. As a result, equilibria that are not perfect are very unstable.

The main difference in the current setting is that there are perfect equilibria that are dominated. We want to put forward a strong version of admissibility for games with population uncertainty. Such a definition comprises items α and γ from the list of possible concepts of admissibility provided by Mertens (2004) and listed at the end of Section 3.3.

DEFINITION 3.7. θ is an admissible best response against σ if it is undominated and there exists a sequence of completely mixed σ^k converging to σ such that θ is a best response against each (σ^k) .

Accordingly, we may say that the strategy σ is admissible if for every t , σ_t is an admissible best response against σ . Therefore, if σ is an admissible strategy it is a perfect equilibrium, and we may talk about the set of admissible equilibria.

We want to propose an equilibrium concept that satisfies admissibility and that generates a nonempty set of equilibria for any game. Such a concept is introduced in Definition 3.8, the admissibility property will come directly from the definition and the existence result is offered in Proposition 3.4. The following Proposition shows that every Poisson game has an equilibrium in undominated strategies. It could have been proposed as a corollary of our main existence result. However, we prefer to invert the order of presentation so that the argument of the main proof can be more easily followed.

We proceed to prove that every Poisson game has an equilibrium in undominated strategies. Lemma 3.2 implies that the set of undominated strategies is not convex and, hence, we could not show existence of undominated equilibria using a standard fixed point argument in this set. A constructive proof shows that:

PROPOSITION 3.3. *Every Poisson game has a Nash equilibrium in undominated strategies.*

PROOF. Consider a Poisson game Γ , with set of choices C and utility vector u . Recall that if θ is an action, $C(\theta)$ denotes the carrier of θ . Notice that if $C(\theta) \subseteq C(\theta')$ then there exist a $\lambda \in (0, 1)$ and an action θ'' such that $\theta' = \lambda\theta + (1 - \lambda)\theta''$. If θ is dominated for players of type t , there exists a $\tilde{\theta}$ that dominates it, and a $\hat{\sigma}$ such that $U_t(\theta, \hat{\sigma}) < U_t(\tilde{\theta}, \hat{\sigma})$. Moreover, if $C(\theta) \subseteq C(\theta')$ then $\theta' = \lambda\theta + (1 - \lambda)\theta''$ is dominated by $\tilde{\theta}' = \lambda\tilde{\theta} + (1 - \lambda)\theta''$ and $U_t(\theta', \hat{\sigma}) < U_t(\tilde{\theta}', \hat{\sigma})$.

This implies that we can talk about dominated carriers and that, given a dominated carrier C there exists a strategy $\hat{\sigma}$ such that any action with carrier that contains C is dominated by an action that is a strictly better response to $\hat{\sigma}$.

Consider the set of all possible carriers, and call D_t the finite set of all dominated carriers for players of type t . For each minimal element of D_t , say d_t , let σ_{d_t} be a strategy such that any action with carrier that contains d_t is dominated by an action that is a strictly better response to such a strategy. Let M_t be the set of minimal elements of D_t .

For $\lambda > 0$, define a new Poisson game Γ^λ , with utility vector given by

$$u_t^\lambda(c, x) = u_t(c, x) + \lambda \sum_{d_t \in M_t} U_t(c, \sigma_{d_t})$$

which implies expected utilities,

$$U_t^\lambda(\theta_t, \sigma) = U_t(\theta_t, \sigma) + \lambda \sum_{d_t \in M_t} U_t(\theta_t, \sigma_{d_t}).$$

This new Poisson game has an equilibrium. Moreover, no dominated action of the original game is used with positive probability in that equilibrium. Take a sequence of $\lambda \rightarrow 0$. There exists a subsequence of equilibria $\{\sigma^\lambda\}_\lambda$ that converges to some $\bar{\sigma}$. By continuity of the utility function, $\bar{\sigma}$ is an equilibrium in undominated strategies of the original game. \square

In Section 3.4 we have defined perturbed Poisson games. In a perturbed game (Γ, η) an action $\theta \in \Sigma_t(\eta_t)$ is dominated for type t if there exists an alternative action $\theta' \in \Sigma_t(\eta_t)$ such that $U_t(\theta, \sigma) \leq U_t(\theta', \sigma)$, for every possible strategy $\sigma \in \Sigma(\eta)$ and $U_t(\theta, \sigma') < U_t(\theta', \sigma')$ for at least one $\sigma' \in \Sigma(\eta)$.

We could strenght the definition of perfection (Definition 3.5), asking the equilibria in the sequence to be undominated:

DEFINITION 3.8. A strategy σ is an undominated perfect equilibrium of a Poisson game Γ if it is the limit point of a sequence $\{\sigma^\eta\}_{\eta \rightarrow 0}$ where σ^η is an undominated equilibrium of (Γ, η) for all η .

Every perturbed Poisson game has an undominated equilibrium.¹² Moreover, for η close to zero the sets of dominated carriers in Γ and in

¹²To see this, a modification of the proof of Proposition 3.3 would do, where the carrier of an action is defined as the set of pure actions that receive strictly more probability than the minimum weight imposed by η .

(Γ, η) coincide for every possible type. Hence, every undominated perfect equilibrium is perfect and undominated (i.e., it satisfies our strong version of admissibility). Since every perturbed Poisson game has an undominated equilibrium and since this equilibrium is contained in the compact set $(\Delta(C))^T$ it follows:¹³

PROPOSITION 3.4. *Every Poisson game has an undominated perfect equilibrium.*

The definition appears to be stronger than requiring separately perfection and undominance because it poses restrictions in the sequence of equilibria of the associated perturbed Poisson games. The next Proposition shows that both definitions are equivalent. This fact, in view of Lemma 3.6, simplifies the analysis of undominated perfect equilibrium in Poisson games.

PROPOSITION 3.5. *The set of undominated perfect equilibria coincides with the intersection of the set of undominated equilibria with the set of perfect equilibria.*

PROOF. Let σ belong both to the set of perfect equilibria and to the set of undominated equilibria of Γ . Since σ is perfect it is the limit point of a sequence $\{\sigma^\eta\}_{\eta \rightarrow 0}$ where σ^η is an equilibrium of (Γ, η) . Because σ is undominated, its carrier is not a dominated one. Moreover, for η close to zero the sets of dominated carriers in Γ and in (Γ, η) coincide for every possible type. For each η , let η' be defined by:

$$\eta'_t(c) = \begin{cases} \sigma_t^\eta(c) & \text{if } \sigma_t(c) = 0 \\ \eta_t(c) & \text{otherwise} \end{cases} \quad \text{for all } c, t.$$

¹³See footnote 5.

Then $\sigma^{\eta'} = \sigma^{\eta}$ is an undominated equilibrium of (Γ, η') . Moreover the sequence of η' converges to zero. Hence, σ is the limit point of the sequence $\{\sigma^{\eta'}\}_{\eta' \rightarrow 0}$ of undominated equilibria for (Γ, η') . \square

CHAPTER 4

Generic Determinacy of Nash Equilibrium in Network Formation Games

4.1. Introduction

A basic tool in applying noncooperative game theory is to have a finite set of probability distributions on outcomes derived from equilibria.¹ When utilities are defined over the relevant outcome space, it is well known that this is generically the case when we can assign a different outcome to each pure strategy profile (Harsanyi, 1973), or to each ending node of an extensive form game (Kreps and Wilson, 1982).²

A *game form* endows players with finite strategy sets and specifies which is the outcome that arises from each pure strategy profile.³ It could identify, for instance, two ending nodes in an extensive game form with the same outcome. Govindan and McLennan (2001) give an example of a game form such that, in an open set of utilities over outcomes, produces infinitely many equilibrium distributions on outcomes. In view of such a negative result, we have to turn to specific classes of games to seek for positive results regarding the generic determinacy of the Nash equilibrium concept. For

¹By *outcomes* we mean the set of physical or economic outcomes of the game (i.e. the set of different economic alternatives that can be found after the game is played) and not the set of probability distribution induced by equilibria. We will refer to the latter concept as the set of equilibrium distributions.

²Harsanyi (1973) actually proves that the set of Nash equilibria is finite for a generic assignment of payoffs to pure strategy profiles.

³More generally, it specifies a probability distribution on the set of outcomes. Game forms are formally defined in Section 4.2.2

some examples, see Park (1997) for sender-receiver games, and De Sinopoli (2001), De Sinopoli and Iannantuoni (2005) for voting games.

This paper studies the generic determinacy of the Nash equilibrium concept when individual payoffs depend on the network connecting them. The network literature has been fruitful to describe social and economic interaction. See for instance Jackson and Wolinsky (1996), Jackson and Watts (2002), Kranton and Minehart (2001), or Calvo-Armengol (2004). It is, therefore, important to have theories about how such networks form. Different network formation procedures have been proposed. For a comprehensive survey of those theories the reader is referred to Jackson (2003).

The current paper is concerned with a noncooperative approach to network formation. We focus on the network formation game proposed by Myerson (1991). It can be described as follows: each player simultaneously proposes a list of players with whom she wants to form a link, and a direct link between two players is formed if and only if both players agree on that. This game is simple and intuitive, however, since it takes two players to form a link, a coordination problem arises which makes the game exhibit multiplicity of equilibria. Nevertheless, we can prove that even though a network formation game may have a large number of equilibria, every probability distribution on networks induced by equilibria is generically isolated.

The network formation game is formally presented in the next section. Section 4.3 discusses an example. Section 4.4 contains the main result and its proof. To conclude, Section 4.5 discusses some extensions of the result to other network formation games as well as a related result for the extensive form game of network formation introduced by Aumann and Myerson (1989).

4.2. Preliminaries

Given a finite set A , denote as $\mathcal{P}(A)$ the power set of A , and as $\Delta(A)$ the set of probability distributions on A .

4.2.1. Networks. Given a set of players N , a *network* g is a collection of direct links. A direct link in the network g between two different players i and j is denoted by $ij \in g$. For the time being we focus on undirected networks. In an undirected network $ij \in g$ is equivalent to $ji \in g$.⁴ The set of i 's direct links in g is $L_i(g) = \{jk \in g : j = i \text{ or } k = i\}$.

The complete network g^N is such that $L_i(g^N) = \{ij : j \neq i\}$, for all $i \in N$. In g^N player i is directly linked to every other player. The set of all undirected networks on N is $\mathcal{G} = \mathcal{P}(g^N)$.

Each player i can be directly linked with $N - 1$ other players. The number of links in the complete network g^N is $N(N - 1)/2$, dividing by 2 not to count links twice. Since \mathcal{G} is the power set of g^N , it has $2^{N(N-1)/2}$ elements.

4.2.2. Game forms. A *game form* is given by a set of players $N = \{1, \dots, n\}$, nonempty finite sets of pure strategies S_1, \dots, S_n , a finite set of outcomes Ω , a function $\theta : S \rightarrow \Delta(\Omega)$, and utilities defined over the outcome space Ω , that is, $u_1, \dots, u_n : \Omega \rightarrow \mathbb{R}$. Once we fix $N, S_1, \dots, S_n, \Omega$, and θ , a game form is given by a point in $(\mathbb{R}^\Omega)^N$.

Utility functions u_1, \dots, u_n over Ω induce utility functions v_1, \dots, v_n over S according to $u_1 \circ \theta, \dots, u_n \circ \theta$. Hence, every game form has associated its finite normal form game.

4.2.3. The Network Formation Game. The following network formation game is due to Myerson (1991). The set of players is N . All players

⁴In a directed network, if i and j are two different agents, the link ij is different from the link ji . This two links can be regarded as different if, for instance, they explain which is the direction of information, or which is the player who is sponsoring the link.

in N simultaneously announce the set of direct links they wish to form. Formally, the set of player i 's pure strategies is $S_i = \mathcal{P}(N \setminus \{i\})$. Therefore, a strategy $s_i \in S_i$ is a subset of $N \setminus \{i\}$ and is interpreted as the set of players other than i with whom player i wishes to form a link. Mutual consent is needed to create a direct link, i.e., if s is played, ij is created if and only if $j \in s_i$ and $i \in s_j$.

We can adapt the previous general description of game forms to the present context in order to specify the game form that structures the network formation game. Let the set of players and the collection of pure strategy sets be as above. The set of outcomes is the set of undirected networks, i.e., $\Omega = \mathcal{G}$. The function θ is a deterministic outcome function, formally, $\theta: S \rightarrow \mathcal{G}$. Given a pure strategy profile, θ specifies which network is formed respecting the rule of mutual consent to create direct links. Utilities are functions $u_1, \dots, u_n: \mathcal{G} \rightarrow \mathbb{R}$. Once the set of players N is given, the pure strategy sets are automatically created and the network formation game is defined by a point in $(\mathbb{R}^{\mathcal{G}})^N$.

If players other than i play according to $s_{-i} \in S_{-i}$,⁵ the utility to player i from playing strategy s_i is equal to $v_i(s_i, s_{-i}) = u_i(\theta(s_i, s_{-i}))$.

Let $\Sigma_i = \Delta(S_i)$ be the set of mixed strategies of player i . Furthermore, let $\Sigma = \Sigma_1 \times \dots \times \Sigma_n$. While a pure strategy profile s results in the network $\theta(s)$ with certainty, a mixed strategy profile σ generates a probability distribution on \mathcal{G} , where the probability that $g \in \mathcal{G}$ forms equals

$$\mathbf{P}(g \mid \sigma) = \sum_{s \in \theta^{-1}(g)} \left(\prod_{i \in N} \sigma_i(s_i) \right).$$

⁵ $S_{-i} = \prod_{j \neq i} S_j$.

If players other than i play according to σ_{-i} in Σ_{-i} ,⁶ the utility to player i from playing the mixed strategy σ_i is equal to $V_i(\sigma_i, \sigma_{-i}) = \sum_{g \in \mathcal{G}} \mathbf{P}(g \mid (\sigma_i, \sigma_{-i})) u_i(g)$.

DEFINITION 4.1 (Nash Equilibrium). The strategy profile $\sigma \in \Sigma$ is a Nash equilibrium of the network formation game if $V_i(\sigma_i, \sigma_{-i}) \geq V_i(\sigma'_i, \sigma_{-i})$ for all σ'_i in Σ_i , and for all i in N .

4.2.4. Generic Finiteness of Equilibrium Distributions. Let us first give the definition of a generic set.

DEFINITION 4.2. For any $m \geq 0$, we say that $G \subset \mathbb{R}^m$ is a generic set, or generic, if $\mathbb{R}^m \setminus \text{int}(G)$ has Lebesgue measure 0.

Govindan and McLennan (2001) give an example of a game form that, in an open set of utilities over outcomes, produces infinitely many equilibrium distributions on the outcome space.⁷ Nevertheless, they also provide a number of positive results. Consider the general specification of game forms given in Section 4.2.2. The following theorem is a slight modification of Theorem 5.3 in Govindan and McLennan (2001).

THEOREM 4.1. *If θ is such that at all completely mixed strategy tuples and for each agent i the set of distributions on Ω that agent i can induce by changing her strategy is $(|S_i| - 1)$ -dimensional, then for generic utilities there are finitely many completely mixed equilibria.*

The proof of Theorem 4.1 is offered in the appendix.

⁶ $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$.

⁷Their counterexample needs at least three players. In a recent paper, Kukushkin et al. (2007) provide a counterexample for the two player case.

4.3. An Example

Consider a 3 person network formation game. The corresponding game form is depicted in Figure 4.1. Player 1 is the row player, player 2 the column player, and player 3 the matrix player. The symbol g^0 denotes the empty network, g^N denotes the complete network, g^{ij} denotes the network that only contains link ij , and g^i denotes the network where player i is connected to every other player and such that there are no further links.⁸

	$\{0\}$	$\{1\}$	$\{3\}$	$\{1,3\}$	$\{0\}$	$\{1\}$	$\{3\}$	$\{1,3\}$
$\{0\}$	g^0	g^0	g^0	g^0	g^0	g^0	g^0	g^0
$\{2\}$	g^0	g^{12}	g^0	g^{12}	g^0	g^{12}	g^0	g^{12}
$\{3\}$	g^0	g^0	g^0	g^0	g^{13}	g^{13}	g^{13}	g^{13}
$\{2,3\}$	g^0	g^{12}	g^0	g^{12}	g^{13}	g^1	g^{13}	g^1
		$\{0\}$				$\{1\}$		
$\{0\}$	g^0	g^0	g^{23}	g^{23}	g^0	g^0	g^{23}	g^{23}
$\{2\}$	g^0	g^{12}	g^{23}	g^2	g^0	g^{12}	g^{23}	g^2
$\{3\}$	g^0	g^0	g^{23}	g^{23}	g^{13}	g^{13}	g^3	g^3
$\{2,3\}$	g^0	g^{12}	g^{23}	g^2	g^{13}	g^1	g^3	g^N
		$\{2\}$				$\{1,2\}$		

FIGURE 4.1. The game form of a network formation game with three players.

Suppose that the utility function of player $i = 1, 2$ is $u_i(g) = |L_i(g)|$, i.e. player $i = 1, 2$ derives an utility from network g equal to the number of direct links that she maintains in g . Suppose also that player 3 has the same utility as players 1 and 2, except that she derives an utility equal to 2 from

⁸This network architecture is often referred to as a *star*, see Bala and Goyal (2000)

network g^2 . Specifically,

$$\begin{aligned}
 u_i(g^0) &= 0 \quad \text{for all } i, \\
 u_i(g^{jk}) &= \begin{cases} 1 & \text{if } i = k \text{ or } i = j \\ 0 & \text{otherwise,} \end{cases} \\
 u_i(g^j) &= \begin{cases} 2 & \text{if } i = j \\ 2 & \text{if } i = 3 \text{ and } j = 2 \\ 1 & \text{otherwise,} \end{cases} \\
 g_i(g^N) &= 2 \quad \text{for all } i.
 \end{aligned}$$

Figure 4.2 displays the set of Nash equilibria of this game. The subset of Nash equilibria of line (i) supports the empty network, the subsets of line (ii) support, respectively, networks g^{12} , g^{13} and g^{23} , the subsets of line (iii) support, respectively, networks g^1 , g^2 and g^3 .

(i)

$$\text{NE} = \{(\{\emptyset\}, \{\emptyset\}, \{\emptyset\})\} \cup$$

(ii)

$$\{(\{2\}, \{1\}, \{\emptyset\})\} \cup \{(\{3\}, \{\emptyset\}, \{1\})\} \cup \{(\{\emptyset\}, \{3\}, \{2\})\} \cup$$

(iii)

$$\{(\{2, 3\}, \{1\}, \{1\})\} \cup \{(\{2\}, \{1, 3\}, \{2\})\} \cup \{(\{3\}, \{3\}, \{1, 2\})\} \cup$$

(iv)

$$\{(\{2, 3\}, \{1, 3\}, \lambda\{2\} + (1 - \lambda)\{1, 2\}) : \lambda \in [0, 1]\}.$$

FIGURE 4.2. Set of Nash equilibria of the 3 person network formation game discussed in Section 4.3.

The subset of equilibria of line (iv) induces a continuum of probability distribution over the set of networks that give probability λ to network g^2 and probability $(1 - \lambda)$ to the complete network g^N for $\lambda \in [0, 1]$.

Now perturb independently the utility that each player obtains from each network. The subsets of strategy profiles of lines (i) through (iii) are still equilibrium strategy profiles. In addition, there are two possibilities:

- Player 3 ranks the complete network g^N over network g^2 . In this case the set of Nash equilibria is composed of lines (i) through (iii) united to

$$\left\{ (\{2, 3\}, \{1, 3\}, \{1, 2\}) \right\},$$

which supports the complete network.

- Player 3 ranks network g^2 over the complete network g^N . Then, no Nash equilibrium gives positive probability to the complete network. The set of Nash equilibria is composed of lines (i) through (iii) united to

$$\left\{ (\lambda\{2\} + (1 - \lambda)\{2, 3\}, \{1, 3\}, \{2\}) : \lambda \in [0, 1] \right\},$$

which supports network g^2 .

In either case, there is a finite number of probability distributions on networks induced by equilibria.

4.4. The Result

PROPOSITION 4.1. *For generic $u \in (\mathbb{R}^G)^N$ the set of probability distributions on networks induced by Nash equilibria of the network formation game is finite.*

PROOF. Given a network formation game, there are a finite number of different normal form games obtained by assigning to each player i an element of $\mathcal{P}(S_i)$ as her strategy set.

Let $T = T_1 \times \cdots \times T_n$, where $T_i \subseteq S_i$. The normal form game Γ_T is defined by the set of players N , the collection of strategy sets $\{T_i\}_{i \in N}$, and the collection of utility functions $\{v_i^T\}_{i \in N}$, where v_i^T is the restriction of v_i to T . Furthermore, let $\mathcal{G}_T = \theta(T)$.

It is enough to prove that for a generic assignment of payoffs to networks, completely mixed Nash equilibria of each of those games induce a finite set of probability distributions on \mathcal{G} . Notice that every equilibrium of any game can be obtained as a completely mixed equilibrium of the modified game obtained by eliminating unused strategies.

Consider the game Γ_T . If there exists a strategy $t_i \in T_i$ with $j \in t_i$ and there does not exist a strategy $t_j \in T_j$ such that $i \in t_j$, replace strategy t_i with $t'_i = t_i \setminus \{j\}$ in case t'_i is not already contained in T_i , otherwise just eliminate strategy t_i from T_i . Notice that by making this change, the set of probability distributions on \mathcal{G}_T that can be obtained through mixed strategies remains unaltered. Most importantly, for every completely mixed Nash equilibrium of Γ_T , there exists a completely mixed Nash equilibrium of the modified game that induces the same probability distribution on \mathcal{G}_T .

Repeat the same procedure with t'_i : if there exists a $k \in t'_i$ and there does not exist a strategy t_k in T_k with $i \in t_k$ substitute t'_i for $t''_i = t'_i \setminus \{k\}$ in case t''_i is not already contained in T_k . Continue eliminating and replacing pure strategies in the same vein, for every t_i in T_i and for every i in N , until every link proposal that any player has in some on her strategies is formed with positive probability under a completely mixed strategy profile. Let \hat{T} denote the resulting set of pure strategy profiles, and notice that $\mathcal{G}_{\hat{T}} = \mathcal{G}_T$.

At every completely mixed strategy profile σ of $\Gamma_{\hat{T}}$, every network in \mathcal{G}_T receives positive probability. At the strategy profile (t_i, σ_{-i}) , only those networks $g \in \mathcal{G}_T$ such that $\{ij : j \in t_i\} \subset g$ receive positive probability, and

since for every player i each of her pure strategies is different, we have that:

$$\text{rank} \frac{\partial \mathbf{P}}{\partial \sigma_i}(\cdot | \sigma) = |\hat{T}_i| - 1.$$

Therefore, at every completely mixed strategy profile of $\Gamma_{\hat{T}}$ the set probability distributions on \mathcal{G}_T that player i can induce by varying her strategy is $(|\hat{T}_i| - 1)$ -dimensional. We can apply Theorem 4.1 to the game form given by \hat{T} and $\theta_{\hat{T}}$, the restriction of θ to \hat{T} . This implies that for generic utilities over \mathcal{G}_T there are finitely many completely mixed equilibria of $\Gamma_{\hat{T}}$, which in turn implies that the set of probability distributions on \mathcal{G}_T induced by completely mixed Nash equilibria of Γ_T is generically finite.

Let $T \subseteq S$, we can write $(\mathbb{R}^{\mathcal{G}})^N = (\mathbb{R}^{\mathcal{G}_T})^N \times (\mathbb{R}^{\mathcal{G} \setminus \mathcal{G}_T})^N$. Let K be a closed set of zero measure in $(\mathbb{R}^{\mathcal{G}_T})^N$, i.e., the closure of the set of payoffs over \mathcal{G}_T such that the set of completely mixed Nash equilibria of Γ_T induces infinitely many probability distributions on \mathcal{G}_T , then for any closed set H in $(\mathbb{R}^{\mathcal{G} \setminus \mathcal{G}_T})^N$ the closed set $K \times H$ has zero measure in $(\mathbb{R}^{\mathcal{G}})^N$. The same is true for any other $T' \subseteq S$. This concludes the proof. \square

4.5. Remarks

4.5.1. Absence of Mutual Consent. Models of network formation can be found in the literature that do not require common agreement between the parties to create a direct link, see for instance Bala and Goyal (2000). Thus, suppose that mutual consent is not needed to create a direct link. Let N be the set of players, let S_1, \dots, S_n be the collection of pure strategy sets, where $S_i = \mathcal{P}(N \setminus \{i\})$ for all i in N , and let \mathcal{G} be the outcome space. In the model analyzed in Section 4.4, a link may not be created even if a player wants it to be created. In the current model, a link may be created even if a player does not want it to be created.

In this modified network formation game, generically, the set of equilibrium distributions on \mathcal{G} is also finite. Notice that we can reinterpret pure

strategies $s_i \in S_i$ as the set of players other than i with whom player i does not want to form a link. The link ij is not created only if player i does not want to be linked with player j and player j does not want to be linked with player i . Define $\theta' : S \rightarrow \mathcal{G}$ according to $\theta'(s) = g^N \setminus \theta(s)$, where θ is the one defined in Section 4.2.3. Now, apply the proof of Section 4.4.

4.5.2. Directed Networks. Sometimes links ij and ji cannot be treated as equivalent for reasons coming from the nature of the phenomena being modeled. Directed networks respond to this necessity, for an example see again Bala and Goyal (2000). Denote the set of directed networks as \mathcal{G}^d . Suppose first that link formation does not need mutual consent. The strategy set of player i is $S_i = \mathcal{P}(N \setminus \{i\})$. A strategy $s_i \in S_i$ is interpreted as the set of players other than i with whom player i wants to start an arrowhead link pointing at herself, i.e. the set of links that player i wishes to receive.⁹

Notice that each pure strategy profile leads to a different element in \mathcal{G}^d : each player has 2^{N-1} pure strategies, and there are $2^{N(N-1)}$ undirected networks. Therefore, we are in the case of normal form payoffs where the generic finiteness of equilibria is guaranteed.

Suppose now that if a player i wants to receive a link from player j , player j needs to declare that she wants to send a link to player i for it to be created. To accommodate for this case, let the strategy set of player i be $S_i = S_i^r \times S_i^s = \mathcal{P}(N \setminus \{i\}) \times \mathcal{P}(N \setminus \{i\})$. A strategy $s_i \in S_i$ has two components, s_i^r and s_i^s . We interpret s_i^r as the set of players other than i from whom player i wishes to receive a link, and s_i^s as the set of players other than i to whom player i wishes to send a link. Suppose that the pure strategy profile s is played. The link ij is created only if $j \in s_i^r$ and $i \in s_j^s$.

⁹We can assume, for instance, that the arrowhead tells which is the direction of the flow of information.

A similar proof to the one used in Section 4.4 establishes the generic determinacy of the Nash equilibrium concept under this setting. The key step that we must change is the following: Let $T = T_1 \times \cdots \times T_n$ where $T_i \subset S_i$ for all i . Consider the normal form game Γ_T . If there exists a strategy $t_i \in T_i$ such that $j \in t_i^r$ (such that $j \in t_i^s$) and there does not exist a strategy $t_j \in T_j$ such that $i \in t_j^s$ (such that $i \in t_j^r$), replace strategy t_i with $t'_i = (t_i^r \setminus \{j\}, t_i^s)$ (with $t'_i = (t_i^r, t_i^s \setminus \{j\})$). Finally, repeat the same procedure for every t_i, t'_i, \dots and for every i until the hypothesis of Theorem 4.1 holds.

4.5.3. A Extensive Form Game of Network Formation. We have focused on normal form games of network formation. However, there exists a prominent extensive game of network formation due to Aumann and Myerson (1989). They proposed the first explicit formalization of network formation as a game. It relies on an exogenously given order over possible links. Let $(i_1 j_1, \dots, i_m j_m)$ be such a ranking.

The game has m stages. In the first stage players i_1 and j_1 play a simultaneous move game to decide whether or not they form link $i_1 j_1$. Each of them chooses an action from the set $\{yes, not\}$. The link $i_1 j_1$ is established if and only if both players choose *yes*. Once the decision on link $i_1 j_1$ is taken, every player gets informed about it, and the play of the game moves to the decision about link $i_2 j_2$. The game evolves in the same fashion, and finishes with the stage where players i_m and j_m decide upon link $i_m j_m$.¹⁰ The resulting network is formed by the set links $i_k j_k$ such that both players i_k and j_k chose *yes* at stage k . Although in the argument we work with

¹⁰If players get informed about which has been the terminal position in the simultaneous move game of every stage, the same argument offered below also goes through.

Several features can be added to this basic model. For instance, two players can be called to reconsider their decision in case some set of links is formed, or two player may not be allowed to decide upon the link connecting them. At this respect, if players are forming an undirected network, m can be different from $2^{\frac{N(N-1)}{2}}$.

undirected networks, the game can be applied to the formation of directed networks.

The argument that follows is a modification of the one used by Govindan and McLennan (2001) to prove that, for a given assignment of outcomes to ending nodes in an extensive game of perfect information, and for utilities such that no player is indifferent between two different outcomes, every Nash equilibrium induces a degenerate probability distribution in the set of outcomes. Such an argument is, in turn, a generalization of the one used by Kuhn (1953) to prove his “backwards induction” theorem that characterizes subgame perfect equilibria for games of perfect information.

Consider the generic set of utilities

$$U_G = \left\{ u \in \left(\mathbb{R}^{\mathcal{G}} \right)^N : u_i(g_1) \neq u_i(g_2) \text{ for all } i \in N \text{ and all } g_1, g_2 \in \mathcal{G} \right\}.$$

The claim is that if the utility vector is $u \in U_G$, every Nash equilibrium induces a probability distribution on \mathcal{G} that assigns probability one to some $g \in \mathcal{G}$.

Let S_i denote the set of pure strategies of player i , where now a pure strategy is a function that assigns one element of $\{yes, not\}$ to each information set of player i . As usual, $\Sigma_i = \Delta(S_i)$ and $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$.

Let $\sigma \in \Sigma$ be a Nash equilibrium for $u \in U_G$. The appropriate modification of σ , say $\bar{\sigma}$, is a completely mixed Nash equilibrium of the extensive form game obtained by eliminating all information sets and branches that occur with zero probability in case σ is played. In this reduced game, every information set has a well defined conditional probability over networks and, obviously, $\bar{\sigma}$ induces the same probability distribution on \mathcal{G} as σ .

If there is a stage where a player randomizes between *yes* and *not* and the other player chooses *yes* with positive probability, there must be a last such stage. But at this last stage, say $i_h j_h$, such an agent, say i_h , cannot be

optimizing, since she is not indifferent between $g \setminus \{i_h j_h\}$ and $g \cup \{i_h j_h\}$ for any $g \in \mathcal{G}$.

We can adapt the previous argument to the case where mutual consent is not needed to create a link. Let $(i_1 j_1, \dots, i_m j_m)$ be an order of links. At stage k , player i_k decides whether or not to create link $i_k j_k$. Her decision becomes publicly known. It is, consequently, a game of perfect information and the argument given by Govindan and McLennan (2001) covers this case.

4.6. Appendix: Proof of Theorem 4.1

The current proof is based on the one offered by Govindan and McLennan (2001). It uses some concepts and results of semi-algebraic theory that we will now revise. Expositions of semi-algebraic geometry in the economic literature occur in Blume and Zame (1994), Schanuel et al. (1991) and Govindan and McLennan (2001). Proofs of major results are omitted.

DEFINITION 4.3. A set A is semi-algebraic if it is the finite union of sets of the form

$$\{x \in \mathbb{R}^m : P(x) = 0 \text{ and } Q_1(x) > 0 \text{ and } \dots \text{ and } Q_k(x) > 0\}$$

where P and Q_1, \dots, Q_k are polynomials in x_1, \dots, x_m with real coefficients. A function (or correspondence) $g : A \rightarrow B$ with semi-algebraic domain $A \subset \mathbb{R}^n$ and range $B \subset \mathbb{R}^m$ is semi-algebraic if its graph is a semi-algebraic subset of \mathbb{R}^{n+m} .

Each semi-algebraic set is the finite union of connected components. Each component is a *semi-algebraic manifold* of a given dimension. A *d-dimensional semi-algebraic manifold* in \mathbb{R}^m is a semi-algebraic set $M \subset \mathbb{R}^m$ such that for each $p \in M$ there exist polynomials P_1, \dots, P_{m-d} and U , a

neighborhood of p , such that $DP_1(p), \dots, DP_{m-d}(p)$ are linearly independent and

$$M \cap U = \{q \in U : P_1(q) = \dots = P_{m-d}(q) = 0\}.$$

THEOREM 4.2 (Stratification, Whitney (1957)). *If A is a semi-algebraic set, then A is the union of a finite number of disjoint, connected semi-algebraic manifolds A^j with $A^j \subset \text{cl}(A^k)$ whenever $A^j \cap \text{cl}(A^k) \neq \emptyset$.*

Henceforth, the superscript of a set indexes components of a decomposition as per Theorem 4.2, while a subscript keeps indexing strategy sets by players. Theorem 4.2 has important consequences. Among those, we will use the following intuitive ones: Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be semi-algebraic sets, then

- the dimension of A , $\dim A$, is equal to the largest dimension of any element of any stratification,
- if A is 0-dimensional then A is finite,
- A is generic if and only if $\dim(\mathbb{R}^m \setminus A) < m$,
- $\dim(A \times B) = \dim A + \dim B$.

We need one additional result. While Theorem 4.2 decomposes semi-algebraic sets, the following one decomposes semi-algebraic functions.

THEOREM 4.3 (Generic Local Triviality, Hardt (1980)). *Let A and B be semi algebraic sets, and let $g: A \rightarrow B$ be a continuous semi-algebraic function. Then there is a relatively closed semi-algebraic set $B' \subset B$ with $\dim B' < \dim B$ such that each component B^j of $B \setminus B'$ has the following property: there is a semi algebraic set F^j and a semi-algebraic homeomorphism $h: B^j \times F^j \rightarrow A^j$, where $A^j = g^{-1}(B^j)$, with $g(h(b, f)) = b$ for all $(b, f) \in B^j \times F^j$.*

We can now proceed to prove Theorem 4.1. Recall that at every completely mixed strategy $\sigma \in \Sigma$, the set of probability distributions on

outcomes that player i can induce by varying her strategy is $(|S_i| - 1)$ -dimensional.

PROOF OF THEOREM 4.1. Let

$$A = \{(\sigma, u) : \sigma \text{ is a completely mixed equilibrium for } u\}.$$

Let π_Σ be the projection of A onto Σ . Apply Theorem 4.3 to π_Σ and choose Σ^j such that $\dim A^j = \dim A$.¹¹ We have that $\dim A = \dim \Sigma^j + \dim F^j \leq \dim \Sigma + \dim F^j$. Let σ belong to Σ^j , then $\dim \pi_\Sigma^{-1}(\sigma) = \dim \{\sigma\} + \dim F^j = \dim F^j$. Now consider a given u , the set

$$\{\tilde{u}_i \in U_i : \sigma \text{ is a completely mixed equilibrium for } (\tilde{u}_i, u_{-i})\}$$

is $(\dim U_i - (|S_i| - 1))$ -dimensional. Consequently, the dimension of $\pi_\Sigma^{-1}(\sigma)$ and F^j is equal to $\dim U - \dim \Sigma$, which implies that $\dim A \leq \dim U$.

Now apply Theorem 4.3 to π_U , the projection of A onto U . Choose U^j to be of the same dimension as U . Therefore, $\dim A^j = \dim U + \dim \pi_U^{-1}(u)$. This implies that $\dim \pi_U^{-1}(u) \leq \dim A - \dim U \leq 0$, i.e. there is a finite set of completely mixed equilibria whenever u belongs to a full dimensional U^j . This concludes the proof since lower dimensional U^j 's are nongeneric. \square

¹¹Such a Σ^j can be found because we can keep applying Theorem 4.3 to $\pi_\Sigma : \pi_\Sigma^{-1}(\Sigma') \rightarrow \Sigma'$, where Σ' plays the role of B' .

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