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MODELLING LONG-MEMORY VOLATILITIES WITH LEVERAGE EFFECT: A-LMSV VERSUS FIEGARCH *

Esther Ruiz¹ and Helena Veiga¹

Abstract

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Keywords: Autocorrelations of squares and of absolute values, Conditional heteroscedasticity, Kurtosis, Whittle estimator.

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Modelling long-memory volatilities with leverage effect: A-LMSV versus FIEGARCH

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Abstract

In this paper, we propose a new stochastic volatility model, called A-LMSV, to cope simultaneously with the leverage effect and long-memory. We derive its statistical properties and compare them with the properties of the FIEGARCH model. We show that the dependence of the autocorrelations of squares on the parameters measuring the asymmetry and the persistence is different in both models. The kurtosis and autocorrelations of squares do not depend on the asymmetry in the A-LMSV model while they increase with the asymmetry in the FIEGARCH model. Furthermore, the autocorrelations of squares increase with the persistence in the A-LMSV model and decrease in the FIEGARCH model. On the other hand, the autocorrelations of absolute returns increase with the magnitude of the asymmetry in the FIEGARCH model while they can increase or decrease depending on the sign of the asymmetry in the L-MSV model. Finally, the cross-correlations between squares and original observations are, in general, larger in the FIEGARCH model than in the A-LMSV model. The results are illustrated by fitting both models to represent the dynamic evolution of volatilities of daily returns of the S&P500 and DAX indexes.

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1 Introduction

One of the main empirical characteristics of financial returns is the dynamic evolution of their volatilities. There are two important properties that characterized this evolution. First of all, power transformations of absolute returns have significant autocorrelations which decay towards zero slower than in a short

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memory process. Many authors have argued that this pattern of the sample autocorrelations suggests that the volatilities of financial returns can be represented by long-memory processes; see Ding et al. (1993) and Lobato and Savin (1998) among many others. The second property that characterizes volatilities is their asymmetric response to positive and negative returns. This property, known as leverage effect, was first described by Black (1976).

There are two main families of econometric models proposed to represent the dynamic evolution of volatilities. The ARCH-type models are mainly characterized by specifying the volatility as a function of powers of past absolute returns and, consequently, the volatility can be observed one-step ahead. On the other hand, Stochastic Volatility (SV) models specify the volatility as a latent variable that is not directly observable. There have been several proposals of ARCH-type models that represent simultaneously leverage effect and long-memory. For example, Hwang (2001) proposed to extend the long-memory FIGARCH model of Baillie et al. (1996) to represent the leverage effect. However, some authors have pointed out some drawbacks of this model. For instance, Davidson (2004) has recently shown that the FIGARCH model has the unpleasant property that the persistence of shocks to volatility decreases as the long-memory parameter increases. Zafaroni (2004) has also showed that the FIGARCH model cannot generate autocorrelations of squares with long memory. Finally, Ruiz and Pérez (2003) showed that the model proposed by Hwang (2001) has identification problems. Consequently, in this paper, we focus on the Fractionally Integrated EGARCH (FIEGARCH) model proposed by Bollerslev and Mikkelsen (1996) which extends the asymmetric EGARCH model of Nelson (1991) to long memory. The statistical properties of the FIEGARCH model have been derived following the arguments given by He et al. (2002) for the short-memory EGARCH model. In particular, we give explicit analytical expressions of the kurtosis and autocorrelations of squares and absolute observations. Furthermore, we also derived the cross-correlations between $|y_{t+k}|^c$ and y_t for $c = 1$ and 2 .

Alternatively, in the context of SV models, Harvey (1998) and Breidt et al. (1998) have independently proposed Long-Memory Stochastic Volatility (LMSV) models in which the underlying log-volatility is modelled as an ARFIMA process. On the other hand, Harvey and Shephard (1996) propose to model the leverage effect of the short memory SV model by introducing correlation between the noises of the level and volatility equations¹. In this paper, it is our aim to go one step further and present an extension of the LMSV model to cope with the leverage effect, that we denote A-LMSV. We derive the statistical properties of the new model and compare them with the properties of the FIEGARCH model. We show that both models explain in a different way the kurtosis and correlations of absolute and squared returns. The kurtosis and autocorrelations of squares of the A-LMSV models are not affected by the presence of the leverage effect. However, in the FIEGARCH model, both moments

¹So et al. (2002) have proposed a threshold SV model which is also able to represent the leverage effect.

increase with the asymmetry. Furthermore, we show that the autocorrelations of squares increase with the persistence in A-LMSV models and decrease in FIEGARCH models. On the other hand, in the A-LMSV models, the autocorrelations of absolute observations increase with respect to the autocorrelations of the symmetric model when the correlation between the volatility and the level of returns is positive. However, if, as it is usual in financial returns, this correlation is negative, the autocorrelations decrease as the magnitude of the correlation increases. With respect to the FIEGARCH model, the autocorrelations of absolute returns always increase with the magnitude of the asymmetry regardless of its sign. Finally, the asymmetric response of volatility to positive and negative returns is reflected in the cross-correlations between $|y_{t+k}|^c$ and y_t for $c = 1$ and 2 . We show that the patterns of these cross-correlations in the two models considered in this paper is rather similar.

The rest of the paper is organized as follows. The description of the statistical properties of the A-LMSV model is done in section 2. We focus on the kurtosis, autocorrelations of absolute and squared observations and cross-correlations between $|y_{t+k}|^c$ and y_t where $c = 1$ and 2 . In Section 3, we describe the properties of the FIEGARCH model and compare them with the properties of the A-LMSV model. Section 4 contains an empirical illustration by fitting both models to daily financial returns of the S&P500 and the DAX indexes. Section 5 concludes the paper.

2 Asymmetric LMSV models

The LMSV model, proposed independently by Harvey (1998) and Breidt et al. (1998), extends the stochastic volatility model of Taylor (1982) by assuming that the volatility follows a weakly stationary fractional integrated process. Therefore, the LMSV model captures the long-memory property often observed in the powers of absolute returns. In this section, we extend the LMSV model to represent the asymmetric response of volatility to positive and negative returns. Following Taylor (1994) and Harvey and Shephard (1996), this asymmetry is introduced by allowing the disturbances of the level and volatility equations to be correlated. If, for example, the log-volatility is an ARFIMA(1, d , 0) process, the A-LMSV model is given by

$$\begin{aligned} y_t &= \sigma_* \sigma_t \varepsilon_t \\ (1 - \phi L)(1 - L)^d \log \sigma_t^2 &= \eta_t \end{aligned} \quad (1)$$

where y_t is the return at time t and σ_t is its volatility. The parameter σ_* is a scale parameter and L is the lag operator such that $Lx_t = x_{t-1}$. The disturbances $(\varepsilon_t, \eta_{t+1})'$ are assumed to have the following bivariate normal distribution

$$\begin{pmatrix} \varepsilon_t \\ \eta_{t+1} \end{pmatrix} \sim NID \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \delta \sigma_\eta \\ \delta \sigma_\eta & \sigma_\eta^2 \end{pmatrix} \right), \quad (2)$$

where δ , the correlation between ε_t and η_{t+1} , induces correlation between the returns, y_t , and the variations of the volatility one period ahead, $\sigma_{t+1} - \sigma_t$. The

dynamic properties of the symmetric LMSV model are described by Ghysels et al. (1996). In particular, the stationarity of y_t depends on the stationarity of the log-volatility, $h_t = \log \sigma_t^2$. Therefore, if $|\phi| < 1$ and $d < 0.5$, y_t is stationary. In this case, its variance and kurtosis are given by

$$Var(y_t) = \sigma_*^2 \exp \left\{ \frac{\sigma_h^2}{2} \right\} \quad (3)$$

and

$$\kappa_y = \frac{E(y_t^4)}{E(y_t^2)^2} = 3 \exp(\sigma_h^2) \quad (4)$$

respectively, where $\sigma_h^2 = \sigma_\eta^2 \frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2} \frac{F(1,1+d;1-d;\phi)}{(1+\phi)}$, $\Gamma(\cdot)$ is the gamma function and $F(\cdot, \cdot; \cdot; \cdot)$ is the hypergeometric function; see Hosking (1981) for the expressions of the variance and autocorrelation function (acf) of an ARFIMA process. From (4), it can be shown that, given σ_η^2 and ϕ , the kurtosis of returns, κ_y , increases with d , the parameter of fractional integration. On the other hand, Harvey and Shephard (1996) have shown, in the context of the symmetric LMSV model, that introducing the correlation between ε_t and η_{t+1} into the LMSV model does not change the marginal moments of y_t with respect to the symmetric model with $\delta = 0$. Therefore, expressions (3) and (4) are also the variance and kurtosis of y_t in the A-LMSV model.

Furthermore, although the series of returns, y_t , is a martingale difference and, consequently an uncorrelated sequence², it is not an independent sequence. There are non-linear transformations of returns, as for example, powers of absolute returns, which are correlated. After some very tedious algebra, and using the fact that ε_t and η_t are Gaussian and mutually independent³, it is possible to derive the following expression of the acf of $|y_t|^c$ for $c = 1$ and 2,

$$\rho_c(k) = \frac{\exp \left(\frac{c^2}{4} \sigma_h^2 \rho_h(k) \right) \left(1 + (\delta \sigma_\eta)^c \left(\frac{4}{\kappa_c} \right)^{\frac{c-2}{2}} \lambda_k^c \right) - 1}{\kappa_c \exp \left(\frac{c^2}{4} \sigma_h^2 \right) - 1}, \quad k \geq 1 \quad (5)$$

where $\rho_h(k)$ is the acf of the log-volatility given by

$$\rho_h(k) = \left(\prod_{i=0}^{k-1} \frac{d+i}{1-d+i} \right) \frac{F(1, d+k; 1-d+k; \phi) + F(1, d-k; 1-d-k; \phi) - 1}{(1-\phi)F(1, 1+d; 1-d; \phi)},$$

$\kappa_c = \frac{E(|\varepsilon_t|^{2c})}{\{E|\varepsilon_t|^c\}^2} = \frac{\Gamma(c+0.5)\Gamma(0.5)}{[\Gamma(0.5(c+1))]^2}$ which takes values $\kappa_1 = \frac{\pi}{2}$ and $\kappa_2 = 3$ respectively. Finally, if $d > 0$, $\lambda_k = \sum_{i=0}^{k-1} \frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} \phi^{k-i-1}$. Note that when $\delta = 0$,

²Yu (2004) points out that if the asymmetry is introduced as in Jacquier et al. (2004) by a correlation between ε_t and η_t , the series y_t is not a martingale difference.

³The expression of the autocorrelations of $|y_t|^c$ in (5) is also valid for non-Normal distributions of the errors ε_t . The only difference is that the value of κ_c depends on this distribution. In this paper, we focus on Gaussian errors because, we want to compare the A-LMSV and the FIEGARCH model in the simplest framework.

expression (5) becomes the acf of $|y_t|^c$ derived by Harvey (1998) for symmetric LMSV models. On the other hand, when $d = 0$, $\lambda_k = \phi^{k-1}$ and expression (5) becomes the acf of a short memory SV model with leverage effect. Taylor (1994) have obtained this expression for $c = 2$.

The acf in expression (5) depends on the parameters d , ϕ and σ_η^2 that affect both the variance and the acf of the underlying log-volatility process, σ_h^2 and $\rho_h(k)$ respectively. The autocorrelations also depend on the correlation between ε_t and η_{t+1} , δ , and on the power parameter, c . For fixed values of the parameters d , σ_η^2 and ϕ , the effect of the asymmetry, δ , on the autocorrelation of order k of y_t^2 is measured by $(\delta\sigma_\eta\lambda_k)^2$. First of all note that this effect is the same regardless of the sign of δ . Furthermore, in empirical applications, the variance of the log-volatility process and the constant λ_k are typically very small and consequently, this effect is also rather small. Therefore, the autocorrelations of squared returns generated by symmetric and asymmetric LMSV processes are very similar; see Carnero et al. (2004) for a similar result in the context of short memory ARSV models. As an illustration, the right column of Figure 1 plots the acfs of squared returns, for the following four A-LMSV models: $\{\phi = 0, d = 0.4, \sigma_\eta^2 = 0.05\}$, $\{\phi = 0, d = 0.4, \sigma_\eta^2 = 0.1\}$, $\{\phi = 0.2, d = 0.4, \sigma_\eta^2 = 0.1\}$ and $\{\phi = 0.98, d = 0, \sigma_\eta^2 = 0.05\}$. For each of these models, we consider $\delta = 0, 0.2, 0.5$ and 0.8 . The models have been chosen to resemble the parameter values often estimated when the LMSV model is fitted to time series of financial returns; see, for example, Pérez and Ruiz (2001). Figure 1 shows that, for the four models considered, the acfs of squares are nearly the same regardless of the value of the asymmetry parameter δ .

We consider now the effect of δ on the autocorrelation of order k of absolute observations which is measured by $0.627\delta\sigma_\eta\lambda_k$. In this case, a positive correlation between ε_t and η_{t+1} increases the autocorrelations while a negative value of δ decreases them. Note that, depending on the parameters that govern the dynamic evolution of the volatility, the autocorrelations of absolute returns can even be negative if δ is large enough. The magnitude of this effect is larger than the corresponding effect on the autocorrelations of squared returns if $|\delta\sigma_\eta\lambda_k| < 0.627$. As an illustration, the left column of Figure 1 plots the acf of absolute returns for the same four models considered above when $\delta = 0, \pm 0.2, \pm 0.5$ and ± 0.8 . This figure shows that the effect of the leverage effect on the acfs of absolute returns is clearly larger than in the acfs of squares. Furthermore, we can also observe that a positive correlation increases the autocorrelations of absolute returns while a negative correlation decreases them. As we mentioned before, the autocorrelations can even be negative. For example, in the first three models considered in Figure 1, $\rho_1(1)$ is negative if $\delta < -0.25, -0.33$ and -0.45 respectively. For the last model, which is a short memory model with $d = 0$, the autocorrelations of absolute values are never negative. On the other hand, when $\delta = 0$, i.e. there is not leverage effect, the autocorrelations of absolute observations are always positive. Therefore, the combination of negative correlations between ε_t and η_{t+1} and long-memory generates the possibility of negative autocorrelations in absolute observations. It is also interesting to observe in Figure 1 that, when the correlation between ε_t and η_{t+1} is negative,

the pattern of the autocorrelations of absolute values is different depending on whether the volatility has short or long memory. In the first case, we observe that, as usual, the autocorrelations decay towards zero monotonically. However, in the presence of long-memory, if the autocorrelation of order one is positive, the autocorrelations increase for the first few lags and then decay towards zero. On the other hand, when there is long-memory and $\rho_1(1)$ is negative, the autocorrelations decrease monotonically towards zero in magnitude. Finally, note that when the correlation between the errors, δ , is positive, we always observe the usual pattern of autocorrelations decreasing monotonically towards zero.

A property that has often interested researches dealing with models for second order moments, is the so-called Taylor effect that states that the autocorrelations of absolute returns are larger than the autocorrelations of squares. In the context of symmetric LMSV models Mora-Galán et al. (2004) show that, when the persistence is large, the autocorrelations of powers of absolute observations are maximized when the power is close to one. The results previously described suggest that a positive correlation between the errors reinforces the Taylor effect while if the correlation is negative, it may disappear. This result is illustrated in Figure 2 which plots the differences $\rho_1(1) - \rho_2(1)$ as a function of δ for the same four models considered before. The first result that emerges from Figure 2 is that there is an approximately linear positive relationship between δ and $\rho_1(1) - \rho_2(1)$. When the correlation between the level and volatility noises is positive, the Taylor effect is stronger for larger values of δ . However, when δ is negative or close to zero, the autocorrelations of squares are larger than the autocorrelations of absolute returns. Figure 2 also illustrates that the range of values of δ for the Taylor effect to disappear is larger in SV models with long memory. In the model with $d = 0$, the correlation between ε_t and η_{t+1} has to be negative and very large in absolute value for the autocorrelations of squares to be larger than the corresponding autocorrelations of absolute returns.

As we mentioned in the Introduction, real time series of financial returns are often characterized by significant autocorrelations of powers of absolute returns that decay very slowly towards zero and by the asymmetric response of volatility to positive and negative returns. On top of this, the series of returns usually have excess kurtosis and the autocorrelations of squares are rather small in magnitude. We now analyze whether the proposed A-LMSV is able to explain simultaneously the excess kurtosis and small autocorrelations of squares. We have seen before that the presence of leverage effect does not have any effect on the kurtosis and only very marginal effects on the autocorrelations of squares. Therefore, whether the parameter δ is zero or not is not going to change the ability of the A-LMSV model to represent simultaneously both effects. However, we want to analyze how the presence of long-memory may change the relationship between kurtosis, κ_y , and $\rho_2(1)$ which is given by

$$\rho_2(1) = \frac{\left(\frac{\kappa_y}{3}\right)^{\rho_h(1)} \left(1 + (\delta\sigma_\eta)^2\right) - 1}{\kappa_y - 1}. \quad (6)$$

Figure 3 plots the relationship between κ_y , and $\rho_2(1)$ as a function of the au-

tocorrelation of order one of the underlying log-volatility, $\rho_h(1)$, when $\delta = 0^4$. Note that $\rho_h(1)$ depends on ϕ and d . For fixed d , $\rho_h(1)$ is a non-monotonous function of ϕ . On the other hand, when ϕ is fixed, the autocorrelations of the log-volatility increase with the long-memory parameter, d . Therefore, when interpreting Figure 3, if we assume that ϕ is fixed, larger values of $\rho_h(1)$ are identified with larger values of d . We can observe that for a given kurtosis, κ_y , the autocorrelation of order one of squares increases with the long memory parameter, d . However, the rate of growth of $\rho_2(1)$ is very small for low values of the kurtosis and increases with the kurtosis. On the other hand, given $\rho_2(1)$, the kurtosis decreases as d increases. Figure 3 also illustrates that there is a wide range of combinations of the parameters that govern the dynamic evolution of the underlying volatilities, ϕ and d able to generate series with large kurtosis and small autocorrelations of squares as the ones usually observed in real time series.

If the distribution of y_t is symmetric, the main difference between heteroscedastic series with and without leverage effect is that, in the latter case the correlations between returns and future powers of absolute returns is zero while in the former they are different from zero. Consequently, another instrument for the identification of the leverage effect is the correlation between returns and future absolute returns to the power c . We consider the more interesting cases from the empirical point of view of $c = 1$ and 2. In these cases, the covariances are given by⁵

$$Cov(y_t, |y_{t+k}|^c) = \begin{cases} 0.5\sigma_*^2 \sqrt{\frac{2}{\pi}} \delta \sigma_\eta \lambda_k \exp\{\frac{\sigma_h^2}{4}(\rho_h(k) + 1)\}, & c = 1 \\ \sigma_*^3 \delta \sigma_\eta \lambda_k \exp\{\frac{\sigma_h^2}{8}(4\rho_h(k) + 5)\}, & c = 2 \end{cases}. \quad (7)$$

The variance of $|y_t|^c$ derived by Harvey (1998) for the symmetric LMSV model with Gaussian errors is given by

$$Var(|y_t|^c) = \sigma_*^{2c} 2^c \exp\left(c^2 \frac{\sigma_h^2}{2}\right) \left(\frac{\Gamma(c + \frac{1}{2})}{\Gamma(\frac{1}{2})} - \left[\frac{\Gamma(\frac{c}{2} + \frac{1}{2})}{\Gamma(\frac{1}{2})}\right]^2\right), \quad (8)$$

Given that the asymmetry does not change the marginal moments of y_t and given the expression of the variance of y_t in (3), it is possible to derive the

⁴Given that neither the kurtosis nor the autocorrelations of squares depend on the asymmetry parameter, the results for other values of δ are similar to the ones plotted in Figure 3.

⁵The expression of the covariance between returns and squared returns has been derived by Taylor (2005) in the short memory case. However, his value of $Cov(y_t, |y_{t+k}|^2)$ is twice the value obtained from (7) with $c = 2$ and $d = 0$.

following expression for the correlation between y_t and $|y_{t+k}|^c$

$$Corr(y_t, |y_{t+k}|^c) = \begin{cases} \frac{0.5\delta\sigma_\eta\lambda_k \exp\{0.25\sigma_h^2\rho_h(k)\}}{\exp\{\frac{\sigma_h^2}{8}\}\sqrt{\frac{\pi}{2}\exp\{0.25\sigma_h^2\}-1}}, & c = 1 \\ \frac{\delta\sigma_\eta\lambda_k \exp\{0.5\sigma_h^2\rho_h(k)\}}{\exp\{\frac{\sigma_h^2}{8}\}\sqrt{3\exp\{\sigma_h^2\}-1}}, & c = 2 \end{cases}. \quad (9)$$

From (9), it is clear that the correlations between y_t and $|y_{t+k}|^c$ have the same sign as δ and that their absolute values increase with the absolute value of δ . As an illustration, Figure 4 plots these correlations for the same four models considered above when $\delta = 0.2, 0.5$ and 0.8 . It is possible to observe that the correlations are only slightly larger between observations and future squares than between observations and future absolute values. It is also interesting to observe that the cross-correlations plotted in Figure 4 show the same pattern as the corresponding autocorrelations in the sense that they decay towards zero hyperbolically when the volatility has long memory while the decay is exponential in the short memory model. Finally, note that although the cross-correlations are typically small in the short-memory model, they can have rather large values in the presence of long-memory.

3 Properties of the FIEGARCH model

The FIEGARCH model was proposed by Bollerslev and Mikkelsen (1996). In its simplest form, the FIEGARCH(1,d,0) model is given by

$$\begin{aligned} y_t &= \sigma_t \varepsilon_t, \\ (1 - \phi L)(1 - L)^d \log \sigma_t^2 &= \omega + g(\varepsilon_{t-1}) \end{aligned} \quad (10)$$

where $g(\varepsilon_t) = \alpha \left(|\varepsilon_t| - \sqrt{\frac{2}{\pi}} \right) + \gamma \varepsilon_t$ and ε_t is a Gaussian white noise with variance 1. The parameter γ measures the leverage effect while, as before, d is the long-memory parameter. When $d = 0$, the short-memory EGARCH model of Nelson (1991) is obtained. Note that the main difference between the A-LMSV model and the FIEGARCH model is the way the noise is defined in the log-volatility equation; see Zaffaroni (2005) who proposes an exponential specification of the volatility that encompasses both models. The FIEGARCH model is stationary if $|\phi| < 1$ and $|d| < 0.5$. He *et al.* (2002) have derived the kurtosis and autocorrelations of absolute and squared observations for the short memory EGARCH model, i.e. model (10) with $d = 0$. Following their arguments, we have obtained the kurtosis and acf of $|y_t|^c$ for $c = 1$ and 2 in the long memory FIEGARCH model. In particular, if the stationarity conditions are satisfied,

the kurtosis is given by

$$k_y = 3 \frac{\prod_{j=1}^{\infty} E\{\exp[2\lambda_j g]\}}{\left\{ \prod_{j=1}^{\infty} E(\lambda_j g) \right\}^2}, \quad (11)$$

where for notational simplicity $g = g(\varepsilon_t)$ and λ_j is defined as in (5). Furthermore, if the errors are Normally distributed, the expectations involved in equation (11) can be evaluated using the following result due to Nelson (1991):

$$E[\exp(bg)] = \{\Phi(bc_1) \exp\{0.5b^2c_1^2\} + \Phi(bc_2) \exp\{0.5b^2c_2^2\}\} \exp\{-b\alpha(2/\pi)^{1/2}\}$$

where $c_1 = \alpha + \gamma$ and $c_2 = \alpha - \gamma$. Although it is not evident from expression (11), given the parameters α , ϕ and d , the kurtosis of y_t increases as the magnitude of the asymmetry parameter increases. This is an important difference with respect to the A-LMSV model in which we have seen that the kurtosis does not depend on the leverage effect.

On the other hand, the acf of $|y_t|^c$ in the FIEGARCH model is given by

$$\rho_c(k) = \frac{E[|\varepsilon_t|^c \exp(0.5c\lambda_k g)]}{\kappa_c \prod_{j=1}^{\infty} E[\exp(c\lambda_j g)] - P_3} \left[\left(\frac{\pi}{2}\right)^{\frac{2-c}{2}} P_1 P_2 - P_3 \right] \quad (12)$$

where $P_1 = \prod_{j=1}^k E[\exp(0.5c\lambda_j g)]$, $P_2 = \prod_{j=1}^{\infty} E[\exp(0.5c(\lambda_{j+1+k} + \lambda_j)) g]$ and $P_3 = \prod_{j=1}^{\infty} (E[\exp(0.5c\lambda_j g)])^2$ and λ_k is defined as in the A-LMSV model. Furthermore, He *et al.* (2002) show that

$$\begin{aligned} E[|\varepsilon_t|^c \exp(bg)] &= (2\pi)^{-1/2} \Gamma(c+1) \exp\{-b\alpha(2\pi)^{1/2}\} D_{(-(c+1))}(-b(\alpha+\gamma)) \\ &\quad + \exp\{-b^2\alpha\gamma\} D_{(-(c+1))}(-b(\alpha-\gamma)) \end{aligned}$$

where $D_{(q)}$ is the parabolic cylinder function and $\Phi(\cdot)$ is the standard normal cumulative function. Nelson (1991) derived this expression for squares, i.e. $c = 2$.

As an illustration, Figure 5 plots the acf of absolute and squared returns for the following FIEGARCH models: $\{\phi = 0, d = 0.4, \alpha = 0.2\}$, $\{\phi = 0.7, d = 0.4, \alpha = 0.1\}$, $\{\phi = 0.7, d = 0.4, \alpha = 0.2\}$ and $\{\phi = 0.7, d = 0, \alpha = 0.2\}$ with the asymmetry parameter $\gamma = 0, 0.2, 0.5, 0.8$. These models have been chosen to resemble the parameter values often estimated when the FIEGARCH model is fitted to real time series; see, for example, Bollerslev and Mikkelsen (1996, 1999). Given that the autocorrelations are the same regardless of the sign of the asymmetry parameter, we only represent the autocorrelations for positive values of γ . It is important to point out that this result is not only

satisfied by the autocorrelations of squares but also by the autocorrelations of absolute observations. This fact can explain the lack of identifiability of the sign of γ in the Whittle estimator observed by Zaffaroni (2005). Comparing the autocorrelations of the FIEGARCH model with the corresponding autocorrelations of the A-LMSV model plotted in Figure 1, we can observe that while in the latter model, the autocorrelations of squares are nearly the same regardless of the asymmetry parameter, they may be different in the FIEGARCH model. However, note that there is not a monotonous dependence of these autocorrelations with respect to the asymmetry parameter. In general, the autocorrelations of squares of order one increase with the magnitude of γ , although this is not always the case. On the other hand, for a particular model, the decay towards zero of the autocorrelations depends on the asymmetry parameter. Similar results can be observed when looking at the autocorrelations of absolute values. Although for each particular model considered, the order one autocorrelation is larger the larger is the magnitude of γ , the decay towards zero may be different and, therefore, for large lags, the autocorrelations can be smaller for larger values of γ . A similar result on the decay of the autocorrelations of squares depending on the asymmetry parameter in the context of short-memory EGARCH models were found by Carnero *et al.* (2004).

Figure 5 also suggests that the Taylor effect is reinforced by the leverage effect in the models with both parameters ϕ and d different from zero. However, it seems that the Taylor effect is not a property of the short memory model and of the model with $\phi = 0$. To have a clearer picture of this phenomena, Figure 6 plots the differences between the first order autocorrelations of absolute and squared returns for the same four FIEGARCH models described above. This figure confirms the suspicion that when $\phi = 0$ or $d = 0$ there is not Taylor effect. In the first case, the Taylor effect may appear if the asymmetry parameter is very large. Finally, in the models with both ϕ and d are different from zero, it is present even if γ is relatively small.

Finally, comparing Figure 6 with Figure 2 that plots the same differences as a function of the asymmetry parameter in A-LMSV models, we can observe that there are clear differences with respect to the Taylor effect in both models. We showed that in A-LMSV models, the Taylor effect is present as far as the asymmetry parameter is positive. Even more, when there is not long-memory, rather larger values of the asymmetry parameter can be allowed before the Taylor effect disappears. However, when looking at FIEGARCH models, the differences between the autocorrelations of absolute values and the autocorrelations of squares are not an increasing function of the asymmetry parameter. On the other hand, there are particular specifications of the FIEGARCH model in which the Taylor effect is not a property unless the magnitude of the asymmetry parameter is very large.

Given that it is rather difficult to find an analytical expression relating the kurtosis and the autocorrelation of order one of squared observations, Figure 7 plots this relationship for different FIEGARCH models. This figure illustrates that as the kurtosis increases (the asymmetry increases), the autocorrelation of order one of squares also increases. On the other hand, given ϕ , α and γ ,

larger values of the long-memory parameter, d , imply smaller autocorrelations of squares except when the kurtosis (the asymmetry) is very large. Carnero et al. (2004) have also shown for the short-memory EGARCH model that, given the kurtosis, the autocorrelation of order one of squared observations decreases with the persistence parameter, ϕ . On the other hand, for fixed ϕ , d and γ , the autocorrelations of squares are larger the larger is the ARCH effect, i.e. α . Once more, this relationship can be reverse for large values of γ . In any case, it is important to point out that the effect of d and α on $\rho_2(1)$ is relatively weak while the effect of the asymmetry parameter, γ , is rather strong.

Comparing now the results illustrated in Figure 7 with the corresponding results for the A-LMSV model illustrated in Figure 3, we can observe that in both models, the autocorrelations of order one of squares increase with the kurtosis. However, while in the A-LMSV model, the kurtosis is the same regardless the asymmetry parameter, in the FIEGARCH model, the kurtosis depends on the asymmetry. Therefore, given the parameters that measure the evolution of volatility, α in the FIEGARCH model and σ_η^2 in the A-LMSV models respectively, its persistence, ϕ , and its memory, d , the kurtosis and the autocorrelations of squares increase with the leverage effect in the FIEGARCH model while they are approximately constant in the A-LMSV model. On the other hand, for fixed α and σ_η^2 , the autocorrelations of squares increase with ϕ and d in the A-LMSV model while they decrease in the FIEGARCH model. In both models it is possible to observe that the variations in the values of $\rho_2(1)$ are small for moderate values of the kurtosis while they are large when the kurtosis is large.

As in the A-LMSV model, we have also derived the cross-correlations between returns and powers of absolute returns. In particular,

$$Corr(y_t, |y_{t+k}|^c) = \frac{E[\exp(0.5c\lambda_k g(\varepsilon_t))\varepsilon_t] \prod_{j=1}^{\infty} \exp[E[0.5c\lambda_j g]] \prod_{j=1}^{\infty} E[\exp(0.5c(\lambda_{j+k} + \lambda_j))g]}{\left[\kappa_c \prod_{j=1}^{\infty} E[\exp(c\lambda_j g)] - \left[\prod_{j=1}^{\infty} E[\exp(0.5c\lambda_j g)] \right]^{c/2} \right]^{1/2} \left[\prod_{j=1}^{\infty} E[\exp(\lambda_j g)] \right]^{1/2}} \quad (13)$$

Once more, we illustrate the shape of the cross-correlations by plotting them in Figure 8 for the same four models considered before and $c = 1, 2$. Given that these cross-correlations are symmetric with respect to the parameter γ , we have only plotted them for $\gamma = 0.2, 0.5$ and 0.8 . As in the A-LMSV model, we can observe that the cross-correlations between y_{t+k}^2 and y_t are larger in magnitude than between $|y_{t+k}|$ and y_t . Furthermore, comparing these cross-correlations with the ones plotted in Figure 4 for the A-LMSV models, it is possible to observe that, in general, the FIEGARCH model generates cross-correlations similar to those of the A-LMSV model.

It is important to note that the expressions of the FIEGARCH model derived in this paper have been obtained under the assumption of Normal errors. The expressions for other distributions could be more complicated. However, generalizing the expressions of the moments in the context of non-Gaussian A-LMSV

models is rather straightforward.

4 Empirical illustration

In this section we evaluate the performance of the A-LMSV and FIEGARCH models in capturing the empirical features of financial data. For this purpose, we analyze daily close prices of the S&P 500 and DAX composite indexes observed from January 3, 1928 to April 20, 2005 and November 26, 1990 to September 6, 2006 respectively. The sample sizes are 19104 and 3980 observations in each case. The series of prices denoted by p_t have been plotted in Figures 9 and 10 respectively together with the corresponding series of returns computed as usual as $y_t = 100(\log p_t - \log p_{t-1})$.

We analyze first the series of S&P500 returns plotted in Figure 9. These returns, denoted by y_t , have a kurtosis of 3.083 and show volatility clustering⁶. The presence of conditional heteroscedasticity can also be observed in the correlograms of the absolute and squared observations also plotted in Figure 9. In both correlograms, the sample autocorrelations are significant and very persistent. The observed decay of the correlations is not compatible with the expected decay if squared and absolute returns could be represented by a short-memory model. In consequence, it seems that the volatility of the S&P500 returns should be approximated by a conditional heteroscedasticity model with long memory. Figure 9 also plots the $Corr(|y_{t+k}|, y_t)$ and $Corr(y_{t+k}^2, y_t)$. In both cases, the cross-correlations are similar in magnitude, being negative and significant. Therefore, it seems that a model with leverage effect could be adequate. We fit the A-LMSV and the FIEGARCH models to the series of S&P500 returns.

The A-LMSV model has been estimated by the Whittle estimator proposed by Zaffaroni (2005) who establishes its asymptotic normality. The scale parameter σ_* is not identified by the Whittle estimator and, consequently, following the suggestion of Zaffaroni (2005) we estimate it by $\hat{\sigma}_* = \exp\{0.5(\hat{\mu} + E(\log \varepsilon_t^2))\}$ where $\hat{\mu}$ is the sample mean of $\log(y_t^2)$ and, assuming Normality of the errors, $E(\log \varepsilon_t^2) = -1.27$. The estimated model is given by

$$(1 - 0.400L)(1 - L)^{0.800} \log \sigma_t^2 = \eta_t \quad (14)$$

where the scale parameter is estimated as $\hat{\sigma}_* = 0.102$ and the variance of the log-volatility noise is estimated as $\hat{\sigma}_\eta^2 = 0.010$ and the correlation between the level and volatility noises is estimated as $\hat{\delta} = -0.700$.

⁶To avoid the pernicious effect of large extreme observations on the sample moments and estimates of the parameters that govern the volatility, all the observations larger than 7 conditional standard deviations have been corrected by substituting them by their estimated conditional standard deviation; see Carnero *et al.* (2006) for the effects of outliers on the identification and estimation of conditional heteroscedasticity. In particular, the observations corrected correspond to 26th June 1950, 26th September 1955, 19th October 1987 and 13th October 1989.

The FIGARCH model has been estimated by QML using the G@RCH package version 4.0 of Laurent and Peters (2005) with the following results

$$(1 - 0.469L)(1 - L)^{\frac{0.581}{(0.083)}} \log \sigma_t^2 = \underset{(0.147)}{-0.893} + \underset{(0.019)}{0.185} \left(|\varepsilon_{t-1}| - \sqrt{\frac{2}{\pi}} \right) - \underset{(0.012)}{0.099} \varepsilon_{t-1} \quad (15)$$

where the quantities in parenthesis are estimated standard deviations. Note that both models have estimates of the long-memory parameter larger than 0.5 implying that the series of S&P500 returns is not stationary. This could be due to the fact that the series has been observed over more than 77 years which is a too long period for assuming stationarity. In these circumstances, the moments implied by the estimated models are not defined. Therefore, it is not possible to compare the sample moments of the S&P500 returns with the moments implied by each of the estimated models.

We consider now, the DAX returns plotted in Figure 10. This series has a kurtosis of 7.365 and also shows clear signs of conditional heteroscedasticity when looking at the correlations of squares and absolute returns plotted in Figure 10. Both correlograms show significant positive correlations which decay slowly towards zero. Finally, the cross-correlations between y_t and y_{t+k}^2 and between y_t and $|y_{t+k}|$ are negative and significant. Therefore, it seems that the dynamic evolution of the DAX returns can be also represented by a conditionally heteroscedastic model with long memory and asymmetry. As before, we fit the A-LMSV and FIGARCH models. The estimated A-LMSV model is given by

$$(1 - 0.900L)(1 - L)^{0.399} \log \sigma_t^2 = \eta_t \quad (16)$$

where the scale parameter is estimated as $\hat{\sigma}_* = 0.317$ and the variance of the log-volatility noise is estimated as $\hat{\sigma}_\eta^2 = 0.010$ and the correlation between the level and volatility noises is estimated as $\hat{\delta} = -0.800$.

The estimated FIGARCH model is given by

$$(1 - 0.789L)(1 - L)^{\frac{0.455}{(0.054)}} \log \sigma_t^2 = \underset{(0.308)}{1.022} + \underset{(0.023)}{0.114} \left(|\varepsilon_{t-1}| - \sqrt{\frac{2}{\pi}} \right) - \underset{(0.012)}{0.050} \varepsilon_{t-1} \quad (17)$$

We now analyze which model implies moments closer to the observed moments of DAX returns. With respect to the kurtosis, the observed kurtosis is 7.365 while the kurtosis implied by the estimated A-LMSV and FIGARCH models are 7.876 and 3.790 respectively. Therefore, the kurtosis of the A-LMSV is clearer closer to the observed kurtosis. Looking at the autocorrelations of absolute and squared returns implied by each of the two models, which have been also plotted in Figure 10 together with the corresponding estimated correlations, it is possible to observe that the correlations of order one implied by the A-LMSV model, 0.183 and 0.147 for absolute and squared returns respectively, are closer to the estimated correlations, 0.216 and 0.183, than the corresponding autocorrelations implied by the FIGARCH model, 0.115 and 0.123. However,

although the decay of the correlations implied by the FIEGARCH model is closer to the one observed in reality, none of the models explain well the slow decay of the autocorrelations towards zero. Finally, it seems that the cross-correlations between y_t and $|y_{t+k}|$ implied by both models and represented in Figure 10, are very similar between them and similar to the correlations truly observed in the DAX returns. However, comparing the implied cross-correlations between y_t and y_{t+k}^2 of the A-LMSV and FIEGARCH models, it is possible to observe that the former are clearly closer to the ones estimated for the original DAX returns.

5 Conclusions

In this paper, we propose an extension of the LMSV to represent the asymmetric response of volatility to positive and negative returns. We compare the statistical properties of the new model with the FIEGARCH model. As a by-product, we derive expressions of the autocorrelations of squares and absolute returns as well as cross-correlations between these transformations of returns and the original returns when they are generated by the FIEGARCH model.

We show that the A-LMSV model reproduces better the empirical features of financial data: volatility persistence, excess kurtosis, autocorrelations of absolute and squared returns and cross-correlations between returns and future squared returns. In fact, the FIEGARCH model needs simultaneously high values of d and ϕ , close to the level of nonstationary, and small values of δ in order to capture at the same time persistence and small first order autocorrelation. On the other hand, this last requirement interferes with generating large kurtosis. The Gaussian FIEGARCH model is only able to reproduce high kurtosis if δ is not close to zero. Therefore, it seems to exist, for this model, a trade off among different moments. Contrarily, the A-LMSV model is able to reproduce these three features of financial data for larger combinations of the parameters.

In an empirical application to daily S&P500 and DAX returns, we show that when both models are fitted to real data, the conclusions in terms of the stationarity of the volatility are similar. When one of the models imply stationarity, the other does and the other way round. If the estimated parameters of both models satisfy the stationarity conditions, then the kurtosis and autocorrelations of order one of absolute and squared returns implied by the A-LMSV model are closer to the estimated sample moments of the real data than the ones implied by the FIEGARCH model. However, in our empirical application to the DAX returns none of the implied autocorrelations explain completely the slow decay of the autocorrelations towards zero. Consequently, and given that, as we have seen in this paper, dealing with the statistical properties of the A-LMSV model is easier than when considering the properties of the FIEGARCH model, we think that the former is a model to be considered when modelling the dynamic evolution of the volatility of series with long-memory and asymmetric response to positive and negative returns.

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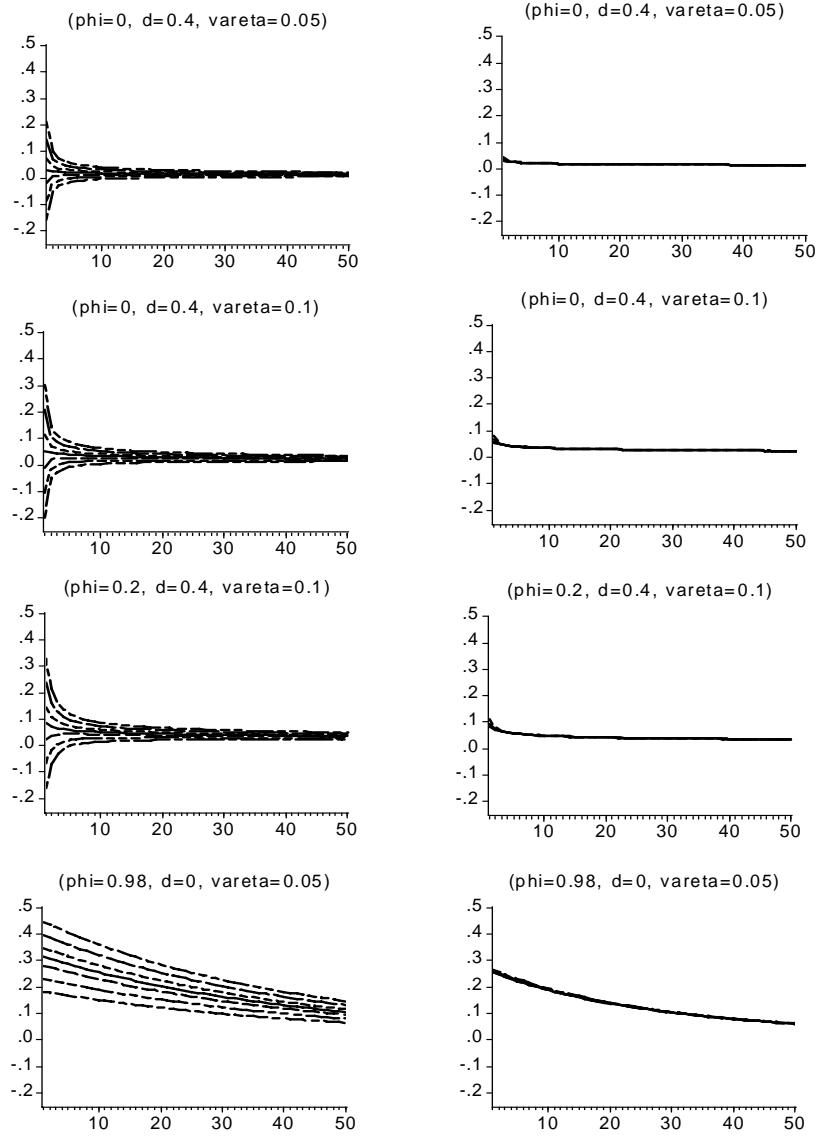


Figure 1. Autocorrelations of $|y_t|$ (left column) and y_t^2 (right column) in four A-LMSV models with different values of the asymmetric parameter: continuous line ($\delta = 0$), dotted ($\delta = 0.2$), discontinuous ($\delta = 0.5$), dotted dotted discontinuous ($\delta = 0.8$), larger dotted ($\delta = -0.2$), dotted dotted dotted discontinuous ($\delta = -0.5$) and dotted discontinuous ($\delta = -0.8$).

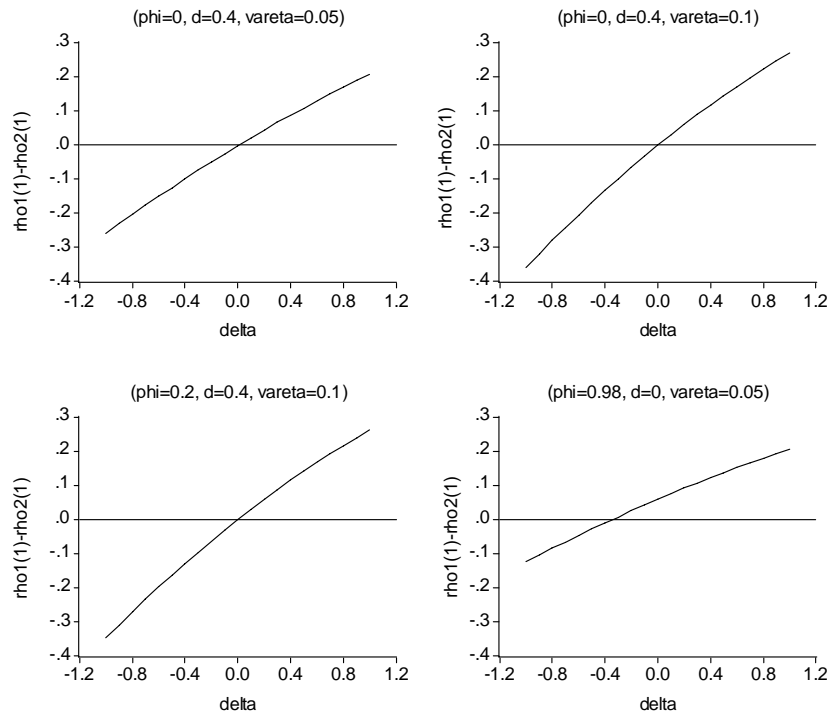


Figure 2. Differences between the autocorrelations of order 1 of absolute and squared returns as a function of the correlation between ε_t and η_{t+1} in four A-LMSV models.

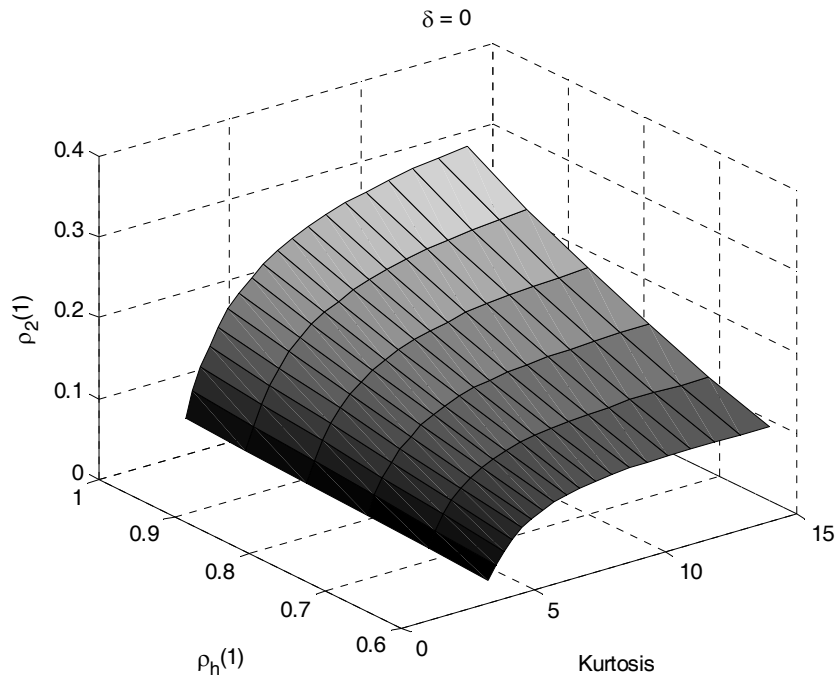


Figure 3. Relationship between kurtosis, first-order autocorrelation of squared observations, $\rho_2(1)$ and the first order autocorrelation of the underlying volatility $\rho_h(1)$, for A-LMSV models with $\delta = 0$.

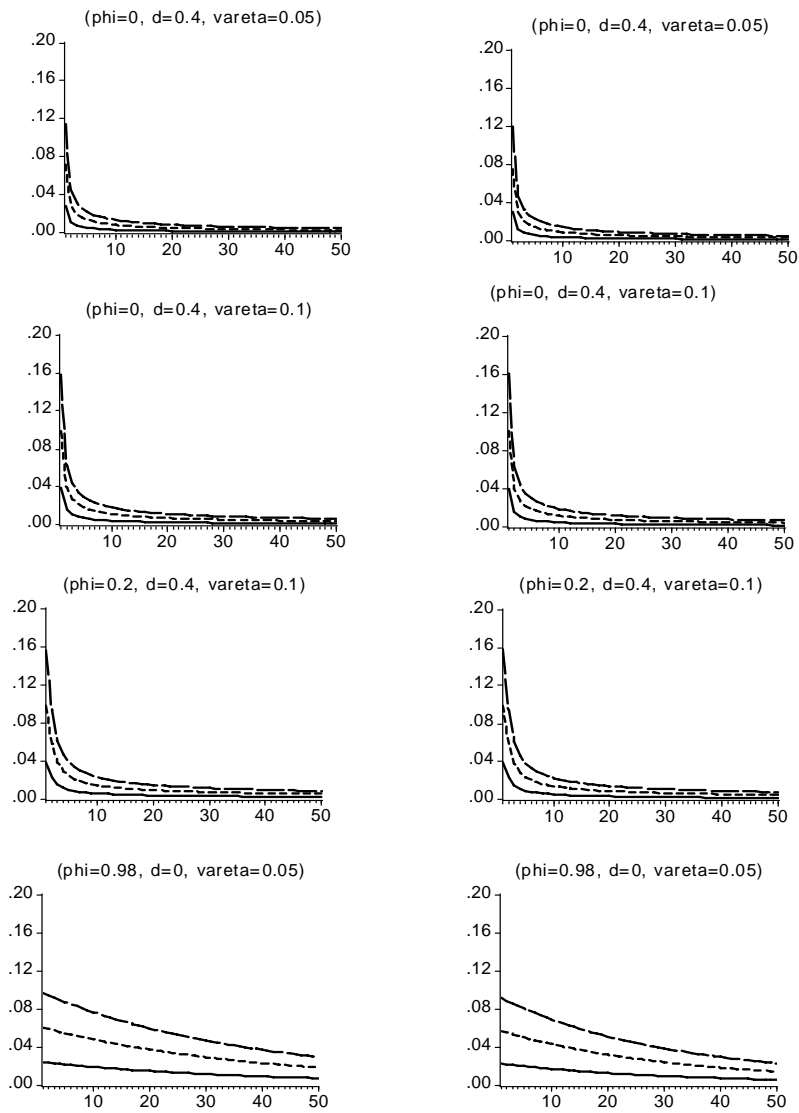


Figure 4. Correlations between y_t and $|y_{t+k}|$ (left column) and y_t and y_{t+k}^2 (right column) in four A-LMSV models with different values of δ : continuous line ($\delta = 0.2$), dotted ($\delta = 0.5$) and discontinuous ($\delta = 0.8$).

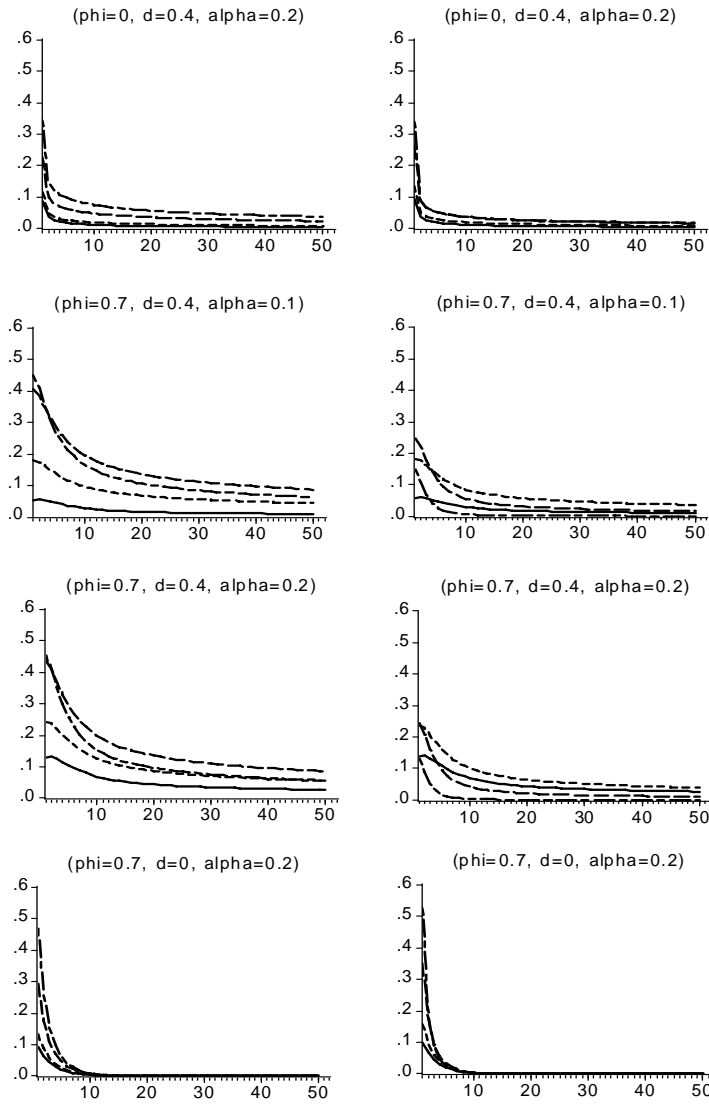


Figure 5. Autocorrelations of $|y_t|$ (left column) and y_t^2 (right column) in four FIEGARCH models with different values of γ : continuous line ($\gamma = 0$), dotted ($\gamma = 0.2$), discontinuous ($\gamma = 0.5$) and dotted discontinuous ($\gamma = 0.8$).

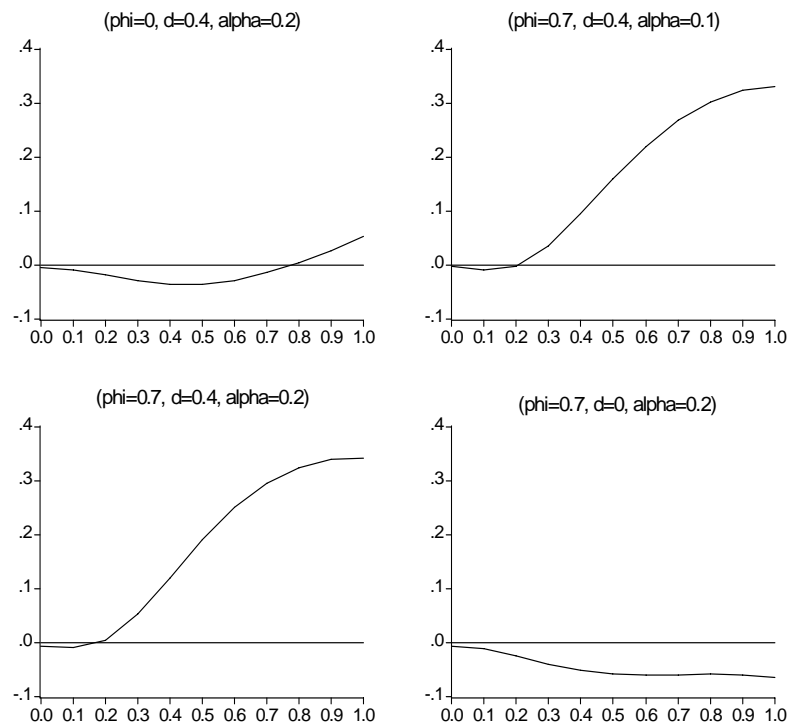


Figure 6. Differences between the autocorrelations of order 1 of absolute and squared returns as a function of γ in FIEGARCH models.

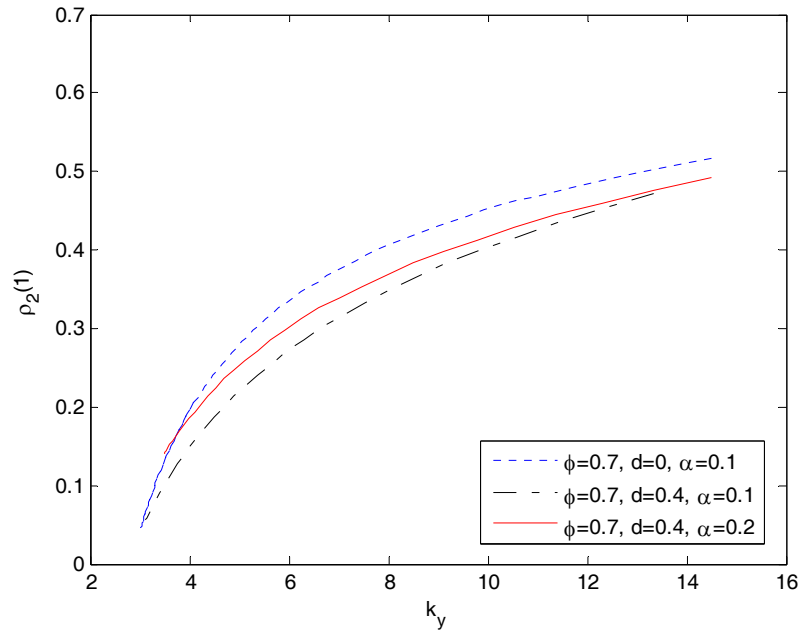


Figure 7. Relationship between kurtosis and first-order autocorrelation of squared observations for different values of the persistence for FIE-GARCH(1,d,0) models.

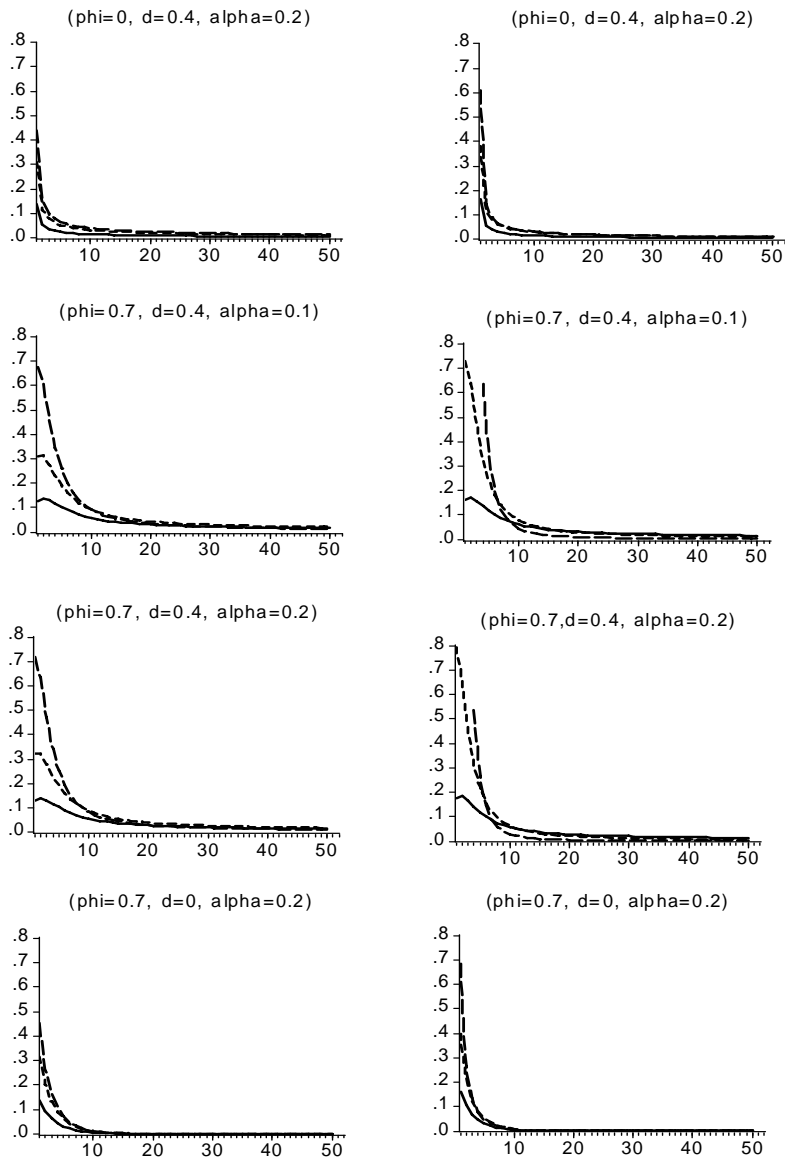


Figure 8. Correlations between y_t and $|y_{t+k}|$ (left column) and y_t and y_{t+k}^2 (right column) in four FIEGARCH models with different values of γ : continuous line ($\delta = 0.2$), dotted ($\delta = 0.5$) and discontinuous ($\delta = 0.8$).

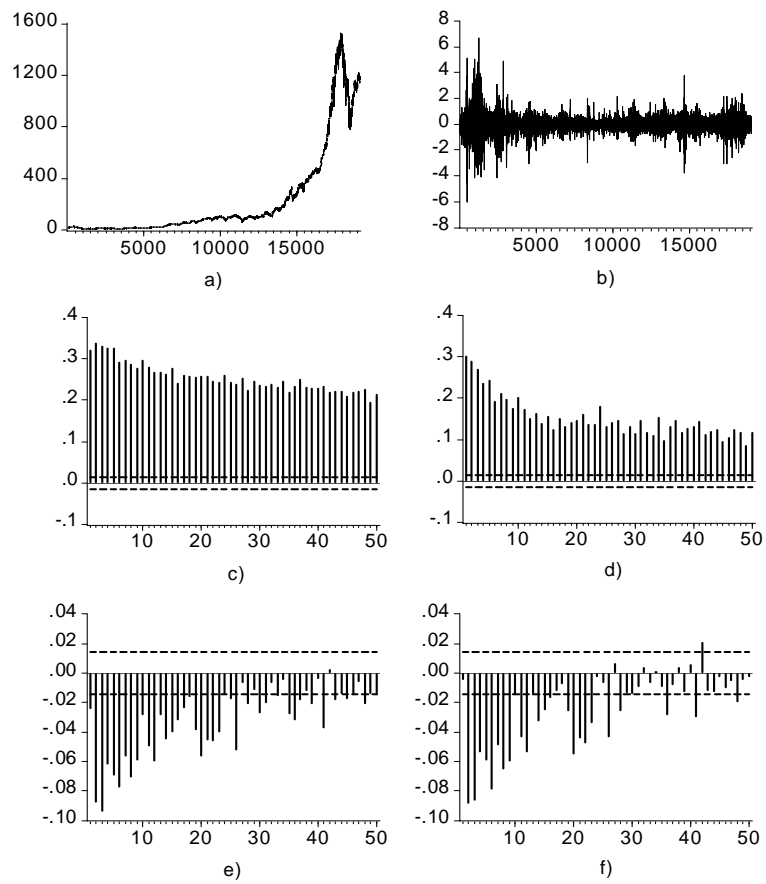


Figure 9. Observations of daily S&P500 prices (a) and returns (b) together with sample autocorrelations of absolute returns (c) and squared returns (d) and cross-correlations between y_t and $|y_{t+k}|$ (e) and between y_t and y_{t+k}^2 (f).

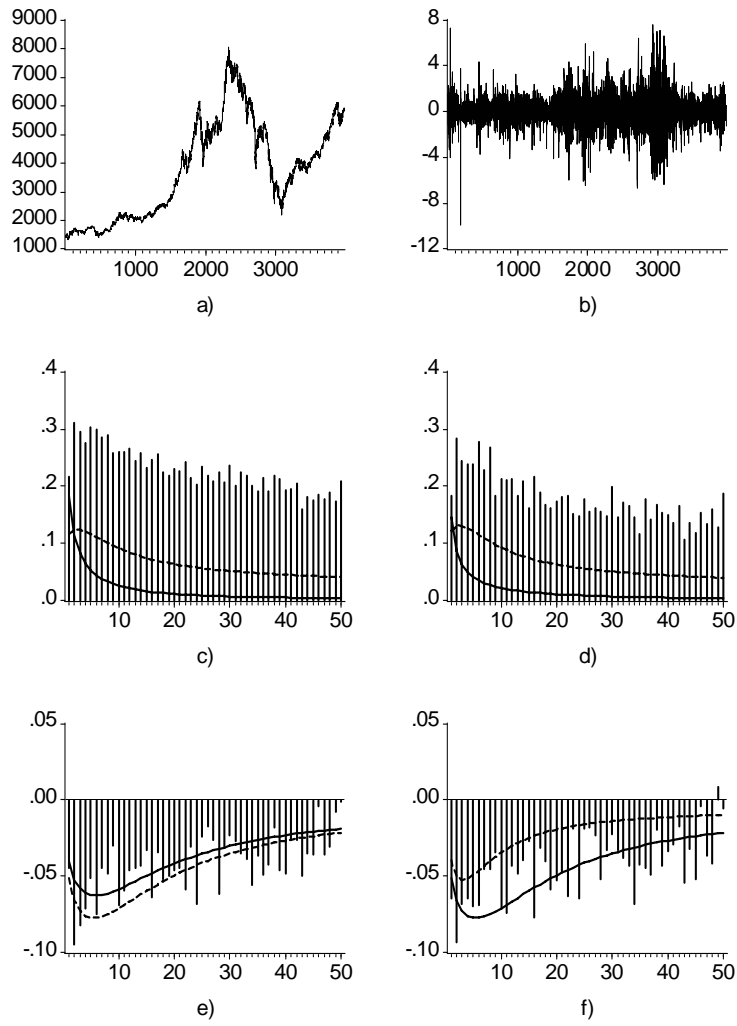


Figure 10. Observations of daily DAX prices (a) and returns (b) together with sample autocorrelations of absolute returns (c) and squared returns (d) and cross-correlations between y_t and $|y_t|$ (e) and between y_t and y_t^2 (f).